Numerical Integration

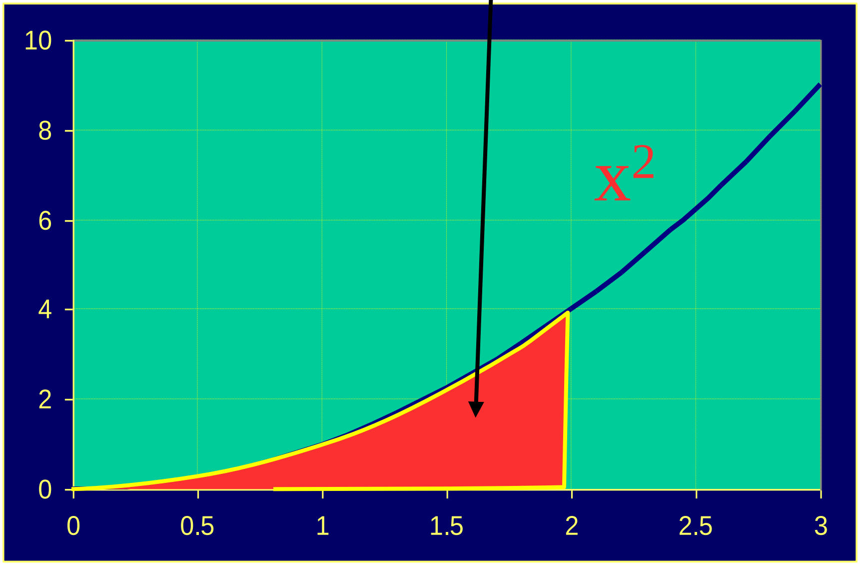
Integrals

Indefinite

$$\int x^2 dx = \frac{1}{3}x^3 + C$$

Definite

$$\int_0^2 x^2 dx = \frac{8}{3}$$



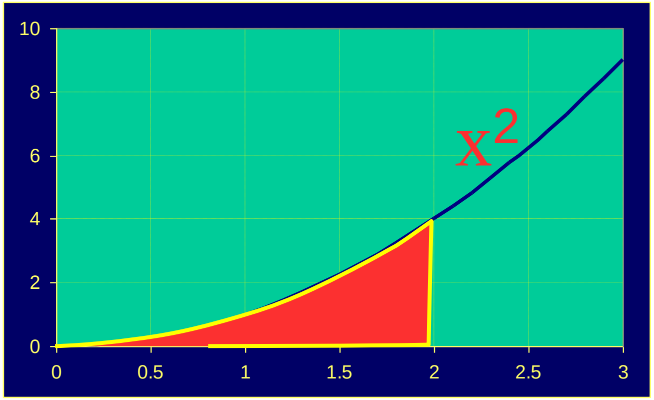
Numerical Integration

- Can be solved exactly, but for various reasons FEA prefers to evaluate integrals like this approximately:
  - Historically, considered more efficient and reduced coding errors.
  - Only possible approach for isoparametric elements.
  - Can actually improve performance in certain cases!

- Definite integrals can be computed numerically

$$\int_a^b f(x)dx \cong \sum_i w_i f(x_i)$$

- Objective:
  - Determine points x_i
  - Determine coefficients w_i



Numerical Integration

- Depending on choice of w_i and x_i
  - Midpoint Rule
  - Trapezoidal Rule
  - Simpson's
  - Gaussian Quadratures
  - etc

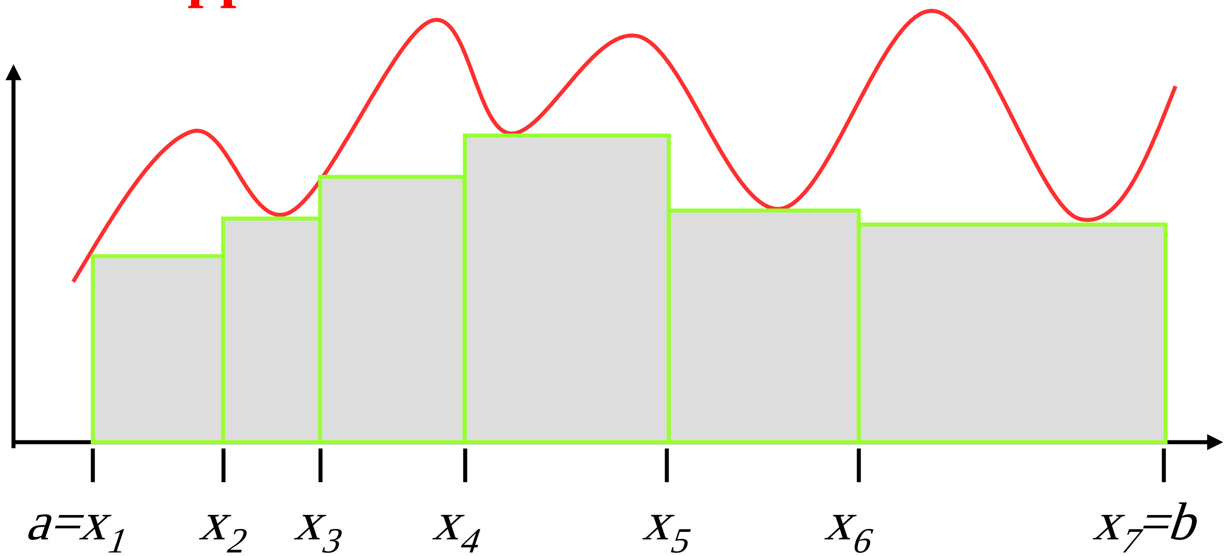
Numerical Integration

Numerical Integration – Upper & Lower Bounds

Depending on choice of w_i and x_i

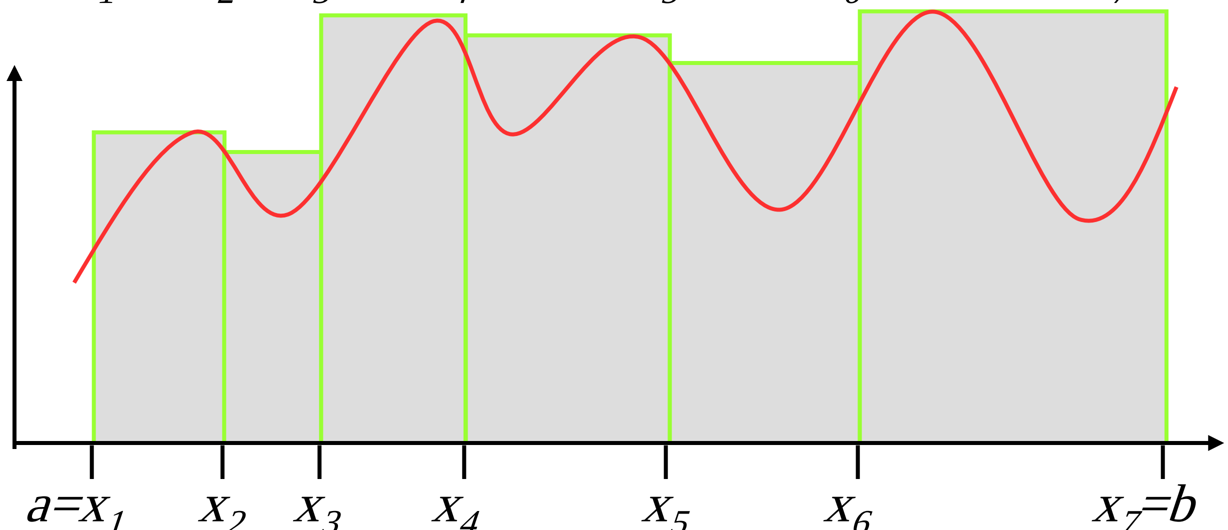
Lower Sum

$L(f;x_i)$



Upper Sum

$U(f;x_i)$



Numerical Integration

It can be shown that

$$L(f;x_i) \le \int_a^b f(x)dx \cong \sum_i w_i f(x_i) \le U(f;x_i)$$

$$\lim_{i\to\infty} L(f;x_i) = \int_a^b f(x)dx = \sum_i w_i f(x_i) = \lim_{i\to\infty} U(f;x_i)$$

Objective

$$\int_a^b f(x)dx \cong \sum_i w_i f(x_i) = w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n)$$

Where do such formulae come from?

Theory of Interpolation….

Let $f(x) \approx p(x) = \sum_{i=1}^{n} l_i(x) f(x_i)$ $l_i(x)$: cardinal functions

Recall Shape Functions

Numerical Integration: Quadratures

$$\int_a^b f(x)dx \approx \int_a^b p(x)dx = \sum_{i=1}^{n} f(x_i)\int_a^b l_i(x)dx = \sum_{i=1}^{n} f(x_i)w_i$$

It will give correct values for the integral of every polynomial of degree ≤ n-1

Gaussian Quadrature:

Karl Friedriech Gauss discovered that by a special placement of nodes the accuracy of the numerical integration could be greatly increased

Theorem on Gaussian nodes

Let q be a polynomial of degree n such that

$$\int_a^b q(x)x^k dx = 0 \qquad k = 0,1,\ldots,n-1$$

Let $x_1, x_2, \ldots, x_n$ be the roots of $q(x)$. Then

$$\int_a^b f(x)dx \cong \sum_i w_i f(x_i) = w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n)$$

with x_i's as nodes is exact for all polynomials of degree ≤2n-1.

Numerical Integration: Gaussian Quadrature

Assume two point formulation, then:

$$\int_{-1}^{1} F(\xi)d\xi = w_1 F(\xi_1) + w_2 F(\xi_2)$$

Four equations are created using Legendre polynomials $(1, \xi, \xi^2, \xi^3)$

$$w_1 F(\xi_1) + w_2 F(\xi_2) = \int_{-1}^{1} 1 d\xi = 2$$

$$w_1 F(\xi_1) + w_2 F(\xi_2) = \int_{-1}^{1} \xi d\xi = 0$$

$$w_1 F(\xi_1) + w_2 F(\xi_2) = \int_{-1}^{1} \xi^2 d\xi = 2/3$$

$$w_1 F(\xi_1) + w_2 F(\xi_2) = \int_{-1}^{1} \xi^3 d\xi = 0$$

$\Rightarrow$

$$w_1(1) + w_2(1) = 2$$

$$w_1(\xi_1) + w_2(\xi_2) = 0$$

$$w_1(\xi_1)^2 + w_2(\xi_2)^2 = 2/3$$

$$w_1(\xi_1)^3 + w_2(\xi_2)^3 = 0$$

$\Rightarrow$

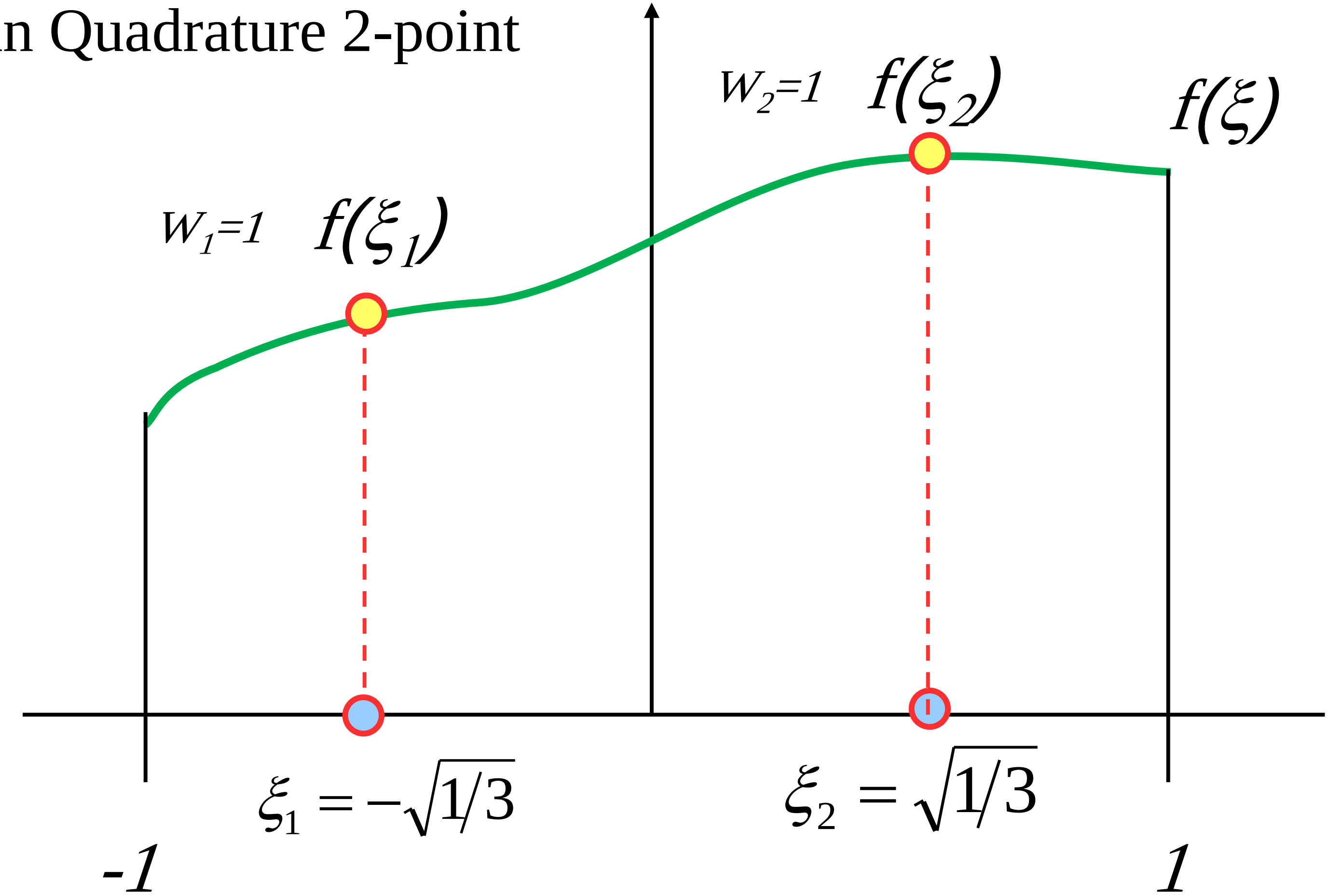
$$w_1 = w_2 = 1$$

$$\xi_1 = -1/\sqrt{3}$$

$$\xi_2 = 1/\sqrt{3}$$

Numerical Integration: Gaussian Quadrature

Gaussian Quadrature 2-point



$$\int_{-1}^{1} f(x)dx \cong 1 * f\left(-\sqrt{1/3}\right) + 1 * f\left(\sqrt{1/3}\right)$$

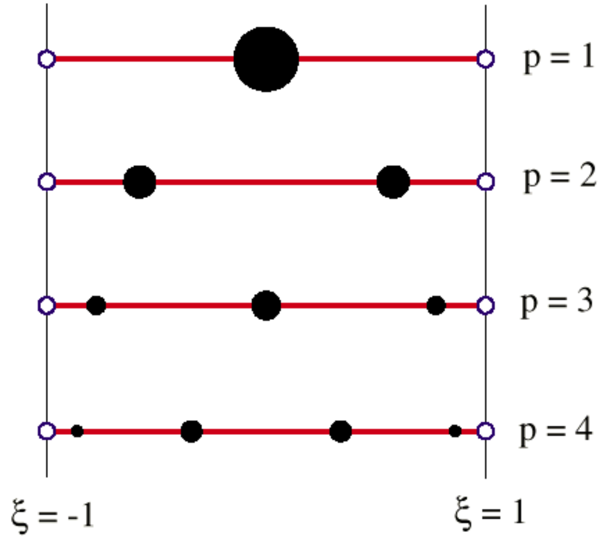
Numerical Integration: Gaussian Quadrature

One Dimensional Gauss Integration Rules:

One point: $$\int_{-1}^{1} F(\xi)\,d\xi \doteq 2F(0),$$

Two points: $$\int_{-1}^{1} F(\xi)\,d\xi \doteq F(-1/\sqrt{3}) + F(1/\sqrt{3}),$$

Three points: $$\int_{-1}^{1} F(\xi)\,d\xi \doteq \frac{5}{9}F(-\sqrt{3/5}) + \frac{8}{9}F(0) + \frac{5}{9}F(\sqrt{3/5})$$



Weighting Factors & Sampling Points for Gauss-Legendre Formula

Points(n)	Weighting Factor (w_i)	Sampling Points (ξ_i)
2	$w_1 = 1.00000000$	$\xi_1 = -.577350269$
	$w_2 = 1.00000000$	$\xi_2 = .577350269$
3	$w_1 = 0.55555556$	$\xi_1 = -.774596669$
	$w_2 = 0.88888889$	$\xi_2 = 0.0$
	$w_3 = 0.55555556$	$\xi_3 = 0.774596669$
4	$w_1 = 0.3478548$	$\xi_1 = -.861136312$
	$w_2 = 0.6521452$	$\xi_2 = -.339981044$
	$w_3 = 0.6521452$	$\xi_3 = 0.339981044$
	$w_1 = 0.3478548$	$\xi_4 = .861136312$
5	$w_1 = 0.2369269$	$\xi_1 = -.906179846$
	$w_2 = 0.4786287$	$\xi_2 = -.538469310$
	$w_3 = 0.5688889$	$\xi_3 = 0.0$
	$w_4 = 0.4786287$	$\xi_4 = .538469310$
	$w_5 = 0.2369269$	$\xi_5 = .906179846$

Example:

$$I=\int_{2}^{6}(x^2+5x+3)dx=?$$

Analytical solution → 161.3333

$$x=4+2\xi \quad \rightarrow \quad I=\int_{-1}^{1}\underbrace{2[(4+2\xi)^2+5(4+2\xi)+3]}_{F(\xi)}d\xi$$

$$I=w_1F(\xi_1)+w_2F(\xi_2)=F(\frac{1}{\sqrt{3}})+F(\frac{-1}{\sqrt{3}})$$

$$I=(1)(50.64445)+(1)(110.68888)=161.3333$$

Numerical Integration: Gaussian Quadrature

Two Dimensional Product Gauss Rules

Canonical form of integral:

$$\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{1} d\eta \int_{-1}^{1} F(\xi,\eta)\,d\xi$$

Gauss integration rules with p_1 points in the ξ direction and p_2 points in the η direction:

$$\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{1} d\eta \int_{-1}^{1} F(\xi,\eta)\,d\xi \approx \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} w_i w_j F(\xi_i,\eta_j)$$

Usually $p1 = p2 = p$

Numerical Integration: Gaussian Quadrature

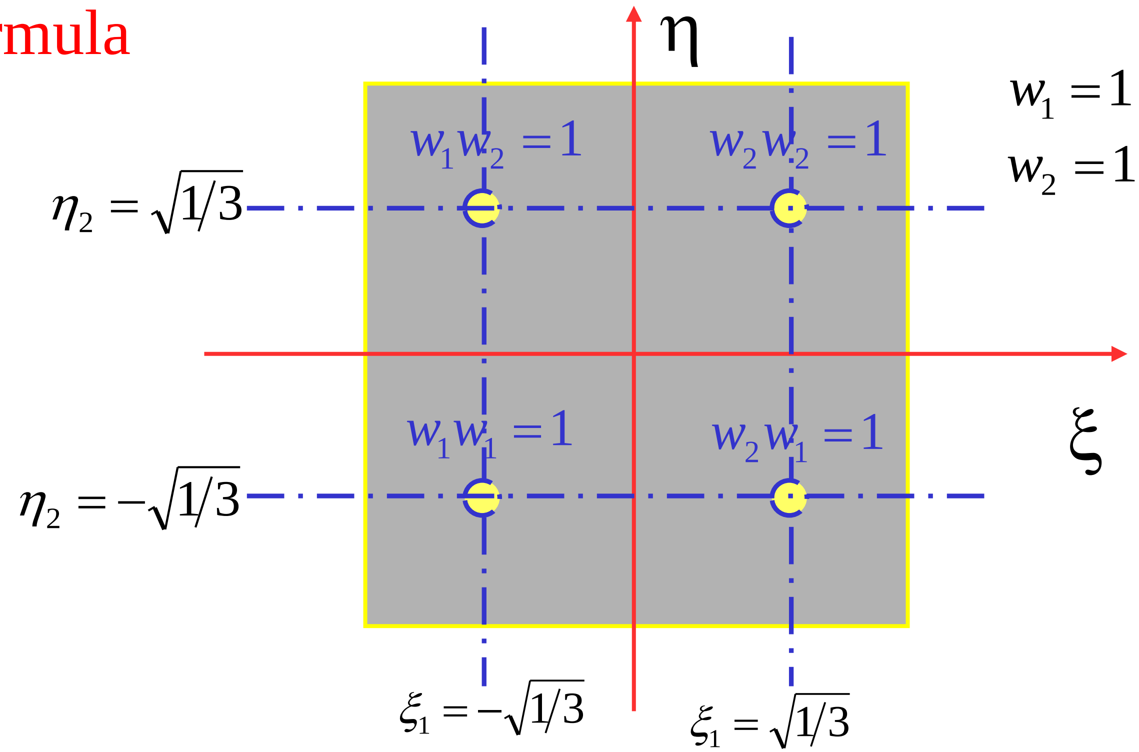
2-Dimensional Integration:Gaussian Quadrature

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta \cong \int_{-1}^{1}\left[\sum_{i=1}^{n} w_i f(\xi_i,\eta)\right]d\eta \approx \sum_{j=1}^{n}\sum_{i=1}^{n} w_j w_i f(\xi_i,\eta_j)$$

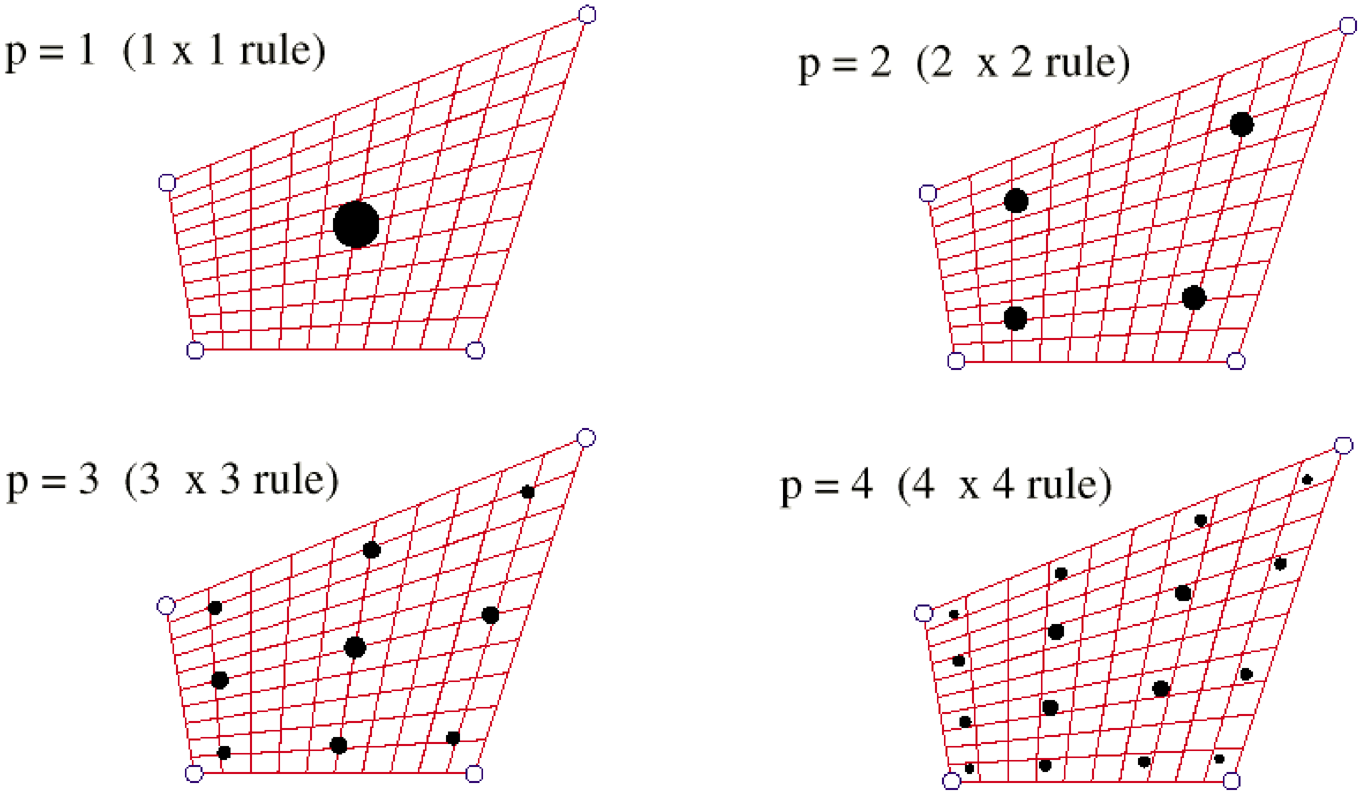
2-D Integration 2-point formula

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta \cong$$

$$w_1 w_1 f(\xi_1,\eta_1)$$
$$+\, w_2 w_1 f(\xi_2,\eta_1)$$
$$+\, w_1 w_2 f(\xi_1,\eta_2)$$
$$+\, w_2 w_2 f(\xi_2,\eta_2)$$



Graphical Representation of the First Four 2D Product-Type Gauss Integration Rules

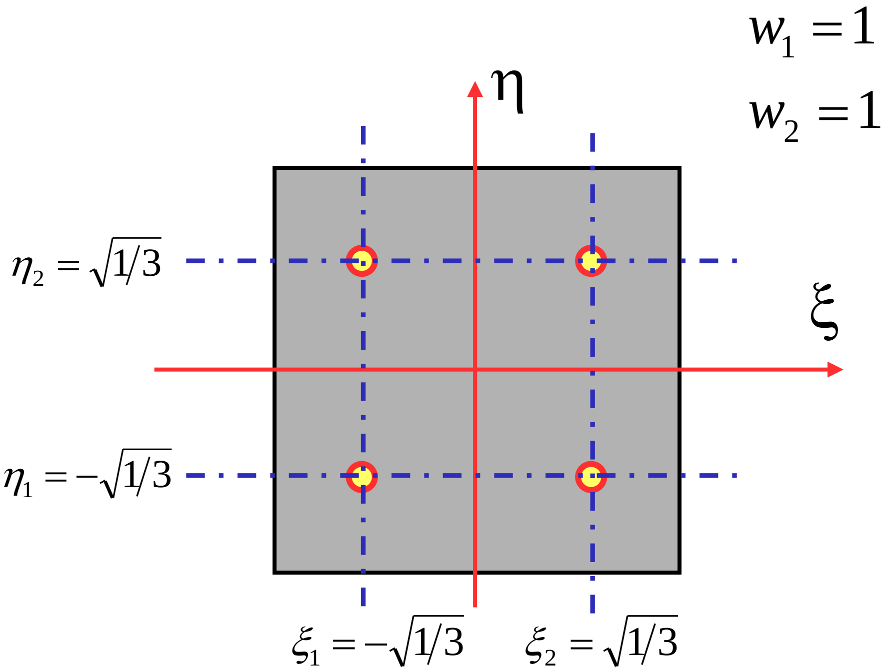


With Equal # of Points p in Each Direction

Integration of Stiffness Matrix

$$\mathbf{k} = t\int_A \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,dA$$

$$= t\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\det J\,d\xi\,d\eta$$



$$k_{ij} = t\int_{-1}^{1}\int_{-1}^{1} g_{ij}(\xi,\eta)\,d\xi\,d\eta$$

$$= w_1 w_1 g_{ij}(\xi_1,\eta_1) + w_2 w_1 g_{ij}(\xi_2,\eta_1) + w_1 w_2 g_{ij}(\xi_1,\eta_2) + w_2 w_2 g_{ij}(\xi_2,\eta_2)$$

$$= g_{ij}(\xi_1,\eta_1) + g_{ij}(\xi_2,\eta_1) + g_{ij}(\xi_1,\eta_2) + g_{ij}(\xi_2,\eta_2)$$

Numerical Integration: Gaussian Quadrature

Modeling Issues: Element Shape

Square : Optimum Shape

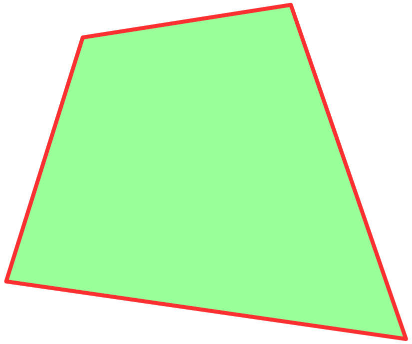
Not always possible to use

Rectangles:

Rule of Thumb

Ratio of sides <2

Larger ratios may be used with caution

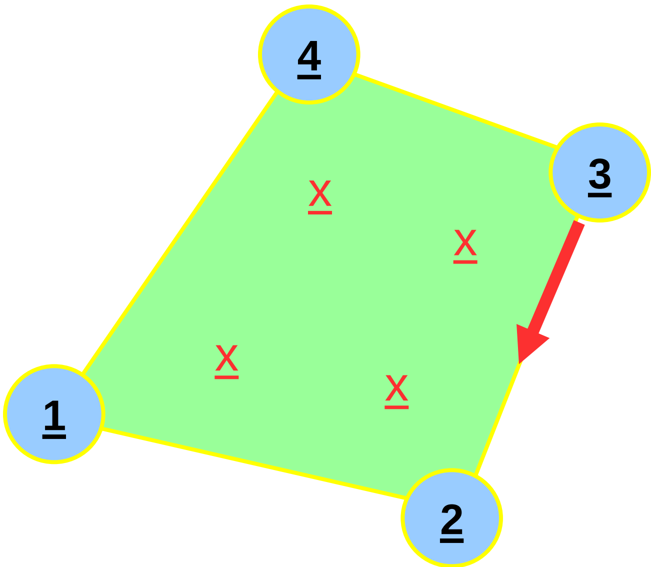


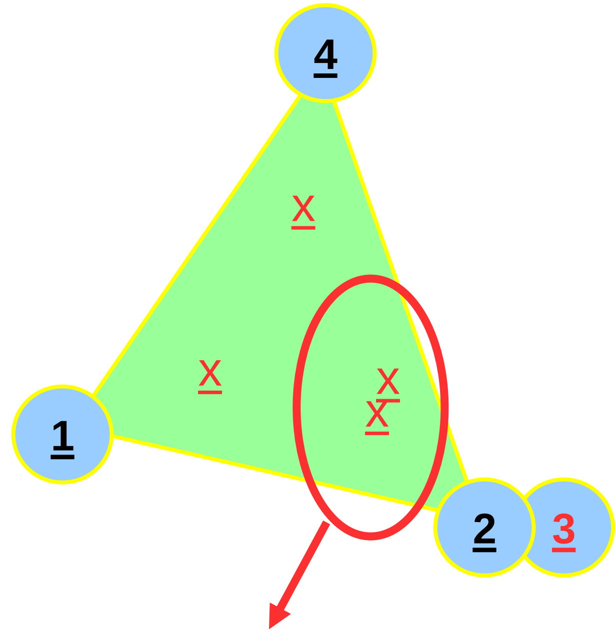
Angular Distortion

Internal Angle $< 180^{o}$

Modeling Issues: Degenerate Quadrilaterals

Coincident Corner Nodes



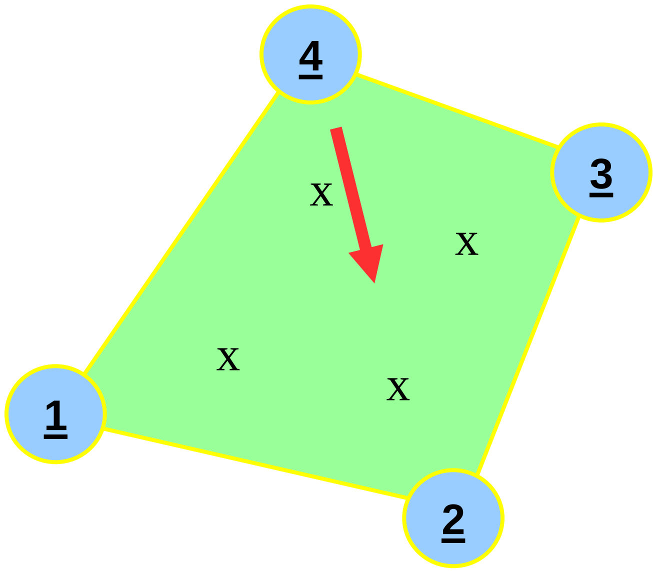


Integration Bias

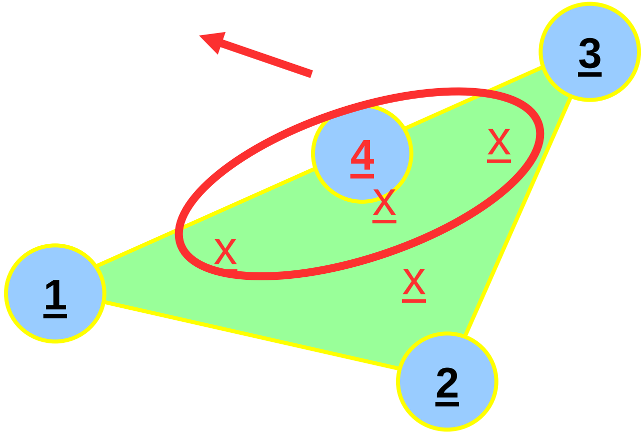
Less accurate

Modeling Issues: Degenerate Quadrilaterals

Three nodes collinear



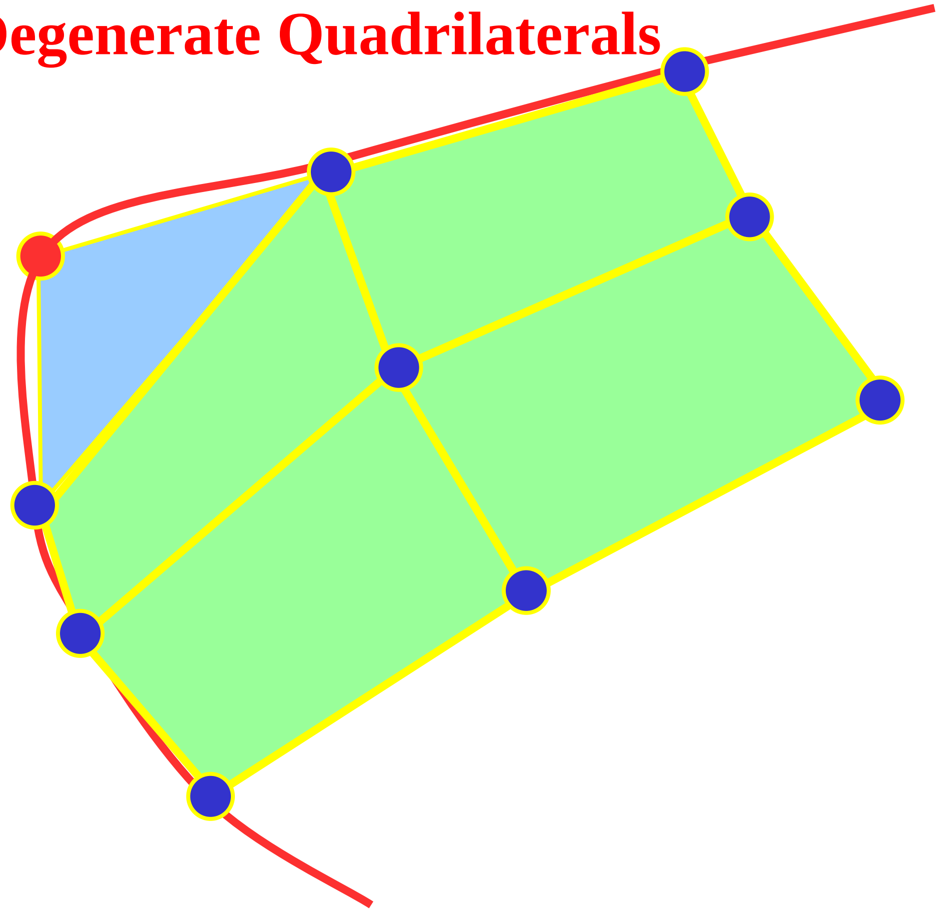
Integration Bias



Less accurate

Modeling Issues: Degenerate Quadrilaterals

2 nodes



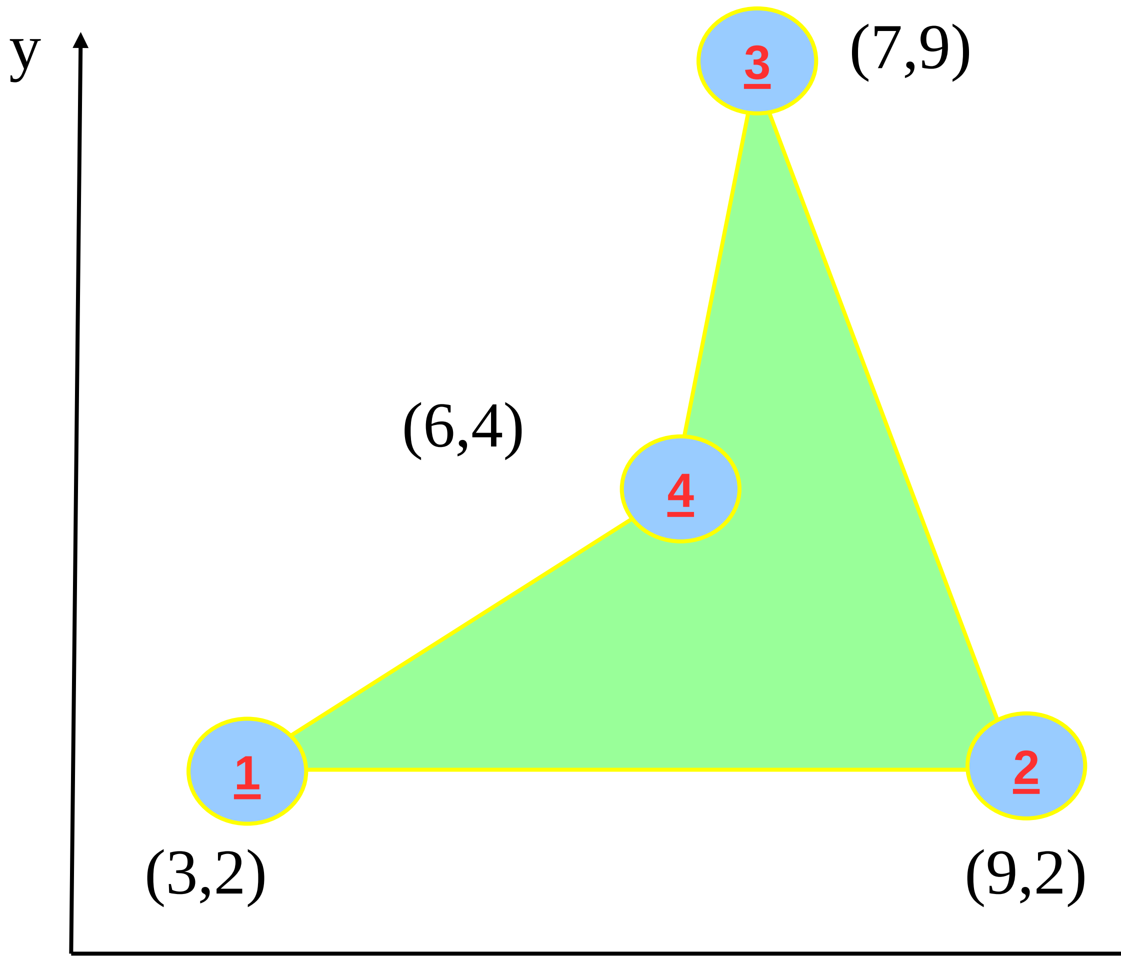
Use only as necessary to improve representation of geometry

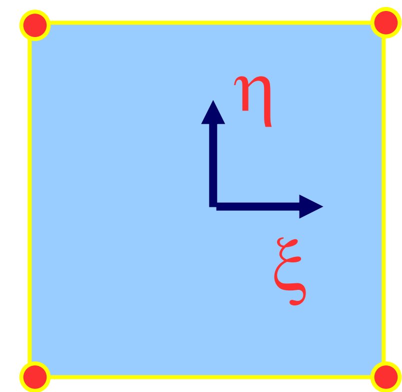
Do not use in place of triangular elements

Numerical Integration: Gaussian Quadrature

Modeling Issues: Degenerate Quadrilaterals

A NoNo Situation





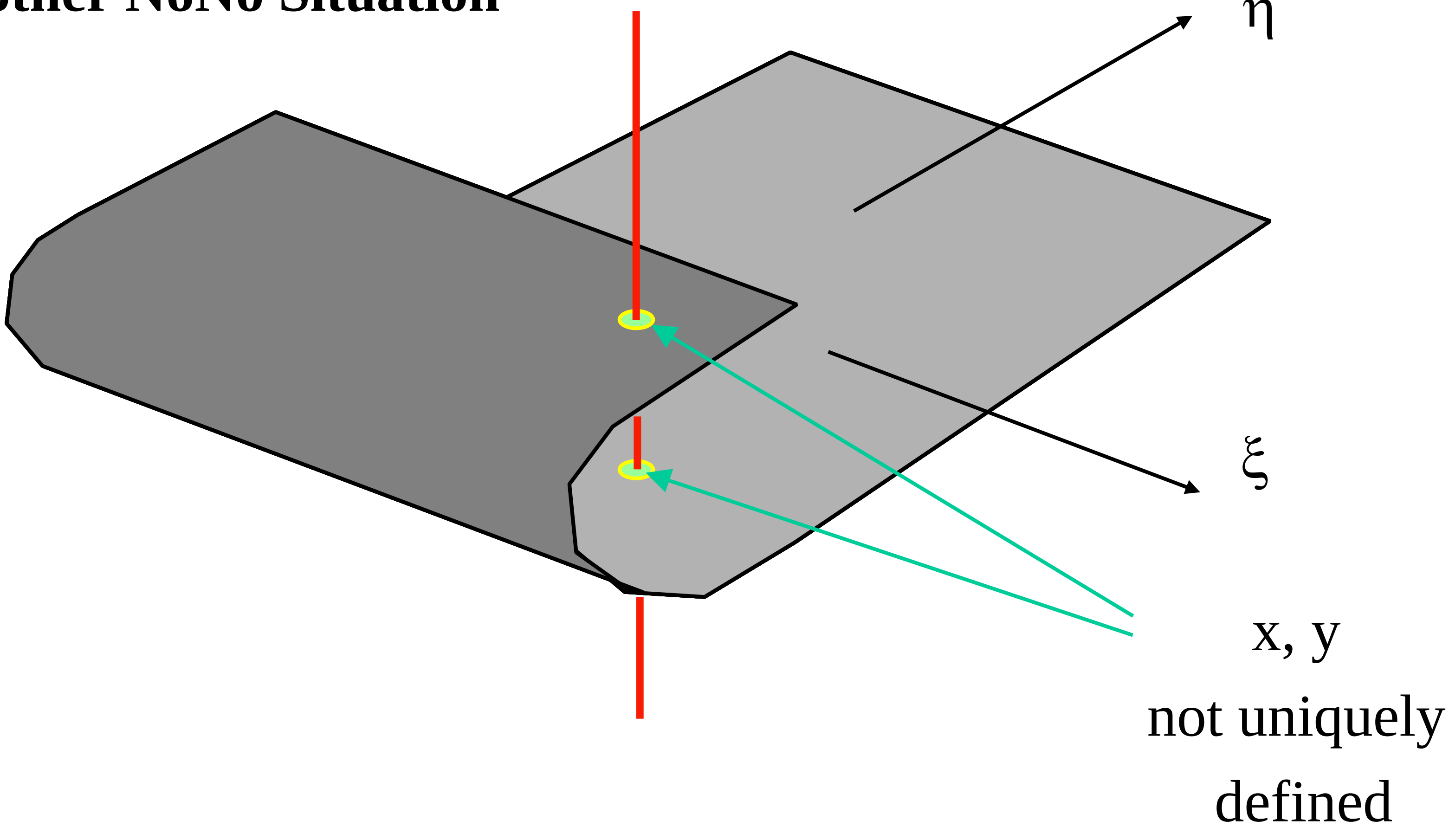
Parent

J singular

All interior angles < 180

Modeling Issues: Degenerate Quadrilaterals

Another NoNo Situation



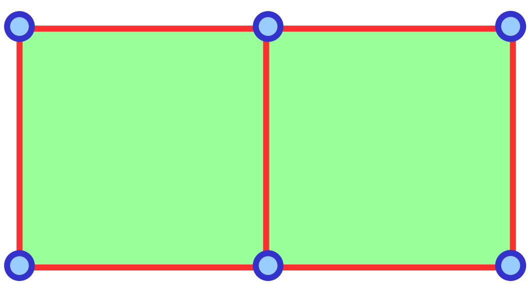
Convergence Considerations

For monotonic convergence of solution; Requirements

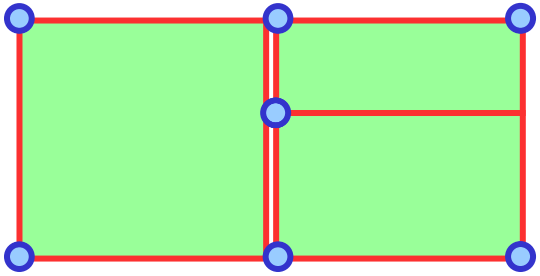
Elements (mesh) must be compatible

Elements must be complete

Mesh Compatibility

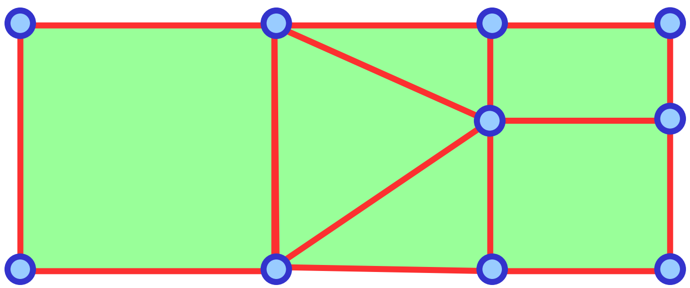


OK



NO!

Mesh compatibility - Refinement



Acceptable Transition

Compatibility of displacements OK

Stresses?

Gauss integration

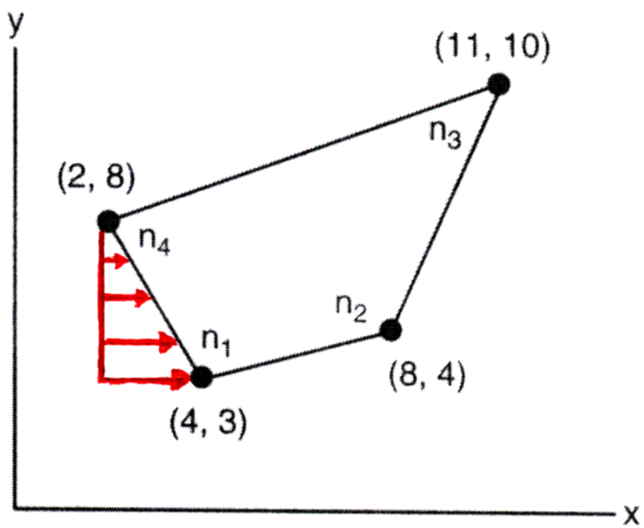
- For evaluation of integrals in **k** (in practice)

In 1 direction: $$I=\int_{-1}^{+1} f(\xi)\mathrm{d}\xi=\sum_{j=1}^{m} w_j f(\xi_j)$$

m gauss points gives exact solution of polynomial integrand of $n = 2m - 1$

In 2 directions: $$I=\int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta=\sum_{i=1}^{n_x}\sum_{j=1}^{n_y} w_i w_j f(\xi_i,\eta_j)$$

Example:



$$\mathbf{t}(x,y)=\begin{pmatrix}0.4*(8-y)\\0\end{pmatrix}\text{ ksi}$$

- Given: 4-node plane stress element has E = 30,000 ksi, ν = 0.25, h = 0.50 in, no body force, and surface traction shown.
- Required: Find **k** and ***f***. Use 2 x 2 Gauss quadrature for **k**.

Solution:

- Isoparametric mapping:

$$x=\tfrac{1}{4}(1-\xi)(1-\eta)x_1+\tfrac{1}{4}(1+\xi)(1-\eta)x_2+\tfrac{1}{4}(1+\xi)(1+\eta)x_3+\tfrac{1}{4}(1-\xi)(1+\eta)x_4$$
$$=\tfrac{1}{4}(1-\xi)(1-\eta)*4+\tfrac{1}{4}(1+\xi)(1-\eta)*8+\tfrac{1}{4}(1+\xi)(1+\eta)*11+\tfrac{1}{4}(1-\xi)(1+\eta)*2$$
$$=\tfrac{25}{4}+\tfrac{13}{4}\xi+\tfrac{1}{4}\eta+\tfrac{5}{4}\xi\eta;$$
$$y=\tfrac{1}{4}(1-\xi)(1-\eta)*3+\tfrac{1}{4}(1+\xi)(1-\eta)*4+\tfrac{1}{4}(1+\xi)(1+\eta)*10+\tfrac{1}{4}(1-\xi)(1+\eta)*8$$
$$=\tfrac{25}{4}+\tfrac{3}{4}\xi+\tfrac{11}{4}\eta+\tfrac{1}{4}\xi\eta;$$

- Jacobian matrix and Jacobian:

$$\mathbf{J}=\begin{bmatrix}\dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi}\\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta}\end{bmatrix}=\begin{bmatrix}\frac{13}{4}+\frac{5}{4}\eta & \frac{3}{4}+\frac{1}{4}\eta\\ \frac{1}{4}+\frac{5}{4}\xi & \frac{11}{4}+\frac{1}{4}\xi\end{bmatrix};\qquad J=\det\mathbf{J}=\tfrac{35}{4}-\tfrac{1}{8}\xi+\tfrac{27}{8}\eta.$$

Numerical Integration: Gaussian Quadrature

- **B** matrix:

$$[B(\xi,\eta)] = \frac{1}{|\mathbf{J}|}\left[B_1 \;\vdots\; B_2 \;\vdots\; B_3 \;\vdots\; B_4\right]$$

$$[B_i] = \begin{bmatrix} a\left(N_{i,\xi}\right) - b\left(N_{i,\eta}\right) & 0 \\ 0 & c\left(N_{i,\eta}\right) - d\left(N_{i,\xi}\right) \\ c\left(N_{i,\eta}\right) - d\left(N_{i,\xi}\right) & a\left(N_{i,\xi}\right) - b\left(N_{i,\eta}\right) \end{bmatrix}$$

Numerical Integration: Gaussian Quadrature

- **B** matrix:

$$N_{1,\xi}=\frac{\partial N_1}{\partial \xi}=\frac{-1(1-\eta)}{4}=\frac{(\eta-1)}{4} \qquad N_{1,\eta}=\frac{\partial N_1}{\partial \eta}=\frac{(1-\xi)(-1)}{4}=\frac{(\xi-1)}{4}$$

$$N_{2,\xi}=\frac{\partial N_2}{\partial \xi}=\frac{(1)(1-\eta)}{4}=\frac{(1-\eta)}{4} \qquad N_{2,\eta}=\frac{\partial N_2}{\partial \eta}=\frac{(1+\xi)(-1)}{4}=\frac{-(\xi+1)}{4}$$

$$N_{3,\xi}=\frac{\partial N_3}{\partial \xi}=\frac{(1)(1+\eta)}{4}=\frac{(1+\eta)}{4} \qquad N_{3,\eta}=\frac{\partial N_3}{\partial \eta}=\frac{(1+\xi)(1)}{4}=\frac{(\xi+1)}{4}$$

$$N_{4,\xi}=\frac{\partial N_4}{\partial \xi}=\frac{(-1)(1+\eta)}{4}=\frac{-(1+\eta)}{4} \qquad N_{4,\eta}=\frac{\partial N_4}{\partial \eta}=\frac{(1-\xi)(1)}{4}=\frac{(1-\xi)}{4}$$

$$a=1/4\left[y_1(\xi-1)+y_2(-\xi-1)+y_3(\xi+1)+y_4(1-\xi)\right]$$

$$b=1/4\left[y_1(\eta-1)+y_2(1-\eta)+y_3(\eta+1)+y_4(-1-\eta)\right]$$

$$c=1/4\left[x_1(\eta-1)+x_2(1-\eta)+x_3(\eta+1)+x_4(-1-\eta)\right]$$

$$d=1/4\left[x_1(\xi-1)+x_2(-\xi-1)+x_3(\xi+1)+x_4(1-\xi)\right]$$

Numerical Integration: Gaussian Quadrature

- **B** matrix:

$$\mathbf{B}=\frac{1}{70-\xi+27\eta}\times$$

$$\begin{bmatrix} -4+6\eta-2\xi & 0 & 7-5\eta+2\xi & 0 & 4+5\eta-\xi & 0 & -7-6\eta+\xi & 0 \\ 0 & -6-3\eta+9\xi & 0 & -7-2\eta-9\xi & 0 & 6+2\eta+4\xi & 0 & 7+3\eta-4\xi \\ -6-3\eta+9\xi & -4+6\eta-2\xi & -7-2\eta-9\xi & 7-5\eta+2\xi & 6+2\eta+4\xi & 4+5\eta-\xi & 7+3\eta-4\xi & -7-6\eta+\xi \end{bmatrix}$$

Numerical Integration: Gaussian Quadrature

- **k** matrix:

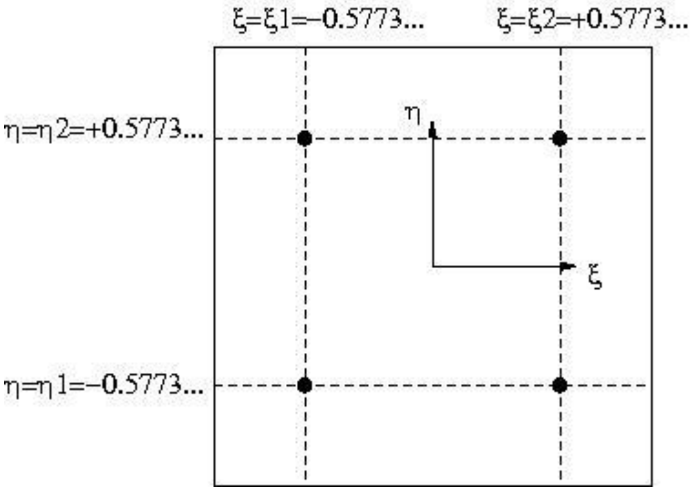
$$\mathbf{E}=\begin{bmatrix}32000 & 8000 & 0\\ 8000 & 32000 & 0\\ 0 & 0 & 12000\end{bmatrix}\text{ksi};$$

$$\mathbf{k}=(0.5\text{ in})\int_{-1}^{1}\int_{-1}^{1}\mathbf{B}^T\,\mathbf{E}\,\mathbf{B}*J\,d\xi\,d\eta$$

$$=\int_{-1}^{1}\int_{-1}^{1}\underbrace{\frac{8}{70-\xi+27\eta}*\begin{bmatrix}31.25*\left(236-276\eta-196\xi+315\eta^2-354\eta\xi+275\xi^2\right) & \cdots & 31.25*\left(70+231\eta-203\xi+90\eta^2-231\eta\xi+43\xi^2\right)\\ \vdots & \ddots & \vdots\\ \text{sym} & \cdots & 31.25*\left(539+588\eta-490\xi+180\eta^2-228\eta\xi+131\xi^2\right)\end{bmatrix}}_{\left[\mathbf{k}'(\xi,\eta)\right]}d\xi\,d\eta$$

Numerical Integration: Gaussian Quadrature

- 2 x 2 Gauss quadrature:



$$W_i = W_j = 1;\ i, j = 1, 2.$$

$$\mathbf{k} \approx \sum_{i=1}^{2}\sum_{j=1}^{2} W_i W_j * \left[\mathbf{k}'\left(\xi=\xi_i, \eta=\eta_j\right)\right] = \left[\mathbf{k}'\left(-\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right)\right] + \left[\mathbf{k}'\left(-\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right)\right] + \left[\mathbf{k}'\left(\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right)\right] + \left[\mathbf{k}'\left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right)\right]$$

$$\therefore \mathbf{k} \approx \begin{bmatrix} 7028.9 & \cdots & 1260.6 \\ \vdots & \ddots & \vdots \\ 1260.6 & \cdots & 8489.9 \end{bmatrix} \text{ kips/in.}$$

$$\text{Note: } \mathbf{k}_{exact} = \begin{bmatrix} 7136.6 & \cdots & 1263.9 \\ \vdots & \ddots & \vdots \\ 1263.9 & \cdots & 8499.0 \end{bmatrix} \text{ kips/in.}$$

- Element nodal forces: . . . .

Problem 1

Figure (1) shows a four-node quadrilateral. The (x,y) coordinates of each node are given in the figure. The element displacement vector u is given as: U=[0,0,0.20,0,0.15,0.10,0,0.05]

Find:

A- The x-y coordinates of a point *P* whose location in the master element is given by

$\xi = 0.5, \eta = 0.5$

B- The *u* and *v* displacement of the point *P*

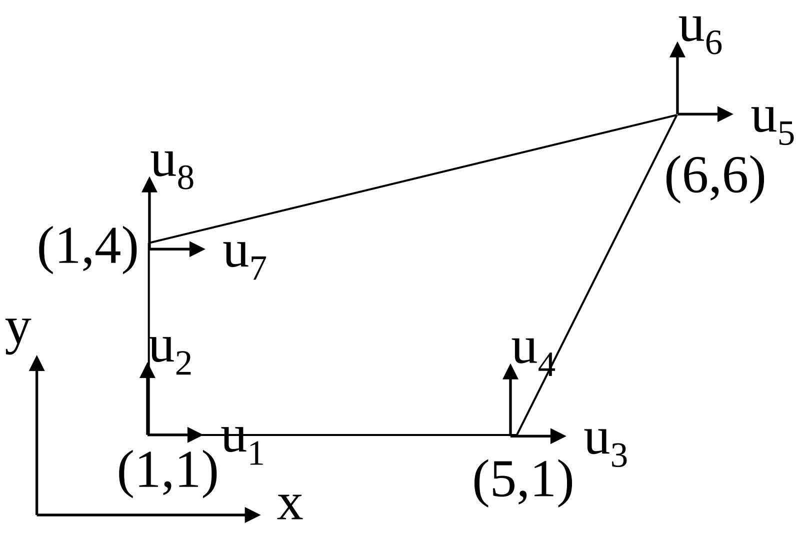


Figure (1)

Problem 2

Using a 2 by 2 rule evaluate the following integral by Gaussian quadrature, where a denotes the region shown in Figure (1).

$$\int\int_A (x^2 + xy^2)dxdy$$

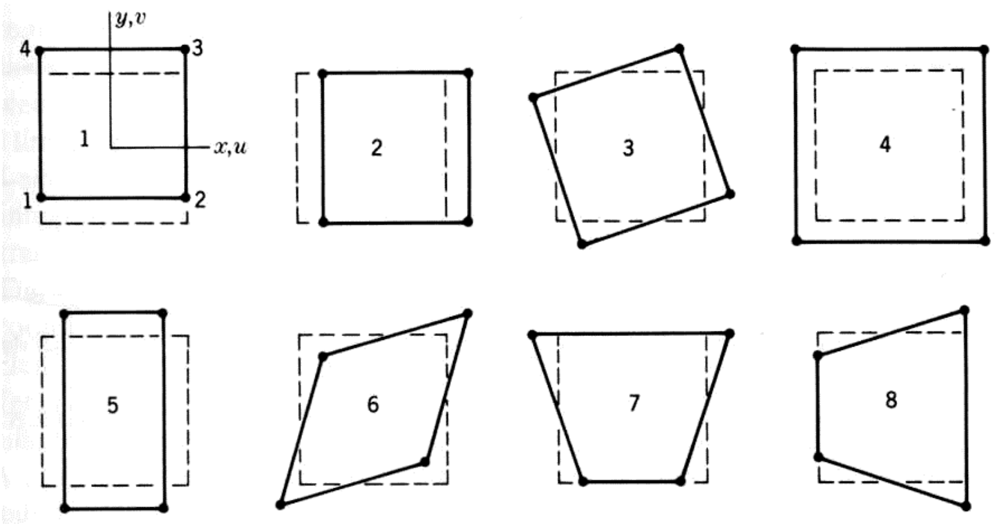
Numerical Integration: Gaussian Quadrature

Zero-Energy Modes (Mechanisms; Kinematic Modes) –

- Instabilities for an element (or group of elements) that produce deformation without any strain energy.
- Typically caused by using an inappropriately low order of Gauss quadrature.
- If present, will dominate the deformation pattern.
- Can occur for all 2D elements except the CST.

Zero-Energy Modes –

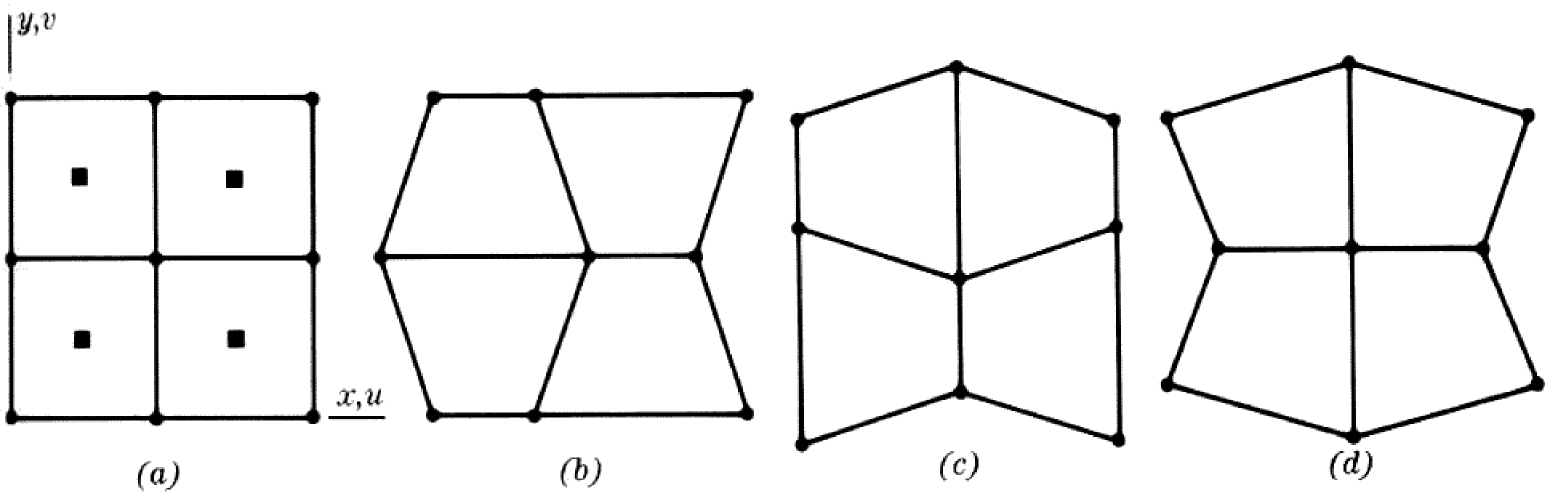
❖ Deformation modes for a bilinear quad:



- #1, #2, #3 = *rigid body modes*; can be eliminated by proper constraints.
- #4, #5, #6 = *constant strain modes*; always have nonzero strain energy.
- #7, #8 = *bending modes*; produce zero strain at origin.

Zero-Energy Modes –

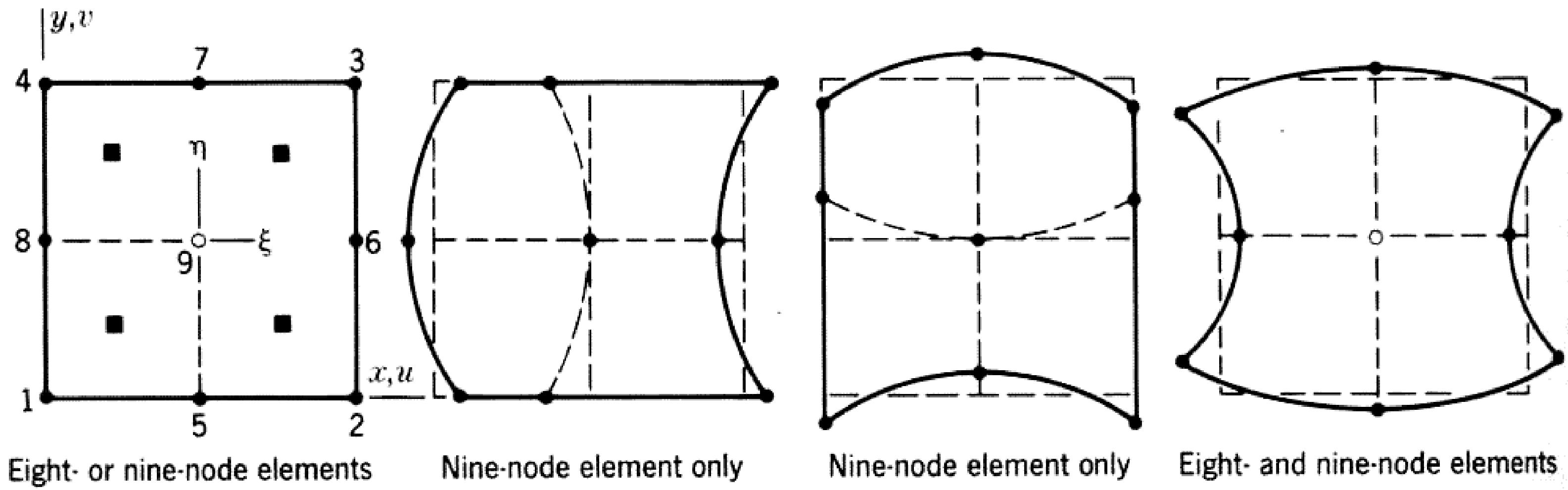
❖ Mesh instability for bilinear quads using order 1 quadrature:



"Hourglass modes"

Zero-Energy Modes –

❖ Element instability for quadratic quadrilaterals using 2x2 Gauss quadrature:



"Hourglass modes"

Zero-Energy Modes –

- How can you prevent this?
  - Use higher order Gauss quadrature in formulation.
  - Can artificially “stiffen” zero-energy modes via penalty functions.
  - Avoid elements with known instabilities!

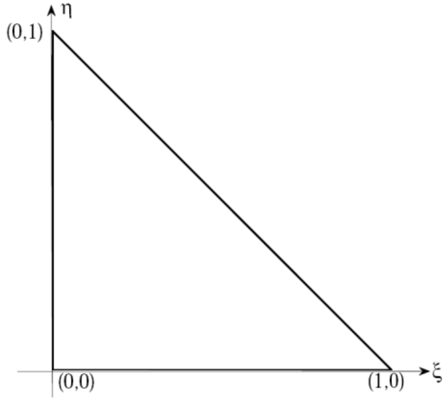
Gauss Integration for Triangular Region

The gauss points for a triangular region differ from the square region considered earlier. The simplest one is the one-point rule at the centroid with weight $w_1=1/2$ and $\xi_1=\eta_1=\zeta_1=1/3$

$$K^{(e)} = t\int_{\Omega^e} B^T E B \det J d\xi d\eta = \frac{1}{2} \underbrace{t\bar{B}^T \bar{E}\bar{B} \det \bar{J}}$$

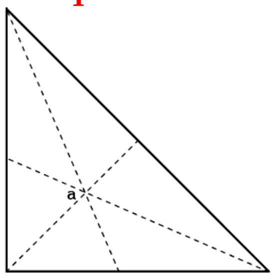
Evaluated at Gauss point

$$I = \int_0^1 \int_0^{1-\xi} f(\xi,\eta) d\eta d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i,\eta_i)$$



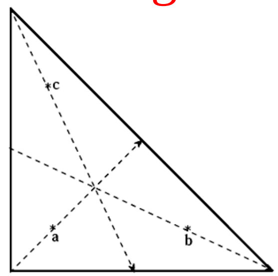
Gauss Integration for Triangular Region

First set of quadrature rules for triangular elements



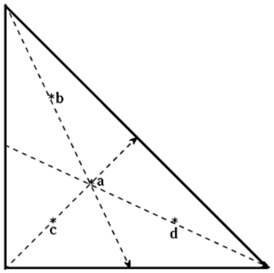
(a) Linear

$a = \left(\frac{1}{3}, \frac{1}{3}\right), \quad w = 1$



(b) Quadratic

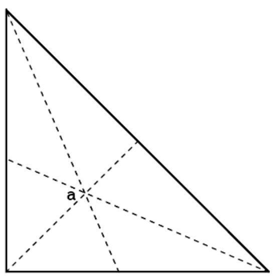
$a = \left(\frac{1}{6}, \frac{1}{6}\right), \quad w = \frac{1}{3}$
$b = \left(\frac{2}{3}, \frac{1}{6}\right), \quad w = \frac{1}{3}$
$c = \left(\frac{1}{6}, \frac{2}{3}\right), \quad w = \frac{1}{3}$



(c) Cubic

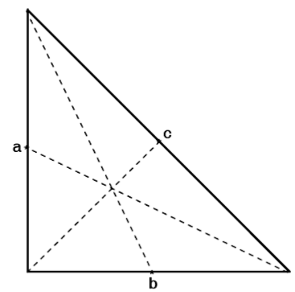
$a = \left(\frac{1}{3}, \frac{1}{3}\right), \quad w = -\frac{27}{48}$
$b = \left(\frac{1}{5}, \frac{3}{5}\right), \quad w = \frac{25}{48}$
$c = \left(\frac{1}{5}, \frac{1}{5}\right), \quad w = \frac{25}{48}$
$d = \left(\frac{3}{5}, \frac{1}{5}\right), \quad w = \frac{25}{48}$

Second set of quadrature rules for triangular elements



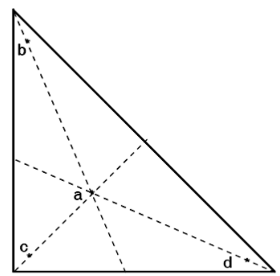
(a) Linear

$a = \left(\frac{1}{3}, \frac{1}{3}\right), \quad w = 1$



(b) Quadratic

$a = \left(0, \frac{1}{2}\right), \quad w = \frac{1}{3}$
$b = \left(\frac{1}{2}, 0\right), \quad w = \frac{1}{3}$
$c = \left(\frac{1}{2}, \frac{1}{2}\right), \quad w = \frac{1}{3}$



(c) Cubic

$a = \left(\frac{1}{3}, \frac{1}{3}\right), \quad w = -\frac{27}{48}$
$b = \left(\frac{2}{15}, \frac{11}{15}\right), \quad w = \frac{25}{48}$
$c = \left(\frac{2}{15}, \frac{2}{15}\right), \quad w = \frac{25}{48}$
$d = \left(\frac{11}{15}, \frac{2}{15}\right), \quad w = \frac{25}{48}$

Gauss Integration for Triangular Region

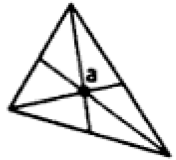
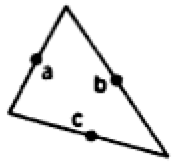
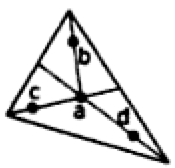
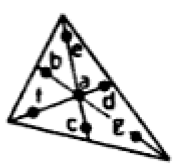
$$\int_0^1 \int_0^{1-\xi} f(\xi,\eta) d\eta d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i,\eta_i)$$

No. of Points (n)	Weight w_i	Multiplicity	ξ_i	η_i	ζ_i
One	1/2	1	1/3	1/3	1/3
Three	1/6	3	2/3	1/6	1/6
Three	1/6	3	1/2	1/2	0
Four	-9/32 25/96	1 3	1/3 3/5	1/3 1/5	1/3 1/5
Six	1/12	6	0.6590276223	0.231933685	0.109039009

Because of triangular symmetry, the Gauss point are occurred in group or *multiplicity* of one, three or six. For multiplicity of three if one Gauss point is at (2/3,1/6,1/6) then the other two Gauss points are located at (1/6,2/3,1/6) and (1/6,1/6,2/3). For multiplicity of six all six possible permutation of three coordinate are used.

Appendix

TABLE 5.3. Numerical Integration Formulas for Triangles

No.	Order	Figure	Rem.	Points	Triangular Coordinates	Weights W_k
1	Linear		$R = 0(h^2)$	a	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	1
2	Quadratic		$R = 0(h^3)$	a	$\frac{1}{2}, \frac{1}{2}, 0$	$\frac{1}{3}$
				b	$0, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{3}$
				c	$\frac{1}{2}, 0, \frac{1}{2}$	$\frac{1}{3}$
3	Cubic		$R = 0(h^4)$	a	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$-27/48$
				b	$0.6, 0.2, 0.2$	$25/48$
				c	$0.2, 0.6, 0.2$	$25/48$
				d	$0.2, 0.2, 0.6$	$25/48$
5	Quintic		$R = 0(h^6)$	a	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	0.225
				b	$\alpha_1, \beta_1, \beta_1$	0.13239415
				c	$\beta_1, \alpha_1, \beta_1$	
				d	$\beta_1, \beta_2, \beta_2$	
				e	$\alpha_2, \beta_2, \beta_2$	0.12593918
				f	$\beta_2, \alpha_2, \beta_2$	
				g	$\beta_2, \beta_2, \alpha_2$	

With:

$\alpha_1 = 0.05961587$
$\beta_1 = 0.47014206$
$\alpha_2 = 0.79742699$
$\beta_2 = 0.10128651$