تدوین یک برنامه در محیط MATLAB برای یک سازه دو بعدی

Preprocessor

Geometry Properties (Nodes, Elements) , Material Properties

Boundary Conditions(displacements, Forces)

↓

Definition of Stiffness Matrices (Local), **k**, for each Element

↓

Definition of Strain-Displacement Matrices, **B**, for each Element

↓

Calculation of Structure Stiffness Matrices (Global), **K**,

↓

Definition of Primary External Nodal Forces Vector

↓

Apply the B.C.

تدوین یک برنامه در محیط MATLAB برای یک سازه دو بعدی

↓

Solution of Equation “**K U =F**”

↓

Calculation of Stress in Gaussian Points for each Element

↓

Transportation of Stress from Gaussian Points to Element Nodes

↓

Calculation of average Stress in Element Nodes

↓

Analysis Results

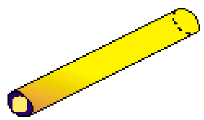
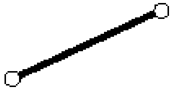
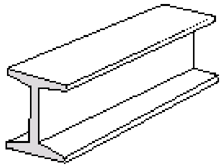
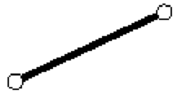
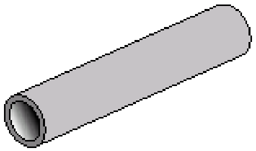
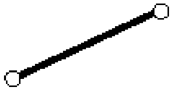
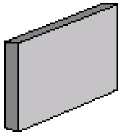
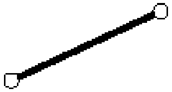
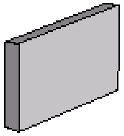
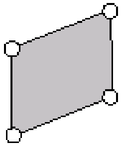
Finite Element Modeling Techniques

Classification of Mechanical Finite Elements

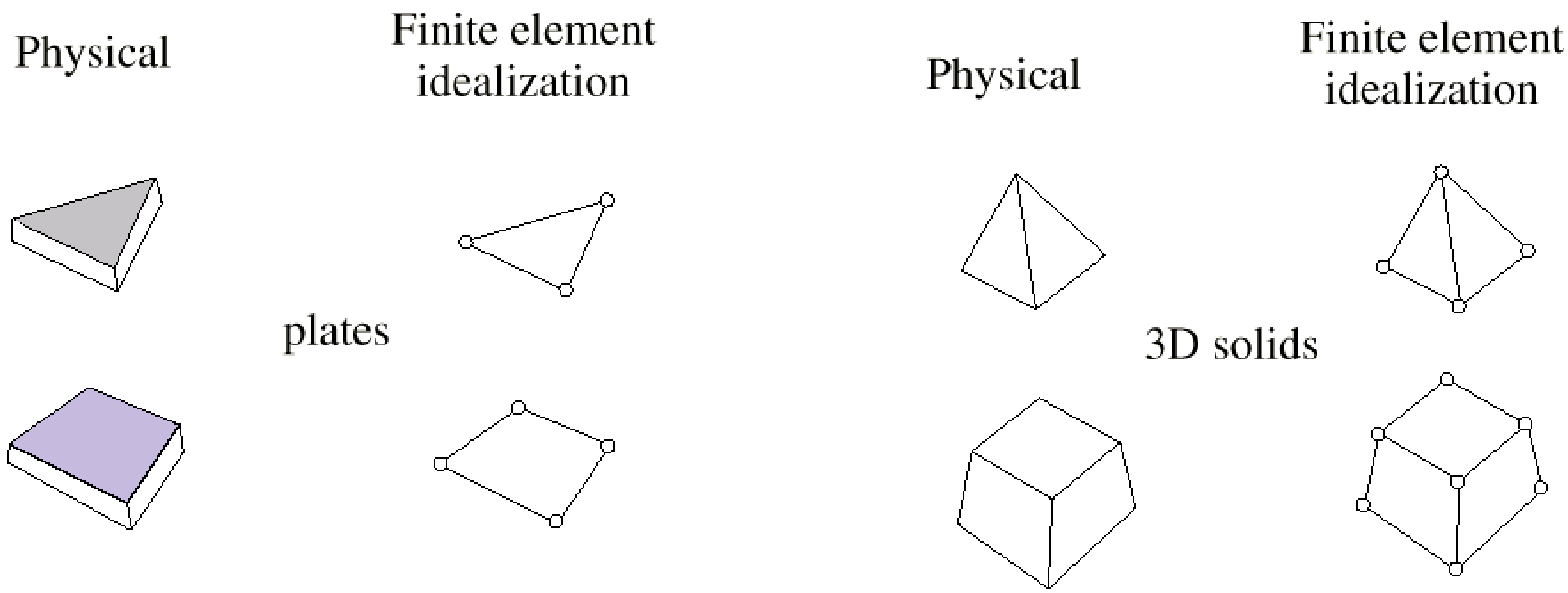
- Primitive Structural
- Continuum
- Special
- Macroelements
- Substructures

} Superelements

Primitive Structural Elements

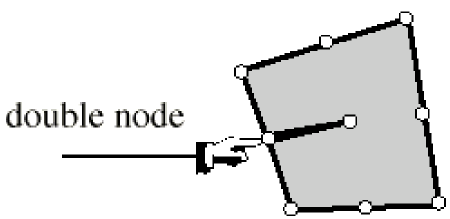
Physical Structural Component	Mathematical Model Name	Finite Element Discretization
	bar	
	beam	
	tube, pipe	
	spar (web)	
	shear panel (2D version of above)	

Continuum Elements

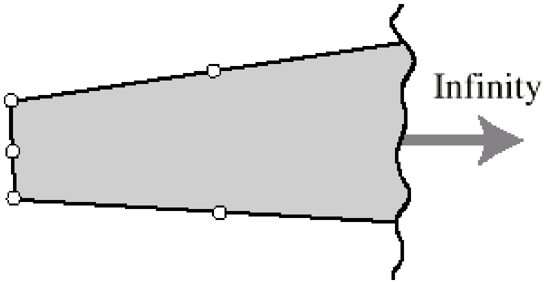


Special Elements

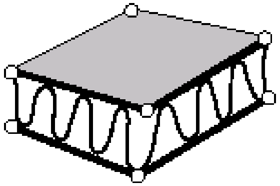
- Crack tip elements
- Infinite elements
- Cohesive elements
- . . .



Crack
element



Infinite
element



Honeycomb
panel

Crack Tip Element

- Fracture mechanics – singularity point at crack tip.
- Conventional finite elements do not give good approximation at/near the crack tip.

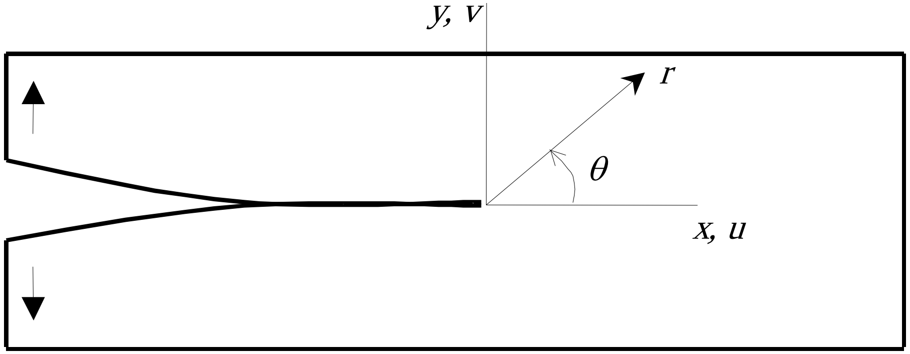
From fracture mechanics,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{yy} \end{bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \begin{bmatrix} 1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2} \\ \sin\frac{\theta}{2}\sin\frac{3\theta}{2} \\ 1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2} \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{K_I\sqrt{r}}{2G\sqrt{2\pi}} \begin{bmatrix} \cos\frac{\theta}{2}(\kappa-1+2\sin^2\frac{\theta}{2}) \\ \sin\frac{\theta}{2}(\kappa+1-2\cos^2\frac{\theta}{2}) \end{bmatrix}$$

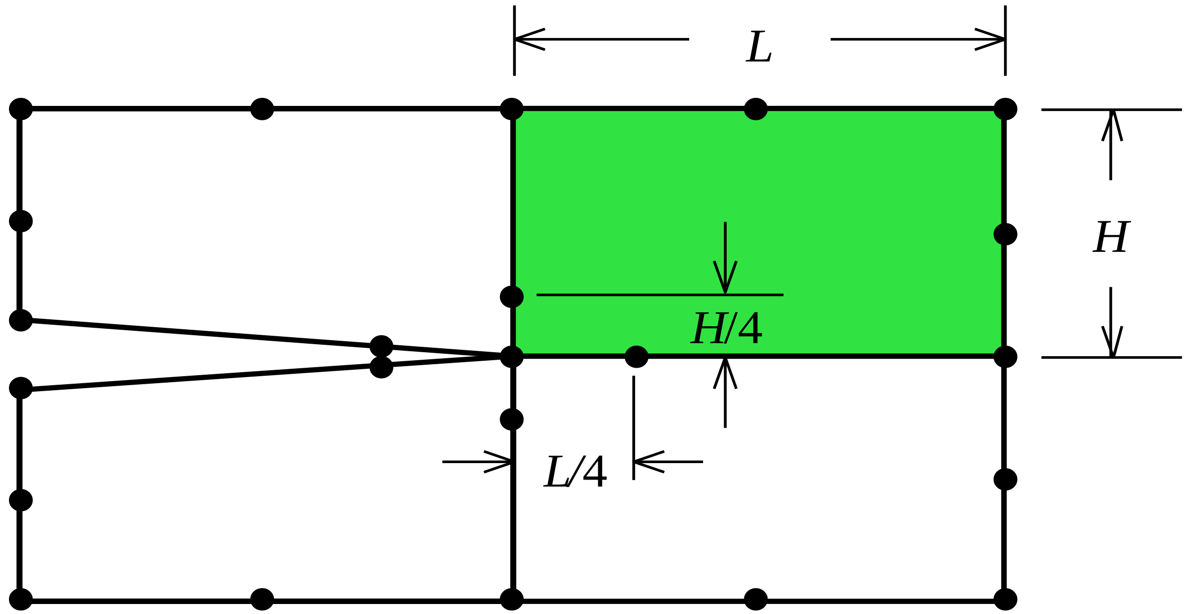
(Near crack tip)

(Mode I fracture)



Crack Tip Element

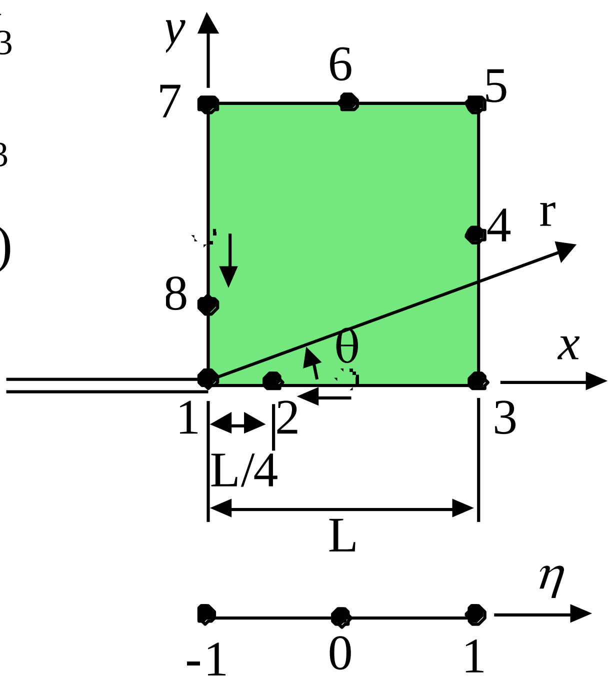
- Special purpose crack tip element with middle nodes shifted to quarter position:



Crack Tip Element

$$\begin{cases} x = -0.5\eta\,(1-\eta)x_1 + (1+\eta)(1-\eta)x_2 + 0.5\eta\,(1+\eta)\,x_3 \\ u = -0.5\eta\,(1-\eta)u_1 + (1+\eta)(1-\eta)u_2 + 0.5\eta\,(1+\eta)\,u_3 \end{cases}$$

(Measured from node 1)



Move node 2 to L/4 position

$$\begin{cases} x_1 = 0,\ x_2 = L/4,\ x_3 = L,\ u_1 = 0 \end{cases}$$

$$\Rightarrow \quad \begin{aligned} x &= 0.25(1+\eta)(1-\eta)L + 0.5\eta\,(1+\eta)L \\ u &= (1+\eta)(1-\eta)u_2 + 0.5\eta\,(1+\eta)\,u_3 \end{aligned}$$

Crack Tip Element

Simplifying,

$$x = 0.25(1+\eta)^2 L$$

$$u = (1+\eta)[(1-\eta)u_2 + 0.5\eta u_3]$$

Along x-axis, $x = r$

$$r = 0.25(1+\eta)^2 L \quad \text{or} \quad (1+\eta) = 2\sqrt{\frac{r}{L}}$$

$$\Rightarrow \quad u = 2(\sqrt{r}/\sqrt{L})\,[(1-\eta)u_2 + 0.5\eta u_3]$$

Note: Displacement is proportional to $\sqrt{r}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \quad \text{where} \quad \frac{\partial x}{\partial \eta} = 0.5(1+\eta)L = \sqrt{r}\sqrt{L}$$

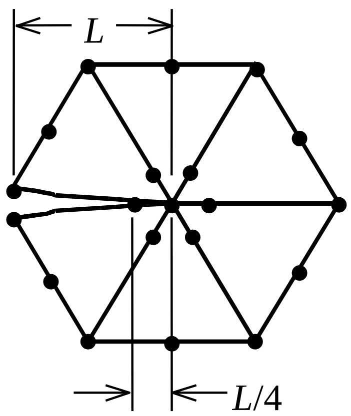
Therefore,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{r}}\frac{1}{\sqrt{L}}[-2\eta u_2 + (\frac{1}{2}+\eta)u_3]$$

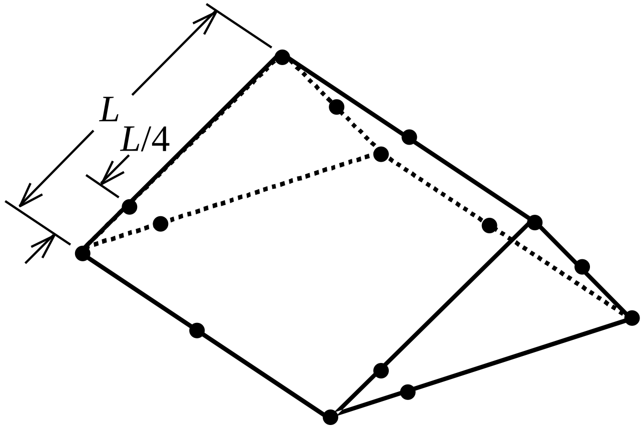
Note: Strain (hence stress) is proportional to $1/\sqrt{r}$

Crack Tip Element

- Therefore, by shifting the nodes to quarter position, we approximating the stress and displacements more accurately.
- Other crack tip elements:



Triangular crack tip elements



A 3-D, wedge crack tip element

Methods for Infinite Domain

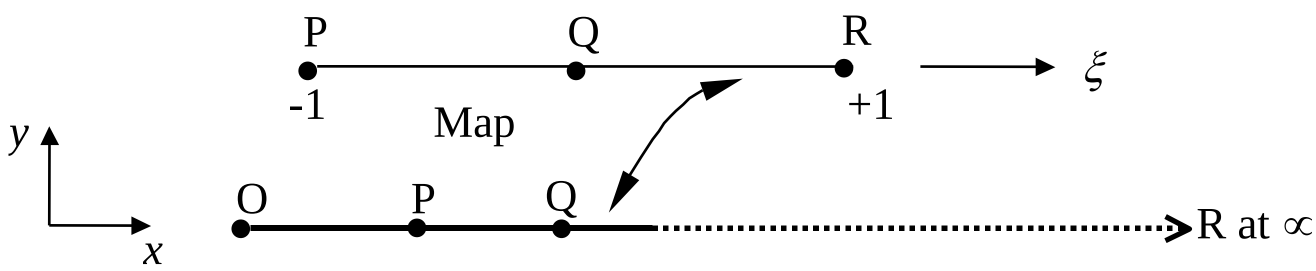
- Infinite elements formulated by mapping (Zienkiewicz and Taylor, 2000)
- Gradual damping elements
- Coupling of FEM and BEM
- Coupling of FEM and SEM

Infinite elements formulated by mapping

Use shape functions to approximate decaying sequence:

$$\frac{C_1}{r}+\frac{C_2}{r^2}+\frac{C_3}{r^3}+\ldots$$

In 1D:



$$x=-\frac{\xi}{1-\xi}x_O+\left(1+\frac{\xi}{1-\xi}\right)x_Q$$

(Coordinate interpolation)

$$r=x-x_O \quad \Rightarrow \quad \xi=1-\frac{x_Q-x_O}{x-x_O}=1-\frac{x_Q-x_O}{r}$$

Infinite elements formulated by mapping

If the field variable is approximated by polynomial,

$$u = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3 + \ldots$$

Substituting ξ will give function of decaying form, $\dfrac{C_1}{r} + \dfrac{C_2}{r^2} + \dfrac{C_3}{r^3} + \ldots$

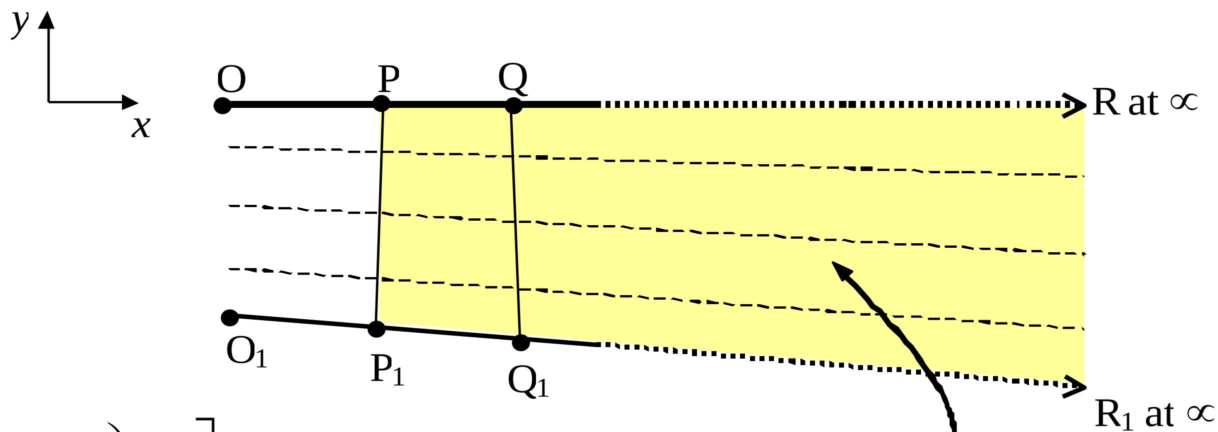
For 2D (3D):

$$x = -\frac{\xi}{1-\xi} x_{O_1} + \left(1 + \frac{\xi}{1-\xi}\right) x_{Q_1}$$

$$y = -\frac{\xi}{1-\xi} y_{O_1} + \left(1 + \frac{\xi}{1-\xi}\right) y_{Q_1}$$

$$z = -\frac{\xi}{1-\xi} z_{O_1} + \left(1 + \frac{\xi}{1-\xi}\right) z_{Q_1}$$

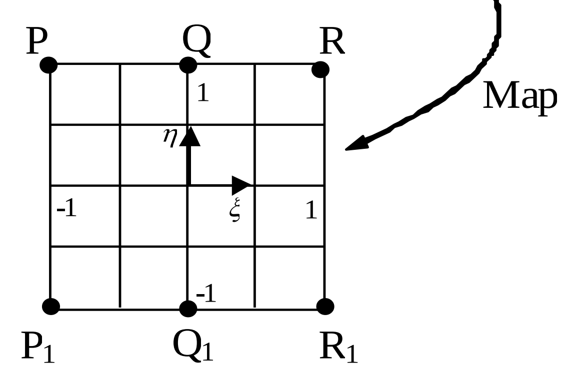
Infinite elements formulated by mapping



Element
$PP_1QQ_1RR_1$:

$$x = N_1(\eta)\left[-\frac{\xi}{1-\xi}x_O\left(1+\frac{\xi}{1-\xi}\right)x_Q\right]$$

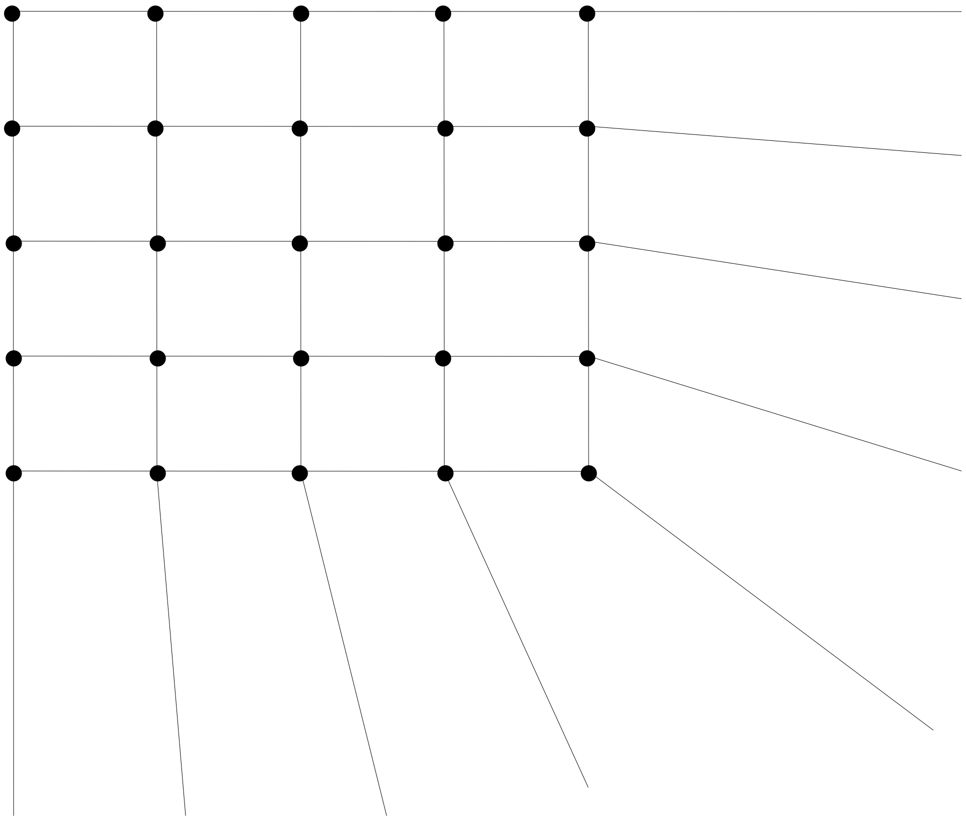
$$+N_0(\eta)\left(-\frac{\xi}{1-\xi}x_{O_1}+\frac{\xi}{1-\xi}x_{Q_1}\right)$$



with $N_1(\eta)=\dfrac{1+\eta}{2},\quad N_0(\eta)=\dfrac{1-\eta}{2}$

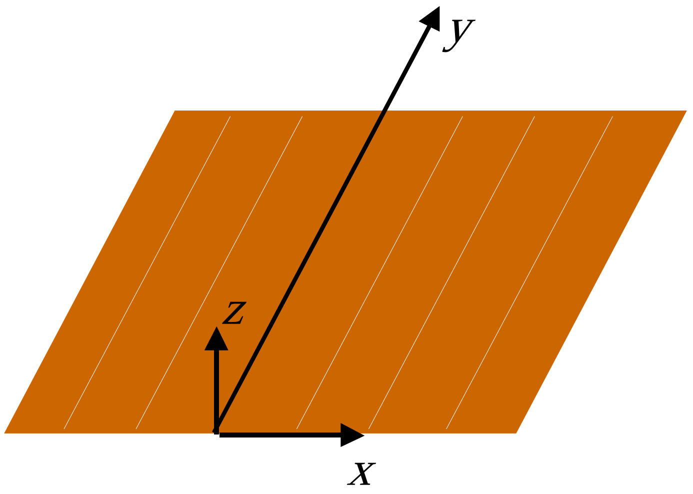
Infinite elements formulated by mapping

Infinite elements are attached to conventional FE mesh to simulate infinite domain.



Finite Strip Elements

- Developed by Y. K. Cheung, 1968.
- Used for problems with regular geometry and simple boundary.
- Key is in obtaining the shape functions.



Finite Strip Elements

$$w = \sum_{m=1}^{r} f_m(x) Y_m$$ (Approximation of displacement function)

(Polynomial) (Continuous series)

Polynomial function must represent state of constant strain in the x direction and continuous series must satisfy end conditions of the strip.

Together the shape function must satisfy compatibility of displacements with adjacent strips.

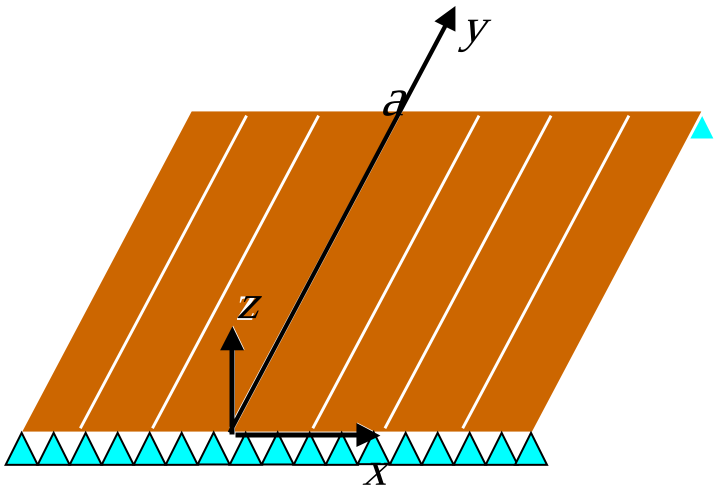
Finite Strip Elements

$Y(0) = 0,\ Y''(0) = 0,\ Y(a) = 0$ and $Y''(a) = 0$

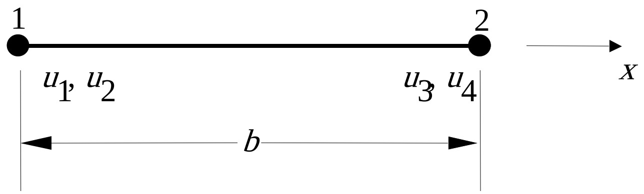
Satisfies

$$Y_m(y) = \sin\left(\frac{\mu_m y}{a}\right)$$

$$\mu_m = \pi, 2\pi, 3\pi, \ldots, m\pi$$



$$f_m(x) = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \end{bmatrix} \begin{Bmatrix} u_1^m \\ u_2^m \\ u_3^m \\ u_4^m \end{Bmatrix}$$



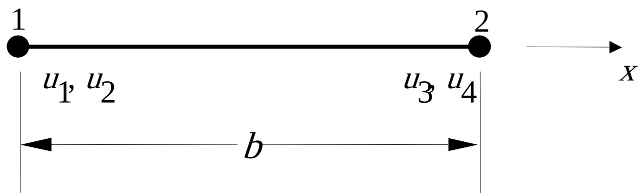
Finite Strip Elements

$$C_1(x) = \frac{2x^3}{b^3} - \frac{3x^2}{b^2} + 1$$

$$C_2(x) = \frac{x^3}{b^2} - \frac{2x^2}{b} + x$$

$$C_3(x) = -\frac{2x^3}{b^3} + \frac{3x^2}{b^2}$$

$$C_4(x) = \frac{x^3}{b^2} - \frac{x^2}{b}$$



Therefore,

$$w(x,y) = \sum_{m=1}^{r} \left(C_1(x) u_1^m + C_2(x) u_2^m + C_3(x) u_3^m + C_4(x) u_4^m \right) Y_m(y)$$

Finite Strip Elements

$$w(x,y)=\sum_{m=1}^{r}\left(C_1(x)u_1^m+C_2(x)u_2^m+C_3(x)u_3^m+C_4(x)u_4^m\right)Y_m(y)$$

or
$$w(x,y)=\sum_{m=1}^{r}\begin{bmatrix}N_1^m & N_2^m & N_3^m & N_4^m\end{bmatrix}\begin{Bmatrix}u_1^m\\u_2^m\\u_3^m\\u_4^m\end{Bmatrix}$$

where $N_i^m(x,y)=C_i(x)Y_m(y)$ $\quad i = 1, 2, 3 ,4$

The remaining procedure is the same as the FEM. The size of the matrix is usually much smaller and makes the solving much easier.

STRIP ELEMENT METHOD (SEM)

- Proposed by Liu and co-workers [Liu *et al.*, 1994, 1995; Liu and Xi, 2001].
- Solving wave propagation in composite laminates.
- Semi-analytic method for stress analysis of solids and structures.
- Applicable to problems of arbitrary boundary conditions including the infinite boundary conditions.
- Coupling of FEM and SEM for infinite domains.

Substructures

Substructuring is a process of analyzing a large structure as a collection of (natural) components. The FE models for these components are called *substructures* or *superelements* (SE).

Physical Meaning:

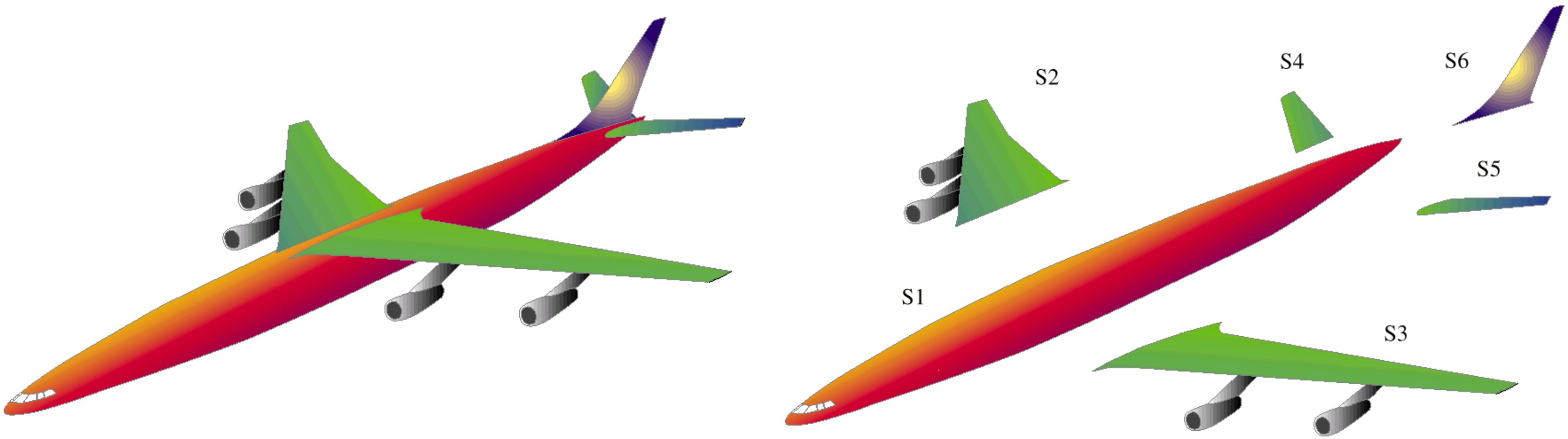
A finite element model of a portion of structure.

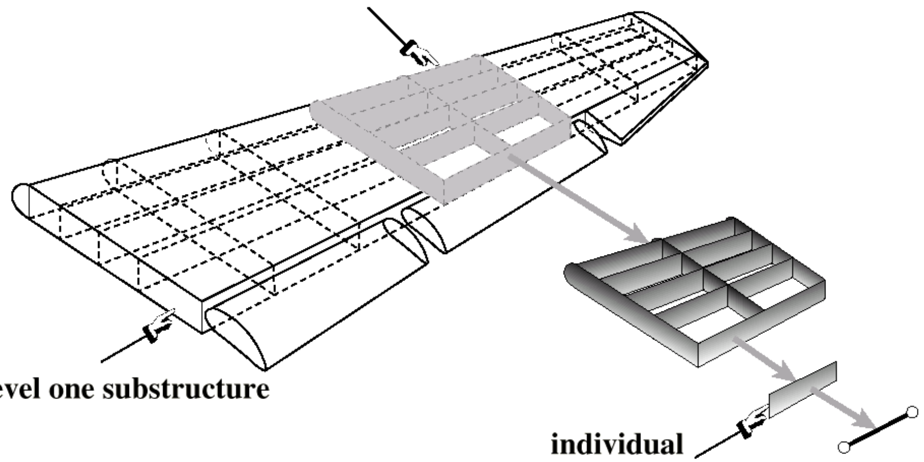
Mathematical Meaning:

Boundary matrices which are load and stiffness matrices reduced (condensed) from the *interior* points to the *exterior* or boundary points.

One obvious advantage of this idea results if the structure is built of several identical units. For example, the wing substructures S2 and S3 are largely identical except for a reflection about the fuselage midplane, and so are the stabilizers S4 and S5.

Substructures



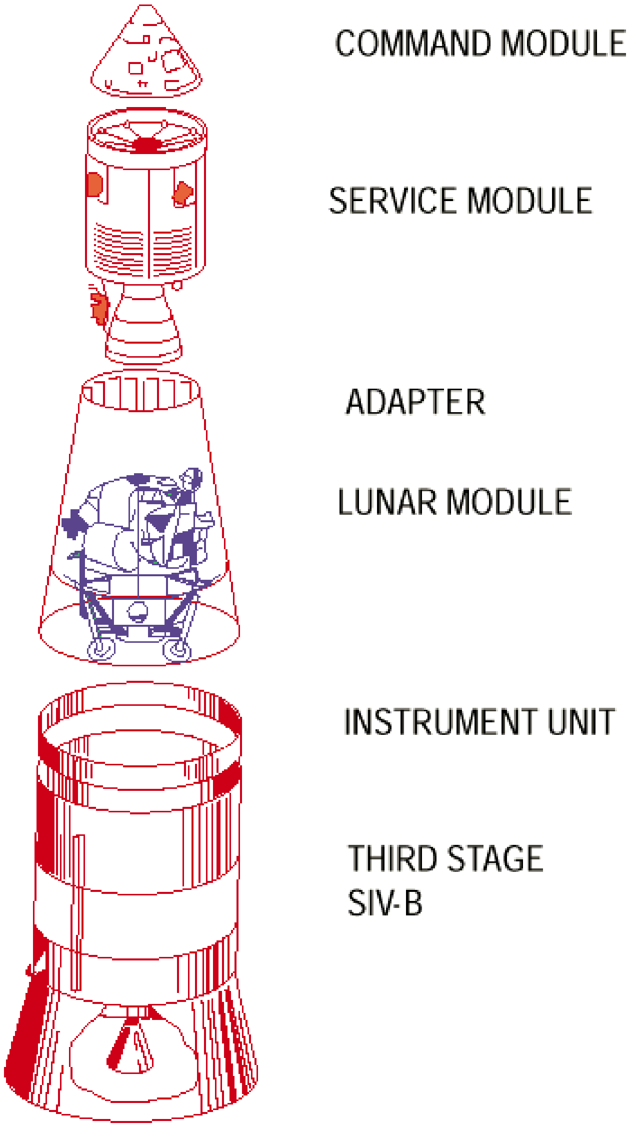


Substructures

Multistage Rockets Naturally Decompose into Substructure

Short stack Apollo/Saturn

Lunar rocket

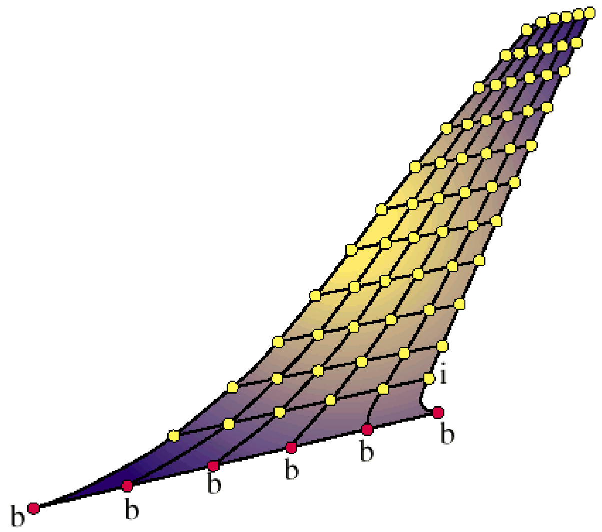


Static Condensation

Degrees of freedom of a superelement are classified into two groups:

Internal Freedoms. Those that are not connected to the freedoms of another superelement. Node whose freedoms are internal are called *internal nodes*.

Boundary Freedoms. These are connected to at least another superelement. They usually reside at *boundary nodes* placed on the periphery of the superelement.



The vertical stabilizer substructure *S*6

Static Condensation

The assembled stiffness equations of the superelement are partitioned as follows:

$$\begin{bmatrix} \boldsymbol{K}_{bb} & \boldsymbol{K}_{bi} \\ \boldsymbol{K}_{ib} & \boldsymbol{K}_{ii} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_b \\ \boldsymbol{u}_i \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_b \\ \boldsymbol{f}_i \end{bmatrix} \qquad (*)$$

where subvectors $\mathbf{u}_b$ and $\mathbf{u}_i$ collect *boundary* and *interior* degrees of freedom, respectively. Take the second matrix equation:

$$\boldsymbol{K}_{ib}\,\boldsymbol{u}_b + \boldsymbol{K}_{ii}\,\boldsymbol{u}_i = \boldsymbol{f}_i$$

Assume $\mathbf{K}_{ii}$ is nonsingular, we can solve for the interior freedoms:

$$\boldsymbol{u}_i = \boldsymbol{K}_{ii}^{-1}(\boldsymbol{f}_i - \boldsymbol{K}_{ib}\,\boldsymbol{u}_b)$$

Replacing into the first matrix equation of (*) yields the *condensed stiffness equations*

$$\tilde{\boldsymbol{K}}_{bb}\,\boldsymbol{u}_b = \tilde{\boldsymbol{f}}_b$$

$$\tilde{\boldsymbol{K}}_{bb} = \boldsymbol{K}_{bb} - \boldsymbol{K}_{bi}\boldsymbol{K}_{ii}^{-1}\,\boldsymbol{K}_{ib}$$ The *condensed* stiffness matrix

$$\tilde{\boldsymbol{f}}_b = \boldsymbol{f}_b - \boldsymbol{K}_{bi}\boldsymbol{K}_{ii}^{-1}\,\boldsymbol{f}_i$$ *The condensed* force vector

Substructures

Advantages of Using Substructures/Superelements:

- Large problems (which will otherwise exceed your computer capabilities)
- Less CPU time per run once the superelements have been processed (i.e., matrices have been saved)
- Components may be modeled by different groups
- Partial redesign requires only partial reanalysis (reduced cost)
- Efficient for problems with local nonlinearities (such as confined plastic deformations) which can be placed in one superelement (residual structure)
- Exact for static stress analysis

Disadvantages:

- Increased overhead for file management
- Matrix condensation for dynamic problems introduce new approximations
- . . .

General FEM Modeling Rules

1- Use the simplest elements that will do the job.

2- *Never, never, never* use complicated or special elements unless you are absolutely sure of what you are doing.

3- Use the coarsest mesh that will capture the dominant behavior of the physical model, particularly in *design* situations.

3 word summary: *Keep It Simple*