FEM FOR HEAT TRANSFER PROBLEMS

Field problems

- General form of system equations of *2D* linear steady state field problems:

$$D_x \frac{\partial^2 \phi}{\partial x^2} + D_y \frac{\partial^2 \phi}{\partial y^2} - g\phi + Q = 0$$

(*Helmholtz equation*)

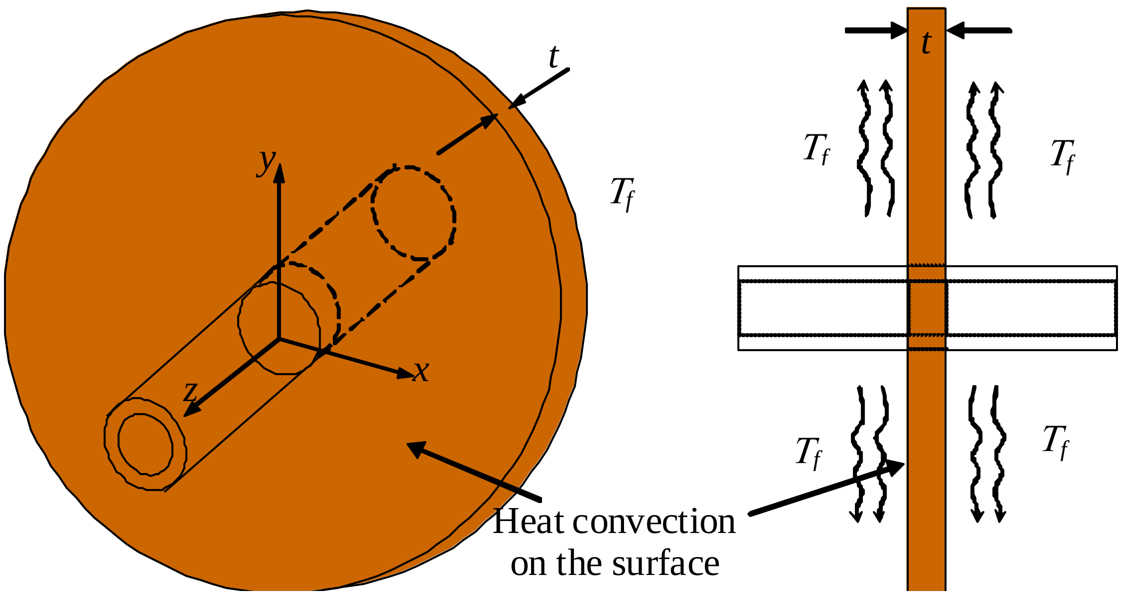
- For *1D* problems:

$$D \frac{d^2 \phi}{dx^2} - g\phi + Q = 0$$

Field problems

- Heat transfer in *2D* fin

$$\underbrace{-\left(D_x \frac{\partial^2 T}{\partial x^2} + D_y \frac{\partial^2 T}{\partial y^2}\right)}_{\text{Heat conduction}} + \underbrace{\left(\frac{2h}{t} T - \frac{2hT_f}{t}\right)}_{\text{Heat convectction}} = \underbrace{q}_{\text{Heat supply}}$$



Note:

$$g = \frac{2h}{t}, \quad Q = q + \frac{2hT_f}{t}$$

Field problems

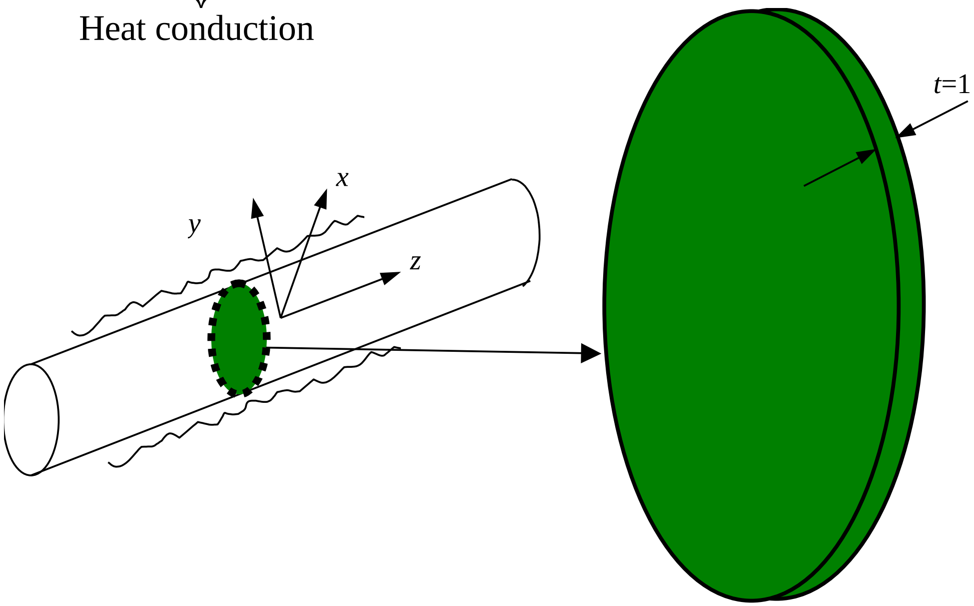
- Heat transfer in long 2D body

$$\underbrace{k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2}}_{\text{Heat conduction}} + \underbrace{q}_{\text{Heat supply}} = 0$$

Note:

$D_x = k_x$, $D_y = k_y$,

$g = 0$ and $Q = q$



Representative plane

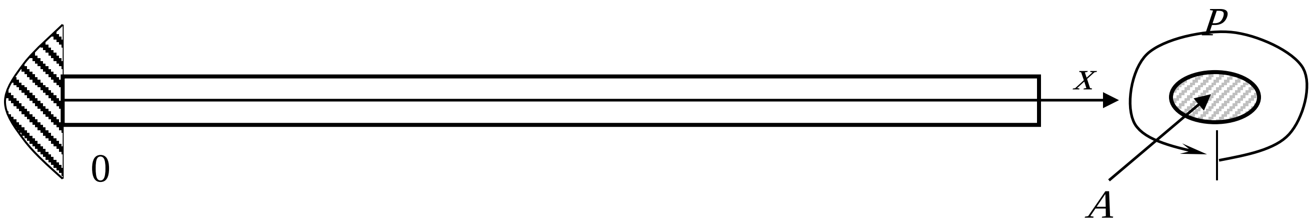
Field problems

- Heat transfer in *1D* fin

$$\underbrace{kA\frac{d^2T}{dx^2}}_{\text{Heat conduction}} \underbrace{-hPT + hPT_f}_{\text{Heat convectoin}} + \underbrace{q}_{\text{Heat supply}} = 0$$

Note:

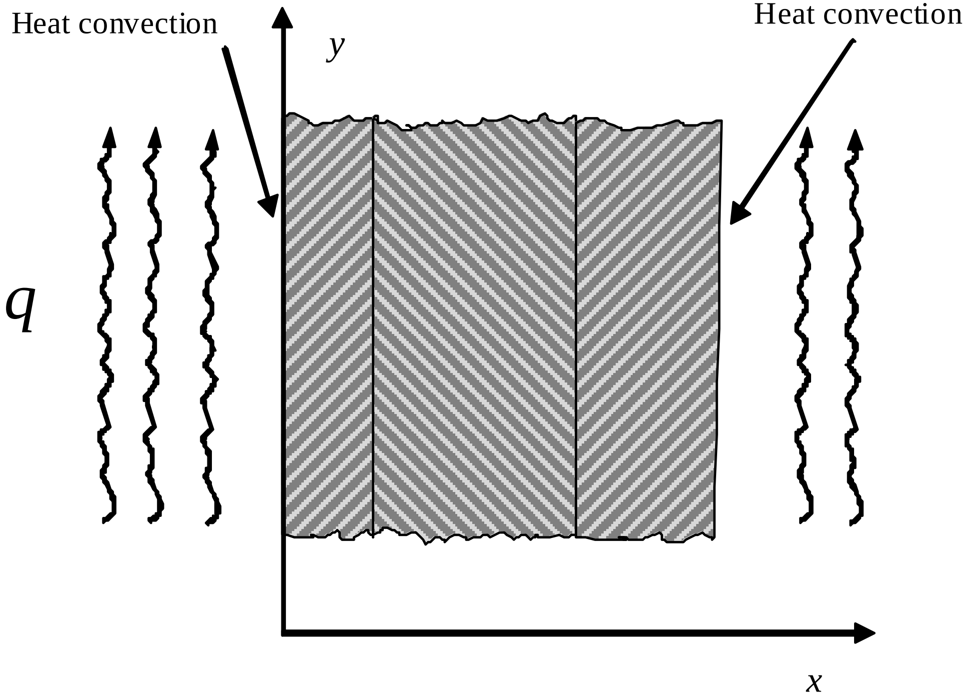
$$D = kA, \quad g = hP, \quad Q = q + hPT_f$$



Field problems

- Heat transfer across composite wall

$$\underbrace{kA\frac{d^2T}{dx^2}}_{\text{Heat conduction}} + \underbrace{q}_{\text{Heat supply}} = 0$$



Note:

$$D = kA, \quad g = 0, \quad Q = q$$

Field problems

- Torsional deformation of bar

$$\frac{1}{G}\frac{\partial^2\phi}{\partial x^2}+\frac{1}{G}\frac{\partial^2\phi}{\partial y^2}+2\theta=0$$

(ϕ - stress function)

Note:

$D_x=1/G,\quad D_y=1/G,\quad g=0,\quad Q=2\theta$

- Ideal irrotational fluid flow

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

Note:

$D_x=D_y=1,\ g=Q=0$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

(ψ - streamline function and
ϕ - potential function)

Field problems

- Accoustic problems

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{w^2}{c^2} P = 0$$

Note:

$$g = -\frac{w^2}{c^2} \ , D_x = D_y = 1, \ Q = 0$$

P - the pressure above the ambient pressure ;

w - wave frequency ;

c - wave velocity in the medium

Weighted Residual Approach For FEM

- Establishing FE equations based on governing equations without knowing the functional.

$$D_x \frac{\partial^2 \phi}{\partial x^2} + D_y \frac{\partial^2 \phi}{\partial y^2} - g\phi + Q = 0 \quad \longrightarrow \quad f(\phi(x,y)) = 0$$

(*Strong form*)

Approximate solution:

$$\int_A wf(\phi(x,y))\mathrm{d}x\mathrm{d}y = 0$$

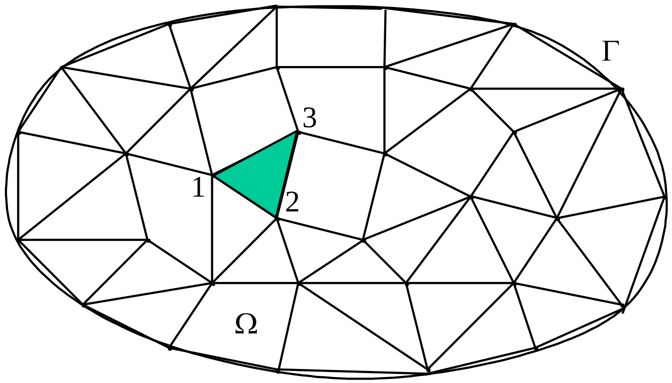
(*Weak form*)

Weight function

Weighted Residual Approach For FEM

- Discretize into smaller elements to ensure better approximation
- In each element, $\phi(x,y) = \mathbf{N}(x,y)\mathbf{\Phi}^{(e)}$

where $\mathbf{N} = \begin{bmatrix} N_1 & N_1 & \cdots & N_{n_d} \end{bmatrix}$



- Using **N** as the weight functions

⟶ Galerkin method

$$\mathbf{R}^{(e)} = \int_{A_e} \mathbf{N}^T f(\phi(x,y))\mathrm{d}x\mathrm{d}y$$

Residuals are then assembled for all elements and enforced to zero.

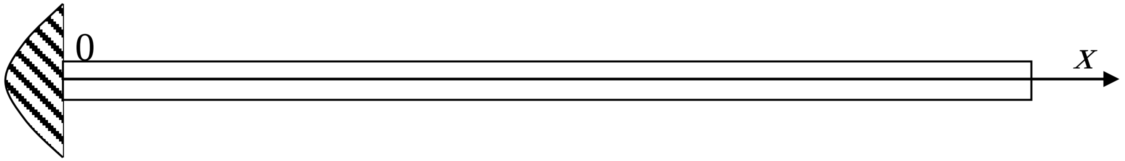
1D fin

$$\underbrace{kA\frac{d^2T}{dx^2}}_{\text{Heat conduction}} \underbrace{-hPT+hPT_f}_{\text{Heat convectoin}} + \underbrace{q}_{\text{Heat supply}} = 0$$

k : thermal conductivity
h : convection coefficient
A : cross-sectional area of the fin
P : perimeter of the fin
T : temperature, and
T_f : ambient temperature in the fluid

$$T(0)=T_0$$

(Specified boundary condition)



$$-kA\frac{dT}{dx}=hA\left(T_b-T_f\right)$$

(Convective heat loss at free end)

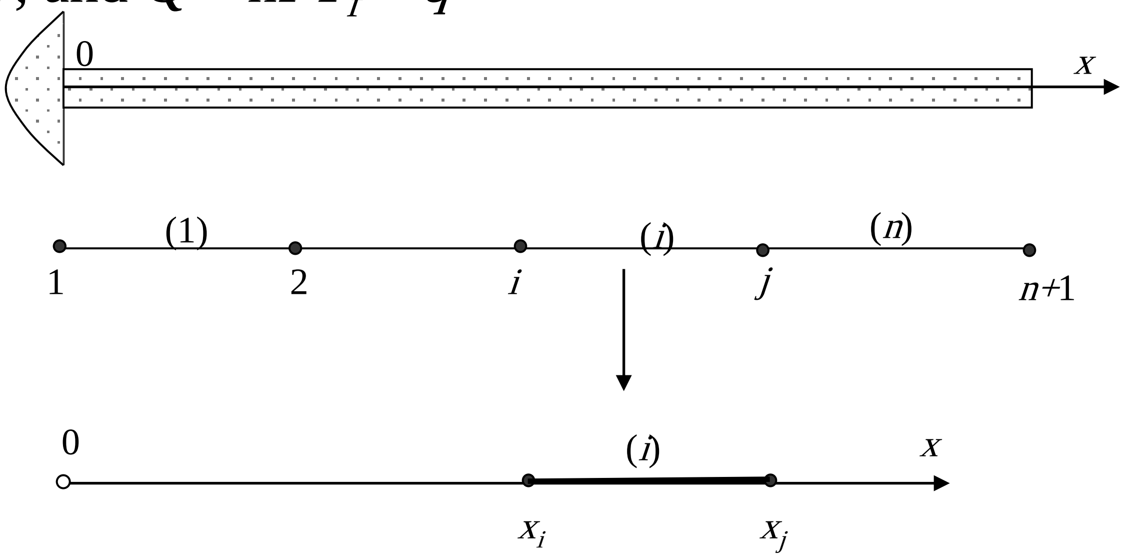
1D fin

Using Galerkin approach,

$$\mathbf{R}^{(e)} = -\int_{x_i}^{x_j} \mathbf{N}^T \left(D\frac{d^2T}{dx^2} - gT + Q \right) \mathrm{d}x$$

$$= -\int_{x_i}^{x_j} \mathbf{N}^T \left(D\frac{d^2T}{dx^2} + Q \right) dx + \int_{x_i}^{x_j} g\mathbf{N}^T T dx$$

where; $D = kA$, $g = hP$, and $Q = hPT_f + q$



1D fin

Integration by parts of first term on right-hand side,

$$\mathbf{R}^{(e)} = -\mathbf{N}^T D \frac{dT}{dx}\bigg|_{x_i}^{x_j} + \int_{x_i}^{x_j} \frac{d\mathbf{N}^T}{dx} D \frac{dT}{dx} \mathrm{d}x - \int_{x_i}^{x_j} Q\mathbf{N}^T \mathrm{d}x + \int_{x_i}^{x_j} g\mathbf{N}^T T \mathrm{d}x$$

Using $T(x) = \mathbf{N}(x)\mathbf{T}^{(e)}$

$$\mathbf{R}^{(e)} = \underbrace{-\mathbf{N}^T D \frac{\mathrm{d}T}{\mathrm{d}x}\bigg|_{x_i}^{x_j}}_{\mathbf{b}^{(e)}} + \underbrace{\left(\int_{x_i}^{x_j} \frac{\mathrm{d}\mathbf{N}^T}{\mathrm{d}x} D \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x} \mathrm{d}x\right)}_{\mathbf{k}_D^{(e)}} \mathbf{T}^{(e)}$$

$$-\underbrace{\left(\int_{x_i}^{x_j} Q\mathbf{N}^T \mathrm{d}x\right)}_{\mathbf{f}_Q^{(e)}} + \underbrace{\left(\int_{x_i}^{x_j} g\mathbf{N}^T \mathbf{N} \mathrm{d}x\right)}_{\mathbf{k}_g^{(e)}} \mathbf{T}^{(e)}$$

1D fin

$$\mathbf{R}^{(e)} = \mathbf{b}^{(e)} + [\mathbf{k}_D^{(e)} + \mathbf{k}_g^{(e)}]\mathbf{T}^{(e)} - \mathbf{f}_Q^{(e)}$$

where

$$\mathbf{B} = \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x} \quad \text{(Strain matrix)}$$

$$\mathbf{k}_D^{(e)} = \int_{x_i}^{x_j} \frac{\mathrm{d}\mathbf{N}^T}{\mathrm{d}x} D \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}\mathrm{d}x = \int_{x_i}^{x_j} \mathbf{B}^T D\mathbf{B}\mathrm{d}x \quad \text{(Thermal conduction)}$$

$$\mathbf{k}_g^{(e)} = \int_{x_i}^{x_j} g\mathbf{N}^T\mathbf{N}\mathrm{d}x \quad \text{(Thermal convection)}$$

$$\mathbf{f}_Q^{(e)} = \int_{x_i}^{x_j} Q\mathbf{N}^T\mathrm{d}x \quad \text{(External heat supplied)}$$

$$\mathbf{b}^{(e)} = -\mathbf{N}^T D \frac{\mathrm{d}T}{\mathrm{d}x}\bigg|_{x_i}^{x_j}$$

(Temperature gradient at two ends of element)

1D fin

For linear elements,

$$\mathbf{N}(x) = \begin{bmatrix} N_i & N_j \end{bmatrix} = \begin{bmatrix} \dfrac{x_j - x}{l} & \dfrac{x - x_i}{l} \end{bmatrix}$$

(Recall *1D* truss element)

$$\mathbf{B} = \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} \dfrac{x_j - x}{l} & \dfrac{x - x_i}{l} \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{l} & \dfrac{1}{l} \end{bmatrix}$$

Therefore,

$$\mathbf{k}_D^{(e)} = \int_{x_i}^{x_j} \begin{bmatrix} \dfrac{-1}{l} \\ \dfrac{1}{l} \end{bmatrix} D \begin{bmatrix} \dfrac{-1}{l} & \dfrac{1}{l} \end{bmatrix} \mathrm{d}x = \frac{kA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\dfrac{AE}{l}$ for truss element

(Recall stiffness matrix of truss element)

1D fin

$A\rho l$ for truss element

$$\mathbf{k}_g^{(e)} = \int_{x_i}^{x_j} g \begin{bmatrix} \dfrac{x_j - x}{l} \\ \dfrac{x - x_i}{l} \end{bmatrix} \begin{bmatrix} \dfrac{x_j - x}{l} & \dfrac{x - x_i}{l} \end{bmatrix} \mathrm{d}x = \frac{hPl}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(Recall mass matrix of truss element)

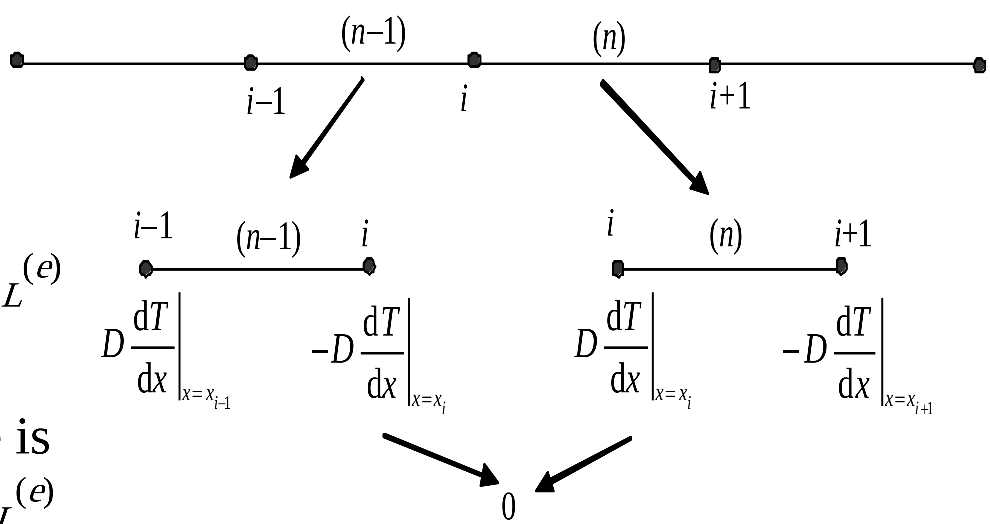
$$\mathbf{f}_Q^{(e)} = \int_{x_i}^{x_j} Q \begin{bmatrix} \dfrac{x_j - x}{l} \\ \dfrac{x - x_i}{l} \end{bmatrix} \mathrm{d}x = \frac{Ql}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = (\underbrace{q}_{\text{Heat supply}} + \underbrace{hPT_f}_{\text{Heat convection}}) \frac{l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

1D fin

$$\mathbf{b}^{(e)} = -\mathbf{N}^T D \frac{\mathrm{d}T}{\mathrm{d}x}\bigg|_{x_i}^{x_j} = \begin{Bmatrix} kA\dfrac{\mathrm{d}T}{\mathrm{d}x}\bigg|_{x=x_i} \\ -kA\dfrac{\mathrm{d}T}{\mathrm{d}x}\bigg|_{x=x_j} \end{Bmatrix} = \underbrace{\begin{Bmatrix} kA\dfrac{\mathrm{d}T}{\mathrm{d}x}\bigg|_{x=x_i} \\ 0 \end{Bmatrix}}_{\mathbf{b}_L^{(e)}} + \underbrace{\begin{Bmatrix} 0 \\ -kA\dfrac{\mathrm{d}T}{\mathrm{d}x}\bigg|_{x=x_j} \end{Bmatrix}}_{\mathbf{b}_R^{(e)}}$$

or $\mathbf{b}^{(e)} = \mathbf{b}_L^{(e)} + \mathbf{b}_R^{(e)}$

(Left end) (Right end)



At the internal nodes of the fin, $\mathbf{b}_L^{(e)}$ and $\mathbf{b}_R^{(e)}$ vanish upon assembly.

At boundaries, where temperature is prescribed, no need to calculate $\mathbf{b}_L^{(e)}$ or $\mathbf{b}_R^{(e)}$ first.

1D fin

When there is heat convection at boundary,

$$-kA\frac{dT}{dx} = hA\left(T_b - T_f\right)$$

e.g. $$\mathbf{b}_R^{(e)} = \begin{Bmatrix} 0 \\ hA\left(T_b - T_f\right) \end{Bmatrix} = \begin{Bmatrix} 0 \\ hAT_j \end{Bmatrix} - \begin{Bmatrix} 0 \\ hAT_f \end{Bmatrix}$$

Since T_b is the temperature of the fin at the boundary point, $T_b = T_j$

Therefore, $$\mathbf{b}_R^{(e)} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & hA \end{bmatrix}}_{\mathbf{k}_M^{(e)}} \begin{Bmatrix} T_i \\ T_j \end{Bmatrix} - \underbrace{\begin{Bmatrix} 0 \\ hAT_f \end{Bmatrix}}_{\mathbf{f}_S^{(e)}}$$

1D fin

$$\mathbf{b}_R^{(e)} = \mathbf{k}_M^{(e)}\mathbf{T}^{(e)} - \mathbf{f}_s^{(e)}$$

where $\mathbf{k}_M^{(e)} = \begin{bmatrix} 0 & 0 \\ 0 & hA \end{bmatrix}$, $\mathbf{f}_s^{(e)} = \begin{Bmatrix} 0 \\ hAT_f \end{Bmatrix}$

For convection on left side,

$$\mathbf{b}_L^{(e)} = \mathbf{k}_M^{(e)}\mathbf{T}^{(e)} - \mathbf{f}_s^{(e)}$$

where $\mathbf{k}_M^{(e)} = \begin{bmatrix} hA & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{f}_s^{(e)} = \begin{Bmatrix} hAT_f \\ 0 \end{Bmatrix}$

1D fin

Therefore,

$$\mathbf{R}^{(e)} = \underbrace{[\mathbf{k}_D^{(e)} + \mathbf{k}_g^{(e)} + \mathbf{k}_M^{(e)}]}_{\mathbf{k}^{(e)}}\mathbf{T}^{(e)} - \underbrace{\left\{\mathbf{f}_Q^{(e)} + \mathbf{f}_S^{(e)}\right\}}_{\mathbf{f}^{(e)}}$$

$$\mathbf{R}^{(e)} = \mathbf{k}^{(e)}\mathbf{T}^{(e)} - \mathbf{f}^{(e)}$$

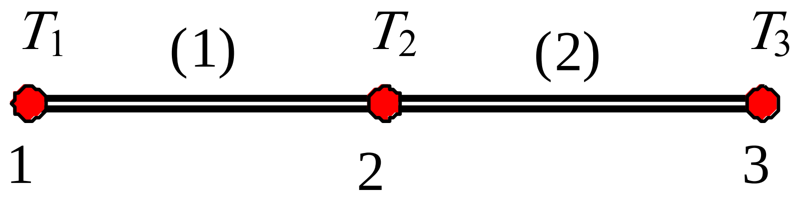
Residuals are assembled for all elements and enforced to zero: $\underbrace{\mathbf{Ku} = \mathbf{F}}$

same form for static mechanics problem

1D fin

- *Direct assembly procedure*

$$\mathbf{R}^{(e)} = \mathbf{k}^{(e)}\mathbf{T}^{(e)} - \mathbf{f}^{(e)}$$



or

$$\begin{Bmatrix} R_1^{(e)} \\ R_2^{(e)} \end{Bmatrix} = \begin{bmatrix} k_{11}^{(e)} & k_{12}^{(e)} \\ k_{21}^{(e)} & k_{22}^{(e)} \end{bmatrix} \begin{Bmatrix} T_1^{(e)} \\ T_2^{(e)} \end{Bmatrix} - \begin{Bmatrix} f_1^{(e)} \\ f_2^{(e)} \end{Bmatrix}$$

Element *1*:

$$R_1^{(1)} = k_{11}^{(1)} T_1 + k_{12}^{(1)} T_2 - f_1^{(1)}$$

$$R_2^{(1)} = k_{21}^{(1)} T_1 + k_{22}^{(1)} T_2 - f_2^{(1)}$$

1D fin

- *Direct assembly procedure* (Cont'd)

Element 2:

$$R_1^{(2)} = k_{11}^{(2)} T_2 + k_{12}^{(2)} T_3 - f_1^{(2)}$$

$$R_2^{(2)} = k_{21}^{(2)} T_2 + k_{22}^{(2)} T_3 - f_2^{(2)}$$

Considering all contributions to a node, and enforcing to zero

$$R_1^{(1)} = 0: \quad k_{11}^{(1)} T_1 + k_{12}^{(1)} T_2 - f_1^{(1)} = 0 \qquad \text{(Node 1)}$$

$$R_1^{(1)} + R_2^{(1)} = 0: \quad k_{21}^{(1)} T_1 + (k_{22}^{(1)} + k_{11}^{(2)}) T_2 + k_{12}^{(2)} T_3 - (f_2^{(1)} + f_1^{(2)}) = 0 \qquad \text{(Node 2)}$$

$$R_2^{(2)} = 0: \quad k_{21}^{(2)} T_2 + k_{22}^{(2)} T_3 - f_2^{(2)} = 0 \qquad \text{(Node 3)}$$

1D fin

- *Direct assembly procedure* (Cont'd)

In matrix form:

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} \end{Bmatrix}$$

(Note: same as assembly introduced before)

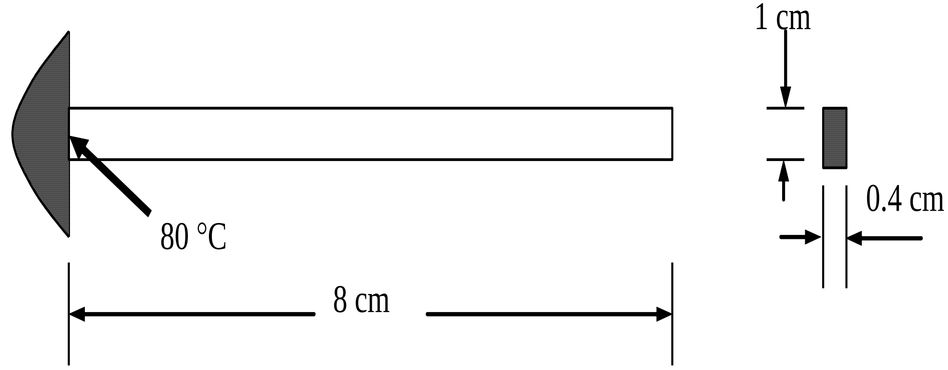
1D Heat Transfer Problems

1D fin

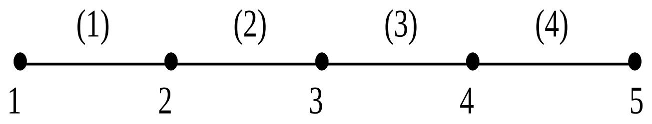
- *example*: Heat transfer in 1D fin

Calculate temperature distribution using FEM.

$$k = 3\frac{W}{cm \cdot {}^0C}, \qquad h = 0.1\frac{W}{cm^2 \cdot {}^0C}, \qquad \phi_f = 20^0 C$$



4 linear elements, 5 nodes



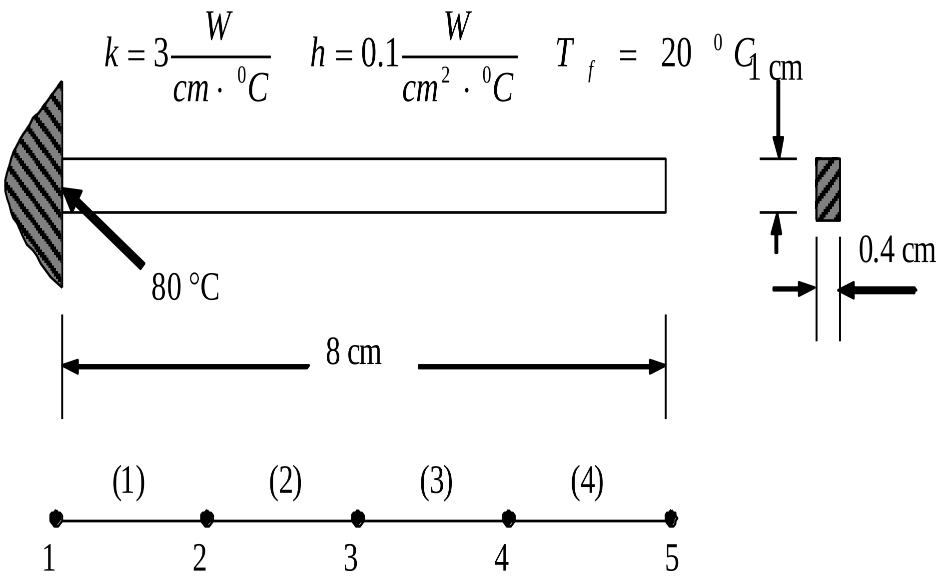
1D fin

Element 1, 2, 3:

$\mathbf{k}_M^{(e)}$, $\mathbf{f}_S^{(e)}$ not required

Element 4:

$\mathbf{k}_M^{(e)}$, $\mathbf{f}_S^{(e)}$ required



$$\frac{kA}{l} = \frac{3(0.4)}{2} = 0.6\frac{W}{{}^0C}$$

$$\frac{hPl}{6} = \frac{0.1(2.8)2}{6} = 0.093\frac{W}{{}^0C}$$

$$hA = 0.1(0.4) = 0.04\frac{W}{{}^0C}$$

$$\frac{hPlT_f}{2} = \frac{0.1(2.8)(20)(2)}{2} = 5.6W$$

$$hAT_f = 0.1(0.4)(20) = 0.8W$$

1D fin

For element 1, 2, 3 $\mathbf{k}^{(e)} = \mathbf{k}_D^{(e)} + \mathbf{k}_g^{(e)}$, $\mathbf{f}^{(e)} = \mathbf{f}_Q^{(e)}$

$$\mathbf{k}^{(e)} = \frac{kA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \longrightarrow \mathbf{k}^{(1,2,3)} = \begin{bmatrix} 0.786 & -0.507 \\ -0.507 & 0.786 \end{bmatrix}$$

$$\mathbf{f}^{(e)} = \frac{hPLT_f}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \longrightarrow \mathbf{f}^{(1,2,3)} = \begin{Bmatrix} 5.6 \\ 5.6 \end{Bmatrix}$$

For element 4 $\mathbf{k}^{(e)} = \mathbf{k}_D^{(e)} + \mathbf{k}_g^{(e)} + \mathbf{k}_M^{(e)}$, $\mathbf{f}^{(e)} = \mathbf{f}_Q^{(e)} + \mathbf{f}_S^{(e)}$

$$\mathbf{k}^{(e)} = \frac{kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & hA \end{bmatrix} \longrightarrow \mathbf{k}^{(4)} = \begin{bmatrix} 0.786 & -0.507 \\ -0.507 & 0.826 \end{bmatrix}$$

$$\mathbf{f}^{(e)} = \frac{hPLT_f}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ hAT_f \end{Bmatrix} \longrightarrow \mathbf{f}^{(4)} = \begin{Bmatrix} 5.6 \\ 6.4 \end{Bmatrix}$$

1D fin

$$\begin{bmatrix} 0.786 & -0.507 & 0 & 0 & 0 \\ -0.507 & 1.572 & -0.507 & 0 & 0 \\ 0 & -0.507 & 1.572 & -0.507 & 0 \\ 0 & 0 & -0.507 & 1.572 & -0.507 \\ 0 & 0 & 0 & -0.507 & 0.826 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 5.6 \\ 11.2 \\ 11.2 \\ 11.2 \\ 6.4 \end{Bmatrix} + \begin{Bmatrix} Q^* \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

← Heat source (Still unknown)

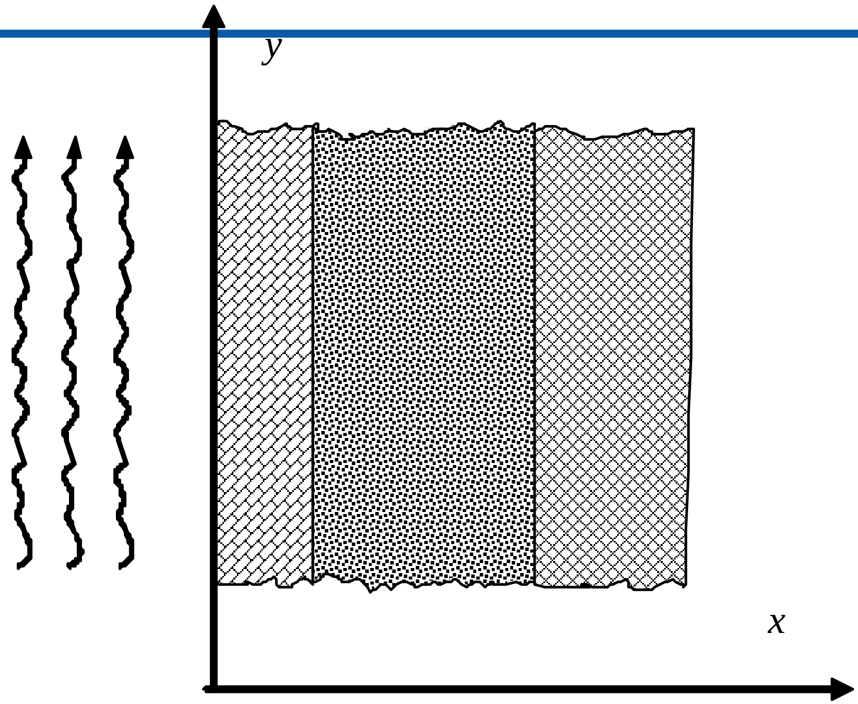
$T_1 = 80$, four unknowns – eliminate Q^*

$$\begin{bmatrix} 0.786 & -0.507 & 0 & 0 & 0 \\ -0.507 & 1.572 & -0.507 & 0 & 0 \\ 0 & -0.507 & 1.572 & -0.507 & 0 \\ 0 & 0 & -0.507 & 1.572 & -0.507 \\ 0 & 0 & 0 & -0.507 & 0.826 \end{bmatrix} \begin{Bmatrix} T_1 = 80 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 5.6 + Q^* \\ 11.2 + 80 \times 0.507 \\ 11.2 \\ 11.2 \\ 6.4 \end{Bmatrix}$$

Solving: $\mathbf{T}^T = \{80.0 \quad 42.0 \quad 28.2 \quad 23.3 \quad 22.1\}$

Composite wall

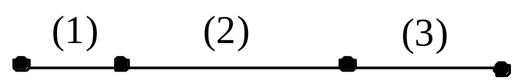
$$\underbrace{kA\frac{d^2T}{dx^2}}_{\text{Heat conduction}} + \underbrace{q}_{\text{Heat supply}} = 0$$



Convective boundary:

$$kA\frac{dT}{dx} = hA\left(T_b - T_f\right) \quad \text{at} \quad x = 0$$

$$-kA\frac{dT}{dx} = hA\left(T_b - T_f\right) \quad \text{at} \quad x = H$$



All equations for 1D fin still applies except $\mathbf{k}_G^{(e)}$ and $\mathbf{f}_Q^{(e)}$ vanish.

Recall: Only for heat convection

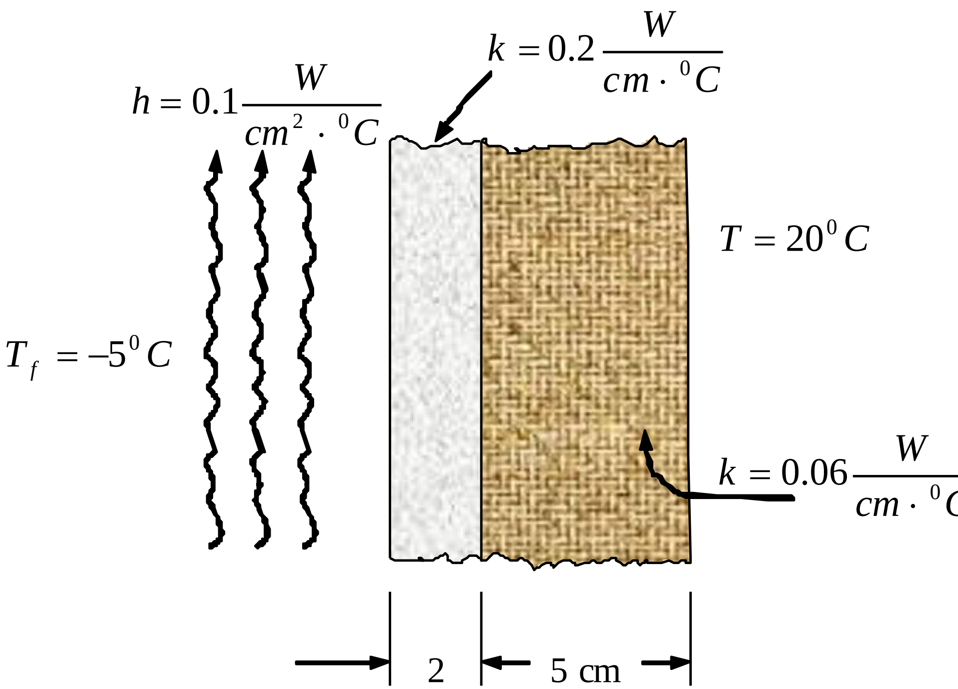
Therefore, $\mathbf{k}^{(e)} = \frac{kA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \mathbf{k}_M^{(e)}$, $\mathbf{f}^{(e)} = \mathbf{f}_S^{(e)}$

1D Heat Transfer Problems

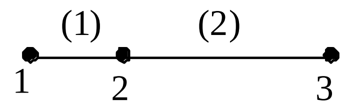
Composite wall

- *example*: Heat transfer through composite wall

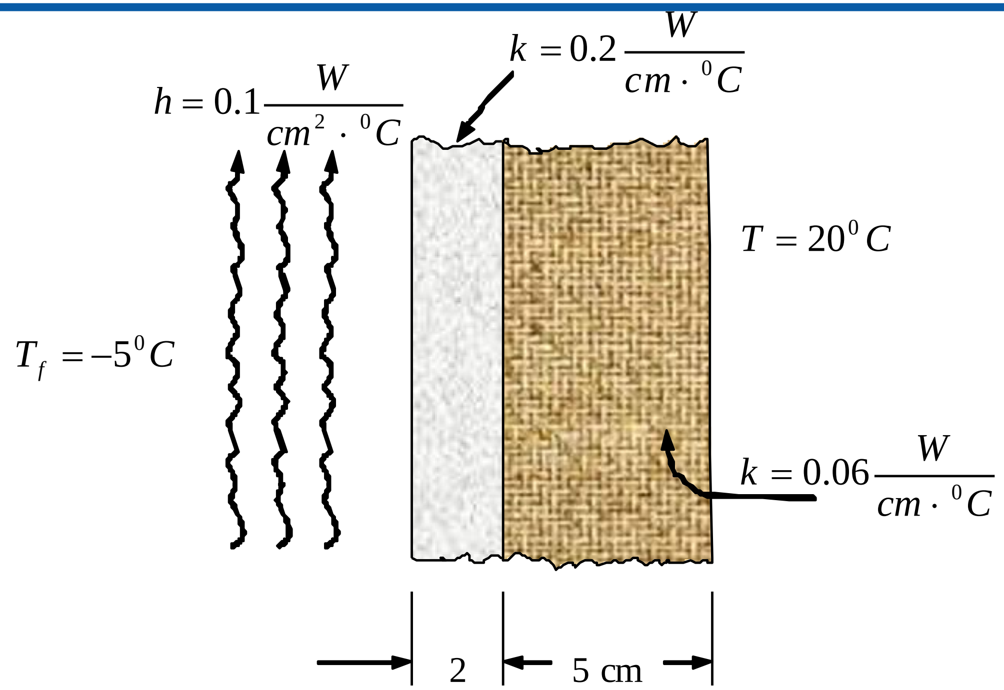
Calculate the temperature distribution across the wall using the FEM.



2 linear elements, 3 nodes



Composite wall



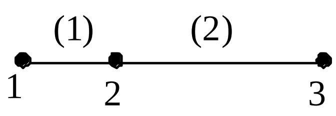
For element 1,

$$\frac{kA}{L} = \frac{0.2(1)}{2} = 0.1 \frac{W}{{}^0C}$$

$$hA = 0.1(1) = 0.1 \frac{W}{{}^0C}$$

$$hAT_f = 0.1(1)(-5) = -0.5W$$

$$\mathbf{k}^{(e)} = \frac{kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & hA \end{bmatrix} \longrightarrow \mathbf{k}^{(1)} = \begin{bmatrix} 0.2 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}$$



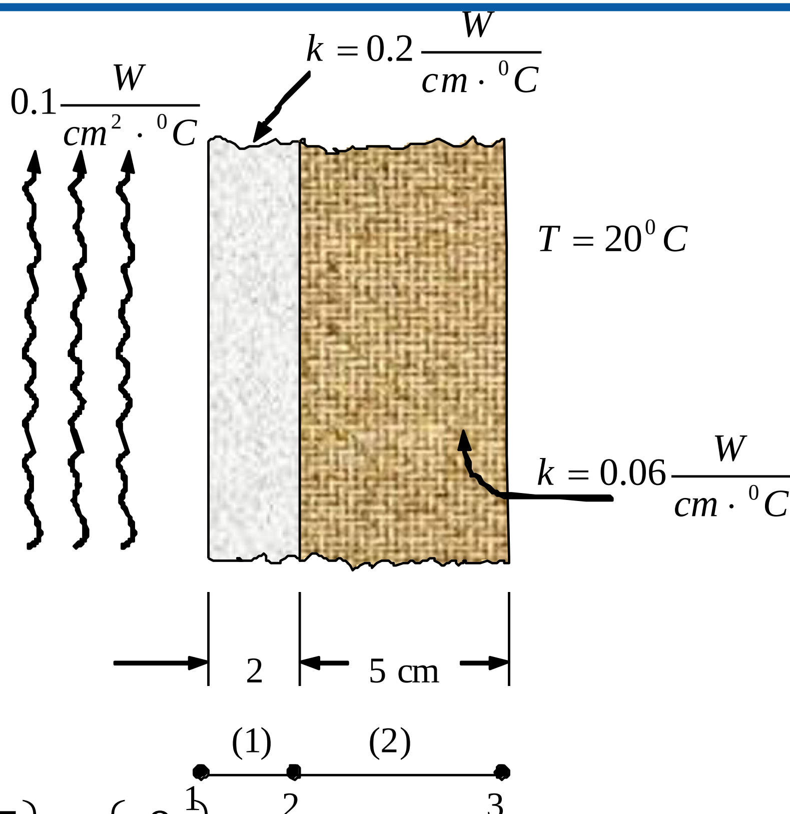
$$\mathbf{f}^{(e)} = \mathbf{f}_s^{(e)} = \begin{Bmatrix} hAT_f \\ 0 \end{Bmatrix} \longrightarrow \mathbf{f}_S^{(1)} = \begin{Bmatrix} -0.5 \\ 0 \end{Bmatrix}$$

Composite wall

For element 2,

$$\frac{kA}{L} = \frac{0.06(1)}{5} = 0.012 \frac{W}{{}^{0}C}$$

$$\mathbf{k}^{(e)} = \frac{kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \longrightarrow \mathbf{k}^{(2)} = \begin{bmatrix} 0.012 & -0.012 \\ -0.012 & 0.012 \end{bmatrix}$$

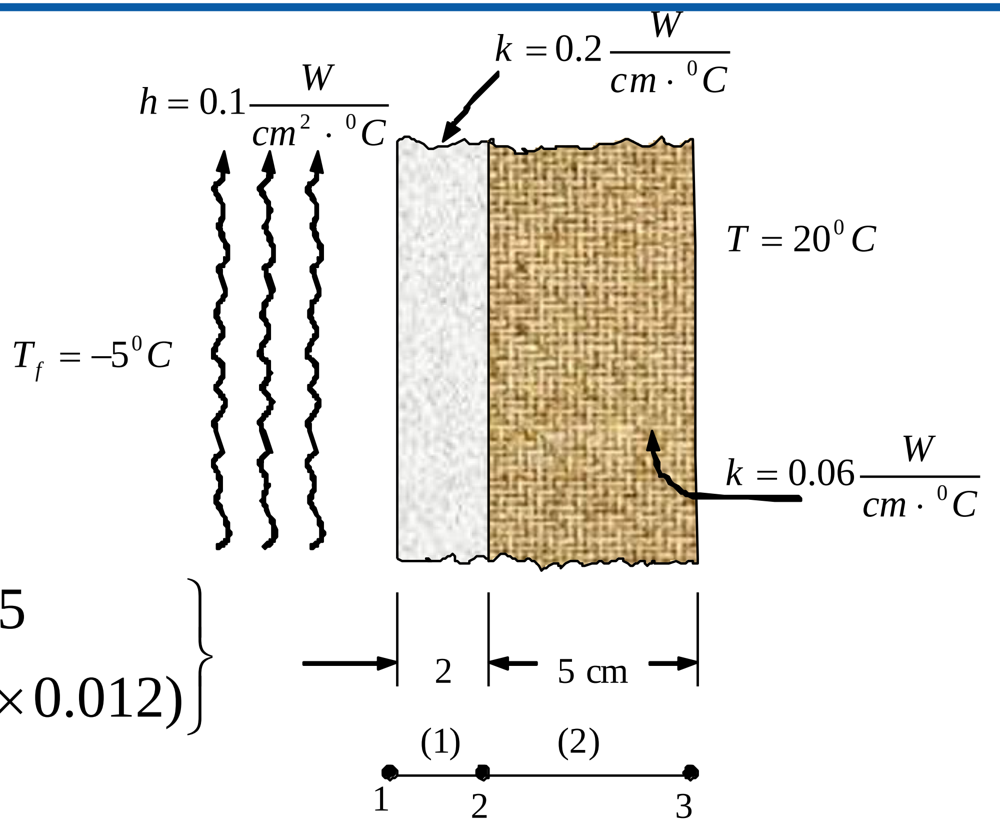


Upon assembly,

$$\begin{bmatrix} 0.20 & -0.10 & 0 \\ -0.10 & 0.112 & -0.012 \\ 0 & -0.012 & 0.012 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3(=20) \end{Bmatrix} = \begin{Bmatrix} -0.5 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ Q^* \end{Bmatrix}$$

(Unknown but required to balance equations)

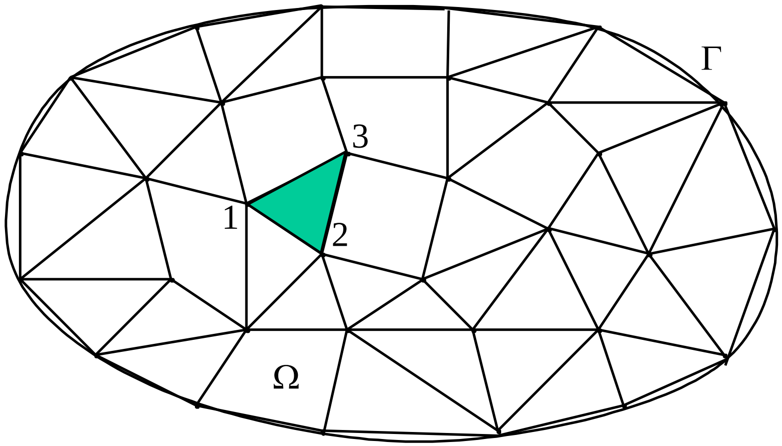
Composite wall



$$\begin{bmatrix} 0.20 & -0.10 \\ -0.10 & 0.112 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} -0.5 \\ 0.24(= 20 \times 0.012) \end{Bmatrix}$$

Solving: $\mathbf{T}^T = \{-2.5806 \quad -0.1613 \quad 20\}$

Element equations



$$D_x \frac{\partial^2 T}{\partial x^2} + D_y \frac{\partial^2 T}{\partial y^2} - gT + Q = 0$$

For one element,

$$\mathbf{R}^{(e)} = -\int_{A_e} \mathbf{N}^T \left(D_x \frac{\partial^2 T}{\partial x^2} + D_y \frac{\partial^2 T}{\partial y^2} - gT + Q \right) \mathrm{d}A$$

Note: $w = \mathbf{N}$: Galerkin approach

Element equations

$$\mathbf{R}^{(e)} = -\int_{A_e} \mathbf{N}^T \left(D_x \frac{\partial^2 T}{\partial x^2} + D_y \frac{\partial^2 T}{\partial y^2} - gT + Q\right) \mathrm{d}A$$

(Need to use divergence theorem to evaluate integral in residual.)

$$\frac{\partial}{\partial x}\left(\mathbf{N}^T \frac{\partial T}{\partial x}\right) = \mathbf{N}^T \frac{\partial^2 T}{\partial x^2} + \frac{\partial \mathbf{N}^T}{\partial x}\frac{\partial T}{\partial x}$$

(Product rule of differentiation)

Therefore,

$$-\int_{A_e} \mathbf{N}^T D_x \frac{\partial^2 T}{\partial^2 x^2} \mathrm{d}A = -\int_{A_e} D_x \frac{\partial}{\partial x}\left(\mathbf{N}^T \frac{\partial T}{\partial x}\right) \mathrm{d}A + \int_{A_e} D_x \frac{\partial \mathbf{N}^T}{\partial x}\frac{\partial T}{\partial x} \mathrm{d}A$$

Divergence theorem:

$$\int_{A_e} \frac{\partial}{\partial x}\left(\mathbf{N}^T \frac{\partial T}{\partial x}\right) \mathrm{d}A = \int_{\Gamma_e} \mathbf{N}^T \frac{\partial T}{\partial x} \cos\theta \, \mathrm{d}\Gamma$$

$$-\int_A \mathbf{N}^T D_x \frac{\partial^2 T}{\partial x^2} \mathrm{d}A = -\int_{\Gamma_e} D_x \mathbf{N}^T \frac{\partial T}{\partial x} \cos\theta \, \mathrm{d}\Gamma + \int_A D_x \frac{\partial \mathbf{N}^T}{\partial x}\frac{\partial T}{\partial x} \mathrm{d}A$$

Element equations

2nd integral:

$$-\int_A \mathbf{N}^T D_y \frac{\partial^2 T}{\partial y^2}\mathrm{d}A = -\int_{\Gamma_e} D_y \mathbf{N}^T \frac{\partial T}{\partial y}\sin\theta \mathrm{d}\Gamma + \int_A D_y \frac{\partial \mathbf{N}^T}{\partial y}\frac{\partial T}{\partial y}\mathrm{d}A$$

Therefore,

$$\mathbf{R}^{(e)} = -\int_{\Gamma_e} \mathbf{N}^T \left(D_x \frac{\partial T}{\partial x}\cos\theta + D_y \frac{\partial T}{\partial y}\sin\theta \right)\mathrm{d}\Gamma$$

$$+\int_{A_e} \left(D_x \frac{\partial \mathbf{N}^T}{\partial x}\frac{\partial T}{\partial x} + D_y \frac{\partial \mathbf{N}^T}{\partial y}\frac{\partial T}{\partial y} \right)\mathrm{d}A$$

$$+\int_{A_e} g\mathbf{N}^T T\mathrm{d}A - \int_{A_e} Q\mathbf{N}^T \mathrm{d}A$$

Element equations

$$T(x) = \mathbf{N}(x)\mathbf{T}^{(e)}$$

$$\mathbf{R}^{(e)} = \underbrace{-\int_{\Gamma_e} \mathbf{N}^T \left(D_x \frac{\partial T}{\partial x} \cos\theta + D_y \frac{\partial T}{\partial y} \sin\theta \right) \mathrm{d}\Gamma}_{\mathbf{b}^{(e)}}$$

$$+ \underbrace{\left(\int_{A_e} \left(D_x \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} + D_y \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y} \right) \mathrm{d}A \right)}_{\mathbf{k}_D^{(e)}} \mathbf{T}^{(e)}$$

$$+ (\underbrace{\int_{A_e} g \mathbf{N}^T \mathbf{N} \mathrm{d}A}_{\mathbf{k}_g^{(e)}}) \mathbf{T}^{(e)} - \underbrace{\int_{A_e} Q \mathbf{N}^T \mathrm{d}A}_{\mathbf{f}_Q^{(e)}}$$

Element equations

$$\mathbf{R}^{(e)} = \mathbf{b}^{(e)} + [\mathbf{k}_D^{(e)} + \mathbf{k}_g^{(e)}]\mathbf{T}^{(e)} - \mathbf{f}_Q^{(e)}$$

where $$\mathbf{b}^{(e)} = -\int_{\Gamma_e} \mathbf{N}^T \left(D_x \frac{\partial T}{\partial x} \cos\theta + D_y \frac{\partial T}{\partial y} \sin\theta \right) \mathrm{d}\Gamma$$

$$\mathbf{k}_D^{(e)} = \int_{A_e} \left(\frac{\partial \mathbf{N}^T}{\partial x} D_x \frac{\partial \mathbf{N}}{\partial x} + \frac{\partial \mathbf{N}^T}{\partial y} D_y \frac{\partial \mathbf{N}}{\partial y} \right) \mathrm{d}A$$

$$\mathbf{k}_g^{(e)} = \int_{A_e} g \mathbf{N}^T \mathbf{N} \mathrm{d}A$$

$$\mathbf{f}_Q^{(e)} = \int_{A_e} Q \mathbf{N}^T \mathrm{d}A$$

Element equations

$$\mathbf{k}_D^{(e)} = \int_{A_e} \left(\frac{\partial \mathbf{N}^T}{\partial x} D_x \frac{\partial \mathbf{N}}{\partial x} + \frac{\partial \mathbf{N}^T}{\partial y} D_y \frac{\partial \mathbf{N}}{\partial y} \right) \mathrm{d}A$$

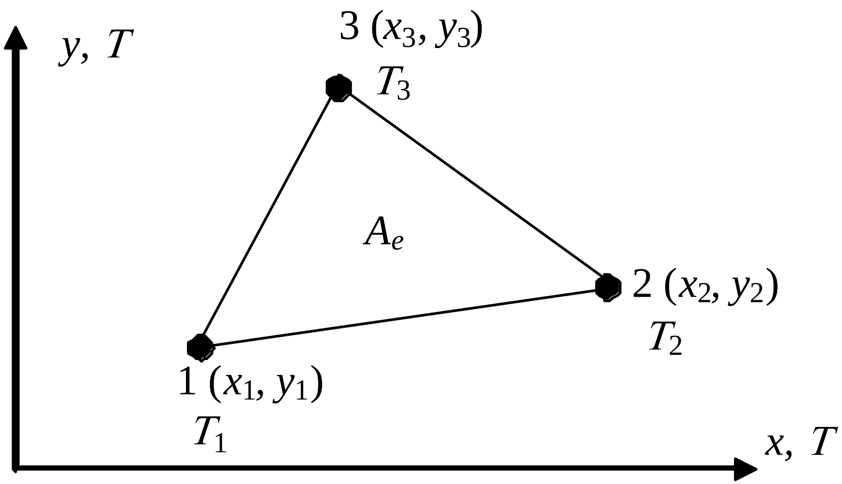
Define $\mathbf{D} = \begin{bmatrix} D_x & 0 \\ 0 & D_y \end{bmatrix}$, $\quad \nabla T = \begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{N}}{\partial x} \\ \frac{\partial \mathbf{N}}{\partial y} \end{bmatrix} \mathbf{T}^{(e)} = \mathbf{B}\mathbf{T}^{(e)}$

$$\mathbf{B} = \begin{bmatrix} \frac{\partial \mathbf{N}}{\partial x} \\ \frac{\partial \mathbf{N}}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_{n_d}}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_{n_d}}{\partial y} \end{bmatrix}$$ (Strain matrix)

$$\mathbf{B}^T\mathbf{D}\mathbf{B} = D_x \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} + D_y \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y} \quad \Rightarrow \quad \mathbf{k}_D^{(e)} = \int_{A_e} \mathbf{B}^T\mathbf{D}\mathbf{B} \, \mathrm{d}A$$

Triangular elements

$$T^{(e)} = \mathbf{N}\mathbf{T}^{(e)} = [N_1 \quad N_2 \quad N_3]\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$



Note: constant strain matrix

$$\mathbf{k}_D^{(e)} = \int_{A_e} \mathbf{B}^T\mathbf{D}\mathbf{B}\ \mathrm{d}A = \mathbf{B}^T\mathbf{D}\mathbf{B}\int_{A_e} \mathrm{d}A = \mathbf{B}^T\mathbf{D}\mathbf{B}A_e$$

$$\mathbf{k}_D^{(e)} = \frac{D_x}{4A}\begin{bmatrix} b_i^2 & b_ib_j & b_ib_k \\ b_ib_j & b_j^2 & b_jb_k \\ b_ib_k & b_jb_k & b_k^2 \end{bmatrix} + \frac{D_y}{4A}\begin{bmatrix} c_i^2 & c_ic_j & c_ic_k \\ c_ic_j & c_j^2 & c_jc_k \\ c_ic_k & c_jc_k & c_k^2 \end{bmatrix}$$

$$N_i = a_i + b_i x + c_i y$$

$$a_i = \frac{1}{2A_e}(x_j y_k - x_k y_j)$$

$$b_i = \frac{1}{2A_e}(y_j - y_k)$$

$$c_i = \frac{1}{2A_e}(x_k - x_j)$$

(or $N_i = L_i$)

Triangular elements

$$\mathbf{k}_g^{(e)} = \int_{A_e} g\mathbf{N}^T\mathbf{N}\mathrm{d}A = g\int_{A_e} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} [N_1 \quad N_2 \quad N_3]\mathrm{d}A$$

$$= g\int_{A_e} \begin{bmatrix} N_1^2 & N_1N_2 & N_1N_3 \\ N_1N_2 & N_2^2 & N_2N_3 \\ N_1N_3 & N_2N_3 & N_3^2 \end{bmatrix} \mathrm{d}A$$

Note:

$$\int_A L_1^m L_2^n L_3^p \mathrm{d}A = \frac{m!\,n!\,p!}{(m+n+p+2)!}2A$$

(Area coordinates)

e.g. $$\int_{A_e} N_1 N_2 \mathrm{d}A = \int_{A_e} L_1^1 L_2^1 L_3^0 \mathrm{d}A = \frac{1!1!0!}{(1+1+0+2)!}2A = \frac{1}{4\times3\times2\times1}2A = \frac{A}{12}$$

Therefore, $$\mathbf{k}_g^{(e)} = \frac{gA}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Triangular elements

Similarly,

$$\mathbf{f}_Q^{(e)} = \int_{A_e} Q\mathbf{N}^T \mathrm{d}A = Q\int_{A_e} \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} \mathbf{d}A = Q\int_{A_e} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} \mathbf{d}A = \frac{QA}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

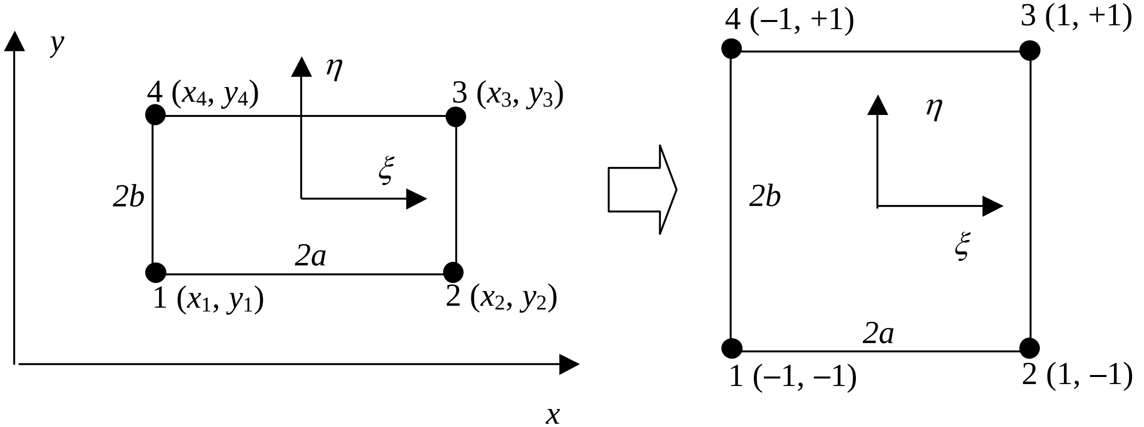
Note: $\mathbf{b}^{(e)}$ will be discussed later

Rectangular elements

$$T^{(e)} = \mathbf{N}\mathbf{T}^{(e)} = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$

$$N_1 = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \tfrac{1}{4}(1+\xi)(1+\eta)$$
$$N_4 = \tfrac{1}{4}(1-\xi)(1+\eta)$$

$$\mathbf{B} = \begin{bmatrix} -\frac{1-\eta}{a} & \frac{1-\eta}{a} & \frac{1+\eta}{a} & -\frac{1+\eta}{a} \\ -\frac{1-\xi}{b} & -\frac{1+\xi}{b} & \frac{1+\xi}{b} & \frac{1-\xi}{b} \end{bmatrix}$$



$$\xi = x/a\,, \ \eta = y/b$$

Rectangular elements

$$\mathbf{k}_D^{(e)} = \int_{A_e} \mathbf{B}^T \mathbf{D} \mathbf{B} \, \mathrm{d}A = \int_{-1}^{+1}\int_{-1}^{+1} ab\mathbf{B}^T \mathbf{D}\mathbf{B} \mathrm{d}\xi \mathrm{d}\eta$$

$$= \frac{D_x b}{6a}\begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} + \frac{D_y a}{6b}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ 1 & -1 & 1 & 2 \end{bmatrix}$$

$$\mathbf{k}_g^{(e)} = \int_{A_e} g\mathbf{N}^T \mathbf{N} \mathrm{d}A = \int_{-1}^{+1}\int_{-1}^{+1} abg\mathbf{N}^T \mathbf{N} \mathrm{d}\xi \mathrm{d}\eta$$

$$= abg\int_{A_e} \begin{bmatrix} N_1^2 & N_1N_2 & N_1N_3 & N_1N_4 \\ N_1N_2 & N_2^2 & N_2N_3 & N_2N_3 \\ N_1N_3 & N_2N_3 & N_3^2 & N_3N_4 \\ N_1N_4 & N_2N_4 & N_3N_4 & N_4^2 \end{bmatrix} \mathrm{d}\xi \mathrm{d}\eta \longrightarrow \mathbf{k}_g^{(e)} = \frac{gA}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}$$

Rectangular elements

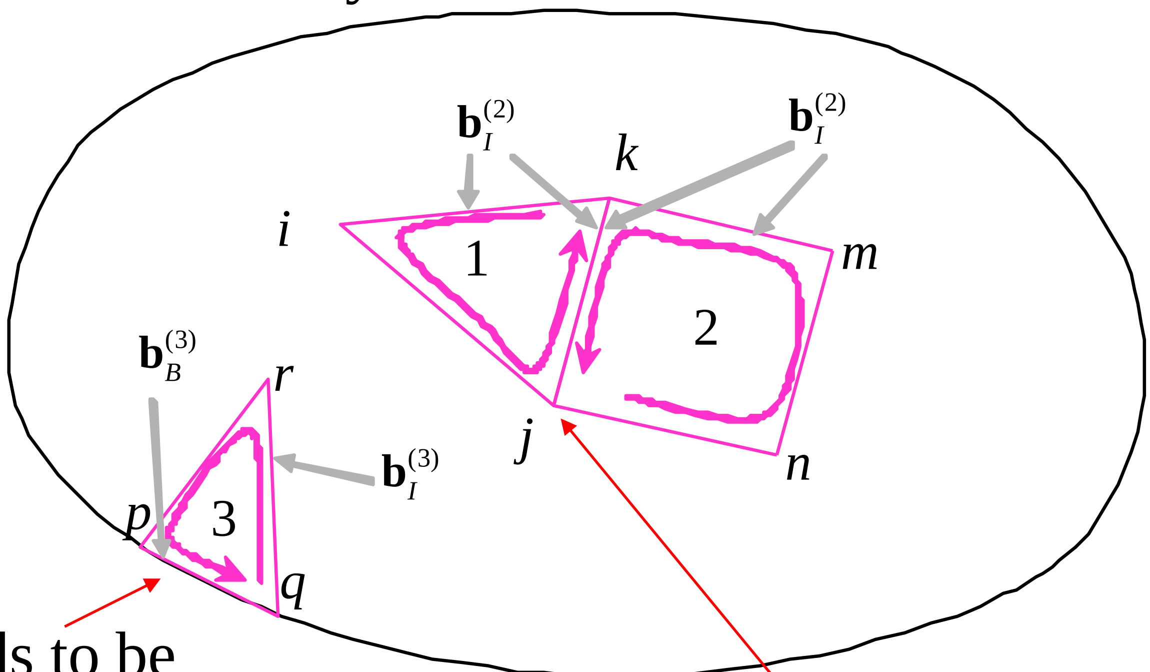
$$\mathbf{f}_Q^{(e)} = \int_{A_e} Q\mathbf{N}^T \mathrm{d}A = Q\int_{A_e} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} \mathbf{d}A = \frac{QA}{4} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

Note: In practice, the integrals are usually evaluated using the Gauss integration scheme

Boundary conditions and vector $\mathbf{b}^{(e)}$

$$\mathbf{b}^{(e)} = \mathbf{b}_I^{(e)} + \mathbf{b}_B^{(e)}$$

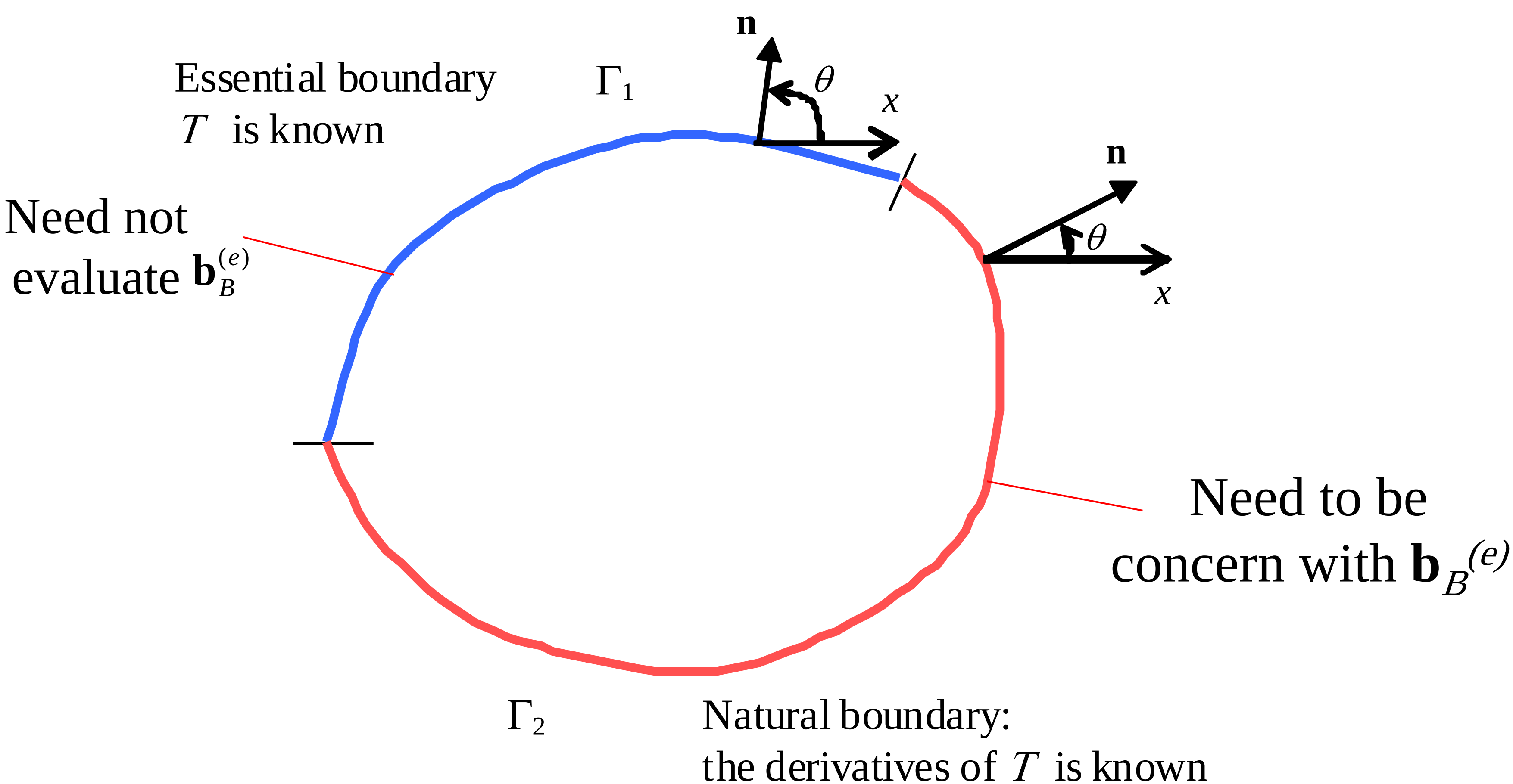
Internal Boundary



$\mathbf{b}_B^{(e)}$ needs to be evaluated at boundary

Vanishing of $\mathbf{b}_I^{(e)}$

Boundary conditions and vector $\mathbf{b}^{(e)}$

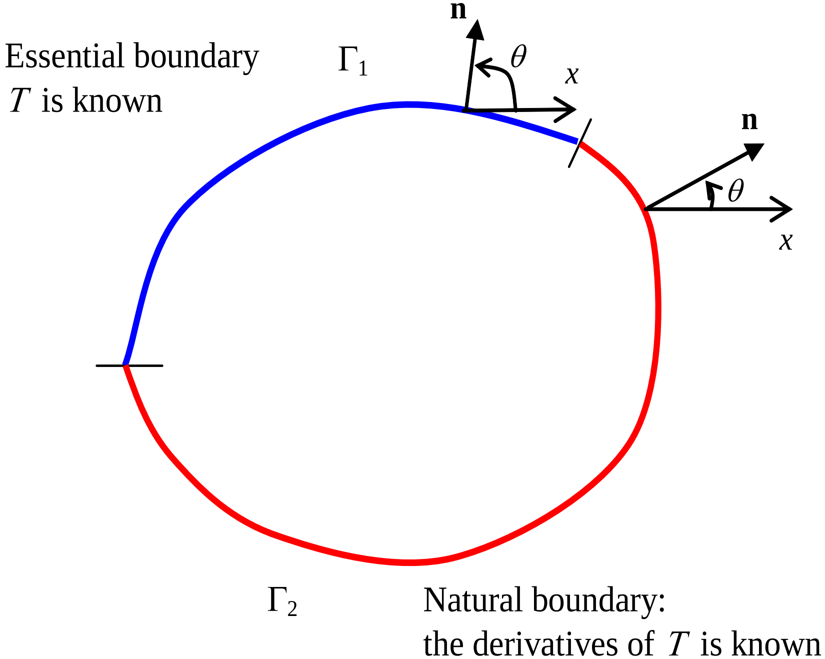


Boundary conditions and vector $\mathbf{b}^{(e)}$

$$\mathbf{b}^{(e)} = -\int_{\Gamma_e} \mathbf{N}^T \left(D_x \frac{\partial T}{\partial x} \cos\theta + D_y \frac{\partial T}{\partial y} \sin\theta \right) \mathrm{d}\Gamma$$

$$D_x \frac{\partial T}{\partial x} \cos\theta + D_y \frac{\partial T}{\partial y} \sin\theta = -MT_b + S$$

on natural boundary Γ_2



$$D_x \frac{\partial T}{\partial x} \cos\theta + D_y \frac{\partial T}{\partial y} \sin\theta = k \frac{\partial T}{\partial n} = -MT_b + S$$

Heat flux across boundary

2D Heat Transfer Problems

Boundary conditions and vector $\mathbf{b}^{(e)}$

Insulated boundary:

$$M = S = 0 \Rightarrow \mathbf{b}_B^{(e)} = 0$$

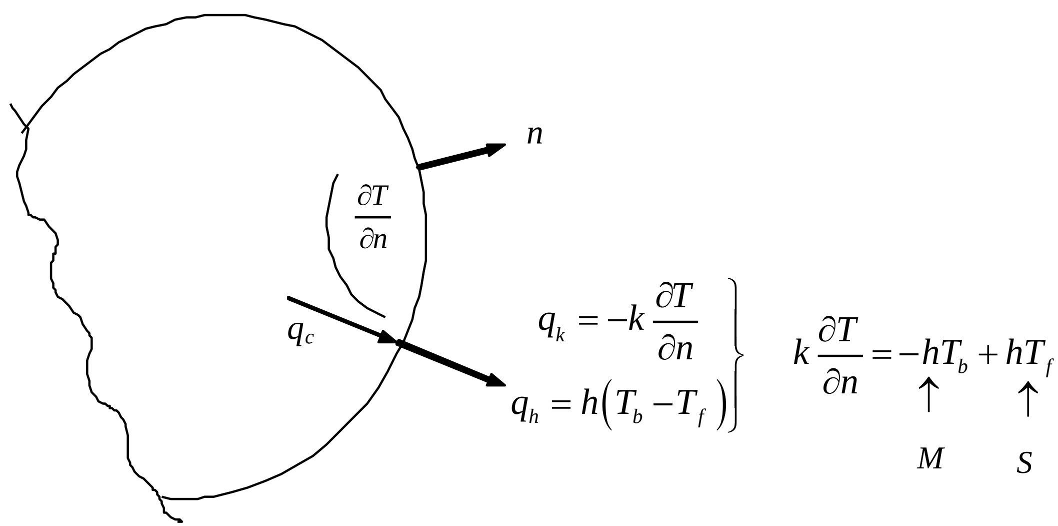
Convective boundary condition:

$$q_k = -k\frac{\partial T}{\partial n}$$

$$q_h = h\left(T_b - T_f\right)$$

$$k\frac{\partial T}{\partial n} = -\underset{M}{h}\,T_b + \underset{S}{hT_f}$$

$$M = h, \quad S = hT_f$$

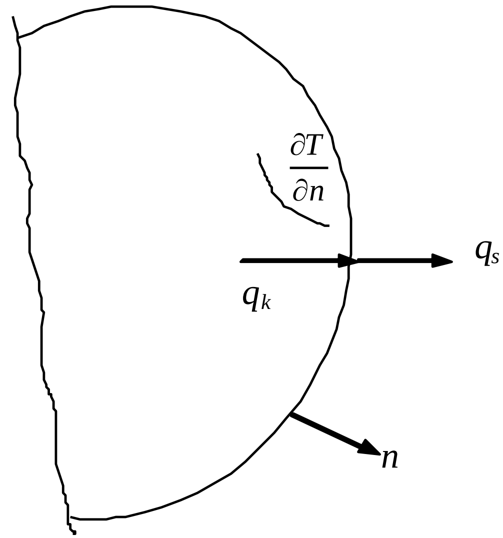


Boundary conditions and vector $\mathbf{b}^{(e)}$

Specified heat flux on boundary:

$$k\frac{\partial T}{\partial n} = \underbrace{0}_{M} \times T_b \underbrace{-q_s}_{S}$$

$$M = 0, \quad S = -q_s$$



$$\left. \begin{aligned} q_k &= -k\frac{\partial T}{\partial n} \\ &q_s \end{aligned} \right\} \qquad k\frac{\partial T}{\partial n} = -q_s$$

$M=0 \quad S=-q_s$

$$S = \begin{cases} \text{Positive} & \text{if heat flows into the boundary} \\ \text{Negative} & \text{if heat flows out off the boundary} \\ 0 & \text{insulated} \end{cases}$$

Boundary conditions and vector $\mathbf{b}^{(e)}$

For other cases whereby $M, S \neq 0$

$$\mathbf{b}_B^{(e)} = -\int_{\Gamma_2} \mathbf{N}^T \left(D_x \frac{\partial T}{\partial x} \cos\theta + D_y \frac{\partial T}{\partial y} \sin\theta \right) \mathrm{d}\Gamma$$

$$= -\int_{\Gamma_2} \mathbf{N}^T (MT_b + S)\mathrm{d}\Gamma$$

$$T_b^{(e)} = \mathbf{NT}$$

$$\mathbf{b}_B^{(e)} = -\int_{\Gamma_2} \mathbf{N}^T (-M\mathbf{NT}^{(e)} + S)\mathrm{d}\Gamma$$

$$= \underbrace{\left(\int_{\Gamma_2} \mathbf{N}^T M\mathbf{N}\mathrm{d}\Gamma\right)}_{\mathbf{k}_M^{(e)}} \mathbf{T}^{(e)} - \underbrace{\int_{\Gamma_2} \mathbf{N}^T S\mathrm{d}\Gamma}_{\mathbf{f}_B^{(e)}}$$

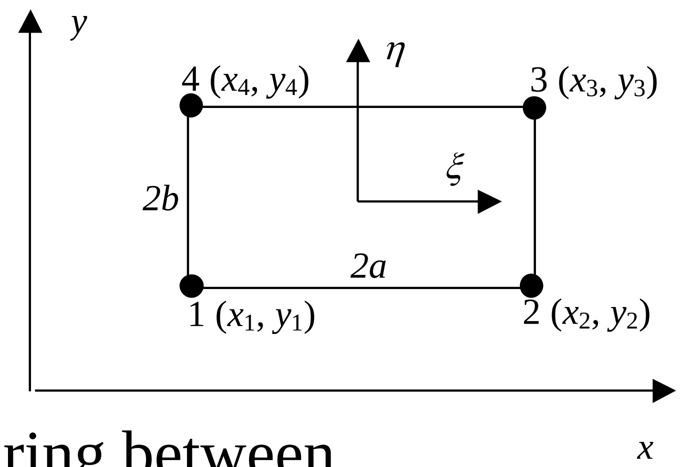
Boundary conditions and vector $\mathbf{b}^{(e)}$

$$\mathbf{b}_B^{(e)} = \mathbf{k}_M^{(e)}\mathbf{T}^{(e)} - \mathbf{f}_S^{(e)}$$

where $\mathbf{k}_M^{(e)} = \int_{\Gamma_2} \mathbf{N}^T M \mathbf{N} \mathrm{d}\Gamma$, $\mathbf{f}_S^{(e)} = \int_{\Gamma_2} \mathbf{N}^T S \mathrm{d}\Gamma$

For a rectangular element,

$$\mathbf{f}_S^{(e)} = \int_{\Gamma_{1-2}} S\mathbf{N}^T d\Gamma = \int_{-1}^{1} S \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} a \mathrm{d}\xi$$



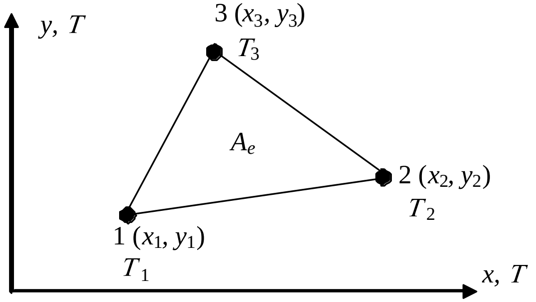
$$\mathbf{f}_S^{(e)} = \int_{-1}^{1} \frac{Sa}{2} \begin{Bmatrix} (1-\xi) \\ (1+\xi) \\ 0 \\ 0 \end{Bmatrix} \mathrm{d}\xi = Sa \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

(Equal sharing between nodes 1 and 2)

Boundary conditions and vector $\mathbf{b}^{(e)}$

Equal sharing valid for all elements with linear shape functions

$$\mathbf{f}_{S,2-3}^{(e)} = Sb\begin{Bmatrix}0\\1\\1\\0\end{Bmatrix} \qquad \mathbf{f}_{S,3-4}^{(e)} = Sa\begin{Bmatrix}0\\0\\1\\1\end{Bmatrix} \qquad \mathbf{f}_{S,1-4}^{(e)} = Sb\begin{Bmatrix}1\\0\\0\\1\end{Bmatrix}$$



Applies to triangular elements too

$$\mathbf{f}_{S,1-2}^{(e)} = \frac{SL_{12}}{2}\begin{Bmatrix}1\\1\\0\end{Bmatrix} \qquad \mathbf{f}_{S,2-3}^{(e)} = \frac{SL_{23}}{2}\begin{Bmatrix}0\\1\\1\end{Bmatrix} \qquad \mathbf{f}_{S,1-3}^{(e)} = \frac{SL_{13}}{2}\begin{Bmatrix}1\\0\\1\end{Bmatrix}$$

Boundary conditions and vector $\mathbf{b}^{(e)}$

$\mathbf{k}_M^{(e)}$ for rectangular element

$$\mathbf{k}_M^{(e)} = \int_{\Gamma_2} M \begin{bmatrix} N_1^2 & N_1N_2 & N_1N_3 & N_1N_4 \\ N_1N_2 & N_2^2 & N_2N_3 & N_2N_4 \\ N_1N_3 & N_2N_3 & N_3^2 & N_3N_4 \\ N_1N_4 & N_2N_4 & N_3N_4 & N_4^2 \end{bmatrix} \mathrm{d}\Gamma$$

$$\mathbf{k}_{M,1-2}^{(e)} = aM \int_{-1}^{1} \begin{bmatrix} N_1^2 & N_1N_2 & 0 & 0 \\ N_2N_1 & N_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\xi$$

Boundary conditions and vector $\mathbf{b}^{(e)}$

$$\int_{-1}^{1} N_1^2 \mathrm{d}\xi = \int_{-1}^{1} \frac{(1-\xi)^2}{4} \mathrm{d}\xi = \frac{2}{3}$$

$$\int_{-1}^{1} N_1 N_2 \mathrm{d}\xi = \int_{-1}^{1} \frac{(1-\xi)(1+\xi)}{4} \mathrm{d}\xi = \frac{2}{6}$$

$$\int_{-1}^{1} N_2^2 \mathrm{d}\xi = \int_{-1}^{1} \frac{(1+\xi)^2}{4} \mathrm{d}\xi = \frac{2}{3}$$

$$\mathbf{k}_{M,1-2}^{(e)} = \frac{2aM}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = (2aM)\begin{bmatrix} \frac{2}{6} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & \frac{2}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Shared in ratio 2/6, 1/6, 1/6, 2/6

$$\mathbf{k}_{M,2-3}^{(e)} = \frac{M\,2b}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{k}_{M,3-4}^{(e)} = \frac{M\,2a}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \mathbf{k}_{M,1-4}^{(e)} = \frac{M\,2b}{6}\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

Boundary conditions and vector $\mathbf{b}^{(e)}$

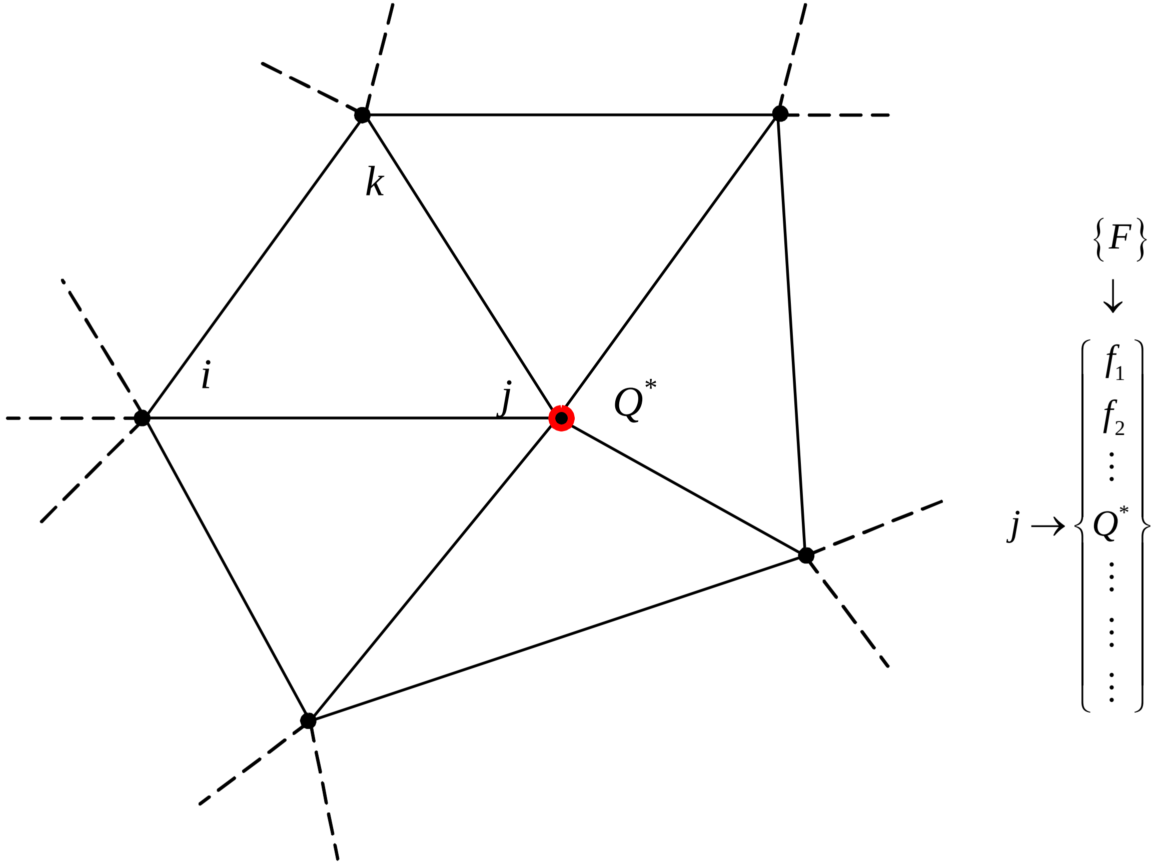
Similar for triangular elements

$$\mathbf{k}_{M,i-j}^{(e)} = \frac{ML_{ij}}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{k}_{M,j-k}^{(e)} = \frac{ML_{jk}}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\mathbf{k}_{M,i-k}^{(e)} = \frac{ML_{ik}}{6}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Point heat source or sink

Preferably place node at source or sink



Point heat source or sink within the element

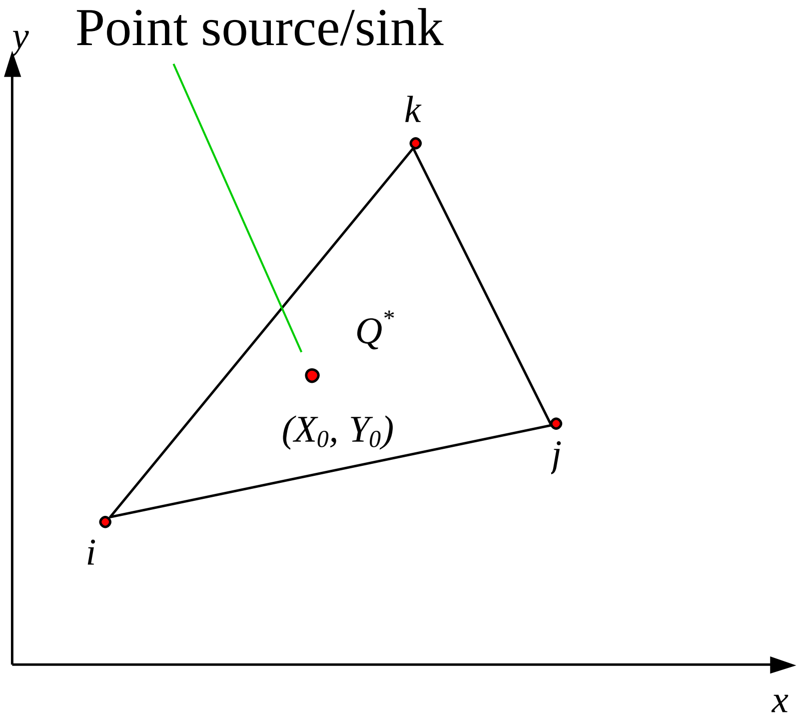
$$\mathbf{f}_Q^{(e)} = \int_{A_e} Q\mathbf{N}^T \mathrm{d}A$$

$$Q = Q^* \delta(x - X_0)\delta(y - Y_0)$$

(Delta function)

$$\mathbf{f}_Q^{(e)} = Q^* \int_{A_e} \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} \delta(x - X_0)\delta(y - Y_0)\,\mathrm{d}x\mathrm{d}y$$

$$\Rightarrow \quad \mathbf{f}_Q^{(e)} = Q^* \begin{Bmatrix} N_i(X_0,\ Y_0) \\ N_j(X_0,\ Y_0) \\ N_k(X_0,\ Y_0) \end{Bmatrix}$$



Field problems

SUMMARY

$$D_x \frac{\partial^2 T}{\partial x^2} + D_y \frac{\partial^2 T}{\partial y^2} - g\,T + Q = 0$$

$k_x t$ $\quad$ $k_y t$ $\quad$ $2h$ $\quad$ $2hT_f$ $\quad$ Source and sink

$$\mathbf{R}^{(e)} = \mathbf{b}^{(e)} + \underbrace{[\mathbf{k}_D^{(e)} + \mathbf{k}_g^{(e)} + \mathbf{k}_M^{(e)}]}_{\mathbf{k}^{(e)}}\mathbf{T}^{(e)} - \underbrace{(\mathbf{f}_Q^{(e)} + \mathbf{f}_{Q^*}^{(e)} + \mathbf{f}_S^{(e)})}_{\mathbf{f}^{(e)}}$$

Only for elements on the derivative boundary

h $\quad$ hT_f

$$\mathbf{b}^{(e)} = -\int_\Gamma \mathbf{N}^T \overbrace{\left(D_x \frac{\partial T}{\partial x}\cos\theta + D_y \frac{\partial T}{\partial y}\sin\theta \right)}^{-M\,T_b + S} \mathrm{d}\Gamma$$