دانشگاه صنعتی اصفهان
دانشکده مکانیک

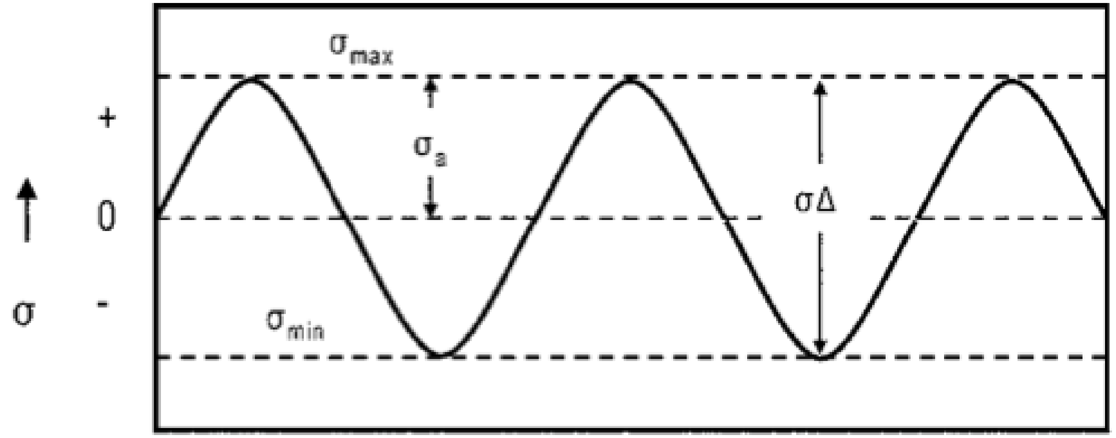
Fatigue-Variable amplitude loading

Tension/compression cyclic loads

$$R = \frac{\sigma_{\min}}{\sigma_{\max}} < 0$$

$$\frac{da}{dN} = C(\Delta K)^m, \quad \Delta K = K_{\max} - \cancel{K_{\min}}$$

$$\boxed{\frac{da}{dN} = C(K_{\max})^m}$$



Fully Reversed Loading

Crack growth rate and number of cycles

- Knowledge of crack growth rate is of assistance in designing components and in nondestructive evaluation to determine if a crack poses imminent danger to the structure.

$$\frac{da}{dN} = C\,(\Delta K)^n \qquad \Delta K = Y\,\Delta\sigma\sqrt{\pi a}$$

$$dN = \frac{da}{CY^n(\Delta\sigma)^n \pi^{n/2} a^{n/2}}$$

Integration between the initial size of a crack and the crack size required for fracture to occur.

$$N = \frac{2\left[a_c^{(2-n)/2} - a_i^{(2-n)/2}\right]}{(2-n)CY^n(\Delta\sigma)^n \pi^{n/2}}$$

a_i is the initial flaw size and a_c is the flaw size required for fracture.

Variable amplitude loading

- Although the constant amplitude of loading occurs frequently in practice, the majority of engineering structures are subjected to complex fluctuating loading. Unlike the case of constant cyclic load where ΔK increases gradually with increasing crack length, abrupt changes take place in ΔK due to changes in applied load. Thus, there occur load interaction effects which greatly influence the fatigue crack propagation behavior.

- Due to the crack retardation phenomenon, the determination of the fatigue life under a variable amplitude loading by simply summing the fatigue lives of the various constant amplitude loads in the loading history leads to conservative predictions. The various methods proposed for this reason, we will present the root-mean-square model.

Variable amplitude loading

- The root-mean-square model proposed by Barsom applies to variable amplitude narrow-band random loading spectra. It is assumed that the average fatigue crack growth rate under a variable amplitude random loading fluctuation is approximately equal to the rate of fatigue crack growth under constant amplitude cyclic load; this is equal to the root-mean-square of the variable amplitude loading. Thus, the fatigue crack propagation laws presented such as Paris' law can be equally applied for a variable amplitude random loading when ΔK is replaced by the root-mean-square value of the stress intensity factor ΔK_{rms} given by:

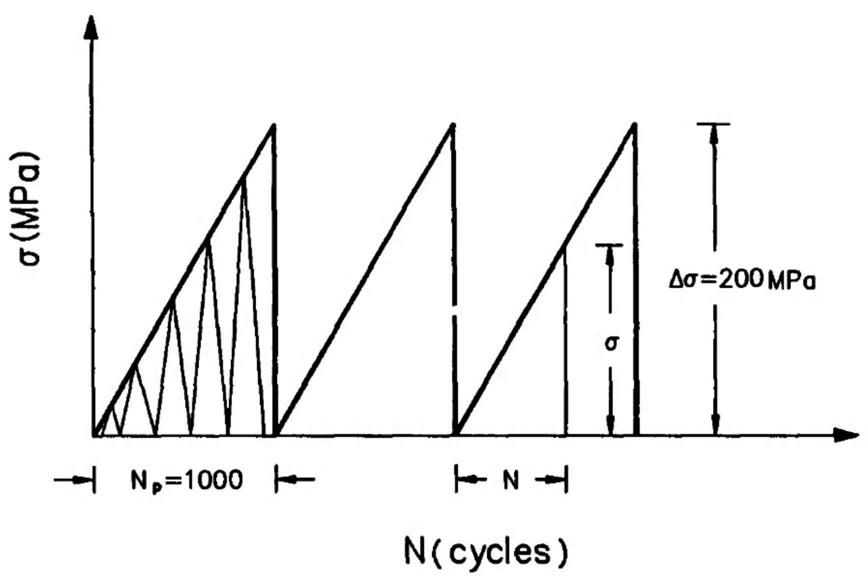
$$\Delta K_{rms} = \sqrt{\frac{\sum (\Delta K_i)^2 n_i}{\sum n_i}}$$

where n_i is the number of loading amplitudes with a stress intensity factor range of ΔK_i .

Variable amplitude loading

➢ Example 1:

A large thick plate contains a crack of length $2a_o$ = 20 mm and is subjected to a series of triangular stress sequences normal to the crack as shown in the figure. The stress varies between σ_{min}=0 and σ_{max}= 200 MPa and it takes 1000 cycles in the triangle to reach σ_{max}. The critical stress intensity factor is K_{Ic} = 100 MPam$^{0.5}$ and fatigue crack growth is governed by equation:



Triangular cyclic stress profile.

$$\frac{da}{dN} = 10^{-12} (\Delta K)^3 \qquad (*)$$

where da/dN is expressed in m/cycle and ΔK in MPa m$^{0.5}$

Variable amplitude loading

- **Example 1**:

Calculate the number of triangular stress sequences $(N_c)_1$ required to grow the crack to instability. Compare $(N_c)_1$ to the number of cycles $(N_c)_2$ required to grow the crack to instability when the plate is subjected to a constant amplitude stress cycle between the stresses σ_{min}=0 and σ_{max}= 200 MPa.

- **Solution**:

To calculate the ΔK_{rms} , we have for the stress σ at N cycles in the triangular stress sequence:

$$\sigma = \Delta\sigma \frac{N}{N_p} \quad \Longrightarrow \quad \Delta K = \Delta\sigma\sqrt{\pi a}\,\frac{N}{N_p}$$

Since we have a large number of cycles (Np = 1000) in each triangular stress sequence ΔK_{rms} can be calculated by integration as:

$$(\Delta K_{rms})^2 = \frac{1}{N_p}\int_0^{N_p} (\Delta K)^2 dN$$

Variable amplitude loading

➢ Solution:

$$(\Delta K_{rms})^2 = \frac{\left(\Delta\sigma\sqrt{\pi a}\right)^2}{N_p}\int_0^{N_p}\frac{N^2}{N_p^2}dN \;\Longrightarrow\; (\Delta K_{rms})^2 = \frac{(\Delta\sigma)^2\,\pi a}{3}$$

The critical crack length a_c at instability is calculated from equation: $K_I = K_{Ic}$

$$200\sqrt{\pi a} = 100 \;\Longrightarrow\; a_c = 79.6 \quad mm$$

Equation (*) with $\Delta K = \Delta K_{rms}$ gives :

$$(N_c)_1 = \int_{0.01}^{0.0796}\frac{da}{10^{-12}\times 200^3(\pi a)^{1.5}(1/3)^{1.5}} = 1.5\times 10^6 \quad cycle$$

The number of cycles $(N_c)_2$ required to grow the crack to instability when the plate is subjected to a constant amplitude stress cycle between the stress σ_{min} =0 and σ_{max} = 200 MPa is calculated from Equation (*) as:

$$(N_c)_2 = \int_{0.01}^{0.0796}\frac{da}{10^{-12}\times 200^3(\pi a)^{1.5}} = 0.29\times 10^6 \quad cycles$$

which is equal to $(N_c)_1$ divided by $3^{1.5}$.

Miner's rule for variable load amplitudes

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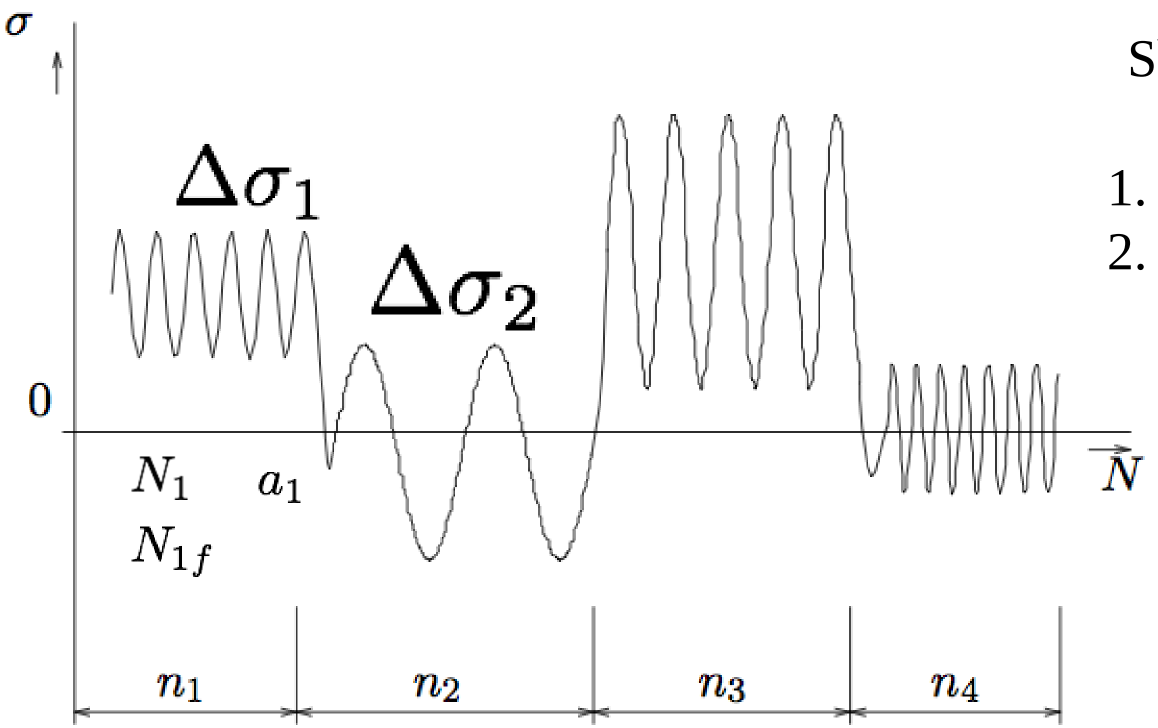


Shortcomings:

1. sequence effect not considered
2. damage accumulation is independent of stress level

N_i/N_{if} : damage

$$\sum_{i=1}^{n} \frac{N_i}{N_{if}} = 1$$

$\Delta\sigma_i$

N_i number of cycles a_0 to a_i

N_{if} number of cycles a_0 to a_c

Miner's rule for variable load amplitudes

➢ Example 2:

A large plate with an initial crack of length $2a_o$ is subjected to a series of cyclic stress amplitudes $\Delta\sigma_i$ (i = 1,2,..., n) normal to the crack. Assume that the final crack length at instability $2a_f$ is the same for all stress amplitudes $\Delta\sigma_i$. If fatigue crack growth is governed by equation:

$$\frac{da}{dN} = C\,(\Delta K)^2 \qquad (*)$$

show that:

$$\sum_{i=1}^{n} \frac{N_i}{N_{if}} = 1$$

where N_i is the number of cycles required to grow the crack from $2a_{i-1}$ to $2a_i$; and N_{if} is the total number of cycles required to grow the crack from its initial length $2a_o$ to its final length $2a_f$ at instability.

Miner's rule for variable load amplitudes

- solution:

From Equation (*) we obtain for the number of cycles N_1 required to grow the crack from its initial length $2a_o$ to a length $2a_1$:

$$dN = \frac{da}{C\left(\Delta\sigma_1\sqrt{\pi a}\right)^2} \qquad N_1 = \frac{1}{\pi C\left(\Delta\sigma_1\right)^2}\int_{a_0}^{a_1}\frac{da}{a} = \frac{1}{\pi C\left(\Delta\sigma_1\right)^2}\ln\left(\frac{a_1}{a_0}\right) \quad (**)$$

$$N_{1f} = \frac{1}{\pi C\left(\Delta\sigma_1\right)^2}\ln\left(\frac{a_f}{a_0}\right) \xrightarrow{(**)} \frac{N_1}{N_{1f}} = \ln\left(\frac{a_1}{a_0}\right) \Big/ \ln\left(\frac{a_f}{a_0}\right)$$

$$\frac{N_2}{N_{2f}} = \ln\left(\frac{a_2}{a_1}\right) \Big/ \ln\left(\frac{a_f}{a_0}\right) \cdots \longrightarrow \frac{N_i}{N_{if}} = \ln\left(\frac{a_i}{a_{i-1}}\right) \Big/ \ln\left(\frac{a_f}{a_0}\right)$$

Miner's rule for variable load amplitudes

- solution:

$$\frac{N_1}{N_{1f}}+\frac{N_2}{N_{2f}}+\cdots+\frac{N_i}{N_{if}}+\cdots+\frac{N_n}{N_{nf}}=$$

$$\left[\ln\left(\frac{a_1}{a_0}\right)+\ln\left(\frac{a_2}{a_1}\right)+\cdots+\ln\left(\frac{a_i}{a_{i-1}}\right)+\cdots+\ln\left(\frac{a_n}{a_{n-1}}\right)\right]\Big/\ln\left(\frac{a_f}{a_0}\right)$$

$$\sum_{i=1}^{n}\frac{N_i}{N_{if}}=\ln\left[\left(\frac{a_1}{a_0}\right)\left(\frac{a_2}{a_1}\right)\cdots\left(\frac{a_i}{a_{i-1}}\right)\cdots\left(\frac{a_n}{a_{n-1}}\right)\right]\Big/\ln\left(\frac{a_f}{a_0}\right)$$

For $a_n=a_f$:

$$\sum_{i=1}^{n}\frac{N_i}{N_{if}}=1$$

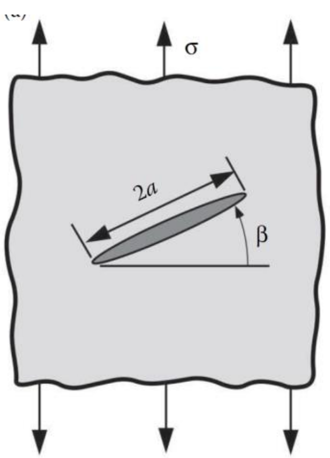
Mixed-Mode Fatigue Loading

- In conducting mixed-mode fatigue studies, a defined effective stress intensity factor ΔK_e may be used in the Paris law for brittle or ductile materials. Effectively, the definition of $\Delta K_e = \Delta K_{eff}$ depends on the mathematical technique and theoretical background one uses. Nevertheless, the Paris law takes the general and empirical form:

$$\frac{da}{dN} = C\,(\Delta K_e)^n$$

- For a mixed-mode I and II interaction, may be used to defined the effective stress intensity factor range as:

$$\Delta K_e^2 = \Delta K_I^2 + \Delta K_{II}^2$$



$$\Delta K_I = \Delta\sigma\sqrt{\pi a}\cos^2\beta$$

$$\Delta K_{II} = \Delta\sigma\sqrt{\pi a}\cos\beta\sin\beta$$

$$\Delta K_e^2 = \pi a(\Delta\sigma)^2\cos^2\beta$$

Mixed-Mode Fatigue Loading

- Combining Eqs. in previous slide yields an expression for calculate $\Delta\sigma$:

$$D_1(\Delta\sigma)^n + D_2(\Delta\sigma)^{n-2} - D_3 = 0$$

$$D_1 = C(n-2)(N - N_0)(K_{Ic})^n$$

$$D_2 = \frac{2}{\pi}\left(\frac{K_{Ic}}{Y \cos\beta}\right)^2$$

$$D_3 = 2a^{1-n/2}\left(\frac{1}{\pi}\right)^{n/2}\left(\frac{K_{Ic}}{Y \cos\beta}\right)^2$$

Prove it!

- Example 3:

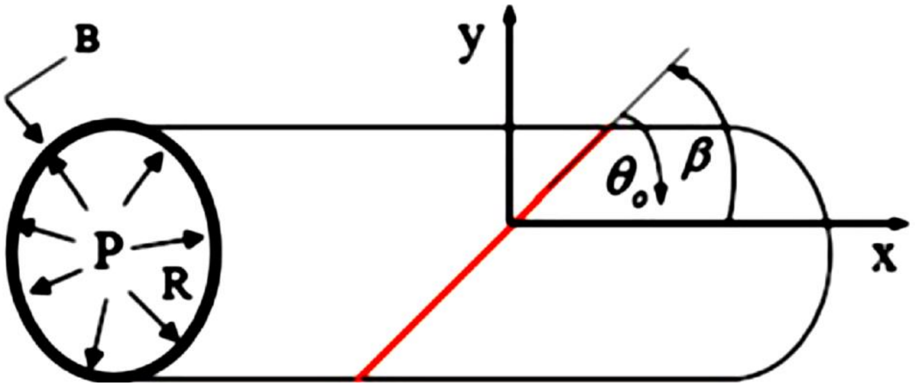
Assume that a solid cylinder of 25 mm in diameter has a round surface crack inclined at $\beta = 20^o$ and that the material has an average plane strain fracture toughness and threshold stress intensity factor of 15 $MPam^{0.5}$ and 5 $MPam^{0.5}$, respectively. If the crack depth is 0.09 mm and the applied cyclic stresses are $\Delta\sigma=\sigma_{max}$ and σ_{min}=0 calculate:

(a) The minimum stress range $\Delta\sigma_{min}$

(b) The applied stress range $\Delta\sigma = \sigma_{max}$

(c) The critical length a_c for a fatigue life of 10^4 cycles. How much will the crack grow? The Paris equation is:

$$\frac{da}{dN} = \left(5\times10^{-12} \frac{MN^{-4}m^{-1}}{cycles}\right)(\Delta K)^4$$



Mixed-Mode Fatigue Loading

➢ solution:

(**a**) Letting $\Delta K_{th} = \Delta K_e$ gives

$$\Delta\sigma_{th} = \frac{\Delta K_{th}}{Y\sqrt{\pi a_0}} = \frac{5\ MPa\sqrt{m}}{1.12\sqrt{\pi(0.09\times10^{-3}\ m)}\cos 20} = 776.25 \quad MPa$$

$$\Delta\sigma_{\min} < \Delta\sigma_{th}$$

(**b**) For maximum stress range: $D_1(\Delta\sigma)^n + D_2(\Delta\sigma)^{n-2} - D_3 = 0$

$$D_1 = C(n-2)(N - N_0)(K_{Ic})^n = 5.0625\times10^{-3} \quad m$$

$$D_2 = \frac{2}{\pi}\left(\frac{K_{Ic}}{Y\ \cos\beta}\right)^2 = 976.16 \quad MPa^2$$

$$D_3 = 2a^{1-n/2}\left(\frac{1}{\pi}\right)^{n/2}\left(\frac{K_{Ic}}{Y\ \cos\beta}\right)^2 = 5.2939\times10^9 \quad MPa^4 m$$

Mixed-Mode Fatigue Loading

- solution:

(**b**) For maximum stress range: $$D_1(\Delta\sigma)^n + D_2(\Delta\sigma)^{n-2} - D_3 = 0$$

$$5.0625\times10^{-3}(\Delta\sigma)^4 + 976.16(\Delta\sigma)^2 - 5.2939\times10^9 = 0$$

$$\Delta\sigma = 964.74 \ \ MPa$$

(**c**) the critical crack size is

$$a_c = \frac{K_{Ic}^2}{\pi(\Delta\sigma)^2\cos^2\beta} = \frac{(15\ MPa\sqrt{m})^2}{\pi(964.74\ MPa)^2\cos^2(\pi/9)} = 0.09\ mm$$

$$\Delta a = a_c - a_0 = 0.09 - 0.09 = 0$$

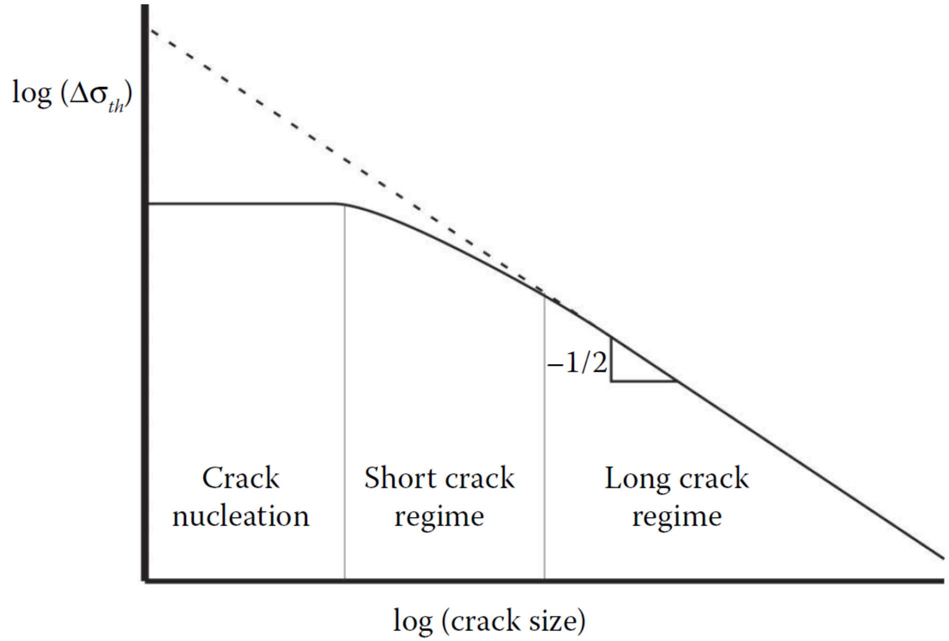
Therefore, the crack grew Δa=0 which represents a pure brittle material.

Growth of Short Cracks

- The concepts of fracture mechanics similitude and a ΔK threshold break down near the point of crack initiation.

$$\Delta\sigma_{th} = \frac{\Delta K_{th}}{Y\sqrt{\pi a}}$$

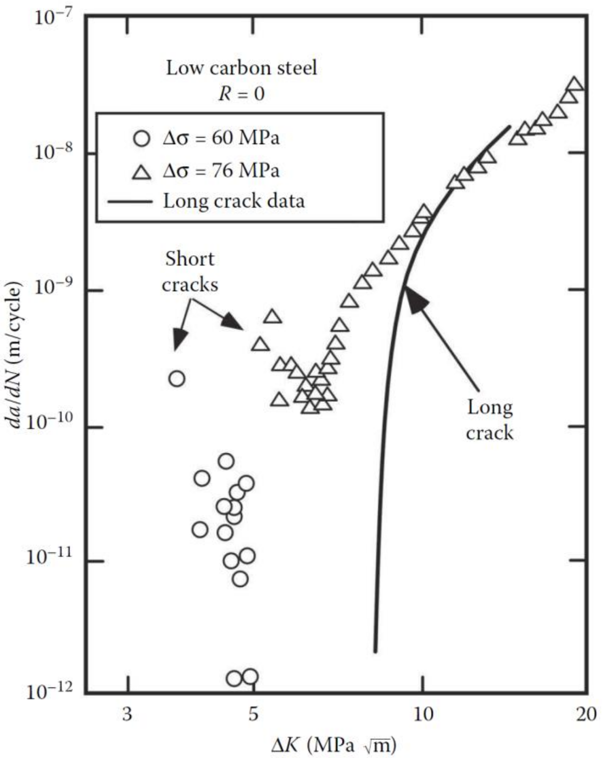
- In reality, $\Delta\sigma_{th}$ reaches a plateau for very small cracks. This plateau represents the minimum cyclic stress required to nucleate a crack from a smooth surface, which is also known as the fatigue endurance limit.



Relationship between threshold cyclic stress, $\Delta\sigma_{th}$, and crack size. Fracture mechanics similitude does not apply in the crack nucleation regime.

Growth of Short Cracks

- The short crack regime corresponds to the transition from the crack nucleation regime to the long crack regime where fracture mechanics similitude applies.
- There is not a precise definition of what constitutes a “short” crack, but most experts consider cracks less than 1 mm deep to be small.
- A number of factors can contribute to the anomalous behavior of small fatigue cracks. The fatigue mechanisms depend on whether the crack is microstructurally short or mechanically short.



Growth of short cracks

Growth of Short Cracks

- **Microstructurally Short Cracks**

- A microstructurally short crack has dimensions that are on the order of the grain size. the growth is strongly influenced by the microstructural features in such cases. The growth of microstructurally short cracks is often very sporadic; the crack may grow rapidly at certain intervals, and then virtually arrest when it encounters barriers such as the grain boundaries and second-phase particles.

- **Mechanically Short Cracks**

- A crack that is between 100 μm and 1 mm in depth is mechanically short. The size is sufficient to apply continuum theory, but the mechanical behavior is not the same as in longer cracks. Mechanically short cracks typically grow much faster than long cracks at the same ΔK level, particularly near the threshold.

- Two factors have been identified as contributing to faster growth of short cracks: *plastic zone size* and *crack closure*.

Growth of Short Cracks

➢ **Mechanically Short Cracks**

- When the plastic zone size is significant compared with the crack length, an elastic singularity does not exist at the crack tip, and K is invalid. The effective driving force can be estimated by incorporating a plastic zone correction. El Haddad introduced an "intrinsic crack length" which, when added to the physical crack size, brings short crack data in line with the corresponding long crack results. The intrinsic crack length is merely a fitting parameter, however, and does not correspond to a physical length scale in the material.
  - According to the closure argument, short cracks exhibit a different crack closure behavior than long cracks, and data for different crack sizes can be rationalized through ΔK_{eff}. The figure shows K_{op} measurements for the short and long crack data in previous figure. The closure loads are significantly higher in the long cracks, particularly at low ΔK levels.
  - The figure shows that the small and large crack data lie on a common curve when da/dN is plotted against ΔK_{eff}, thereby lending credibility to the closure theory of short crack behavior.

Growth of Short Cracks

➢ Mechanically Short Cracks

Short crack fatigue crack growth, corrected for closure: (a) crack closure data for short and long cracks, (b) closure-corrected data.