Finite Element Method
Integral Formulation

Some Mathematical Concepts

Simply connected domain: If any two points of the domain can be Joint by a line lying entirely within the domain

Class of a domain: A function of several variables is said to be of Class $C^m(\Omega)$ in a domain if all its partial derivatives up to and including the *m*th order exist and are continuous in Ω

C^0 ➡ F is continuous (i.e. $\partial f/\partial x$, $\partial f/\partial y$ exist but may not be continuous.)

Boundary Value Problems: A differential equation (*DE*) is said to be a BVP if the dependent variable and possibly its derivatives are required to take specified values on the boundary.

Example: $$-\frac{d}{dx}(a\frac{du}{dx}) = f \quad 0 < x < 1, \qquad u(0) = d_0, \left(x\frac{du}{dx}\right)_{x=1} = g_0$$

Some Mathematical Concepts

Initial Value Problem: An IVP is one in which the dependent variable and possibly its derivatives are specified initially at $t = 0$

Example: $$\rho \frac{d^2u}{dt^2} + au = f \quad 0 < t \le t_0, \qquad u(0) = u_0, \left(\frac{du}{dt}\right)_{t=0} = v_0$$

Initial and Boundary Value Problem:

Example: $$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + \rho\frac{\partial u}{\partial t} = f(x,t) \quad for \;\; 0 < x < 1 and \quad 0 < t \le t_0$$

$$u(0,t) = d_0(t), \left(a\frac{\partial u}{\partial x}\right)_{x=1} = g_0(t), \;\; u(x,0) = u_0(x)$$

Eigenvalue Problem: the problem of determining value λ of such that

Example:

λ *Eigenvalue*

u *Eigenfunction*

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) = \lambda u \quad 0 < x < 1$$

$$u(0) = 0, \left(\frac{du}{dx}\right)_{x=1} = 0$$

Some Mathematical Concepts

Integration-by-Part Formula:

First

$$\frac{d}{dx}(wv)=\frac{dw}{dx}v+w\frac{dv}{dx}\Rightarrow\int_a^b w\frac{dv}{dx}dx=-\int_a^b v\frac{dw}{dx}dx+w(b)v(b)-w(a)v(a)$$

Next

$$\int_a^b w\frac{d^2u}{dx^2}dx=-\int_a^b \frac{du}{dx}\frac{dw}{dx}dx+w(b)\frac{du}{dx}(b)-w(a)\frac{du}{dx}(a)$$

Similarly

$$\int_a^b v\frac{d^4w}{dx^4}dx=\int_a^b \frac{d^2w}{dx^2}\frac{d^2v}{dx^2}dx+\frac{d^2w}{dx^2}(a)\frac{dv}{dx}(a)-\frac{d^2w}{dx^2}(b)\frac{dv}{dx}(b)+v(b)\frac{d^3w}{dx^3}(b)-v(a)\frac{d^3w}{dx^3}(a)$$

Gradient Theorem

$$\int_{\Omega} grad\,F\,dxdy = \int_{\Omega} \nabla F\,dxdy = \oint_{\Gamma} \hat{n} F ds$$

But
$$\nabla F = \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j, \quad \hat{n} = n_x i + n_y j$$

Thus
$$\int_{\Omega} \left(\frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j \right) dxdy = \oint_{\Gamma} \left(n_x i + n_y j \right) F ds$$

or
$$\int_{\Omega} \left(\frac{\partial F}{\partial x} \right) dxdy = \oint_{\Gamma} F n_x ds$$

$$\int_{\Omega} \left(\frac{\partial F}{\partial y} \right) dxdy = \oint_{\Gamma} F n_y ds$$

Divergence Theorem

$$\int_{\Omega} divG\ dxdy = \int_{\Omega} \nabla . G\, dxdy = \oint_{\Gamma} \hat{n}.G ds$$

$$\int_{\Omega} \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} \right) dxdy = \oint_{\Gamma} (n_x G_x + n_y G_y) ds$$

Using gradient and divergence theorem, the following relations can Be derived! (Exercise)

$$\int_{\Omega} (\nabla G) w\, dxdy = -\int_{\Omega} (\nabla w) G dxdy + \oint_{\Gamma} \hat{n} w G ds \qquad (*)$$

and

$$-\int_{\Omega} (\nabla^2 G) w\, dxdy = \int_{\Omega} (\nabla w)\ .(\nabla G) dxdy - \oint_{\Gamma} \frac{\partial G}{\partial n} w ds$$

The components of equation (*) are:

$$\int_{\Omega} \frac{\partial G}{\partial x} w\,dxdy = -\int_{\Omega} \frac{\partial w}{\partial x} G\,dxdy + \oint_{\Gamma} n_x wG\,ds$$

$$\int_{\Omega} \frac{\partial G}{\partial y} w\,dxdy = -\int_{\Omega} \frac{\partial w}{\partial y} G\,dxdy + \oint_{\Gamma} n_y wG\,ds$$

Functionals

An integral in the form of

$$I(u)=\int_a^b F(x,u,u')dx, \quad u=u(x), \quad u'=\frac{du}{dx}$$

where integrand $F(x,u,u')$ is a given function of arguments x, u, u' is called a *functional* (a function of function) .

A functional is said to be *linear* if and only if:

$$I(\alpha u+\beta v)=\alpha I(u)+\beta I(v) \qquad \alpha,\beta \ \ are\ scalars$$

A functional $B(u,v)$ is said to be *bilinear* if it is linear in each of its arguments

$B(\alpha u_1+\beta u_2,v)=\alpha B(u_1,v)+\beta B(u_2,v)$ Linearity in the first argument

$B(u,\alpha v_1+\beta v_2)=\alpha B(u,v_1)+\beta B(u,v_2)$ Linearity in the second argument

Functionals

A *bilinear* form $B(u,v)$ is *symmetric* in its arguments if

$$B(u,v) = B(v,u)$$

Example of linear functional is

$$I(v) = \int_0^L v f dx + \frac{dv}{dx}(L) M_0$$

Example of bilinear functional is

$$B(v,w) = \int_0^L a \frac{dv}{dx} \frac{dw}{dx} dx$$

The Variational Symbol

Consider the function $F = F(x,u,u')$ for fixed value of x, F only depends on u,u'

The change αv in u, where α is constant and v is a function, is called variation of u and denoted by:

Variational Symbol → $\delta u = \alpha v$

In analogy with the total differential of a function

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'$$

Note that

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du'$$

Some Mathematical Concepts

The Variational Symbol

Also

$$\delta(F_1 \pm F_2) = \delta F_1 \pm \delta F_2$$

$$\delta(F_1 F_2) = F_2 \delta F_1 + F_1 \delta F_2$$

$$\delta\left(\frac{F_1}{F_2}\right) = \frac{F_2 \delta F_1 - F_1 \delta F_2}{F_2^{\,2}}$$

$$\delta\left[(F_1)^n\right] = n(F_1)^{n-1} \delta F_1$$

Furthermore

$$\frac{d}{dx}(\delta u) = \frac{d}{dx}(\alpha v) = \alpha \frac{dv}{dx} = \alpha v' = \delta u' = \delta(\frac{du}{dx})$$

$$\delta \int_a^b u(x)dx = \int_a^b \delta u(x)dx$$

Weighted – integral and weak formulation

Consider the following DE

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right]=q(x)\quad 0<x<L$$

$$u(0)=u_0,\ \left(a\frac{du}{dx}\right)_{x=L}=Q_0$$

- *Transverse deflection of a cable*
- *Axial deformation of a bar*
- *Heat transfer*
- *Flow through pipes*
- *Flow through porous media*
- *Electrostatics*

Weak Formulation of BVP

There are *3* steps in the development of a weak form, if exists, of any DE.

STEP 1:

Move all expression in DE to one side, multiply by w (weight function) and integral over the domain.

$$\int_0^L w\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right)-q\right]dx=0 \qquad (+)$$

Weighted-integral or *weighted-residual*

$$u=U_N=\sum_{j=1}^{N} c_j\phi_j+\phi_0$$

N linearly independent equation for w and obtain N equation for $c_1,\ldots,c_N$

Weak Formulation of BVP

STEP 2

1-The integral (+) allows to obtain N independent equations

2- The approximation function, ϕ, should be differentiable as many times as called for the original DE.

3- The approximation function should satisfy the BCs.

4- If the differentiation is distributed between w and ϕ then the resulting integral form has weaker continuity conditions.

Such a weighted-integral statement is called ***weak form***.

The weak form formulation has two main characteristics:

-requires weaker continuity on the dependent variable and often results in a symmetric set of algebraic equations.

- The *natural BCs* are included in the weak form, and therefore the approximation function is required to satisfy only the *essential BCs*.

Weak Formulation of BVP

Returning to our example:

$$\int_0^L \left\{ w\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right)\right] - wq \right\} dx = 0 \Rightarrow \int_0^L \left(\frac{dw}{dx} a \frac{du}{dx} - wq \right) dx - \left[wa\frac{du}{dx} \right]_0^L = 0$$

Secondary Variable (SV):

Coefficient of weight function and its derivatives

$$Q = \left(a\frac{du}{dx}\right)n_x$$ ⇨ Natural Boundary Conditions (NBC)

Primary Variable (PV): The dependent variable of the problem

u ⇨ Essential Boundary Conditions (EBC)

Weak Formulation of BVP

$$\int_0^L \left(\frac{dw}{dx} a \frac{du}{dx} - wq \right) dx - \left[wa \frac{du}{dx} \right]_0^L = 0$$

$$\int_0^L \left(\frac{dw}{dx} a \frac{du}{dx} - wq \right) dx - \left[wa \frac{du}{dx} n_x \right]_{x=0} - \left[wa \frac{du}{dx} n_x \right]_{x=L} = 0$$

$$\int_0^L \left(\frac{dw}{dx} a \frac{du}{dx} - wq \right) dx - (wQ)_0 - (wQ)_L = 0$$

Note that

$$\begin{array}{ll} n_x = -1 & x = 0 \\ n_x = 1 & x = L \end{array}$$

Weak Formulation of BVP

STEP 3:

The last step is to impose the actual BCs of the problem w has to satisfy the *homogeneous form* of specified EBC.

In weak formulation w has the meaning of a virtual change in PV. If PV is specified at a point, its variation is zero.

$$u(0)=u_0 \Rightarrow w(0)=0$$

$$\left(a\frac{du}{dx}n_x\right)_{x=L}=\left(a\frac{du}{dx}\right)_{x=L}=Q_0 \quad \text{NBC}$$

Thus

$$\int_0^L\left(\frac{dw}{dx}a\frac{du}{dx}-wq\right)dx-\left[wa\frac{du}{dx}n_x\right]_{x=0}-\left[wa\frac{du}{dx}n_x\right]_{x=L}=0$$

$$\int_0^L\left(\frac{dw}{dx}a\frac{du}{dx}-wq\right)dx-w(L)Q_0=0$$

Linear and Bilinear Forms

$$\underbrace{\int_0^L \left(\frac{dw}{dx} a \frac{du}{dx} \right) dx}_{B(w,u)} - \underbrace{\int_0^L wq\,dx - w(L)Q_0}_{l(w)} = 0 \quad \Rightarrow \quad B(w,u) - l(w) = 0$$

$B(w,u)$ Bilinear and symmetric in w and u

$l(w)$ Linear

Therefore, problem associated with the DE can be stated as one of finding the solution u such that $B(w,u) = l(w)$

holds for any w satisfies the homogeneous form of the EBC and continuity condition implied by the weak form

Linear and Bilinear Forms

Assume $u = u^* + w$ → Satisfy the homogeneous Form of EBC

Variational solution
Satisfy EBC

Actual solution
Satisfy EBC+NBC

Looking at the definition of the variational symbol, w is the variation of the solution, i.e. $w = \delta u$

Then $B(w,u) = l(w) \Rightarrow B(\delta u, u) = l(\delta u)$ (#)

$$B(\delta u, u) = \int_0^L a \frac{d\delta u}{dx}\frac{du}{dx}dx = \delta \int_0^L \frac{a}{2}\left[\left(\frac{du}{dx}\right)^2\right]dx = \frac{1}{2}\delta \int_0^L a \frac{du}{dx}\frac{du}{dx}dx = \frac{1}{2}\delta[B(u,u)]$$

$$l(\delta u) = \int_0^L \delta u q dx + \delta u(L) Q_0 = \delta\left[\int_0^L u q dx + u(L) Q_0\right] = \delta[l(u)]$$

Linear and Bilinear Forms

Substituting in (#), we have:

$$B(\delta u,u)-l(\delta u)=0 \Rightarrow \delta\left[\frac{1}{2}B(u,u)-l(u)\right]=0 \Rightarrow \delta I(u)=0$$

$$I(u)=\frac{1}{2}B(u,u)-l(u) \qquad (\#\#)$$

In general, the relation $B(\delta u,u)=\frac{1}{2}\delta B(u,u)$ holds only if

$B(w,u)$ is bilinear and symmetric and $l(w)$ is linear

If $B(w,u)$ is not linear but symmetric the functional $I(u)$ can be derived but not from (##). (see Oden & Reddy, 1976, Reddy 1986)

Linear and Bilinear Forms

Equation $\delta I(u) = 0$ represents the necessary condition for the functional $I(u)$ to have an extremum value. For solid mechanics, $I(u)$ represents the total potential energy functional and the statement of the *total potential energy principle*.

Of all admissible function u, that which makes the total potential energy $I(u)$ a minimum also satisfies the differential equation and natural boundary condition in (+).

Some Examples

Example 1

Consider the following *DE* which arise in the study of the deflection of a cable or heat transfer in a fin (when *c* = *0*).

$$-\frac{d}{dx}(a\frac{du}{dx})-cu+x^2=0 \qquad for \quad 0<x<1$$

$$u(0)=0, \qquad \left(a\frac{du}{dx}\right)_{x=1}=1$$

Step 1

$$\int_0^1 w\left[-\frac{d}{dx}(a\frac{du}{dx})-cu+x^2\right]dx=0$$

Step 2

$$\int_0^1\left(a\frac{dw}{dx}\frac{du}{dx}-cuw+wx^2\right)dx-\left(wa\frac{du}{dx}\right)_0^1=0 \Rightarrow$$

$u(0)=0$ *EBC*

$\left(a\frac{du}{dx}\right)_{x=1}=1$ *NBC*

$w(0)=0$

Some Examples

- Example 1

Step 3 $$\int_0^1 \left(a\frac{dw}{dx}\frac{du}{dx} - cuw \right) dx + \int_0^1 wx^2 dx - w(1) = 0$$

or $$B(w,u) = \int_0^1 \left(a\frac{dw}{dx}\frac{du}{dx} - cuw \right) dx$$

$$l(w) = -\int_0^1 wx^2 dx + w(1)$$

$\Rightarrow$ $$B(w,u) - l(w) = 0$$

B is bilinear and symmetric and l is linear! (prove)

Thus we can compute the quadratic functional form

$$I(u) = \frac{1}{2}\int_0^1 \left(a\left(\frac{du}{dx}\right)^2 - cu^2 + 2ux^2 \right) dx - u(1)$$

Some Examples

Example 2

Consider the following fourth-order *DE* (elastic bending of beam)

$$\frac{d^2}{dx^2}\left(b\frac{d^2w}{dx^2}\right) - f(x) = 0 \qquad for \quad 0 < x < L$$

$$w(0) = \frac{dw(0)}{dx} = 0, \quad \left(b\frac{d^2w}{dx^2}\right)_{x=L} = M_0, \quad \frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)_{x=L} = 0$$

Step 1

$$\int_0^L v\left[\frac{d^2}{dx^2}\left(b\frac{d^2w}{dx^2}\right) - f\right]dx = 0$$

Step 2

$$\int_0^L \left[\left(-\frac{dv}{dx}\right)\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right) - vf\right]dx + \left[v\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_0^L = 0$$

Some Examples

- Example 2

$$\int_0^L \left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} - vf \right) dx + \left[v\frac{d}{dx}\left(b\frac{d^2w}{dx^2} \right) - \frac{dv}{dx} b\frac{d^2w}{dx^2} \right]_0^L = 0$$

$$\frac{d}{dx}\left(b\frac{d^2w}{dx^2} \right) = V \quad (Shear \quad force)$$

$$b\frac{d^2w}{dx^2} = M \quad (Bending \quad moment)$$

B.C $\Rightarrow$

$$w(0) = \frac{dw(0)}{dx} = 0$$

$$v(0) = \frac{dv(0)}{dx} = 0$$

$$\frac{d}{dx}\left(b\frac{d^2w}{dx^2} \right)_{x=L} = 0$$

$$\left(b\frac{d^2w}{dx^2} \right)_{x=L} = M_0$$

Some Examples

- Example 2

Step 3

$$\int_0^L \left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} - vf \right)dx - \left[\frac{dv}{dx}\right]_{x=L} M_0 = 0$$

$$B(v,w) = \int_0^L \left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} \right)dx$$

or $B(v,w) = l(v)$ where

$$l(v) = \int_0^L vf dx + \left[\frac{dv}{dx}\right]_{x=L} M_0$$

Symmetric&Bilinear **Linear**

The functional $I(w)$ can be written as:

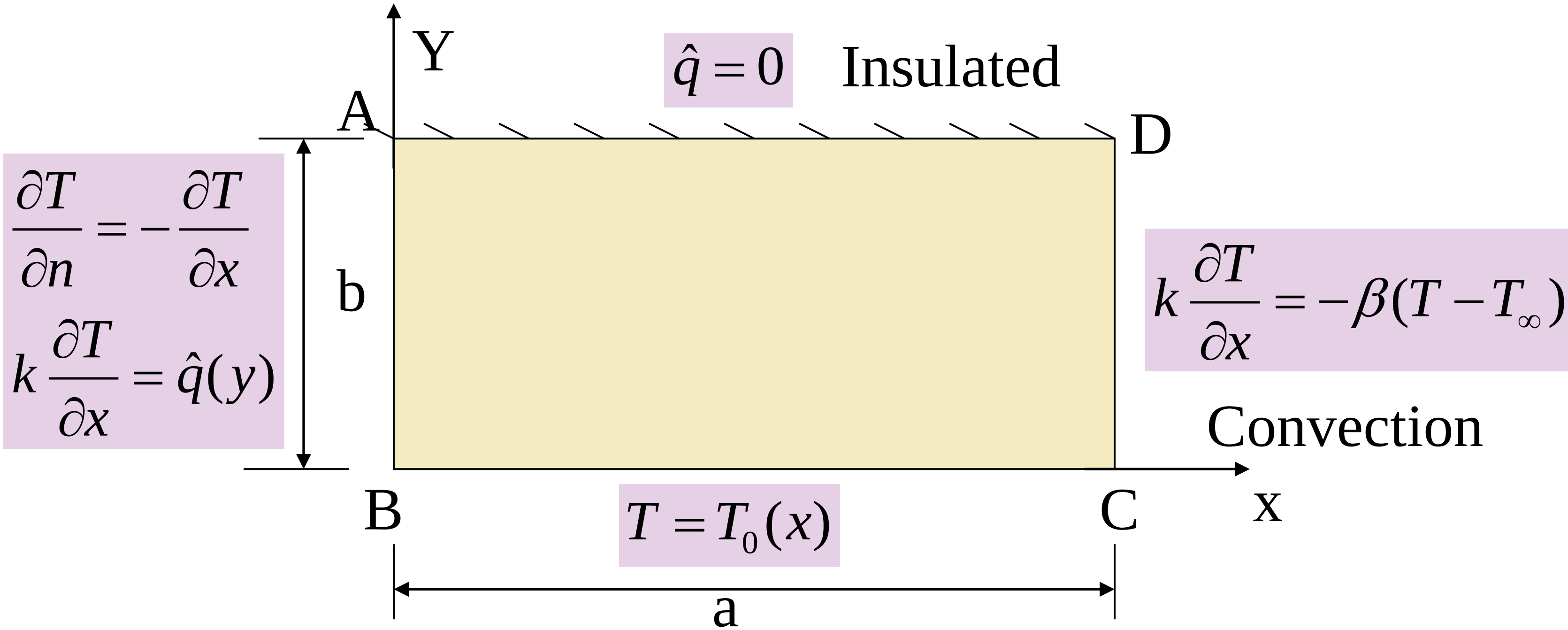
$$I(w) = \int_0^L \left[\frac{b}{2}\left(\frac{d^2w}{dx^2}\right)^2 - wf \right]dx + \left[\frac{dw}{dx}\right]_{x=L} M_0$$

- **Example 3** Steady heat conduction in a two-dimensional domain Ω

Consider a 2D heat transfer problem

$$-k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=q_0 \quad in \ \ \Omega$$

q_0: uniform heat generation
k : conductivity of the isotropic material
T : temperature



Some Examples

- Example 3

Step 1

$$\int_{\Omega} w\left[-k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)-q_0\right]dxdy=0$$

Step 2

$$\int_{\Omega}\left[k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial T}{\partial y}\right)-wq_0\right]dxdy-\oint_{\Gamma} wk\left(\frac{\partial T}{\partial x}n_x+\frac{\partial T}{\partial y}n_y\right)ds=0 \quad (*)$$

$$k\left(\frac{\partial T}{\partial x}n_x+\frac{\partial T}{\partial y}n_y\right)=k\frac{\partial T}{\partial n}=q_n$$

T=Primary variable

q_n =Secondary variable (heat flux)

$$on\ \Gamma_1 = AB\ (n_x=-1, n_y=0) \Rightarrow \hat{q}(y)$$

$$on\ \Gamma_2 = BC\ (n_x=0, n_y=-1) \Rightarrow T_0(x)$$

$$on\ \Gamma_3 = CD\ (n_x=1, n_y=0) \Rightarrow k\frac{\partial T}{\partial n}+\beta(T-T_\infty)=0$$

$$on\ \Gamma_4 = DA\ (n_x=0, n_y=1) \Rightarrow \frac{\partial T}{\partial n}=0$$

Some Examples

- Example 3

Step 3

$$\oint_{\Gamma} wk\left(\frac{\partial T}{\partial x}n_x + \frac{\partial T}{\partial y}n_y\right)ds = \oint_{\Gamma} wk\left(\frac{\partial T}{\partial n}\right)ds =$$

$$\oint_{\Gamma_1} wq_n ds + \oint_{\Gamma_2} 0k\left(\frac{\partial T}{\partial n}\right)ds - \oint_{\Gamma_3} w[\beta(T - T_\infty)]ds + \oint_{\Gamma_4} w(0)ds =$$

$$-\int_0^b w(0,y)\hat{q}(y)dy - \beta\int_0^b w(a,y)[T(a,y) - T_\infty]dy$$

w should satisfy the EBC

Substituting in (*) we have

$$\int_{\Omega}\left[k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial T}{\partial y}\right) - wq_0\right]dxdy + \int_0^b w(0,y)\hat{q}(y)dy + \beta\int_0^b w(a,y)[T(a,y) - T_\infty]dy = 0$$

$$B(w,T) = l(w)$$

Some Examples

- Example 3

$$B(w,T)=\int_{\Omega}\left[k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial T}{\partial y}\right)\right]dxdy+\beta\int_0^b w(a,y)T(a,y)dy$$

$$l(w)=\int_{\Omega} wq_0 dxdy-\int_0^b w(0,y)\hat{q}(y)dy+\beta\int_0^b w(a,y)T_\infty dy$$

The quadratic functional is given by:

$$I(T)=\frac{k}{2}\int_{\Omega}\left[\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2\right]dxdy-\int_{\Omega} Tq_0 dxdy+\int_0^b T(0,y)\hat{q}(y)dy+\beta\int_0^b \frac{1}{2}\left[T^2(a,y)-2T(a,y)T_\infty\right]dy$$

Conclusions

1- The weak form of a *DE* is the same as the statement of the total potential energy.

2- Outside solid mechanics $I(u)$ may not have meaning of energy but it is still a use mathematical tools.

3- Every *DE* admits a weighted-integral statement, or a weak form exists for every DE of order two or higher.

4- Not every DE admits a functional formulation. For a DE to have a functional formulation, its bilinear form should be symmetric in its argument.

5- Variational or FE methods do not require a functional, a weak form of the equation is sufficient.

6- If a DE has a functional, the weak form is obtained by taking its first variation.

References

1- An Introduction to the Finite Element Method, by: J. N. Reddy, 3rd ed., McGraw-Hill Education (2005). (chapter 2)

2- Energy Principles and Variational Methods in Applied Mechanics, by: J. N. Reddy, 2nd ed., John Wiley (2002). (chapter 7)