Finite Element Analysis of Boundary Value Problem

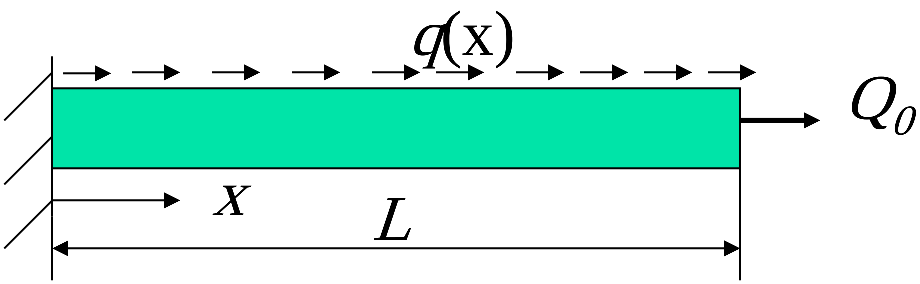
FE Analysis of 1D Bars

The DE is in the form of

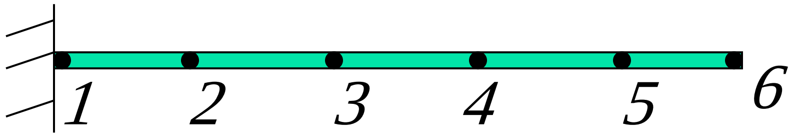
$$-\frac{d}{dx}(EA\frac{du}{dx})-q=0$$

q is the distributed load and Q_0 is the axial force.

$$u(0)=u_0, \quad (EA\frac{du}{dx})_{x=L}=Q_0$$



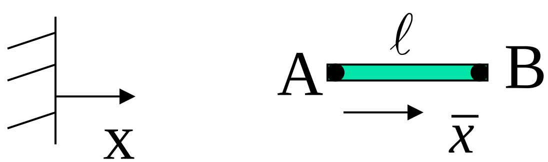
Physical Model



FE Model

Weak form

In FE analysis, we seek an approximation solution over each element.



$$Q_1 = -EA\frac{du}{dx}\bigg|_{x=x_A}$$

$$Q_2 = EA\frac{du}{dx}\bigg|_{x=x_B}$$

$$\int_{x_A}^{x_B}\left(EA\frac{dw}{dx}\frac{du}{dx} - wq\right)dx - w(x_A)Q_A - w(x_B)Q_B = 0$$

$$B(w,u) = \int_{x_A}^{x_B}\left(EA\frac{dw}{dx}\frac{du}{dx}\right)dx$$

$$\Rightarrow \quad B(w,u) = l(w)$$

$$l(w) = \int_{x_A}^{x_B} wq\,dx + w(x_A)Q_A + w(x_B)Q_B$$

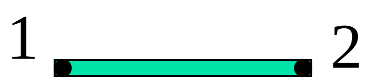
FE Analysis of 1D Bars

Approximation of the solution

1- The approximation solution should be continuous and differentiable as required by the weak form. (nonzero coefficient matrix)

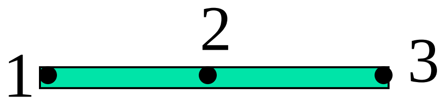
2- It should be a complete polynomial (capture all possible States, e.g. constant, linear, ….)

3- It should be an interpolant of variables at the nodes (satisfy EBCs)



First order

$$\begin{cases} U = a + bx, \quad U(x_1) = u_1, U(x_2) = u_2 \\ U = N_1 u_1 + N_2 u_2 \\ N_1 = 1 - \bar{x}/\ell, \quad N_2 = \bar{x}/\ell \end{cases}$$



Second Order

$$\begin{cases} U = a + bx + cx^2, \quad U(x_1) = u_1, U(x_2) = u_2, U(x_3) = u_3 \\ U = N_1 u_1 + N_2 u_2 + N_3 u_3 \\ N_1 = (1 - \bar{x}/\ell)(1 - 2\bar{x}/\ell), \quad N_2 = 4\bar{x}/\ell(1 - \bar{x}/\ell), \quad N_3 = -\bar{x}/\ell(1 - 2\bar{x}/\ell) \end{cases}$$

FE Analysis of 1D Bars

FE Model

$$u \approx U = \sum_{j=1}^{n} u_j N_j \qquad \text{and} \qquad \int_{x_A}^{x_B} \left(EA \frac{dw}{dx}\frac{du}{dx} - wq \right) dx - w(x_A)Q_A - w(x_B)Q_B = 0$$

$$w = N_j$$

If $n > 2$ then the above integral should modify to include interior nodal forces

$$\begin{cases} \int_{x_A}^{x_B} \left(EA \frac{dN_1}{dx} \sum_{j=1}^{n} u_j \frac{dN_j}{dx} - N_1 q \right) dx - \sum_{j=1}^{n} N_1(x_j) Q_j = 0 \\ \int_{x_A}^{x_B} \left(EA \frac{dN_2}{dx} \sum_{j=1}^{n} u_j \frac{dN_j}{dx} - N_2 q \right) dx - \sum_{j=1}^{n} N_2(x_j) Q_j = 0 \\ \vdots \\ \int_{x_A}^{x_B} \left(EA \frac{dN_n}{dx} \sum_{j=1}^{n} u_j \frac{dN_j}{dx} - N_n q \right) dx - \sum_{j=1}^{n} N_n(x_j) Q_j = 0 \end{cases} \Rightarrow \sum_{j=1}^{n} K_{ij} u_j - f_i - Q_i = 0$$

$$i = 1,2,\ldots,n$$

Stiffness matrix → K_{ij}

Force vector → f_i

Primary nodal DOF → u_j

Secondary nodal DOF → Q_i

FE Model

where
$$K_{ij} = \int_{x_A}^{x_B} \left(EA \frac{dN_i}{dx} \frac{dN_j}{dx} \right) dx = B(N_i, N_j)$$

$$f_i = \int_{x_A}^{x_B} q N_i dx = l(N_i)$$

Note ⟶ $\sum_{j=1}^{n} N_j(x_i) Q_j = Q_i$

Note that the problem has *2n* unknowns for each element, i.e. u_i and Q_i, so it cannot be solved without having another *n* conditions. Some of these will be provided by BCs and the remainder by balance of the secondary variables (forces) at node common to several element. (assembling process)

FE Model (Linear Element)

$$U = N_1 u_1 + N_2 u_2$$

$$N_1 = 1 - \bar{x}/\ell, \quad N_2 = \bar{x}/\ell$$

$$K_{11} = \int_0^\ell (EA)(-1/\ell)(-1/\ell)dx = AE/\ell$$

$$K_{12} = \int_0^\ell (EA)(-1/\ell)(1/\ell)dx = -AE/\ell$$

$$K_{22} = \int_0^\ell (EA)(1/\ell)(1/\ell)dx = AE/\ell$$

$$f_1 = \int_0^\ell q(1 - x/\ell)dx = 1/2q\ell$$

$$f_2 = \int_0^\ell q(x/\ell)dx = 1/2q\ell$$

Eventually for
Linear shape function

$$[K^e] = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad \{f\} = \frac{q\ell}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

FE Model (Quadratic Element)

$$U = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$N_1 = (1 - x/\ell)(1 - 2x/\ell), \quad N_2 = 4x/\ell(1 - x/\ell), \quad N_3 = -x/\ell(1 - 2x/\ell)$$

$$K_{11} = \int_0^\ell (EA)(-3/\ell + 4x/\ell^2)(-3/\ell + 4x/\ell^2)dx = 7AE/3\ell$$

$$K_{12} = K_{21} = \int_0^\ell (EA)(-3/\ell + 4x/\ell^2)(4/\ell - 8x/\ell)dx = -8AE/3\ell$$

. . . .

$$f_1 = \int_0^\ell q(1 - 3x/\ell + 2(x/\ell)^2)dx = 1/6q\ell$$

$$f_2 = \int_0^\ell q(4x/\ell)(1 - x/\ell)dx = 4/6q\ell$$

. . . .

FE Model (Quadratic Element)

For quadratic Shape function

$$[K^e] = \frac{AE}{3\ell}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}; \quad \{f\} = \frac{q\ell}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

FE Analysis of 1D Bars

Assembly (or connectivity) of elements

In driving the element equation

-Isolate the element from mesh

-Formulate weak form (variational form)

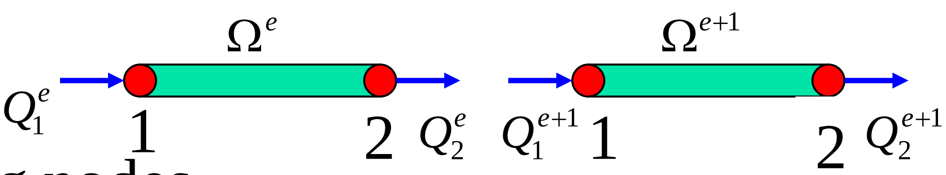
-Developed its finite element model

To solve the total problem

-put the element in its original position

-Impose continuity of PVs at nodal points

$$u_n^e = u_1^{e+1}$$



-Balance of SVs at connecting nodes

$$Q_n^e + Q_1^{e+1} = \begin{cases} 0 & \text{if no external point source is applied} \\ Q_0 & \text{if an external point source of } Q_0 \text{ is applied.} \end{cases} \qquad (*)$$

Assembly (or connectivity) of elements (For linear element *n=2*)

The interelement continuity of the primary variables is imposed by renaming the two variable u_n^e and u_1^{e+1} at $x=x_N$ as one and same, namely the value of u at the global node N:

$$u_n^e = u_1^{e+1} = U_N$$

where $N = (n-1)e + 1$

For a mesh of E linear finite elements ($n=2$):

$$u_1^1 = U_1$$
$$u_2^1 = u_1^2 = U_2$$
$$u_2^2 = u_1^3 = U_3$$
$$\vdots$$
$$u_2^{E-1} = u_1^E = U_E$$
$$u_2^E = U_{E+1}$$

Assembly (or connectivity) of elements (For linear element $n=2$)

To enforce balance of secondary variables Q_i^e , eq. (*), we must add nth equation of the element Ω^e to the first equation of the element Ω^{e+1} :

$$\sum_{j=1}^{n} K_{nj}^{e} u_j^{e} = f_n^{e} + Q_n^{e}$$

and

$$\sum_{j=1}^{n} K_{1j}^{e+1} u_j^{e+1} = f_1^{e+1} + Q_1^{e+1}$$

to give

$$\sum_{j=1}^{n} (K_{nj}^{e} u_j^{e} + K_{1j}^{e+1} u_j^{e+1}) = f_n^{e} + f_1^{e+1} + (Q_n^{e} + Q_1^{e+1})$$

$$= f_n^{e} + f_1^{e+1} + Q_0$$

This process reduces the number of equations from $2E$ to $E+1$.

Assembly (or connectivity) of elements (For linear element *n=2*)

The first equation of the first element and the last equation of the last element will remain unchanged, except for renaming of the primary variables. The left-hand of the equation can be written in terms of the global nodal values as

$$
\begin{aligned}
&(K_{n1}^{e}u_{1}^{e}+K_{n2}^{e}u_{2}^{e}+\cdots+K_{nn}^{e}u_{n}^{e})+(K_{11}^{e+1}u_{1}^{e+1}+K_{12}^{e+1}u_{2}^{e+1}+\cdots+K_{1n}^{e+1}u_{n}^{e+1})\\
&=(K_{n1}^{e}U_{N}+K_{n2}^{e}U_{N+1}+\cdots+K_{nn}^{e}U_{N+n-1})+\\
&\quad(K_{11}^{e+1}U_{N+n-1}+K_{12}^{e+1}U_{N+n}+\cdots+K_{1n}^{e+1}U_{N+2n-2})\\
&=K_{n1}^{e}U_{N}+K_{n2}^{e}U_{N+1}+\cdots+K_{n(n-1)}^{e}U_{N+n-2}+\\
&\quad(K_{nn}^{e}+K_{11}^{e+1})U_{N+n-1}+K_{12}^{e+1}U_{N+n}+\cdots+K_{1n}^{e+1}U_{N+2n-2}
\end{aligned}
$$

where $N=(n-1)e+1$

Assembly (or connectivity) of elements (For linear element *n=2*)

For a mesh of *E* linear finite elements (*n=2*):

$$K_{11}^{1}U_{1}+K_{12}^{1}U_{2}=f_{1}^{1}+Q_{1}^{1}\qquad (unchanged)$$

$$K_{21}^{1}U_{1}+(K_{22}^{1}+K_{11}^{2})U_{2}+K_{12}^{2}U_{3}=f_{2}^{1}+f_{1}^{2}+Q_{2}^{1}+Q_{1}^{2}$$

$$K_{21}^{2}U_{2}+(K_{22}^{2}+K_{11}^{3})U_{3}+K_{12}^{3}U_{4}=f_{2}^{2}+f_{1}^{3}+Q_{2}^{2}+Q_{1}^{3}$$

$$\vdots$$

$$K_{21}^{E-1}U_{E-1}+(K_{22}^{E-1}+K_{11}^{E})U_{E}+K_{12}^{E}U_{E+1}=f_{2}^{E-1}+f_{1}^{E}+Q_{2}^{E-1}+Q_{1}^{E}$$

$$K_{21}^{E}U_{E}+K_{22}^{E}U_{E+1}=f_{2}^{E}+Q_{2}^{E}\qquad (unchanged)$$

Assembly (or connectivity) of elements (For linear element *n=2*)

In matrix form

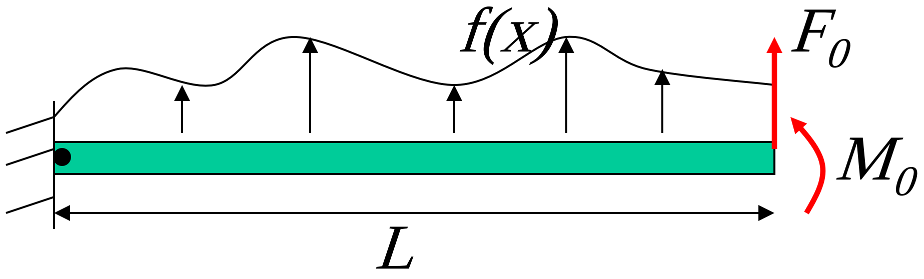
$$\begin{bmatrix} K_{11}^{1} & K_{12}^{1} & & & \\ K_{12}^{1} & K_{22}^{1}+K_{11}^{2} & K_{12}^{2} & & 0 \\ & K_{21}^{2} & K_{22}^{2}+K_{11}^{3} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & 0 & & K_{22}^{E-1}+K_{11}^{E} & K_{12}^{E} \\ & & & & K_{22}^{E} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \cdots \\ U_E \\ U_{E+1} \end{Bmatrix}$$

$$= \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ \cdots \\ f_2^{E-1} + f_1^E \\ f_2^E \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ \cdots \\ Q_2^{E-1} + Q_1^E \\ Q_2^E \end{Bmatrix}$$

FE Analysis of BEAM

The DE is in the form of

$$\frac{d^2}{dx^2}\left(b\frac{d^2w}{dx^2}\right) = f(x) \quad 0 < x < L$$



FE Analysis of BEAM

Weak form

$$\int_{x_e}^{x_{e+1}} v\left(\frac{d^2}{dx^2}\left(b\frac{d^2w}{dx^2}\right)-f\right)dx=0$$

$$Q_1^e=\left[\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_{x_e};\quad Q_2^e=\left[b\frac{d^2w}{dx^2}\right]_{x_e}$$

$$Q_3^e=-\left[\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_{x_{e+1}};\quad Q_4^e=-\left[b\frac{d^2w}{dx^2}\right]_{x_{e+1}}$$

or

BCs

$$\int_{x_e}^{x_{e+1}}\left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}-vf\right)dx-v(x_e)Q_1^e-\left(-\frac{dv}{dx}\right)_{x_e}Q_2^e-v(x_{e+1})Q_3^e-\left(-\frac{dv}{dx}\right)_{x_{e+1}}Q_4^e=0$$

where

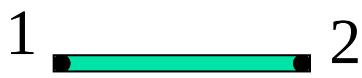
$$B(v,w)=\int_{x_e}^{x_{e+1}}\left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}\right)dx$$

$$l(v)=\int_{x_e}^{x_{e+1}}vf\,dx+v(x_e)Q_1^e+\left(-\frac{dv}{dx}\right)_{x_e}Q_2^e+v(x_{e+1})Q_3^e+\left(-\frac{dv}{dx}\right)_{x_{e+1}}Q_4^e$$

FE Analysis of BEAM

Approximation of the solution

1- The approximation solution should be continuous and differentiable as required by the weak form. (nonzero coefficient matrix)
2- It should be a complete polynomial (capture all possible States, e.g. constant, linear, ….)
3- It should be an interpolant of variables at the nodes (satisfy EBCs)



First order

$$w = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

$$w(x_e) = w_1, w(x_{e+1}) = w_2, \theta(x_e) = \theta_1, \theta(x_{e+1}) = \theta_2$$

or

$$w = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

$$u_1^e = w(x_e), u_2^e = -\left.\frac{dw}{dx}\right|_{x_e} ; u_3^e = w(x_{e+1}), u_4^e = -\left.\frac{dw}{dx}\right|_{x_{e+1}}$$

Shape Functions

Calculating Ci and substituting in the equation for w

$$w^e(x) = \sum_{j=1}^{4} u_j^e N_j$$

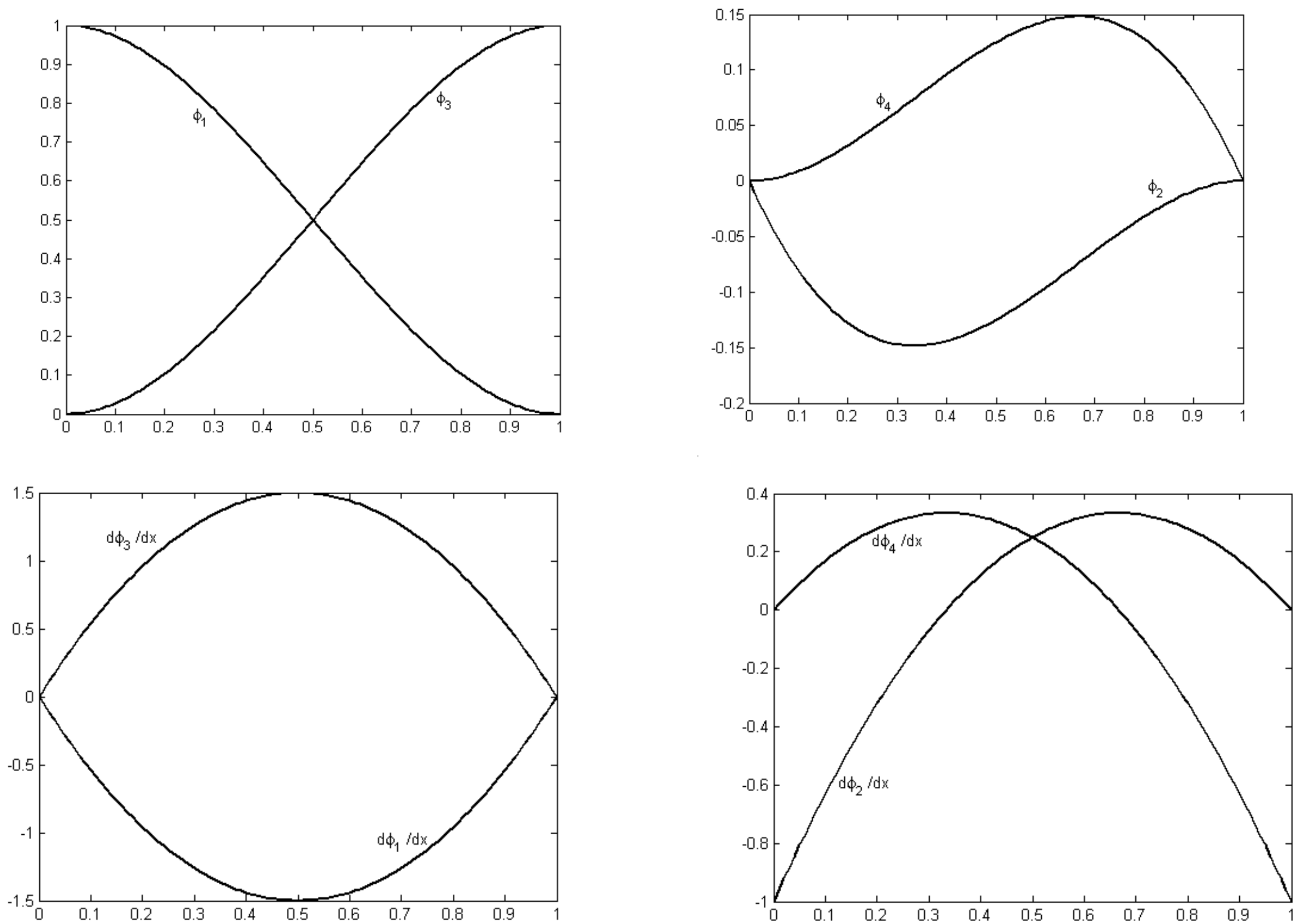
The interpolation functions in term of local coordinates are

$$N_1 = 1 - 3\left(\frac{x}{h}\right)^2 + 2\left(\frac{x}{h}\right)^3 ; N_2 = -x\left(1 - \frac{x}{h}\right)^2$$

$$N_3 = 3\left(\frac{x}{h}\right)^2 - 2\left(\frac{x}{h}\right)^3 ; N_4 = -x\left[\left(\frac{x}{h}\right)^2 - \frac{x}{h}\right]$$

FE Analysis of BEAM

Hermite cubic interpolation function



FE Model

$$\sum_{j=1}^{4}\left(\int_{x_e}^{x_{e+1}} b\frac{d^2N_i}{dx^2}\frac{d^2N_j}{dx^2}dx\right)u_j-\left(\int_{x_e}^{x_{e+1}} N_i f dx+Q_i^e\right)=0$$

or

$$\sum_{j=1}^{4}K_{ij}u_j-F_i=0$$

For b=EI constant and also a constant f over the element.

$$[K]=\frac{2EI}{h^3}\begin{bmatrix}6 & -3h & -6 & -3h\\ -3h & 2h^2 & 3h & h^2\\ -6 & 3h & 6 & 3h\\ -3h & h^2 & 3h & 2h^2\end{bmatrix};\quad \{F\}=\frac{fh}{12}\begin{Bmatrix}6\\ -h\\ 6\\ h\end{Bmatrix}+\begin{Bmatrix}Q_1\\ Q_2\\ Q_3\\ Q_4\end{Bmatrix}$$

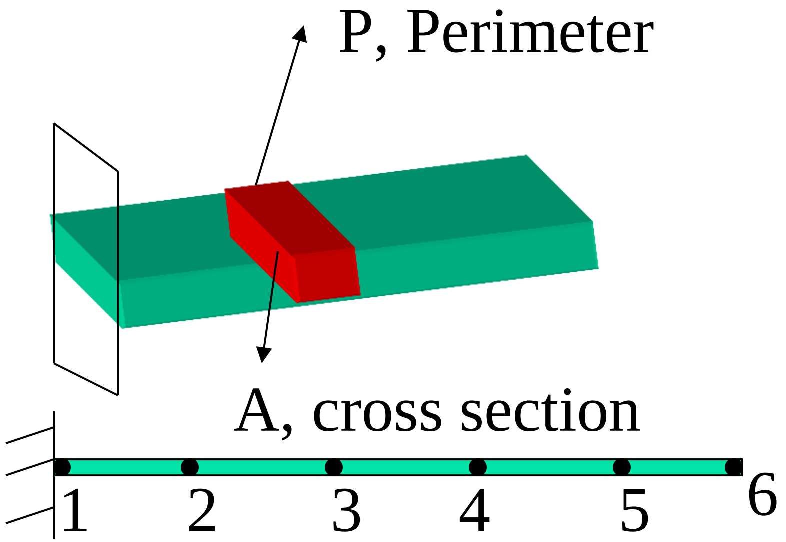
Model Boundary Value Problem

The DE is in the form of

$$-\frac{d}{dx}(kA\frac{dT}{dx}) + P\beta T = Aq + P\beta T_\infty$$

$$T(0) = T_0, \quad Q = -kA\frac{\partial T}{\partial x} = Q_0$$

k is thermal conductivity
β is convection heat transfer coefficient
T_∞ is the ambient temperature
q is the heat energy generated per unit volume



Physical Model

FE Model

Weak form

$$K_{ij} = \int_{x_A}^{x_B} \left(kA \frac{dN_i}{dx} \frac{dN_j}{dx} + P\beta N_i N_j \right) dx$$

$$f_i = \int_{x_A}^{x_B} N_i (qA + P\beta T_\infty) dx$$

$$Q_1^e = \left(-kA \frac{dT}{dx} \right)_{x_A} \quad ; \quad Q_2^e = \left(-kA \frac{dT}{dx} \right)_{x_B}$$

Assume the lateral surfaces of the bar are isolated and the BCs

$$-\frac{d}{dx}(kA\frac{dT}{dx}) = Aq$$

$$T(0) = T_1, \quad T(L) = T_2$$

FE Analysis of 1D FIN

Approximation of the solution

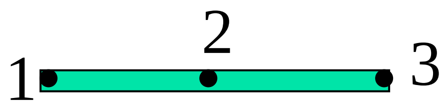
1- the approximation solution should be continuous and differentiable as required by the weak form. (nonzero coefficient matrix)

2- it should be a complete polynomial (capture all possible States, e.g. constant, linear, ….)

3- it should be an interpolant of variables at the nodes (satisfy EBCs)

First order

$$\begin{cases} T = a + bx, \quad T(x_1) = T_1, T(x_2) = T_2 \\ T = N_1 T_1 + N_2 T_2 \\ N_1 = 1 - \bar{x}/\ell, \quad N_2 = \bar{x}/\ell \end{cases}$$



Second Order

$$\begin{cases} T = a + bx + cx^2, \quad T(x_1) = T_1, T(x_2) = T_2, T(x_3) = T_3 \\ T = N_1 T_1 + N_2 T_2 + N_3 T_3 \\ N_1 = (1 - \bar{x}/\ell)(1 - 2\bar{x}/\ell), \quad N_2 = 4\bar{x}/\ell(1 - \bar{x}/\ell), \quad N_3 = -\bar{x}/\ell(1 - 2\bar{x}/\ell) \end{cases}$$

FE Model

Evaluating the integral using linear shape function

$$[K^e]\{T^e\} = \{f^e\} + \{Q^e\}$$

$$\frac{kA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1^e \\ T_2^e \end{Bmatrix} = \frac{Aq\ell}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

For a uniform mesh $\ell = L/N$ and after assembling

$$\frac{kA}{\ell}\begin{bmatrix} 1 & -1 & 0 & \dots \\ -1 & 2 & -1 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{N+1} \end{Bmatrix} = \frac{Aq\ell}{2}\begin{Bmatrix} 1 \\ 2 \\ \vdots \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ \vdots \\ Q_1^N \end{Bmatrix}$$

FE Model

Boundary conditions at nodes *1* and *N+1*

$$T_1 = T_1$$

$$T_{N+1} = T_{N+1}$$

Heat balance at global nodes *2,3,….,N*

$$Q_2^{e-1} + Q_1^e = 0 \quad for \quad e = 2,3,\dots, N$$

After applying the above conditions:

$$\frac{kA}{\ell}\begin{bmatrix} 1 & -1 & 0 & \dots \\ -1 & 2 & -1 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{n+1} \end{Bmatrix} = \frac{Aq\ell}{2}\begin{Bmatrix} 1 \\ 2 \\ \vdots \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ 0 \\ \vdots \\ Q_1^N \end{Bmatrix}$$

Virtual work as the ‘weak form’ of equilibrium equations for analysis of solids

In a general three-dimensional continuum the equilibrium equations of an elementary volume can be written in terms of the components of the symmetric cartesian stress tensor as

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} + b_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} + b_y = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + b_z = 0 \end{Bmatrix} \Rightarrow \boldsymbol{L}\left(\mathbf{u}\left(\mathbf{x}\right)\right) = \mathbf{0},$$

$\boldsymbol{b} = \begin{bmatrix} b_x & b_x & b_x \end{bmatrix}^T$ The body forces acting per unit volume

$\mathbf{u} = \begin{bmatrix} u & v & w \end{bmatrix}^T$ The displacement vector

Virtual work as the 'weak form' of equilibrium equations for analysis of solids

The weighting function vector defined as $\delta\mathbf{u} = [\delta u \quad \delta v \quad \delta w]^T$

We can now write the integral statement of equilibrium equations as

$$\int_V \delta\mathbf{u}^T \boldsymbol{L}(\mathbf{u})d\mathrm{v} = -\int_V \left[\delta u \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + b_x \right) + \delta v(L_2) + \delta w(L_3) \right] d\mathrm{v}$$

$$= 0$$

Integrating each term by parts and rearranging we can write this as

$$\int_V \left[\frac{\partial \delta u}{\partial x}\sigma_x + \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \right)\tau_{xy} + \cdots - \delta u\, b_x - \delta v\, b_y - \delta w\, b_z \right] d\mathrm{v}$$

(*)

$$+\int_\Gamma \left[\delta u(\sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z) + \delta v(..) + \delta w(..) \right] d\Gamma = 0$$

Virtual work as the 'weak form' of equilibrium equations for analysis of solids

where $\mathbf{t} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{Bmatrix} \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z \\ \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z \\ \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z \end{Bmatrix}$ are tractions acting per unit area of external boundary surface Γ of the solid

In the first set of bracketed terms in eq. (*) we can recognize immediately the small strain operators acting on $\delta\mathbf{u}$, which can be termed a virtual displacement.

We can therefore introduce a virtual strain defined as

$$\delta\boldsymbol{\varepsilon}^T = \left\{ \frac{\partial \delta u}{\partial x}, \frac{\partial \delta v}{\partial y}, \frac{\partial \delta w}{\partial z}, \frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x}, \frac{\partial \delta v}{\partial z} + \frac{\partial \delta w}{\partial y}, \frac{\partial \delta v}{\partial z} + \frac{\partial \delta w}{\partial y} \right\}^T = \mathbf{D}\delta\mathbf{u}^{\ T}$$

Arranging the six stress components in a vector $\boldsymbol{\sigma}$ in an order corresponding to that used for $\delta\boldsymbol{\varepsilon}$, we can write Eq. (*) simply as

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$$\int_V \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\text{v} - \int_V \delta\mathbf{u}^T \mathbf{b} d\text{v} - \int_\Gamma \delta\mathbf{u}^T \mathbf{t} d\Gamma = 0$$

we see from the above that the virtual work statement is precisely the weak form of equilibrium equations and is valid for non-linear as well as linear stress–strain (or stress–strain rate) relations.