Weighted Residual Methods

Formulation of FEM Model

Formulation of FEM Model
- *Direct Method*
- *Variational Method*
- *Weighted Residuals*

• Several approaches can be used to transform the physical formulation of a problem to its finite element discrete analogue.

• If the physical formulation of the problem is described as a differential equation, then the most popular solution method is the *Method of Weighted Residuals*.

• If the physical problem can be formulated as the minimization of a functional, then the *Variational Formulation* is usually used.

Formulation of FEM Model

Finite element method is used to solve physical problems

Solid Mechanics
Fluid Mechanics
Heat Transfer
Electrostatics
Electromagnetism
….

Physical problems are governed by **differential equations** which satisfy

Boundary conditions

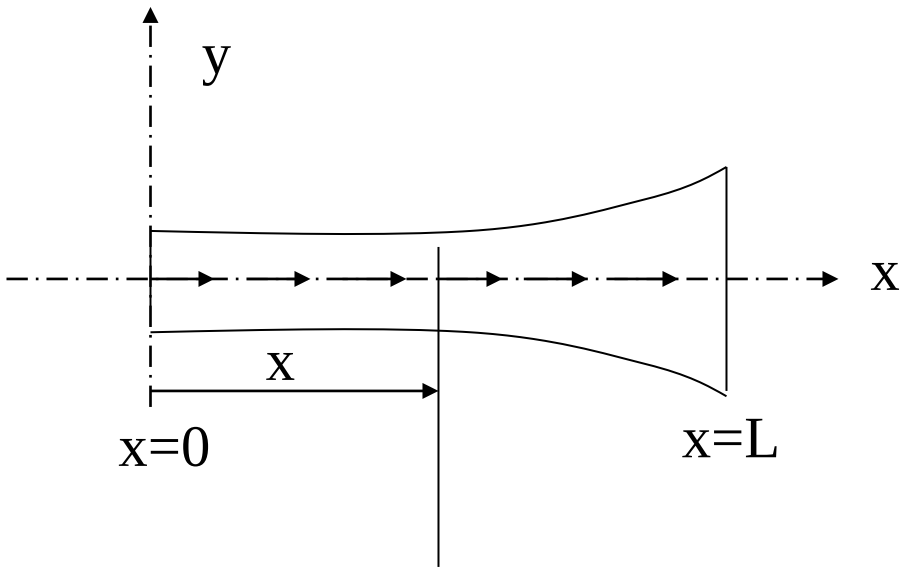
Initial conditions

One variable: Ordinary differential equation (ODE)

Multiple independent variables: Partial differential equation (PDE)

Physical problems

Axially loaded elastic bar



$A(x)$ = cross section at x
$b(x)$ = body force distribution (force per unit length)
$E(x)$ = Young's modulus
$u(x)$ = displacement of the bar at x

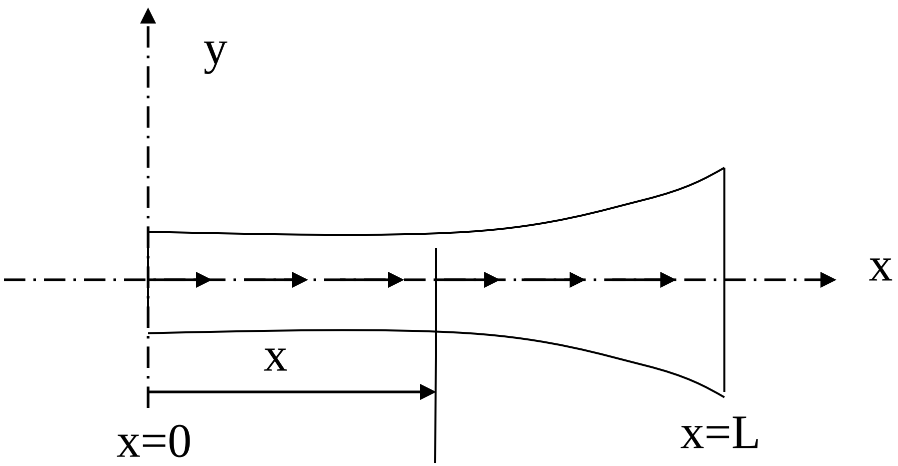
Differential equation governing the response of the bar

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b=0; \qquad 0<x<L$$

Second order differential equations
Requires 2 boundary conditions for solution

Physical problems

Axially loaded elastic bar



Boundary conditions (examples)

$$u = 0 \quad at \quad x = 0$$ Dirichlet/ displacement bc

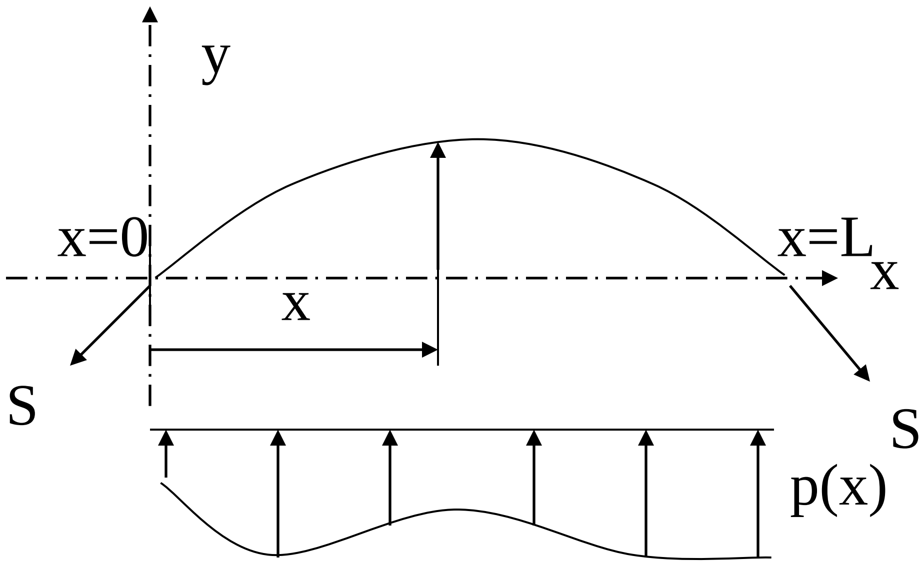
$$u = 1 \quad at \quad x = L$$

$$u = 0 \quad at \quad x = 0$$

$$EA\frac{du}{dx} = F \quad at \ x = L$$ Neumann/ force bc

Differential equation + **Boundary conditions** = **Strong form** of the "boundary value problem"

Flexible string



S = tensile force in string
p(x) = lateral force distribution (force per unit length)
w(x) = lateral deflection of the string in the y-direction

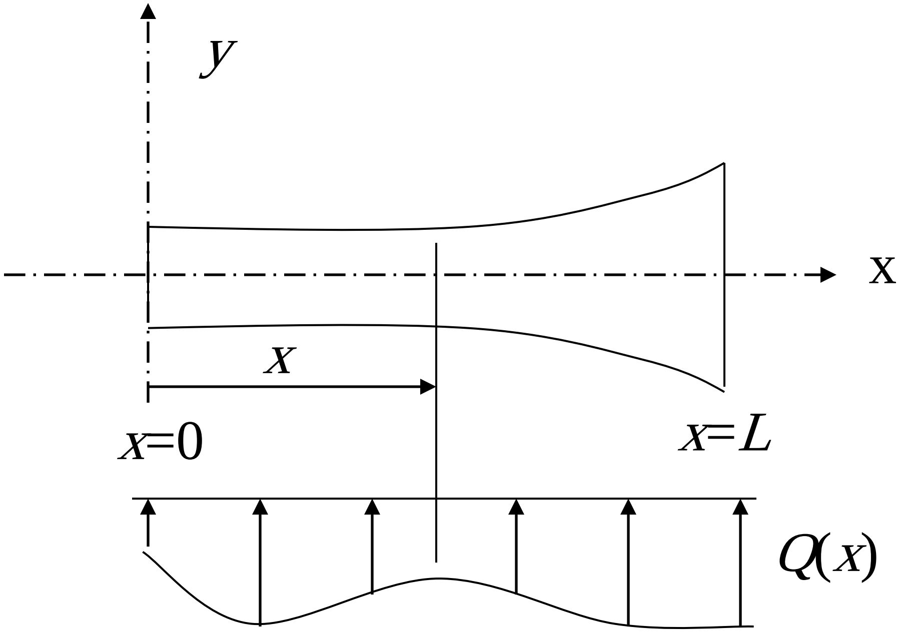
Differential equation governing the response of the bar

$$S\frac{d^2u}{dx^2} + p = 0; \qquad 0 < x < L$$

Second order differential equations
Requires 2 boundary conditions for solution

Physical problems

Heat conduction in a fin



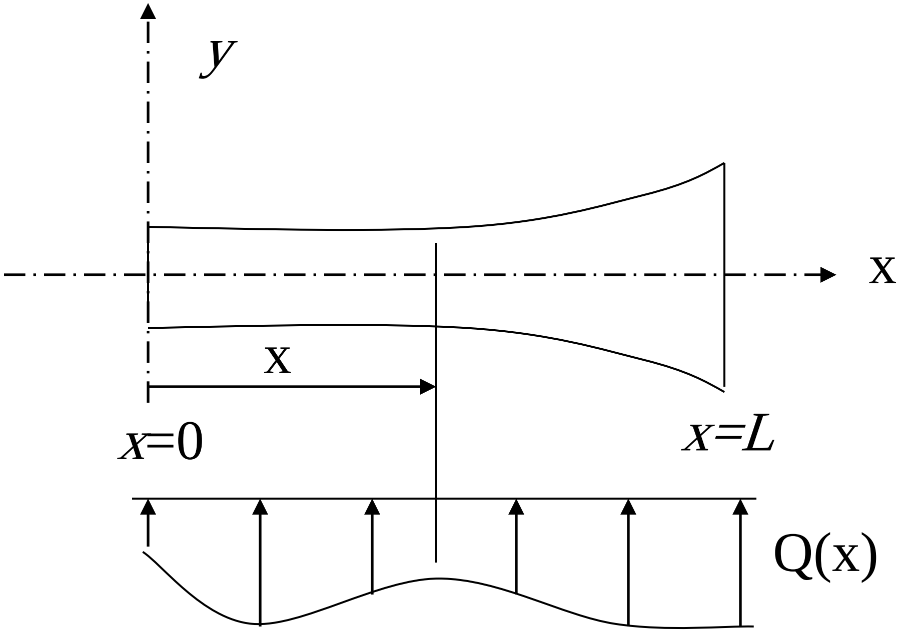
$A(\mathrm{x})$ = cross section at x
$Q(\mathrm{x})$ = heat input per unit length per unit time [J/sm]
$k(\mathrm{x})$ = thermal conductivity [J/^{o}C ms]
$T(\mathrm{x})$ = temperature of the fin at x

Differential equation governing the response of the fin

$$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right)+Q=0; \qquad 0<x<L$$

Second order differential equations
Requires 2 boundary conditions for solution

Heat conduction in a fin



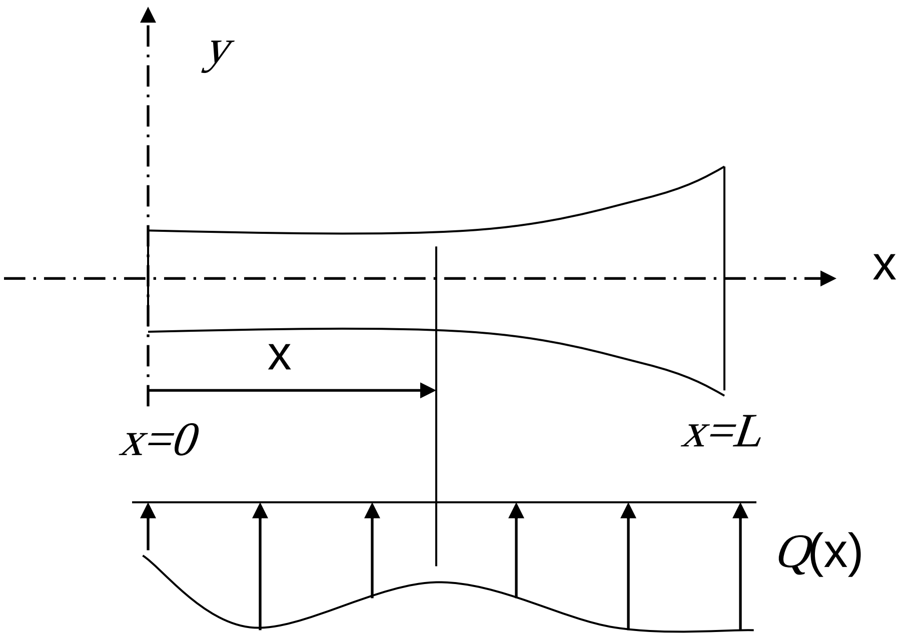
Boundary conditions (examples)

$T = 0 \quad at \quad x = 0$ Dirichlet/ displacement bc

$-k\dfrac{dT}{dx} = h \quad at \;\; x = L$ Neumann/ force bc

Physical problems

Fluid flow through a porous medium (e.g., flow of water through a dam)



A(x) = cross section at x
Q(x) = fluid input per unit volume per unit time
k(x) = permeability constant
φ(x) = fluid head

Differential equation

$$\frac{d}{dx}\left(k\frac{d\varphi}{dx}\right)+Q=0; \qquad 0<x<L$$

Boundary conditions (examples)

$\varphi=0 \quad at \quad x=0$ Known head

$-k\frac{d\varphi}{dx}=h \quad at \quad x=L$ Known velocity

Second order differential equations
Requires 2 boundary conditions for solution

Physical problems

Table 4.1 Examples of second-order differential equations

Differential equation	Physical problem	Quantities	Constitutive law
$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right)+Q=0$	One-dimensional heat flow	T = temperature A = area k = thermal conductivity Q = heat supply	Fourier $q=-k\,dT/dx$ q = heat flux
$\frac{d}{dx}\left(AE\frac{du}{dx}\right)+b=0$	Axially loaded elastic bar	u = displacement A = area E = Young's modulus b = axial loading	Hooke $\sigma=E\,du/dx$ σ = stress
$S\frac{d^2w}{dx^2}+p=0$	Transversely loaded flexible string	w = deflection S = string force p = lateral loading	
$\frac{d}{dx}\left(AD\frac{dc}{dx}\right)+Q=0$	One-dimensional diffusion	c = iron concentration A = area D = diffusion coefficient Q = ion supply	Fick $q=-D\,dc/dx$ q = ion flux

Physical problems

Table 4.1 Examples of second-order differential equations

Differential equation	Physical problem	Quantities	Constitutive law
$\frac{d}{dx}\left(A\gamma\frac{dV}{dx}\right)+Q=0$	One-dimensional electric current	V = voltage A = area γ = electric conductivity Q = electric charge supply	Ohm $q=-\gamma\, dV/dx$ q = electric charge flux
$\frac{d}{dx}\left(A\frac{D^2}{32\mu}\frac{dp}{dx}\right)+Q=0$	Laminar flow in pipe (Poiseuille flow)	p = pressure A = area D = diameter μ = viscosity Q = fluid supply	$q=-(D^2/32\mu)\, dp/dx$ q = volume flux q = mean velocity

Formulation of FEM Model

Observe:

1. All the cases we considered lead to very similar differential equations and boundary conditions.
2. In *1D* it is easy to analytically solve these equations
3. Not so in 2 and 3D especially when the geometry of the domain is complex: need to solve **approximately**
4. We'll learn how to solve these equations in 1D. The approximation techniques easily translate to 2 and 3D, no matter how complex the geometry

A generic problem in 1D

$$\frac{d^2u}{dx^2} + x = 0; \qquad 0 < x < 1$$

$$u = 0 \quad at \;\; x = 0$$

$$u = 1 \quad at \;\; x = 1$$

Analytical solution

$$u(x) = -\frac{1}{6}x^3 + \frac{7}{6}x$$

Assume that we **do not know** this solution.

Formulation of FEM Model

A generic problem in 1D

A general algorithm for approximate solution:

Guess $u(x) \approx a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots$

where $\varphi_0(x)$, $\varphi_1(x)$,… are "known" functions and a_o, a_1, etc are constants chosen such that the approximate solution

Satisfies the differential equation

Satisfies the boundary conditions

i.e.,

$$a_0 \frac{d^2\varphi_o(x)}{dx^2} + a_1 \frac{d^2\varphi_1(x)}{dx^2} + a_2 \frac{d^2\varphi_2(x)}{dx^2} + \ldots + x = 0; \qquad 0 < x < 1$$

$$a_0\varphi_o(0) + a_1\varphi_1(0) + a_2\varphi_2(0) + \ldots = 0$$

$$a_0\varphi_o(1) + a_1\varphi_1(1) + a_2\varphi_2(1) + \ldots = 1$$

Solve for unknowns a_o, a_1, etc and plug them back into

$$u(x) \approx a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots$$

This is your **<u>approximate solution</u>** to the strong form

Formulation of FEM Model

Solution of Continuous Systems – Fundamental Concepts

Exact solutions

limited to simple geometries and boundary & loading conditions

Approximate Solutions

Reduce the continuous-system mathematical model to a discrete idealization

Variational

- *Rayleigh Ritz Method*

Weighted Residual Methods

- *Galerkin*
- *Least Square*
- *Collocation*
- *Subdomain*

Weighted Residual Formulations

Consider a general representation of a governing equation on a region V

$$L\bar{u} = 0$$

L is a differential operator

eg. For Axial element $$\frac{d}{dx}\left(EA\frac{du}{dx}\right) - P(x) = 0$$

$$L = \frac{d}{dx}EA\frac{d}{dx}(\ \ ) - P(x)$$

Assume approximate solution u

then $$Lu = R$$

Weighted Residual Formulations

$$L\bar{u} = 0 \qquad\qquad Lu = R$$

Exact — Approximate

$$ERROR \;=\; L(u) = R$$

Objective:

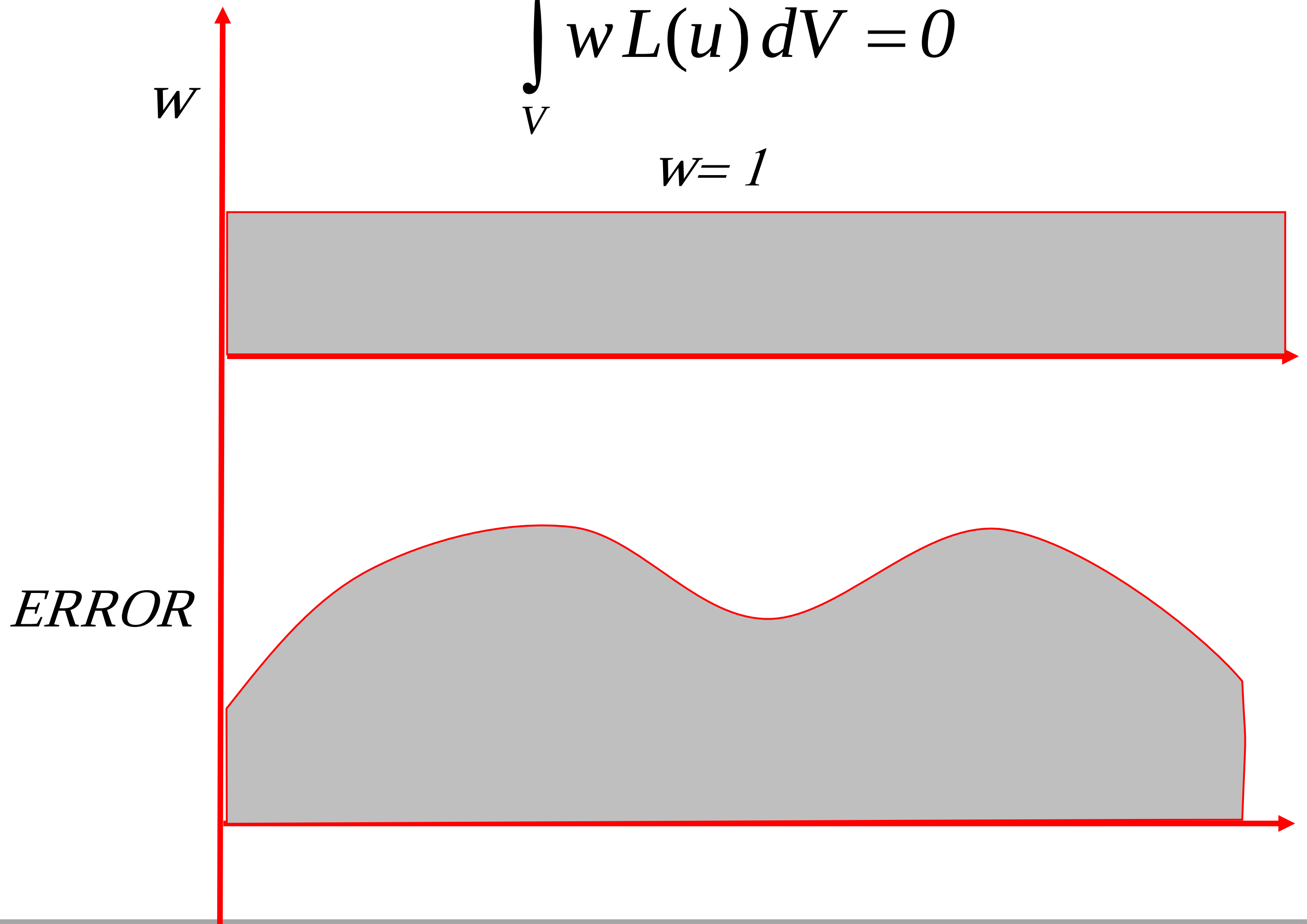
Define u so that weighted average of Error vanishes

Set Error relative to a weighting function w

$$\int_V w\,L(u)\,dV = 0 \qquad or \qquad \int_V w\,R\,dV = 0$$

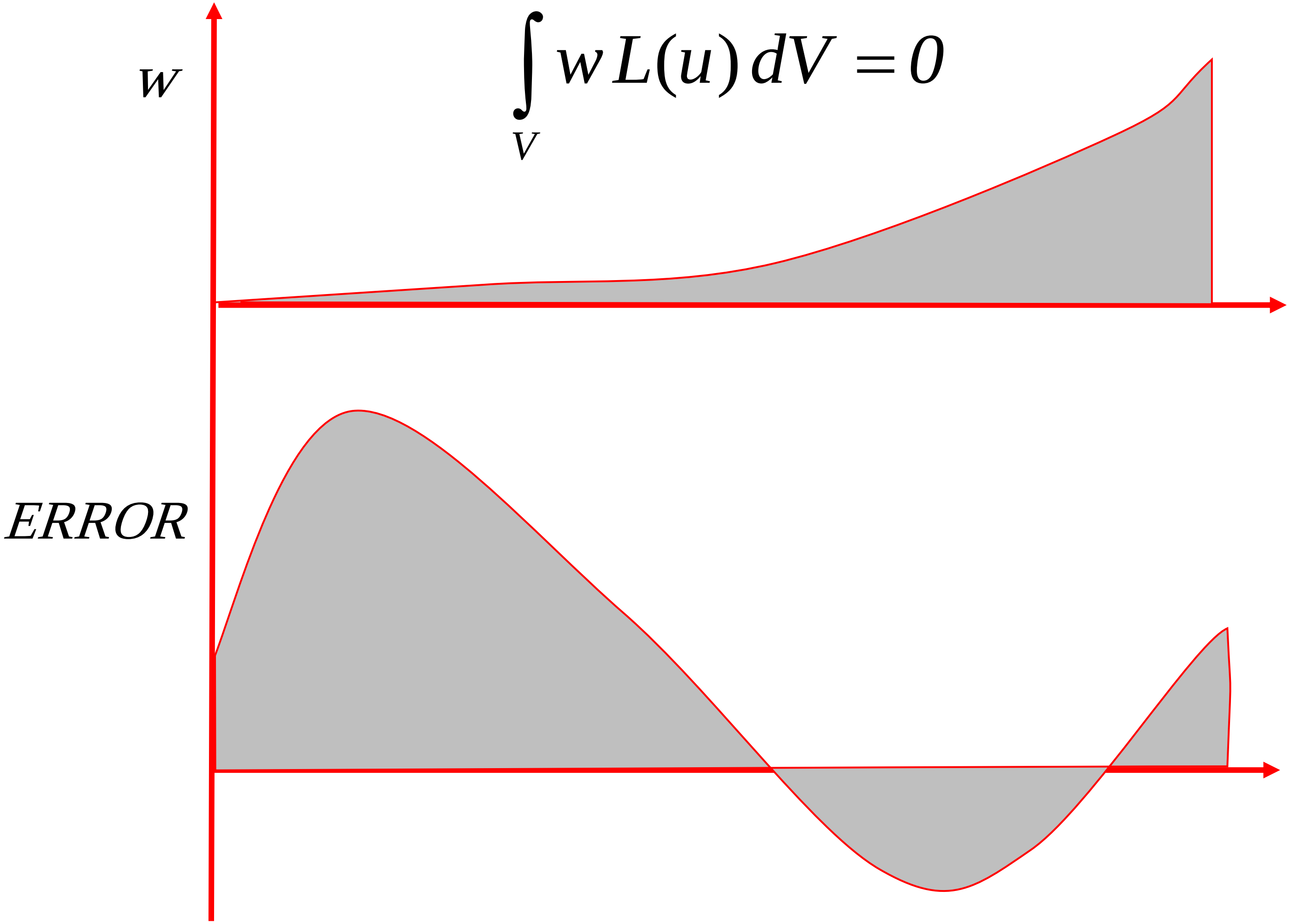
Weighted Residual Formulations

$$\int_V w\,L(u)\,dV = 0$$



Weighted Residual Formulations

$$\int_V w\, L(u)\, dV = 0$$



Weighted Residual Methods

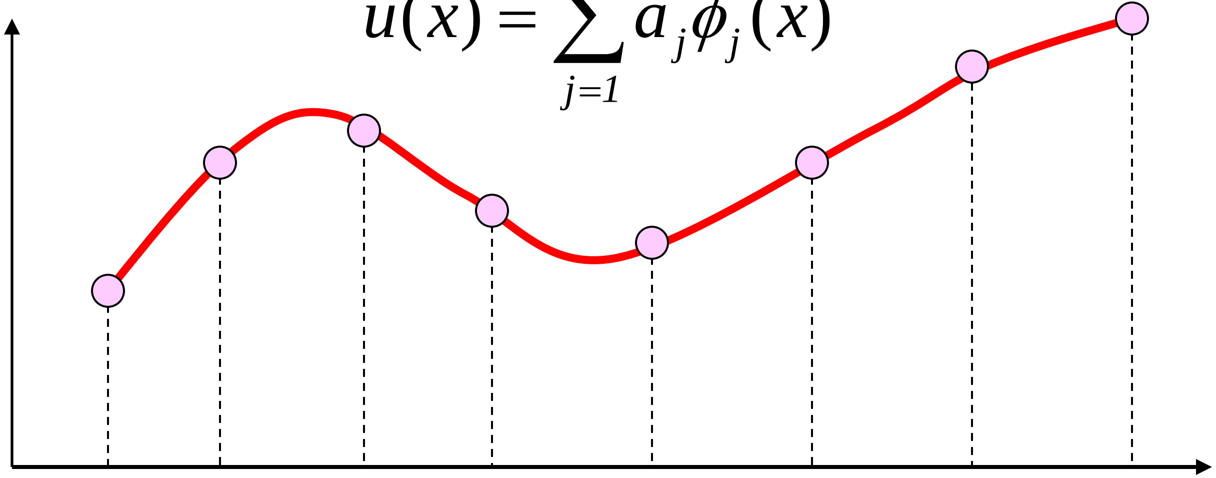
- Start with the integral form of governing equations
- Assume functional form for trial (interpolation, shape) functions
- Minimize errors (residuals) with selected weighting functions

$$w_j(x) = \begin{cases} x^j & \text{Power series} \\ \sin jx, \cos jx & \text{Fourier series} \\ L_j(x) & \text{Lagrange} \\ H_j(x) & \text{Hermite} \\ T_j(x) & \text{Chebychev} \end{cases}$$

Weighted Residual Methods

- Assume certain profile (trial or shape function) between nodes

$$u(x) = \sum_{j=1}^{J} a_j \phi_j(x)$$



$$\begin{cases} L(u) = R(x) \neq 0 \,, & \text{but} \\ \int w R \, dx = \int w L(u) dx = 0 \end{cases}$$

Residual

Weighted Residual

Weighted Residual Methods

- In general, we deal with the numerical integration of trial or interpolation functions
- Trial functions:

  constant, linear, quadratic, sinusoidal, Chebychev polynomial, ….
- Weighting functions:

  subdomain, collocation, least square, Galerkin, ….

$$\int_V w(x,y,z)R(x,y,z)dxdydz = \int_V wL(u)dv = 0$$

General Formulation

- Weighted Residual Methods (WRMs)
- Construct an approximate solution

$$u(x,y,z) = u_o(x,y,z) + \sum_{j=1}^{J} a_j \phi_j(x,y,z)$$

Chosen to satisfy I.C./B.C.s if possible

- Steady problems – system of algebraic equations for trial function $\phi_j(x,y,z)$
- Transient problems – system of ODEs in time

Weighted Residual Methods

- Consider one-dimensional diffusion equation

$$\begin{cases} L(\bar{u}) = 0 & \text{Exact solution} \\ L(u) = R(x) \neq 0 & \text{Approximation} \end{cases}$$

- In general, R$\rightarrow$ 0 with increasing J (higher-order)

$$\iiint w_m(x,y,z)R(x,y,z)dxdydz = 0, \quad m = 1,2,\cdots,M$$

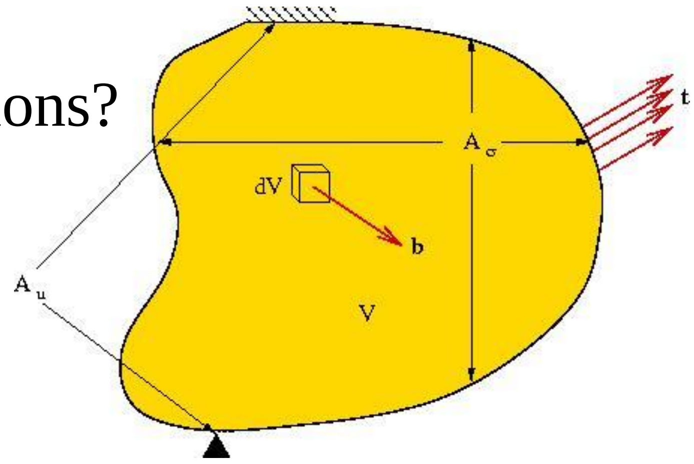
- Weak form – integral form, discontinuity allowed (discontinuous function and/or slope)

Weighted Residual Methods

- Weak form – integral formulation

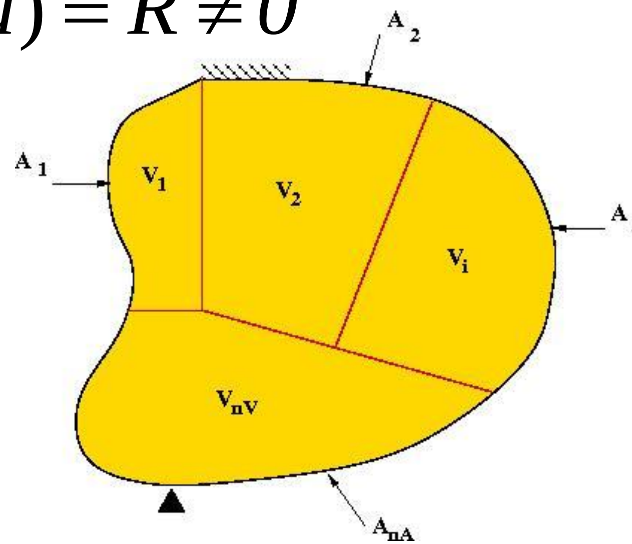
$$\begin{cases} \text{Differential Form:} & L(\bar{u}) = 0 \\ \text{Exact Integral Form:} & \iiint w(x,y,z) L(\bar{u}) dxdydz = 0 \\ \text{Discretization :} & \iiint w_m(x,y,z) L(u) dxdydz = 0 \end{cases}$$

- $R \neq 0$, but "weighted R" = 0
- Choices of shape or interpolation functions?
- Choices of weighting functions?



Subdomain Method

$$\iiint w_m(x,y,z)L(u)dxdydz = 0 \quad ; \quad L(u) = R \neq 0$$



- Equivalent to finite volume method

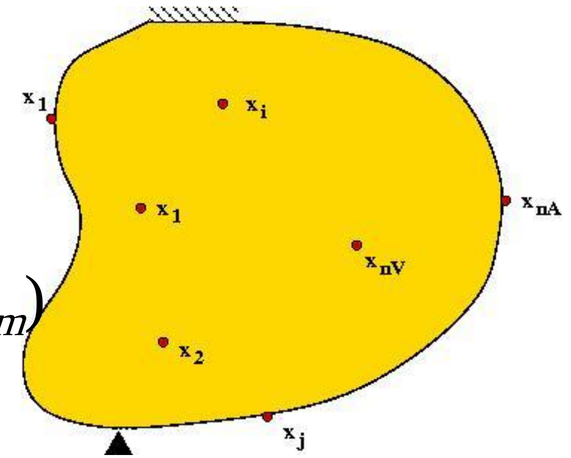
$$w_m = \begin{cases} 1, & \text{in } D_m \\ 0, & \text{outside } D_m \end{cases}$$

$$\iiint w_m L(u)dxdydz = \iiint_{D_m} R(x,y,z)dv = 0$$

- D_m : numerical element (arbitrary control volume)
- D_m may be overlapped

Collocation Method

$$\iiint w_m(x,y,z)L(u)dxdydz = 0 \quad ; \quad L(u) = R \neq 0$$



- Zero residuals at selected locations (x_m, y_m, z_m)

$$w_m(\vec{x}) = \delta(\vec{x} - \vec{x}_m)$$

$$\iiint w_m(\vec{x})R(\vec{x})dv = \iiint \delta(\vec{x} - \vec{x}_m)R(\vec{x})dv$$

$$= R(\vec{x}_m) = R(x_m, y_m, z_m)$$

- No control on the residuals between nodes

Least Square Method

$$\iiint w_m(x,y,z)L(u)dxdydz = 0 \quad ; \quad L(u) = R \neq 0$$

- Minimize the square error

$$\frac{\partial}{\partial a_m}\int_V R^2(x,y,z,a_m)dxdydz = 0 \Rightarrow \int_V \frac{\partial R}{\partial a_m} R dxdydz = 0$$

$$w_m(\vec{x}) = \frac{\partial R}{\partial a_m}$$

Square error
$R^2 \neq 0$

$$\iiint w_m(\vec{x})R(\vec{x})d\vec{x} = \frac{1}{2}\frac{\partial}{\partial a_m}\iiint R^2 d\vec{x} = 0$$

$$R^2 \geq 0$$

Galerkin Method

$$\iiint w_m(x,y,z)L(u)\,dxdydz = 0 \quad ; \quad L(u) = R \neq 0$$

- Weighting function = trial (interpolation) function

$$w_m(\vec{x}) = \phi_m(\vec{x})$$

$$\iiint w_m(\vec{x})R(\vec{x})\,d\vec{x} = \iiint \phi_m(\vec{x})R(\vec{x})\,d\vec{x}$$

- For orthogonal polynomials, the residual *R* is orthogonal to every member of a complete set!

Numerical Accuracy

- How do we determine the most accurate method?
- How should the error be "weighted"?
- Zero average error?
- Least square error?
- Least rms error?
- Minimum error within selected domain?
- Minimum (zero) error at selected points?
- Minimax – minimize the maximum error?
- Some functions have fairly uniform error distributions comparing to the others

Application to an ODE

- Consider a simple ODE (Initial value problem)

$$\begin{cases} \dfrac{d\bar{y}}{dx} - \bar{y} = 0 \;, \quad 0 \le x \le 1 \\ \bar{y}(0) = 1 \end{cases} \Rightarrow \quad \bar{y} = e^{x}$$

- Use global method with only one element
- Select a trial function of the form of

$$y = 1 + \sum_{j=1}^{N} a_j x^j$$

- Automatically satisfy the auxiliary condition
- a_j = constant, not a function of time

Application to an ODE

- Consider a cubic interpolation function with N = 3

$$y = 1 + \sum_{j=1}^{N} a_j x^j = 1 + a_1 x + a_2 x^2 + a_3 x^3$$

- QUESTION: Which cubic polynomial gives the best fit to the exact (exponential function) solution?
- Definition of best fit?
- Zero average error, least square, least rms, …?

Residual

- Substitute the trial function into governing equation

$$R = L(y) = \frac{dy}{dx} - y = \sum_{j=1}^{N} j a_j x^{j-1} - \left(1 + \sum_{j=1}^{N} a_j x^j \right) = -1 + \sum_{j=1}^{N} a_j x^{j-1} (j - x)$$

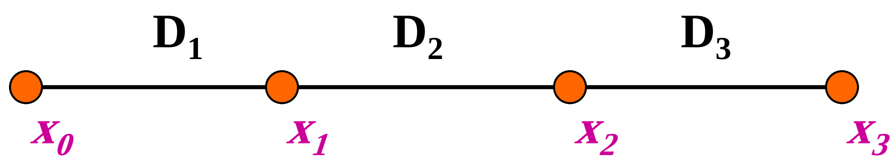
- For cubic interpolation function N = 3

$$R(x) = -1 + a_1(1-x) + a_2(2x - x^2) + a_3(3x^2 - x^3)$$
$$= (a_1 - 1) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 - a_3 x^3 \neq 0$$

- The residual is a cubic polynomial $\Rightarrow R \neq 0$
- Determine the optimal values of a_j to minimize the error (under pre-selected weighting functions)

Subdomain Method

- Zero average error in each subdomain



Uniform spacing

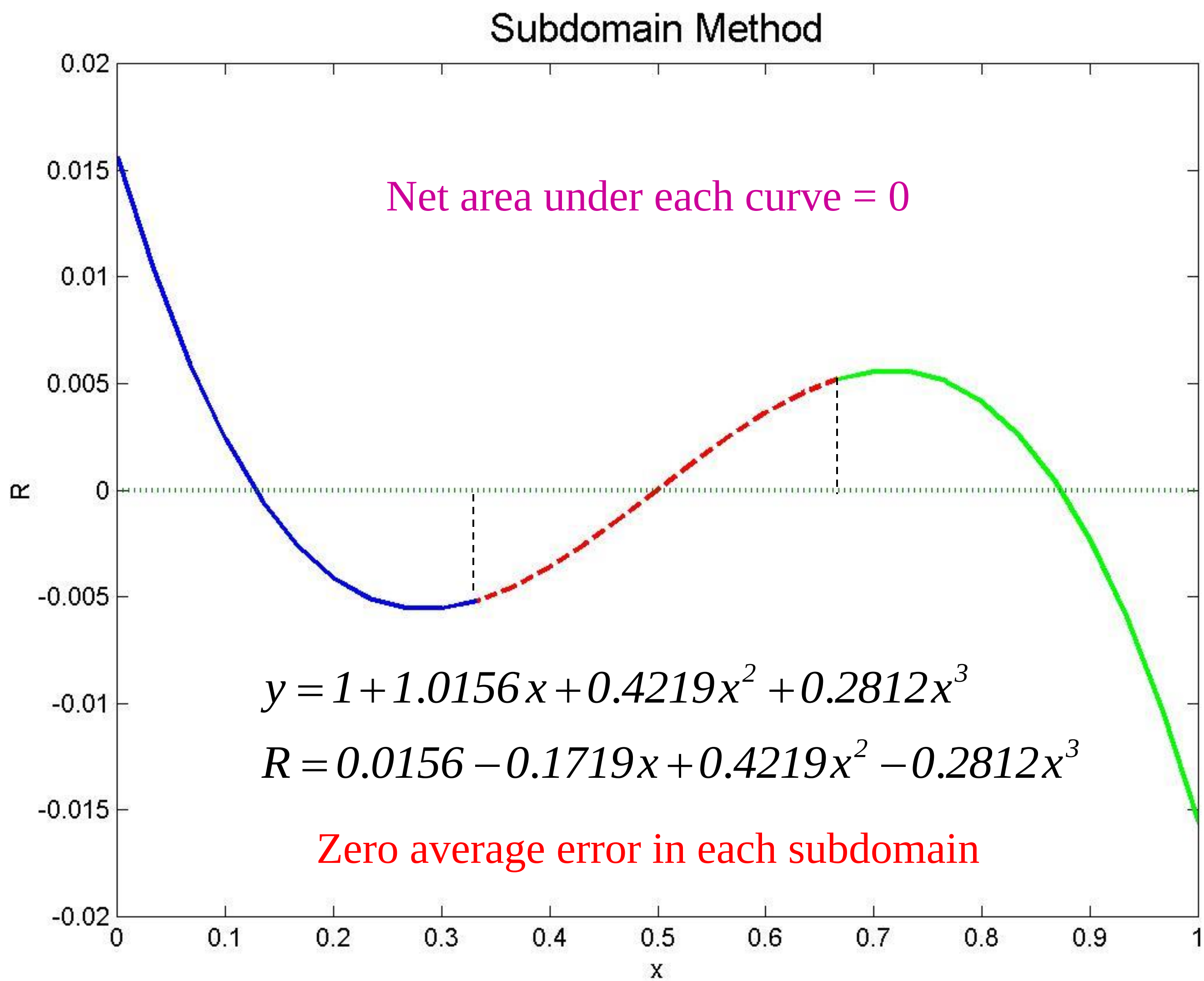
$$\int_0^1 w_m R dx = \int_{x_{m-1}}^{x_m} R(x)dx = 0 = \left\{(a_1 - 1)x + (a_2 - \frac{a_1}{2})x^2 + (a_3 - \frac{a_2}{3})x^3 - \frac{a_3}{4}x^4\right\}\Bigg|_{x_{m-1}}^{x_m}$$

$$\begin{cases} m=1: & \int_0^{1/3} R dx = 0 & \Rightarrow \frac{5}{18}a_1 + \frac{8}{81}a_2 + \frac{11}{324}a_3 = \frac{1}{3} \\ m=2: & \int_{1/3}^{2/3} R dx = 0 & \Rightarrow \frac{3}{18}a_1 + \frac{20}{81}a_2 + \frac{69}{324}a_3 = \frac{1}{3} \\ m=3: & \int_{2/3}^{1} R dx = 0 & \Rightarrow \frac{1}{18}a_1 + \frac{26}{81}a_2 + \frac{163}{324}a_3 = \frac{1}{3} \end{cases}$$

$$\vec{a}_i = \begin{bmatrix} 1.0156 \\ 0.4219 \\ 0.2813 \end{bmatrix}$$

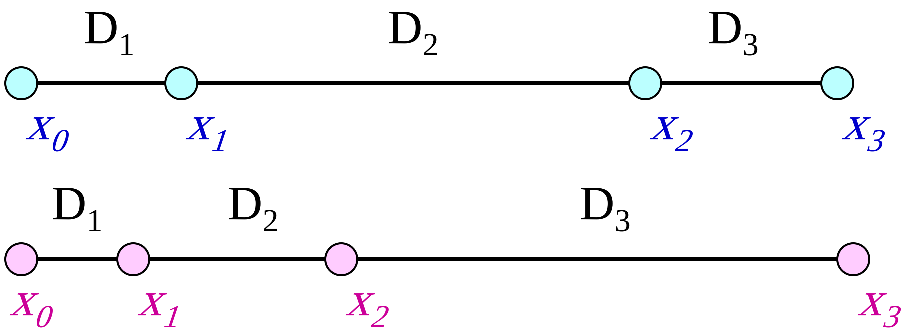
- Note: $R(0) = 0.0156 \neq 0$, $R(1) = -0.0155 \neq 0$

Subdomain Method



Subdomain Method

- Nonuniform subdomains?



Grid clustering in high-gradient regions

$$\begin{cases} m=1: & \int_{x_0}^{x_1} R\,dx = 0 \\ m=2: & \int_{x_1}^{x_2} R\,dx = 0 \\ m=3: & \int_{x_2}^{x_3} R\,dx = 0 \end{cases}$$

Different coefficients for different choices of subdomains

Least Square Method

- Minimum square errors over the entire domain

$$R(x) = -1 + \sum_{m=1}^{N} a_m (mx^{m-1} - x^m) \qquad \Rightarrow \frac{\partial R}{\partial a_m} = mx^{m-1} - x^m$$

$$\int_0^1 w_m R dx = \int_0^1 \frac{\partial R(x)}{\partial a_m} R(x) dx = 0$$

$$= -\int_0^1 (mx^{m-1} - x^m) dx + \sum_{j=1}^{N} a_j \int_0^1 \left[mj\ x^{m+j-2} - (j+m) x^{m+j-1} + x^{m+j} \right] dx$$

- For arbitrary N (symmetric matrix)

$$\sum_{j=1}^{N} a_j \left(\frac{mj}{m+j-1} - 1 + \frac{1}{m+j+1} \right) = -1 + \frac{1}{m+1} = -\frac{m}{m+1}$$

Least Square Method

- For cubic interpolation function (N=3)

$$R=\left(a_1-1\right)+\left(2a_2-a_1\right)x+\left(3a_3-a_2\right)x^2-a_3x^3$$

$$\begin{cases} m=1, \quad w_1=\dfrac{\partial R}{\partial a_1}=1-x & \Rightarrow \int_0^1 \left(1-x\right)Rdx=0 \\ m=2, \quad w_2=\dfrac{\partial R}{\partial a_2}=2x-x^2 & \Rightarrow \int_0^1 \left(2x-x^2\right)Rdx=0 \\ m=3, \quad w_3=\dfrac{\partial R}{\partial a_3}=3x^2-x^3 & \Rightarrow \int_0^1 \left(3x^2-x^3\right)Rdx=0 \end{cases}$$

- Nonuniform weighting of residuals over the domain

Least Square Method

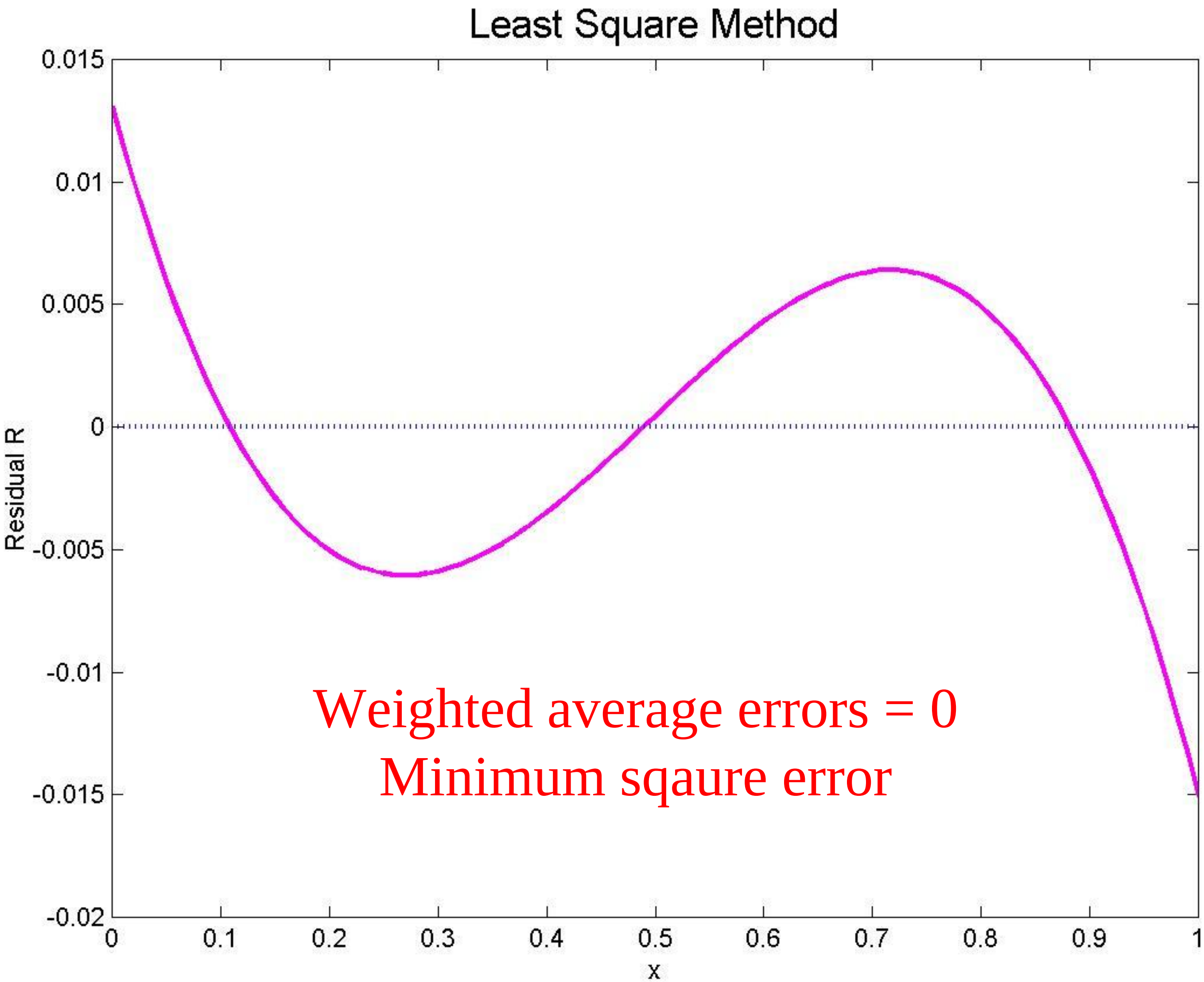
- Cubic trial function

$$\sum_{j=1}^{3} a_j \left(\frac{mj}{m+j-1} - \frac{m+j}{m+j+1} \right) = -\frac{m}{m+1}$$

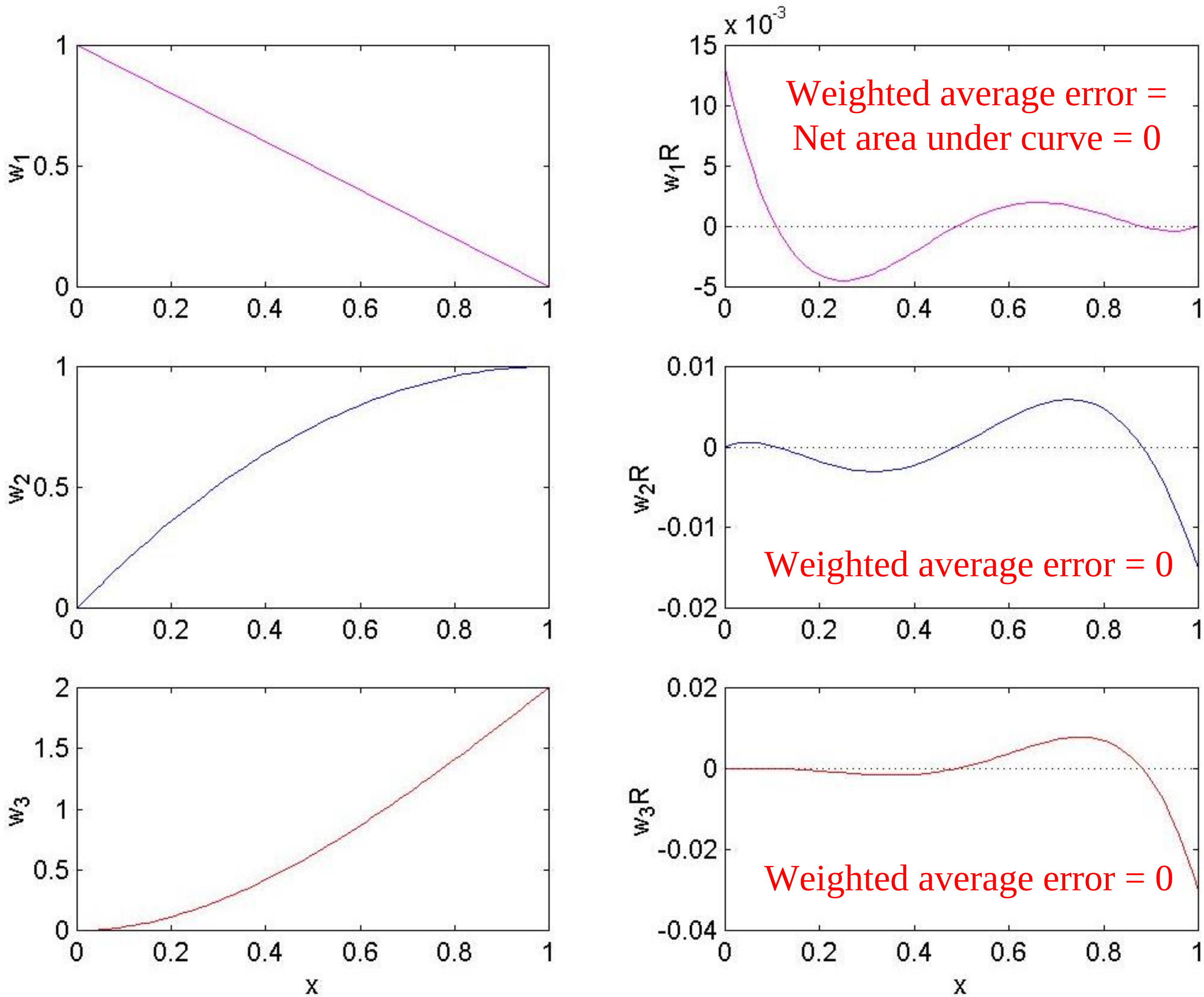
$$\begin{cases} m=1, & \frac{1}{3}a_1 + \frac{1}{4}a_2 + \frac{1}{5}a_3 = \frac{1}{2} \\ m=2, & \frac{1}{4}a_1 + \frac{8}{15}a_2 + \frac{2}{3}a_3 = \frac{2}{3} \\ m=3, & \frac{1}{5}a_1 + \frac{2}{3}a_2 + \frac{33}{35}a_3 = \frac{3}{4} \end{cases} \Rightarrow \quad \vec{a}_i = \begin{bmatrix} 1.0131 \\ 0.4255 \\ 0.2797 \end{bmatrix}$$

- $R(0) = 0.0131 \neq 0, \quad R(1) = -0.0151 \neq 0$

Least Square Method



Least Square Method



Galerkin Method

- Weighting function = Trial function

$$\begin{cases} \phi_m(x) = x^0, x^1, x^2, x^3, \cdots, x^{N-1} \\ w_m(x) = \phi_m(x) = x^{m-1} \end{cases}$$

$$R(x) = -1 + \sum_{m=1}^{N} a_m (m x^{m-1} - x^m)$$

$$\int_0^1 w_m R dx = \int_0^1 x^{m-1} R(x) dx = 0 = -\int_0^1 x^{m-1} dx + \sum_{j=1}^{N} a_j \int_0^1 \left(j x^{m+j-2} - x^{m+j-1} \right) dx$$

$$\sum_{j=1}^{N} \left(\frac{j}{j+m-1} - \frac{1}{j+m} \right) a_j = \frac{1}{m}$$

$$\sum_{j=1}^{N} S_{mj} a_j = d_m \quad \Leftrightarrow \quad \bar{\bar{S}} \vec{A} = \vec{D}$$

Galerkin Method

- For cubic interpolation function (N=3)

$$R=(a_1-1)+(2a_2-a_1)x+(3a_3-a_2)x^2-a_3x^3$$

$$\begin{cases} m=1, \quad W_1=x^0=1 & \Rightarrow \int_0^1 R\,dx=0 \\ m=2, \quad W_2=x^1=x & \Rightarrow \int_0^1 xR\,dx=0 \\ m=3, \quad W_3=x^2 & \Rightarrow \int_0^1 x^2R\,dx=0 \end{cases}$$

- Small weighting of residuals near x = 0
- Largest weight for residuals near x = 1

Galerkin Method

- Cubic trial function

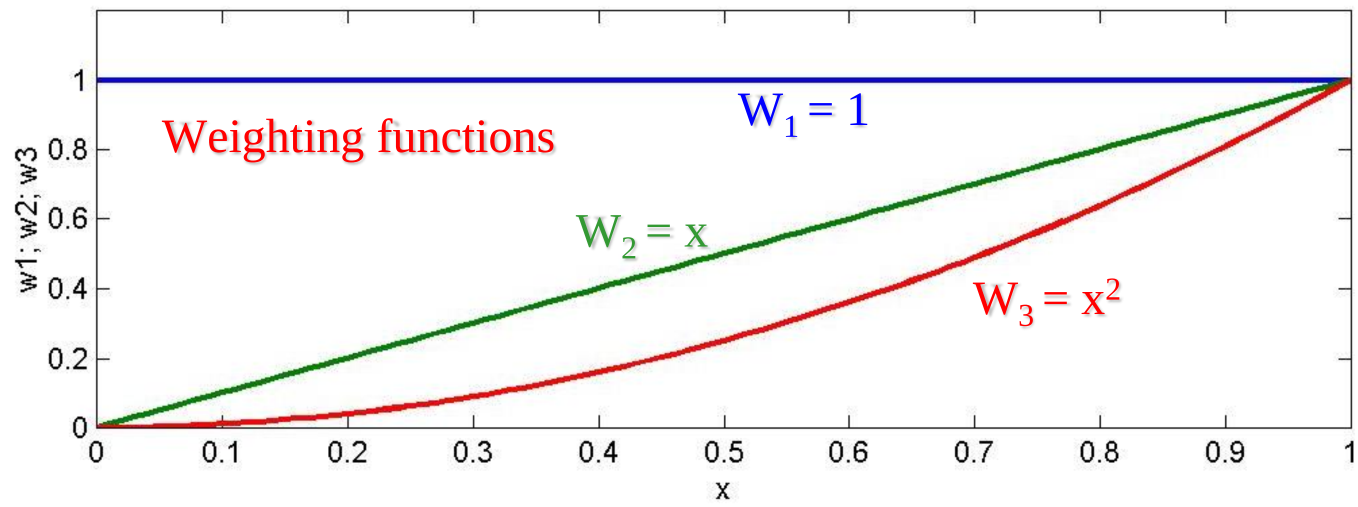
$$\sum_{j=1}^{3} a_j \left(\frac{j}{m+j-1} - \frac{1}{m+j} \right) = \frac{1}{m}$$

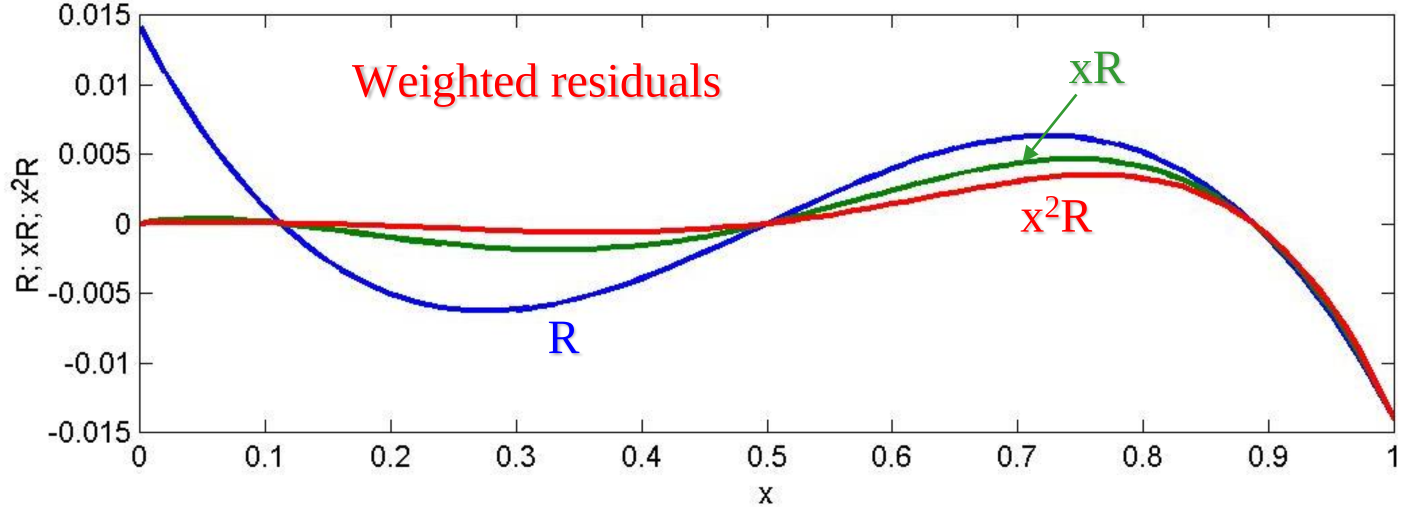
$$\begin{cases} m=1, & \frac{1}{2}a_1 + \frac{2}{3}a_2 + \frac{3}{4}a_3 = 1 \\ m=2, & \frac{1}{6}a_1 + \frac{5}{12}a_2 + \frac{11}{20}a_3 = \frac{1}{2} \\ m=3, & \frac{1}{12}a_1 + \frac{3}{10}a_2 + \frac{13}{30}a_3 = \frac{1}{3} \end{cases} \Rightarrow \vec{a}_i = \begin{bmatrix} 1.0141 \\ 0.4225 \\ 0.2817 \end{bmatrix}$$

$$\begin{cases} y(x) = 1 + 1.0141x + 0.4225x^2 - 0.2817x^3 \\ R(x) = 0.0141 - 0.1691x + 0.4226x^2 - 0.2817x^3 \end{cases}$$

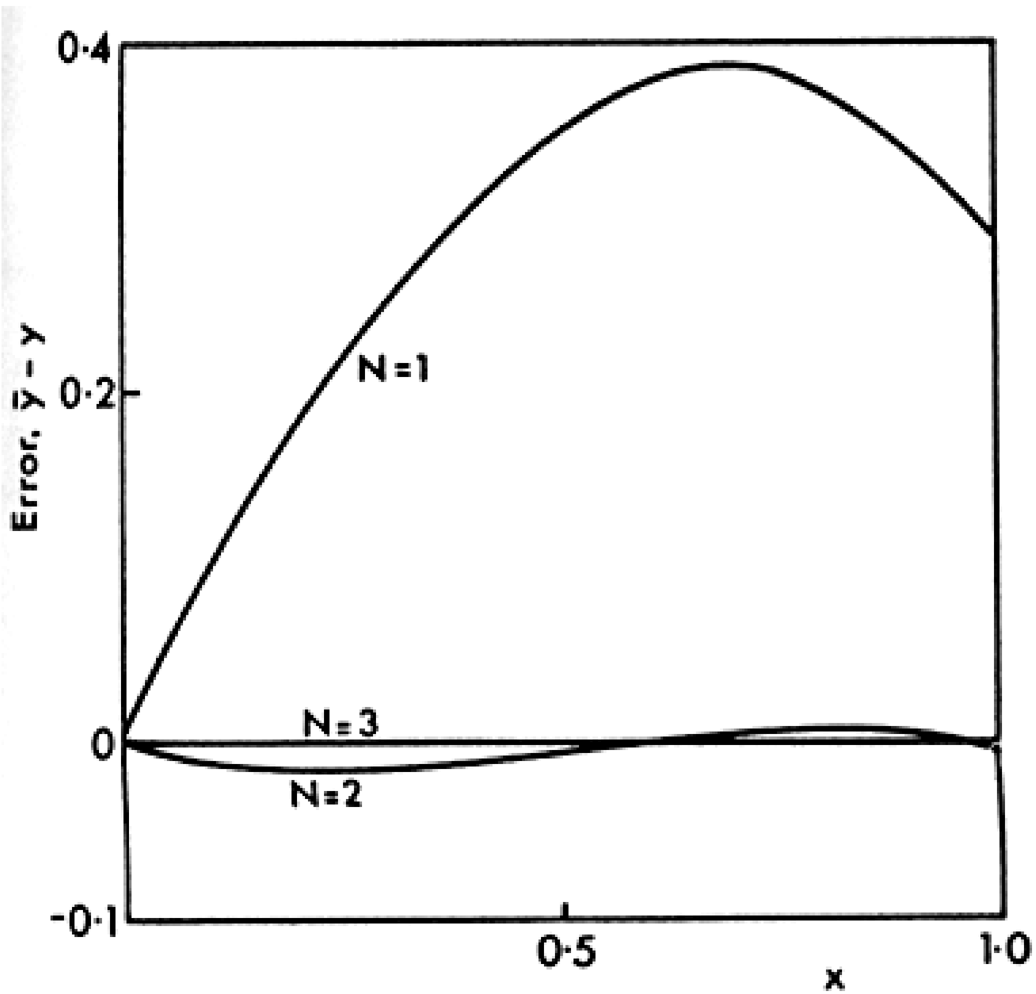
- $R(0) = 0.0141 \neq 0, \quad R(1) = -0.0141 \neq 0$

Galerkin Method





Galerkin Method



Order of approximation:

Linear, Quadratic, and Cubic Trial functions

Galerkin Method

Table 5.1. Galerkin solutions of $dy/dx - y = 0$

x	Approximate solution: Linear ($N=1$)	Quadratic ($N=2$)	Cubic ($N=3$)	Exact solution, $\bar{y} = \exp(x)$
0.	1.0	1.0	1.0	1.0
0.2	1.4	1.2057	1.2220	1.2214
0.4	1.8	1.4800	1.4913	1.4918
0.6	2.2	1.8229	1.8214	1.8221
0.8	2.6	2.2349	2.2259	2.2251
1.0	3.0	2.7143	2.7183	2.7183
Solution err. (rms)	0.2857	0.00886	0.00046	—
R_{rms}	0.5271	0.0583	0.00486	—

Galerkin Method

- Alternative choice of weighting functions

$$R=(a_1-1)+(2a_2-a_1)x+(3a_3-a_2)x^2-a_3x^3$$

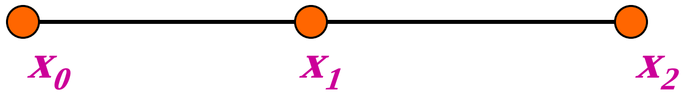
$$=-1+(1-x)a_1+(2x-x^2)a_2+(3x^2-x^3)a_3=-1+\sum_{j=1}^{3}a_j\phi_j$$

$$\begin{cases} m=1, \quad W_1=1-x & \Rightarrow \int_0^1 (1-x)R\,dx=0 \\ m=2, \quad W_2=2x-x^2 & \Rightarrow \int_0^1 (2x-x^2)R\,dx=0 \\ m=3, \quad W_3=3x^2-x^3 & \Rightarrow \int_0^1 (3x^2-x^3)R\,dx=0 \end{cases}$$

- More uniform weighting functions
- Identical to the least square method

Collocation Method

- R = L(u) = 0 at collocation points



$R(u) = 0$ but $R(\bar{u}) \neq 0$

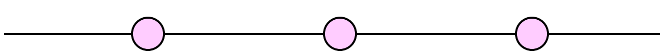
$$\int_0^1 w_m R dx = R(x_m) = (a_1 - 1) + (2a_2 - a_1)x_m + (3a_3 - a_2)x_m^2 - a_3 x_m^3$$

$$\begin{cases} m = 1: \quad x_1 = 0 \quad \Rightarrow a_1 = 1 \\ m = 2: \quad x_2 = \dfrac{1}{2} \quad \Rightarrow \dfrac{1}{2}a_1 + \dfrac{3}{4}a_2 + \dfrac{5}{8}a_3 = 1 \\ m = 3: \quad x_3 = 1 \quad \Rightarrow a_2 + 2a_3 = 1 \end{cases}$$

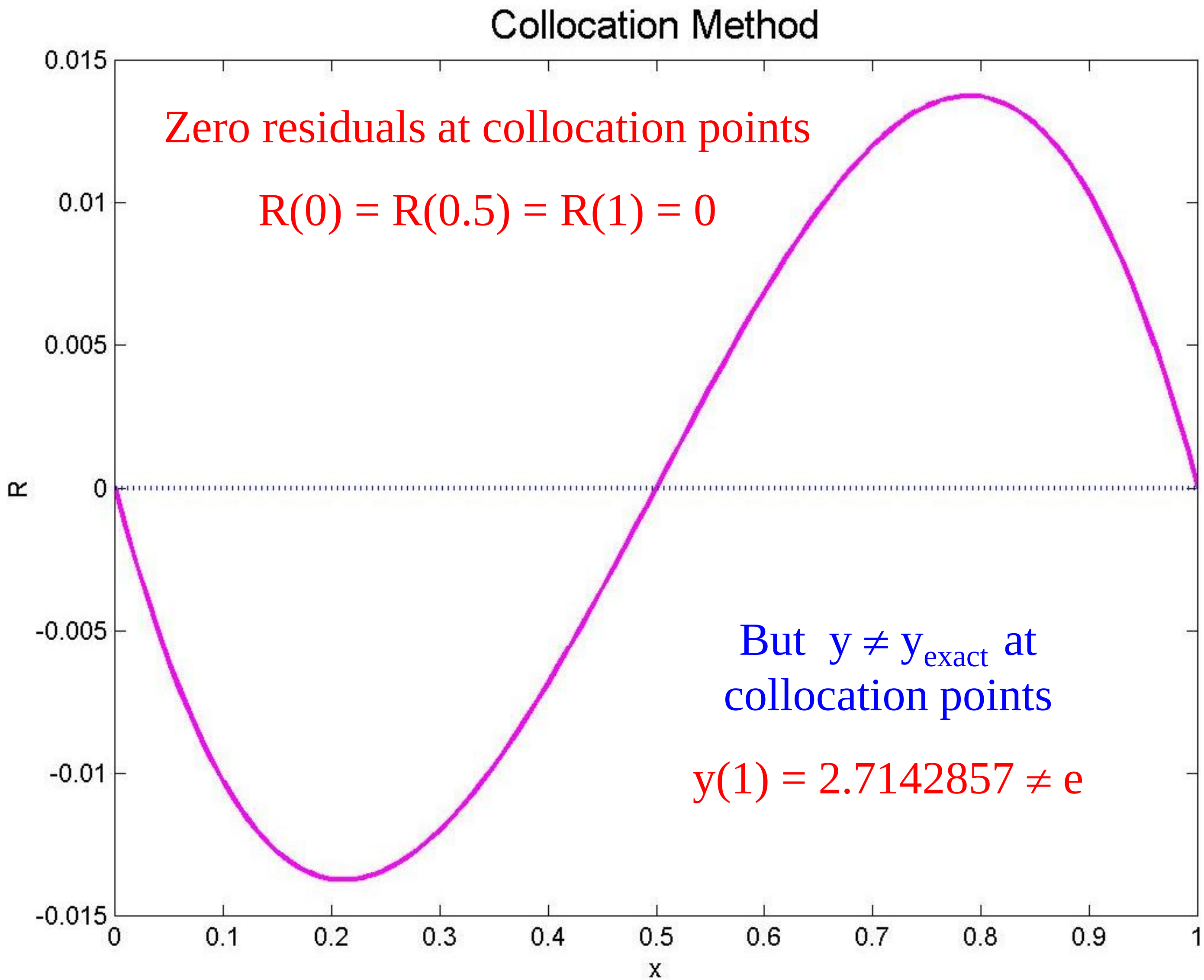
$$\vec{a}_i = \begin{bmatrix} 1 \\ 3/7 \\ 2/7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.4286 \\ 0.2857 \end{bmatrix}$$

$$y(x) = 1 + x + \frac{3}{7}x^2 + \frac{2}{7}x^3 \Rightarrow y(1) = 2\frac{5}{7} \neq e = 2.71828\cdots$$

- Identical to Galerkin method if the residuals are evaluated at x = 0.1127, 0.5, 0.8873



Collocation Method



Taylor-series Expansion

- Truncated Taylor-series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \quad \Rightarrow \vec{a}_i = \begin{bmatrix} 1 \\ 1/2 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ 0.1667 \end{bmatrix}$$

- $R(0) = 0$, $R(1) = -1/6 = -0.1667 \neq 0$
- Poor approximation at $x = 1$
- Power series has highly nonuniform error distribution

Interpolation Functions

Table 5.2. Comparison of coefficients for approximate solutions of $dy/dx - y = 0$

Scheme \ Coefficient	a_1	a_2	a_3
Galerkin	1.0141	0.4225	0.2817
Least squares	1.0131	0.4255	0.2797
Subdomain	1.0156	0.4219	0.2813
Collocation	1.0000	0.4286	0.2857
Optimal rms	1.0138	0.4264	0.2781
Taylor series	1.0000	0.5000	0.1667

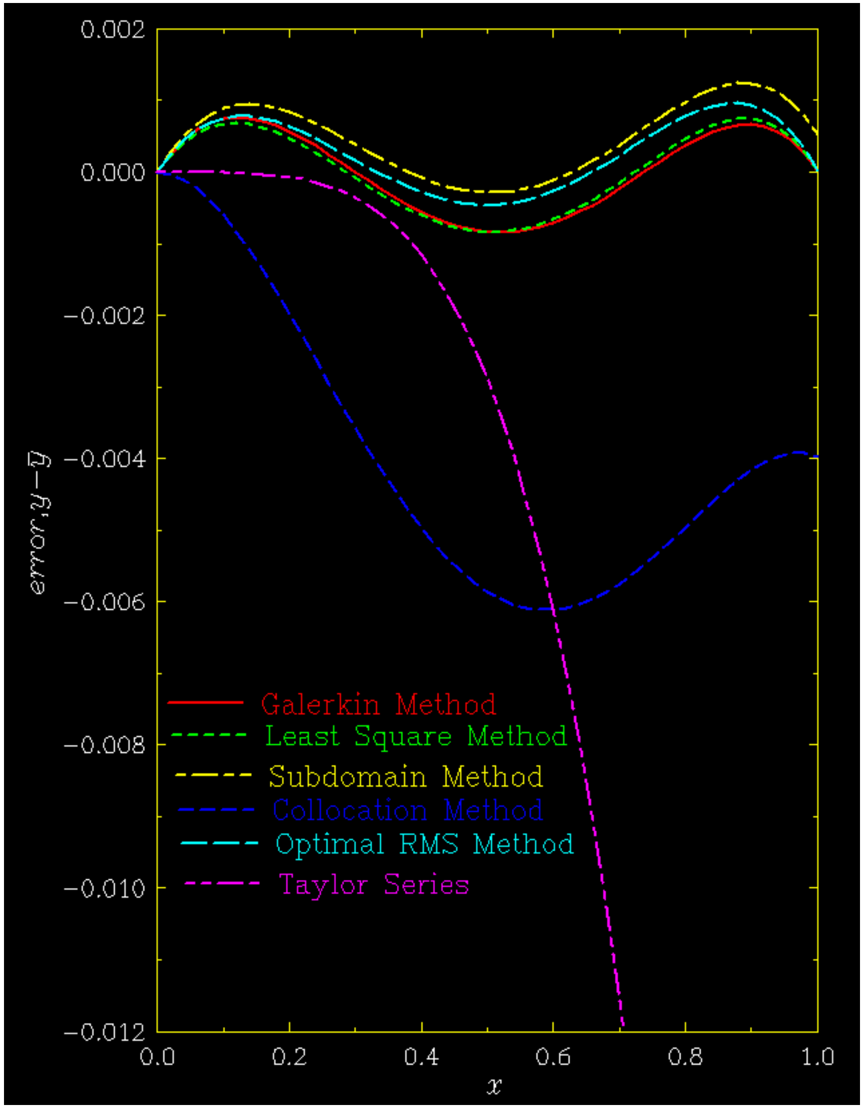
Numerical Accuracy

Table 5.3. Comparison of approximate solutions of $dy/dx - y = 0$

x	Galerkin	Least squares	Sub-domain	Collo-cation	Optimal rms	Taylor series	Exact
0.	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	1.2220	1.2219	1.2213	1.2194	1.2220	1.2213	1.2214
0.4	1.4913	1.4912	1.4917	1.4869	1.4915	1.4907	1.4918
0.6	1.8214	1.8214	1.8220	1.8160	1.8219	1.8160	1.8221
0.8	2.2259	2.2260	2.2265	2.2206	2.2263	2.2053	2.2255
1.0	2.7183	2.7183	2.7187	2.7143	2.7183	2.6667	2.7183
Solution error (rms)	0.000458	0.000474	0.000576	0.004188	0.000434	0.022766	—

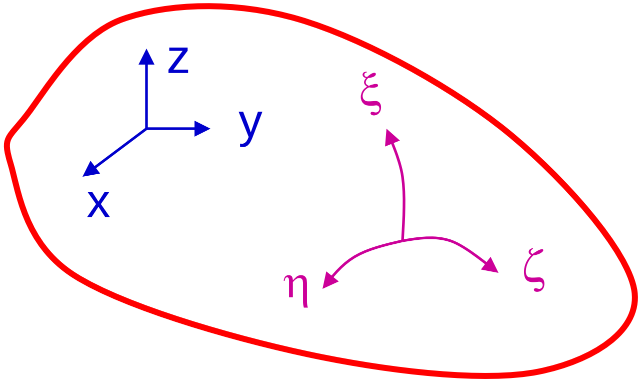
Interpolation Functions

Comparison of numerical errors for weighted residual methods



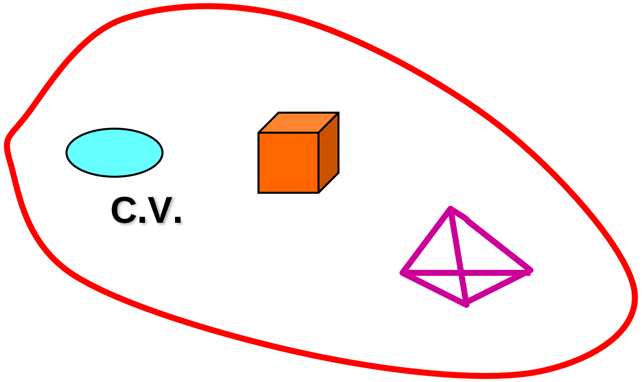
Weighted Residual Methods

- **Finite Differences**



- Discrete nodal values
- Differential formulation for each node
- Taylor series expansion on a structured grid
- Truncation errors
- Accuracy: reduce truncation error

- **Weighted residuals**



- Continuous shape function
- Integral formulation for each element
- Minimize weighted residual for an arbitrary control volume
- Interpolation errors
- Accuracy: higher-order interpolation, optimize coefficients for minimum residuals

Weighted Residual Formulations

Assumption for approximate solution

(Recall shape functions)

$$\tilde{u} = \sum_{i=1}^{n} N_i u_i \qquad \text{ERROR} = R = L\left(\sum_{i=1}^{n} N_i u_i\right) - P$$

Assumption for weighting function

$$w = \sum_{i=1}^{n} N_i w_i$$

w_i are arbitrary and $\neq 0$

GALERKIN FORMULATION

Weighted Residual Formulations

$$\int_V w\left(L\tilde{u}-P\right)dV=0 \qquad w=\sum_{i=1}^{n}N_i w_i$$

$$\int_V N_1\left(L\tilde{u}-P\right)dV w_1+\int_V N_2\left(L\tilde{u}-P\right)dV w_2$$

$$+\cdots+\int_V N_n\left(L\tilde{u}-P\right)dV w_n=0$$

Galerkin Formulation

$$\int_V N_1\left(L\tilde{u} - P\right)dV = 0$$

$$\int_V N_2\left(L\tilde{u} - P\right)dV = 0$$

$$\vdots$$

$$\int_V N_n\left(L\tilde{u} - P\right)dV = 0$$

Algebraic System of

n Equations and n unknowns

$$\tilde{u} = \sum_{i=1}^{n} N_i u_i$$