Variational Methods of Approximation

Rayleigh-Ritz Method

The approximation solution satisfies:

1- The weighted-integral form

2- The weak form

3- Minimizes the quadratic functional

$$u \approx U_N = \sum_{j=1}^{N} c_j \phi_j + \phi_0$$

Different methods differ from
1- the choice of weight function
2- the approximation function

Rayleigh-Ritz Method

1- the method uses weak form integral

2- the weight function $w = \phi_i$

General → $B(w,u) = l(w)$

Bilinear+Symm. → $I(u) = \frac{1}{2} B(u,u) - l(u)$ (*)

The solution is $U_N = \sum_{j=1}^{N} c_j \phi_j + \phi_0$

c_j → Ritz coefficient

Substituting in (*)

$$B(\phi_i, \sum_{j=1}^{N} c_j \phi_j + \phi_0) = l(\phi_i) \qquad i = 1,2,\ldots,N$$

<u>If B is bilinear then</u>

$$\sum_{j=1}^{N} B(\phi_i, \phi_j) c_j = l(\phi_i) - B(\phi_i, \phi_0) \qquad i = 1,2,\ldots,N$$

Rayleigh-Ritz Method

$$[B]\{C\}=\{F\}$$

$$B_{ij}=B(\phi_i,\phi_j) \quad , \quad F_i=l(\phi_i)-B(\phi_i,\phi_0)$$

<u>If B is bilinear and symmetric then</u>

One can seek a solution in the form of $U_N=\sum_{j=1}^{N}c_j\phi_j+\phi_0$ and the parameters can be found by minimizing the quadratic functional.

$$I(U_N)=I(c_1,c_2,\ldots,c_N)$$

Thus

$$\frac{\partial I}{\partial c_1}=0,\ \frac{\partial I}{\partial c_2}=0,\ \ldots,\ \frac{\partial I}{\partial c_N}=0$$

Rayleigh-Ritz Method

$$U_N = \sum_{j=1}^{N} c_j \phi_j$$

U_N :Should satisfy EBC

e.g. $U_N(x_0) = u_0 \Rightarrow \sum_{j=1}^{N} c_j \phi_j(x_0) = u_0$ But c_j are unknowns

It is better to write the approximation function as

$$U_N = \sum_{j=1}^{N} c_j \phi_j + \phi_0 \Rightarrow U_N(x_0) = \sum_{j=1}^{N} c_j \phi_j(x_0) + \phi_0(x_0)$$

$$= 0 + u_0$$

or

$$\phi_j(x_0) = 0$$

$$\phi_0(x_0) = u_0$$

Rayleigh-Ritz Method

The approximation function should satisfy the following conditions

1- a- ϕ_i should be such that $B(\ldots,\ldots)$ is well defined and nonzero

b- ϕ_i must satisfy at least the homogeneous form of the EBCs.

2- For any N, the set $\{\phi_i\}_{i=1}^{N}$ along with the column of the B must be linearly independent.

3- $\{\phi_i\}$ must be complete.

Rayleigh-Ritz Method

Example 1

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \qquad 0 < x < 1$$

$$u(0) = 0, \quad u(1) = 0$$

Using weak form $\quad B(w,u) = \int_0^1 \left(\frac{dw}{dx}\frac{du}{dx} - wu \right) dx, \quad l(w) = -\int_0^1 wx^2 dx$

Assume $\quad U_N = \sum_{j=1}^{N} c_j \phi_j + \phi_0$

$$\phi_0 = 0, \quad \phi_j(0) = \phi_j(1) = 0$$

We choose $\quad \phi_1 = x(1-x), \quad \phi_2 = x^2(1-x), \ \ldots, \ \phi_N = x^N(1-x),$

Rayleigh-Ritz Method

Example 1

$$U_N = \sum_{j=1}^{N} c_j \phi_j \Rightarrow \sum_{j=1}^{N} c_j \int_0^1 \left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} - \phi_i\phi_j \right) dx = -\int_0^1 \phi_i x^2 dx \Rightarrow \sum_{j=1}^{N} c_j B(\phi_i,\phi_j) = F(\phi_i)$$

where $B(\phi_i,\phi_j) = \int_0^1 \left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} - \phi_i\phi_j \right) dx,\qquad F(\phi_i) = -\int_0^1 \phi_i x^2 dx$

The same result can be obtained using quadratic functional:

$$I(u) = \frac{1}{2}\int_0^1 \left[\left(\frac{du}{dx} \right)^2 - u^2 + 2x^2 u \right] dx$$

Substituting for $u \approx U_n$, we obtain

$$I(c_j) = \frac{1}{2}\int_0^1 \left[\left(\sum_{j=1}^{N} c_j \frac{d\phi_j}{dx} \right)^2 - \left(\sum_{j=1}^{N} c_j \phi_j \right)^2 + 2x^2 \left(\sum_{j=1}^{N} c_j \phi_j \right) \right] dx$$

Example 1

The necessary conditions for the minimization of I,

$$\frac{\partial I(u)}{\partial c_i} = 0 = \int_0^1 \left[\frac{d\phi_i}{dx}\left(\sum_{j=1}^{N} c_j \frac{d\phi_j}{dx}\right) - \phi_i \left(\sum_{j=1}^{N} c_j \phi_j\right) + \phi_i x^2 \right] dx$$

$$= \sum_{j=1}^{N} B_{ij} c_j - F_i$$

where

$$B_{ij} = \int_0^1 \left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} - \phi_i \phi_j \right) dx, \qquad F_i = -\int_0^1 x^2 \phi_i dx$$

$$[B]\{c\} = \{F\}$$

Rayleigh-Ritz Method

Example 1

$$\phi_i = x^i - x^{i+1} \quad , \frac{d\phi_i}{x} = ix^{i-1} - (i+1)x^i$$

$$B_{ij} = \int_0^1 \left\{ \left[ix^{i-1} - (i+1)x^i\right]\left[jx^{j-1} - (j+1)x^j\right] - (x^i - x^{i+1})(x^j - x^{j+1}) \right\} dx$$

$$B_{ij} = \frac{2ij}{(i+j)[(i+j)^2 - 1]} - \frac{2}{(i+j+1)(i+j+2)(i+j+3)}$$

$$F_i = -\frac{1}{(3+i)(4+i)}$$

For $N=2$, $$\{c\} = [B]^{-1}\{F\} = \begin{bmatrix} 126 & 63 \\ 63 & 52 \end{bmatrix}^{-1} \begin{Bmatrix} -21 \\ -14 \end{Bmatrix}$$

$c_1=-0.0813$, $c_2=-0.1707$

$$u_2 = c_1\phi_1 + c_2\phi_2 = -0.0813(x - x^2) + -0.1707(x^2 - x^3)$$

Rayleigh-Ritz Method

Example 1

$$u_{exact}(x) = \frac{\sin x + 2\sin(1-x)}{\sin 1} + x^2 - 2$$

x	N=1	N=2	N=3	*Exact*
0.0	0.0	0.0	0.0	0.0
0.1	0.1500	0.0885	0.0954	0.0955
0.2	0.2667	0.1847	0.1890	0.1890
0.3	0.3500	0.2783	0.2766	0.2764
0.4	0.4000	0.3590	0.3520	0.3518
0.5	0.4167	0.4167	0.4076	0.4076
0.6	0.4000	0.4410	0.4340	0.4342
0.7	0.3500	0.4217	0.4200	0.4203
0.8	0.2667	0.3486	0.3529	0.3530
0.9	0.1500	0.2115	0.2183	0.2182
1.0	0.0	0.0	0.0	0.0

Example 2

Consider the Poisson equation in a unit square

$$-k\nabla^2 T = q_0 \qquad 0<(x,y)<1 \qquad \begin{cases} T=0 & x=1,\quad y=1 \\ \dfrac{\partial T}{\partial n}=0 & x=0,\quad y=0 \end{cases}$$

Variational formulation is in the form of:

$$B(w,T)=l(w); \qquad B(w,T)=\int_0^1\int_0^1 k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial T}{\partial y}\right)dxdy$$

$$l(w)=\int_0^1\int_0^1 wq_0\,dxdy$$

Rayleigh-Ritz Method

Example 2

$$T_N=\sum_{i=1}^{N}\sum_{j=1}^{N}C_{ij}\cos\alpha_i x\cos\alpha_j y \qquad , \qquad \alpha_i=\frac{1}{2}(2i-1)\pi$$

$$B_{(ij)(kl)}=k\int_0^1\int_0^1\Big[(\alpha_i\sin\alpha_i x\cos\alpha_j y)(\alpha_k\sin\alpha_k x\cos\alpha_l y)$$

$$+(\alpha_j\sin\alpha_i x\cos\alpha_j y)(\alpha_l\sin\alpha_k x\cos\alpha_l y)\Big]dxdy$$

$$=\begin{cases}0 & if \quad i\neq k \ or \quad j\neq l\\ 1/4k(\alpha_i^2+\alpha_j^2) & if \quad i=k \ and \ j=l\end{cases}$$

$$F_{ij}=q_0\int_0^1\int_0^1\cos\alpha_i x\cos\alpha_j y\,dx\,dy=\frac{q_0}{\alpha_i\alpha_j}\sin\alpha_i\sin\alpha_j$$

Example 2

In evaluating the integrals, the following orthogonality conditions are used

$$\int_0^1 \cos\alpha_i x \cos\alpha_j x\, dx = \begin{cases} 0 & \text{if } \ i \neq j \\ \dfrac{1}{2} & \text{if } \ i = j \end{cases} \quad , \quad \int_0^1 \sin\alpha_i x \sin\alpha_j x\, dx = \begin{cases} 0 & \text{if } \ i \neq j \\ \dfrac{1}{2} & \text{if } \ i = j \end{cases}$$

Finally,

$$C_{ij} = \frac{F_{ij}}{B_{(ij)(ij)}} = \frac{4q_0}{k} \frac{\sin\alpha_i \sin\alpha_j}{(\alpha_i^2 + \alpha_j^2)\alpha_i \alpha_j}$$

One- and two-parameter Rayleigh-Ritz solutions are:

$$T_1 = \frac{32q_0}{k\pi^4} \cos\frac{1}{2}\pi x \cos\frac{1}{2}\pi y$$

(+)

$$T_2 = \frac{q_0}{k}\Big[0.3285 \cos(\pi x/2)\cos(\pi y/2) - 0.0219\big(\cos(\pi x/2)\cos(3\pi y/2) + \cos(3\pi x/2)\cos(\pi y/2)\big) + 0.0041 \cos(3\pi x/2)\cos(3\pi y/2)\Big]$$

Rayleigh-Ritz Method

Example 2

If polynomials are to be used in the approximation of T

$$\phi_1 = (1-x^2)(1-y^2)$$

One-parameter Ritz solution:

$$T_1(x,y) = \frac{5q_0}{16k}(1-x^2)(1-y^2) \qquad (*)$$

The exact solution:

$$T(x,y) = \frac{q_0}{2k}\left[(1-y^2) + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos\alpha_n y \cosh\alpha_n x}{\alpha_n^3 \cosh\alpha_n}\right], \qquad \alpha_n = \frac{1}{2}(2n-1)\pi$$

$$(\#)$$

Rayleigh-Ritz Method

Example 2