Multi-Point Constraints

Multi-Point Constraints

- Multi-Point Constraints
  - Single point constraint examples

$$u_{x4} = 0$$ *linear, homogeneous*

$$u_{y9} = 0.6$$ *linear, non-homogeneous*

  - Multi-Point constraint examples

$$u_{x2} = \frac{1}{2} u_{y2}$$ *linear, homogeneous*

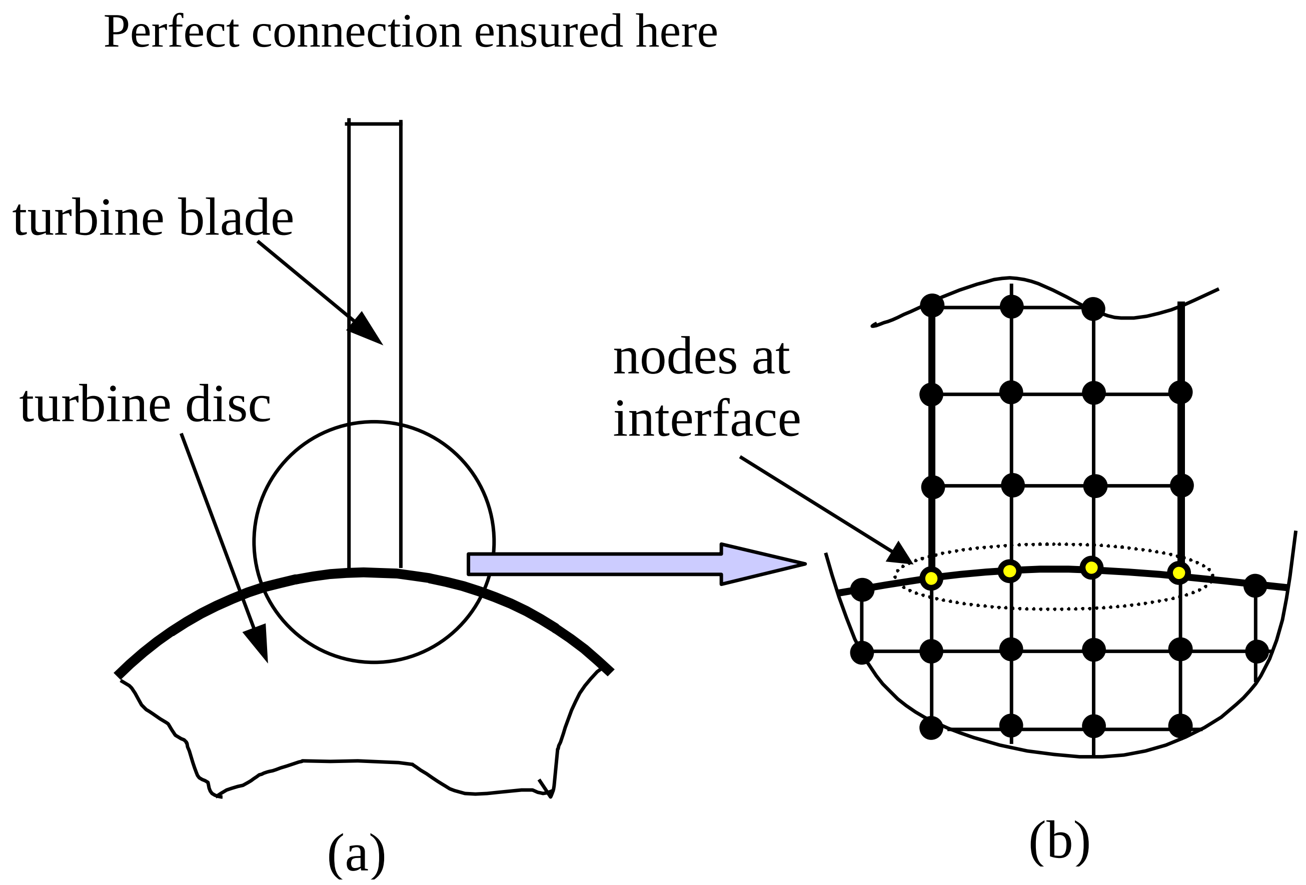
$$u_{x2} - 2u_{x4} + u_{x6} = 0.25$$ *linear, non-homogeneous*

$$(x_5 + u_{x5} - x_3 - u_{x3})^2 + (y_5 + u_{y5} - y_3 - u_{y3})^2 = 0$$

nonlinear, homogeneous

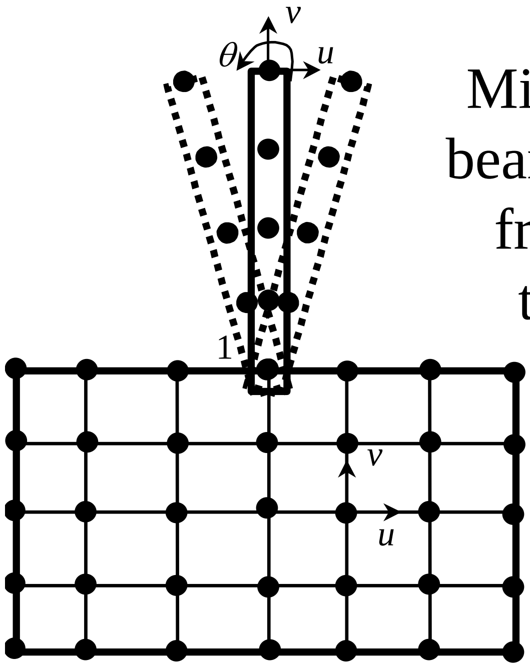
Multi-Point Constraints (Example)

- Modelling of joints

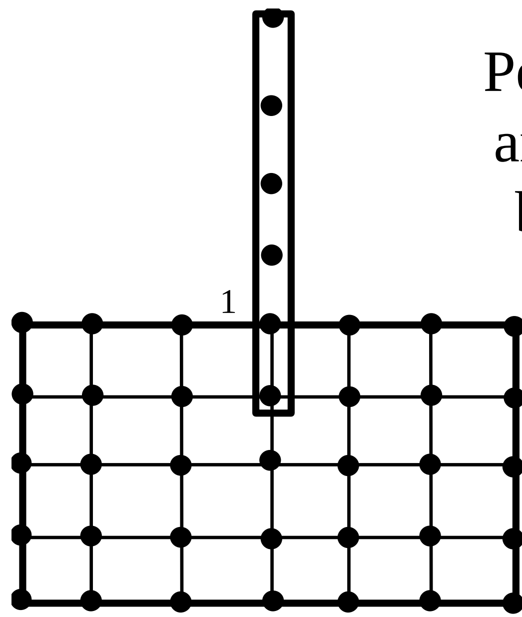


Multi-Point Constraints (Example)

- Modelling of joints



Mismatch between DOFs of beams and 2D solid – beam is free to rotate (rotation not transmitted to 2D solid)



Perfect connection by artificially extending beam into 2D solid (Additional mass)

Multi-Point Constraints (Example)

❖ Modelling of joints

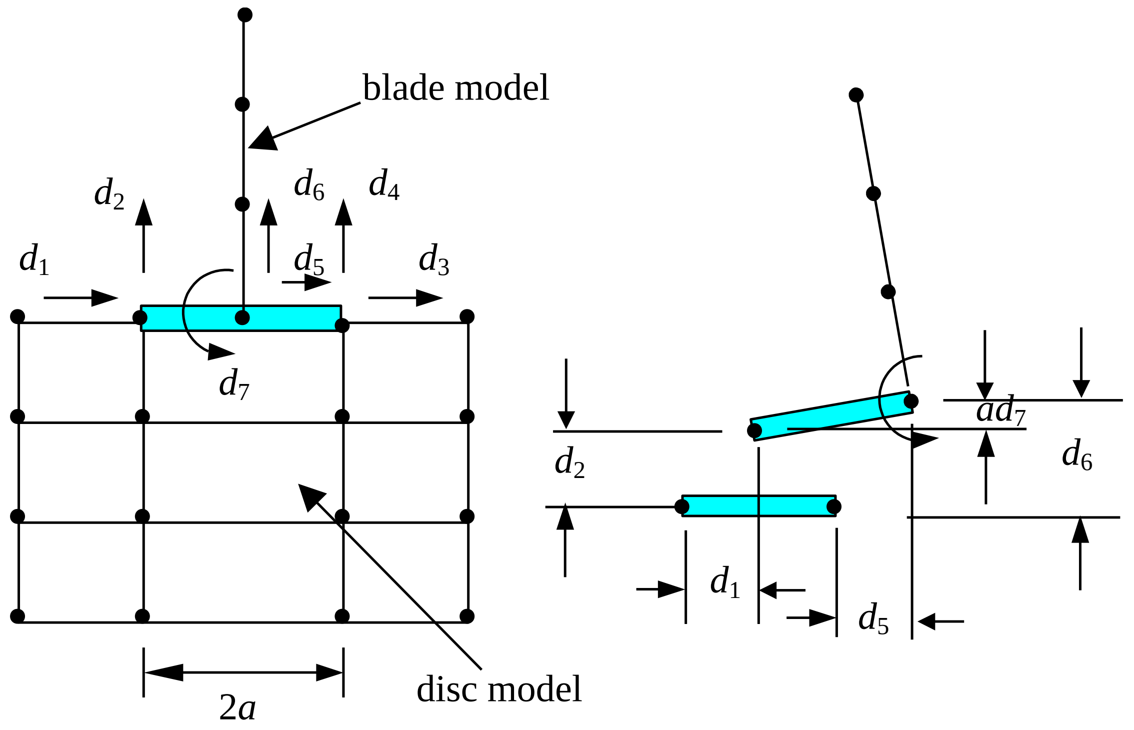
- Using MPC equations

$$d_1 = d_5$$

$$d_2 = d_6 - ad_7$$

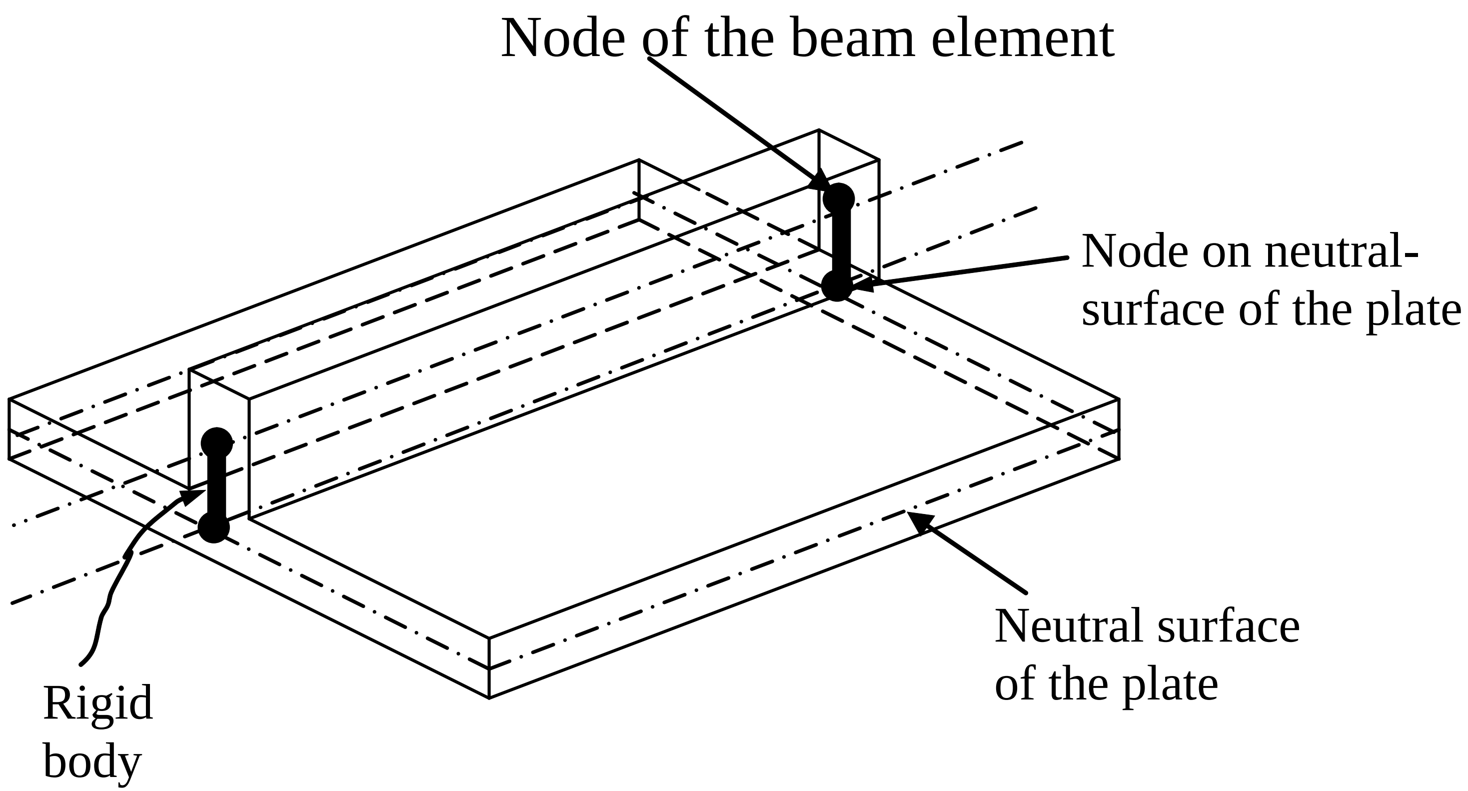
$$d_3 = d_5$$

$$d_4 = d_6 + ad_7$$



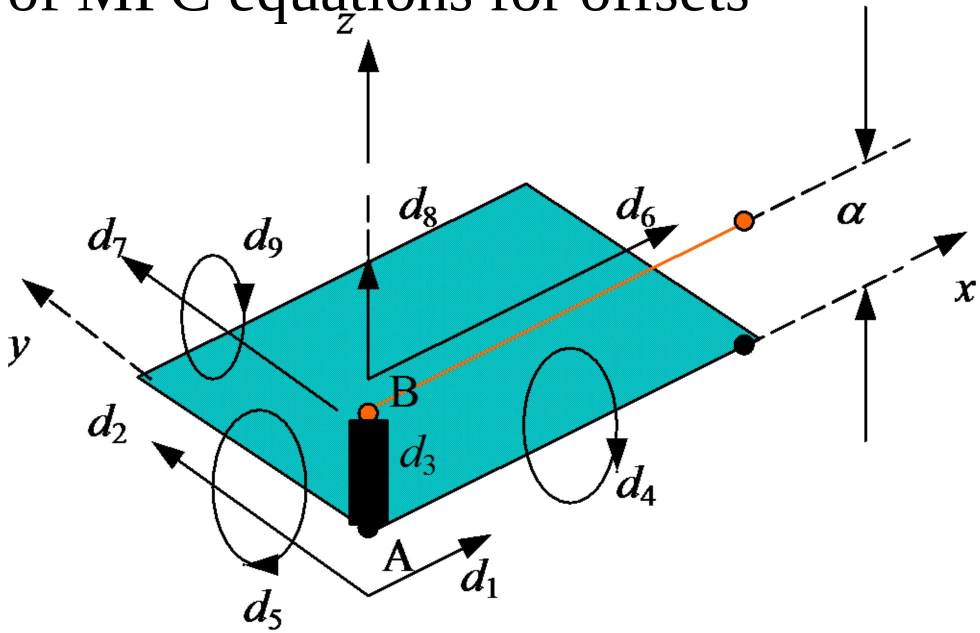
Multi-Point Constraints (Example)

- Creation of MPC equations for offsets



Multi-Point Constraints (Example)

- Creation of MPC equations for offsets



$$d_6 = d_1 + \alpha\, d_5 \quad \text{or} \quad d_1 + \alpha\, d_5 - d_6 = 0$$

$$d_7 = d_2 - \alpha\, d_4 \quad \text{or} \quad d_2 - \alpha\, d_4 - d_7 = 0$$

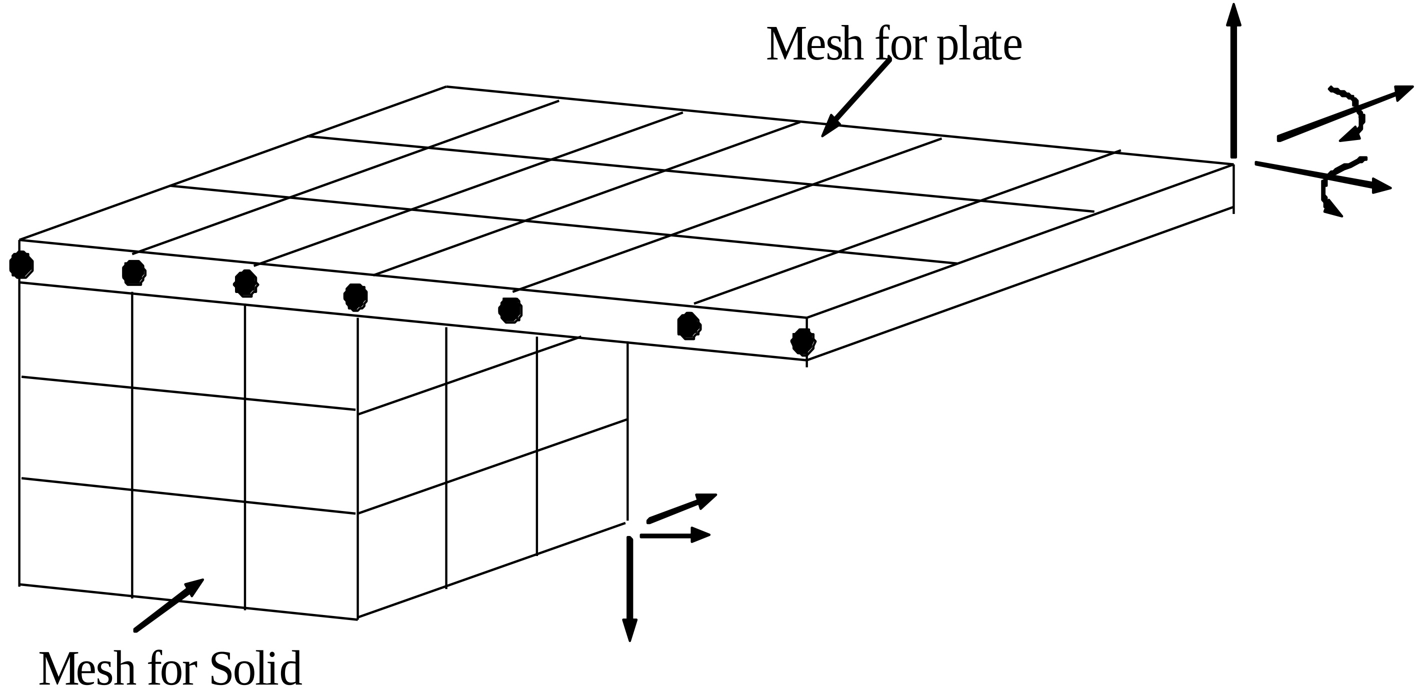
$$d_8 = d_3 \quad \text{or} \quad d_3 - d_8 = 0$$

$$d_9 = d_5 \quad \text{or} \quad d_5 - d_9 = 0$$

Multi-Point Constraints (Example)

- Modelling of joints

Similar for plate connected to 3D solid



Multi-Point Constraints (Example)

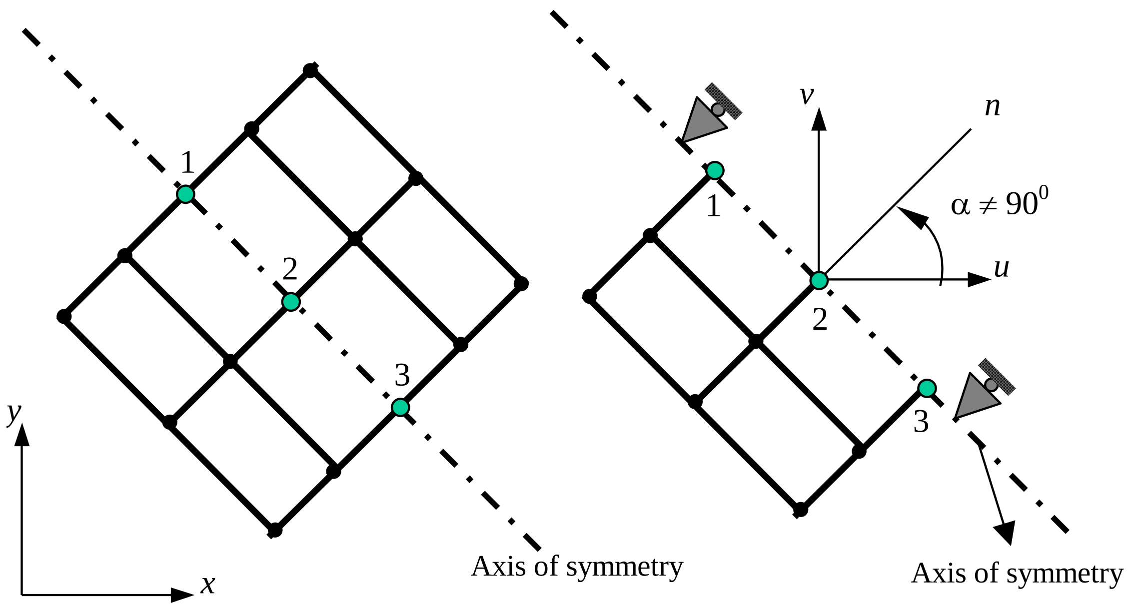
- Modelling of symmetric boundary conditions

$d_n = 0$

$u_i \cos\alpha + v_i \sin\alpha = 0$

or $\quad u_i + v_i \tan\alpha = 0$

for i=1, 2, 3



Multi-Point Constraints (Example)

- Enforcement of mesh compatibility

Use lower order shape function to interpolate

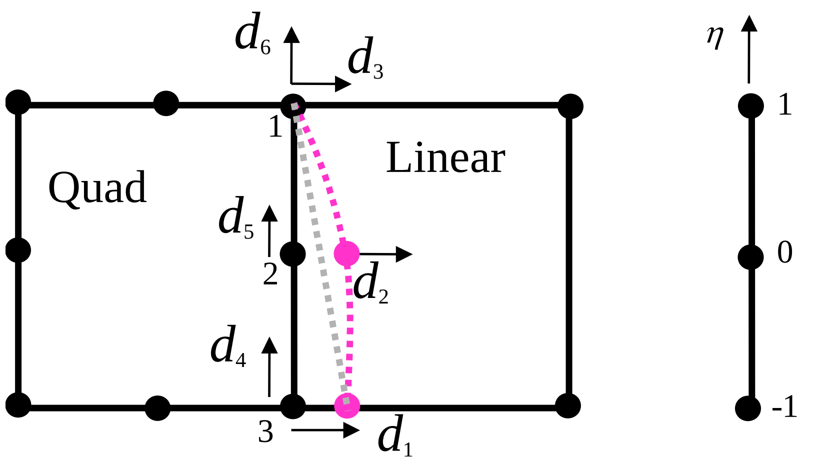
$$d_x = 0.5(1-\eta)\, d_1 + 0.5(1+\eta)\, d_3$$

$$d_y = 0.5(1-\eta)\, d_4 + 0.5(1+\eta)\, d_6$$

Substitute value of η at node 2

$$0.5\, d_1 - d_2 + 0.5\, d_3 = 0$$

$$0.5\, d_4 - d_5 + 0.5\, d_6 = 0$$



Multi-Point Constraints (Example)

- Enforcement of mesh compatibility

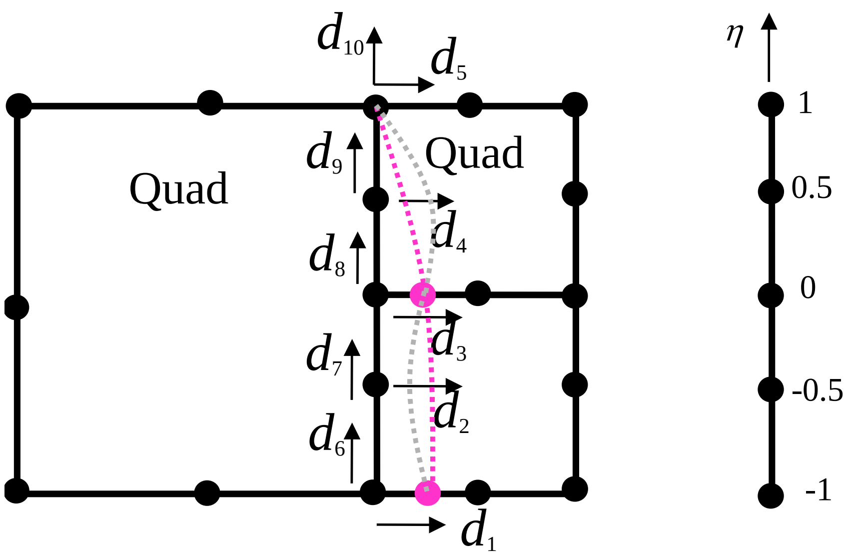
Use shape function of longer element to interpolate

$$d_x = -0.5\eta\,(1-\eta)\;d_1 + (1+\eta)(1-\eta)\;d_3 + 0.5\eta\,(1+\eta)\;d_5$$

Substituting the values of η for the two additional nodes

$$d_2 = 0.25\times1.5\;d_1 + 1.5\times0.5\;d_3 - 0.25\times0.5\;d_5$$

$$d_4 = -0.25\times0.5\;d_1 + 0.5\times1.5\;d_3 + 0.25\times1.5\;d_5$$



Multi-Point Constraints (Example)

- Enforcement of mesh compatibility

In x direction,

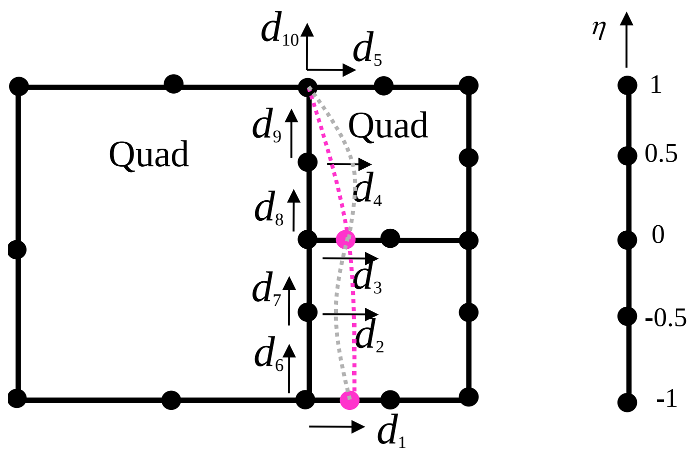
$$0.375\, d_1 - d_2 + 0.75\, d_3 - 0.125\, d_5 = 0$$

$$-0.125\, d_1 + 0.75\, d_3 - d_4 + 0.375\, d_5 = 0$$

In y direction,

$$0.375\, d_6 - d_7 + 0.75\, d_8 - 0.125\, d_{10} = 0$$

$$-0.125\, d_6 + 0.75\, d_8 - d_9 + 0.375\, d_{10} = 0$$



Multi-Point Constraints (Example)

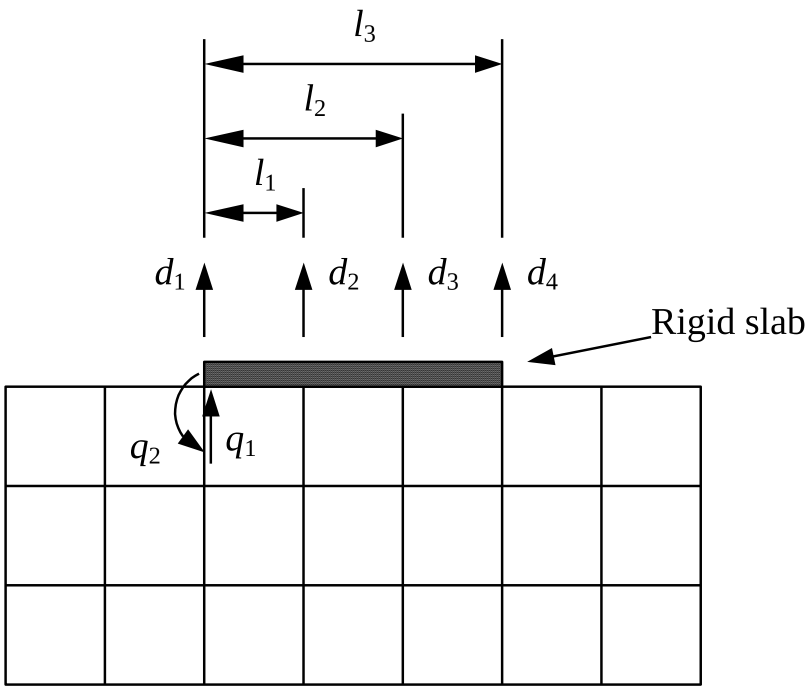
- Modelling of constraints by rigid body attachment

$d_1 = q_1$

$d_2 = q_1 + q_2\, l_1$

$d_3 = q_1 + q_2\, l_2$

$d_4 = q_1 + q_2\, l_3$



Eliminate q_1 and q_2

$(l_2/l_1 - 1)\, d_1 - (l_2/l_1)\, d_2 + d_3 = 0$

$(l_3/l_1 - 1)\, d_1 - (l_3/l_1)\, d_2 + d_4 = 0$

(DOF in x direction not considered)

Multi-Point Constraints

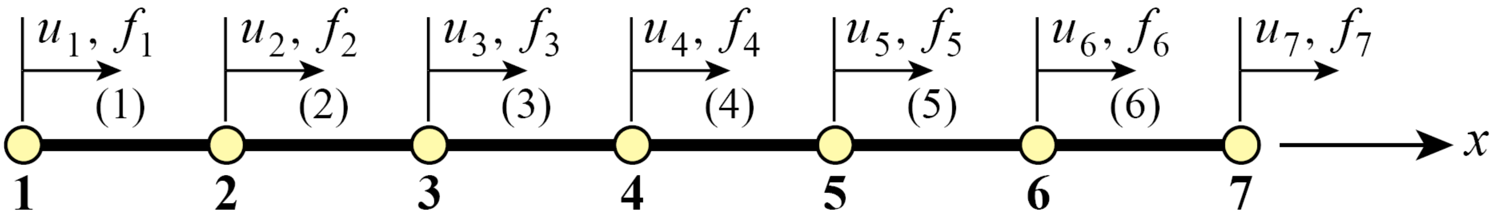
❖ Sources of Multi-Point Constraints

- Skew displacement BCs
- Coupling nonmatched FEM meshes
- Global-local and multiscale analysis
- Incompressibility

Multi-Point Constraints

- MPC Application Methods
  - Master-Slave Elimination
  - Penalty Function Augmentation
  - Lagrange Multiplier Adjunction

❖ Example 1D Structure to Illustrate MPCs



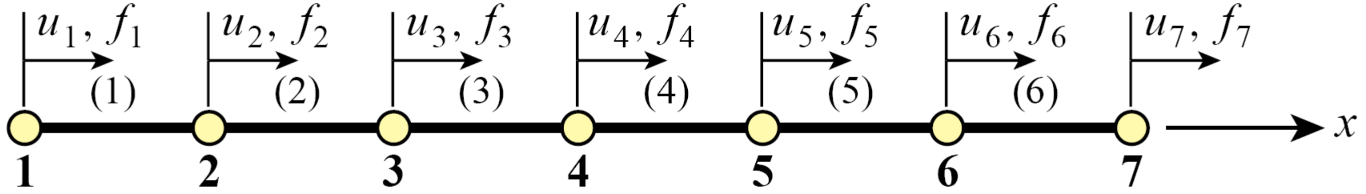
Multi-Point constraint:

$$u_2 = u_6 \quad \text{or} \quad u_2 - u_6 = 0$$

Linear homogeneous MPC

Multi-Point Constraints

❖ Example *1D* Structure to Illustrate MPCs



Unconstrained master stiffness equations

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$

Multi-Point Constraints- Master Slave Method

- Master Slave Method for Example Structure

Recall: $u_2 = u_6$ or $u_2 - u_6 = 0$

Taking u as master:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix}$$

or

$$\mathbf{u} = \mathbf{T}\hat{\mathbf{u}}.$$

Multi-Point Constraints- Master Slave Method

❖ Forming the Modified Stiffness Equations

Unconstrained master stiffness equations: $$\mathbf{K}\mathbf{u} = \mathbf{f}$$

Master-slave transformation: $$\mathbf{u} = \mathbf{T}\hat{\mathbf{u}}$$

Congruential transformation: $$\hat{\mathbf{K}} = \mathbf{T}^T\mathbf{K}\mathbf{T}$$

$$\hat{\mathbf{f}} = \mathbf{T}^T\mathbf{f}$$

Modified stiffness equations: $$\hat{\mathbf{K}}\hat{\mathbf{u}} = \hat{\mathbf{f}}$$

❖ Modified Stiffness Equations for Example Structure

u_2 as master and u_6 as slave DOF.

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 \\ K_{12} & K_{22}+K_{66} & K_{23} & 0 & K_{56} & K_{67} \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 \\ 0 & K_{56} & 0 & K_{45} & K_{55} & 0 \\ 0 & K_{67} & 0 & 0 & 0 & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2+f_6 \\ f_3 \\ f_4 \\ f_5 \\ f_7 \end{bmatrix}$$

❖ Modified Stiffness Equations for Example Structure

u_6 as master and u_2 as slave DOF.

$$\begin{bmatrix} K_{11} & 0 & 0 & 0 & K_{12} & 0 \\ 0 & K_{33} & K_{34} & 0 & K_{23} & 0 \\ 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ K_{12} & K_{23} & 0 & K_{56} & K_{22}+K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_3 \\ f_4 \\ f_5 \\ f_2+f_6 \\ f_7 \end{bmatrix}$$

Although they are algebraically equivalent, the latter would be processed faster if a skyline solver is used for the modified equations.

Multi-Point Constraints- Master Slave Method

- Multiple MPCs

Suppose

$$u_2 - u_6 = 0, \qquad u_1 + 4u_4 = 0, \qquad 2u_3 + u_4 + u_5 = 0$$

take 3, 4 and 6 as slaves:

$$u_6 = u_2, \qquad u_4 = -\tfrac{1}{4}u_1, \qquad u_3 = -\tfrac{1}{2}(u_4 + u_5) = \tfrac{1}{8}u_1 - \tfrac{1}{2}u_5$$

and put in matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{8} & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_5 \\ u_7 \end{bmatrix}$$

Multi-Point Constraints- Master Slave Method

- Nonhomogeneous MPCs

$$u_2 - u_6 = 0.2$$

In matrix form

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.2 \\ 0 \end{bmatrix}$$

$$\mathbf{u} = \mathbf{T}\hat{\mathbf{u}} + \mathbf{g}$$

Multi-Point Constraints- Master Slave Method

- Nonhomogeneous MPCs

modified system: $$\hat{\mathbf{K}}\,\hat{\mathbf{u}} = \hat{\mathbf{f}}$$

in which: $$\hat{\mathbf{K}} = \mathbf{T}^T\mathbf{K}\mathbf{T}, \qquad \hat{\mathbf{f}} = \mathbf{T}^T\mathbf{f} - \mathbf{T}^T\mathbf{K}\mathbf{g}$$

For the example structure

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 \\ K_{12} & K_{22}+K_{66} & K_{23} & 0 & K_{56} & K_{67} \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 \\ 0 & K_{56} & 0 & K_{45} & K_{55} & 0 \\ 0 & K_{67} & 0 & 0 & 0 & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + f_6 - 0.2K_{66} \\ f_3 \\ f_4 \\ f_5 - 0.2K_{56} \\ f_7 - 0.2K_{67} \end{bmatrix}$$

Multi-Point Constraints- Master Slave Method

❖ The General Case of MPCs

For implementation in general-purpose programs the master-slave method can be described as follows. The degrees of freedoms in **u** are classified into three types: independent or uncommitted, masters and slaves.

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{um} & \mathbf{K}_{us} \\ \mathbf{K}_{um}^T & \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{us}^T & \mathbf{K}_{ms}^T & \mathbf{K}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{u}_u \\ \mathbf{u}_m \\ \mathbf{u}_s \end{bmatrix} = \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_m \\ \mathbf{f}_s \end{bmatrix}$$

The MPCs may be written in matrix form as

$$\mathbf{A}_m \mathbf{u}_m + \mathbf{A}_s \mathbf{u}_s = \mathbf{g} \quad \Rightarrow \quad \mathbf{u}_s = -\mathbf{A}_s^{-1} \mathbf{A}_m \mathbf{u}_m + \mathbf{A}_s^{-1} \mathbf{g} = \mathbf{T} \mathbf{u}_m + \mathbf{g}$$

Inserting into the partitioned stiffness matrix and symmetrizing

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{um} \mathbf{T} \\ \mathbf{T}^T \mathbf{K}_{um}^T & \mathbf{T}^T \mathbf{K}_{mm} \mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{u}_u \\ \mathbf{u}_m \end{bmatrix} = \begin{bmatrix} \mathbf{f}_u - \mathbf{K}_{us} \mathbf{g} \\ \mathbf{f}_m - \mathbf{K}_{ms} \mathbf{g} \end{bmatrix}$$

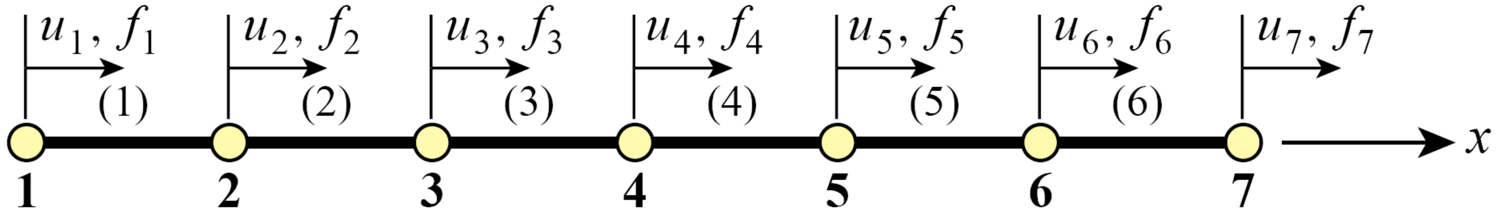
Multi-Point Constraints- Master Slave Method

Assessment of Master-Slave Method

- ADVANTAGES
  - exact if precautions taken
  - easy to understand
  - retains positive definiteness
  - important applications to model reduction
- DISADVANTAGES
  - requires user decisions
  - messy implementation for general MPCs
  - sensitive to constraint dependence
  - restricted to linear constraints

❖ Penalty Function Method, Physical Interpretation

Recall the example structure

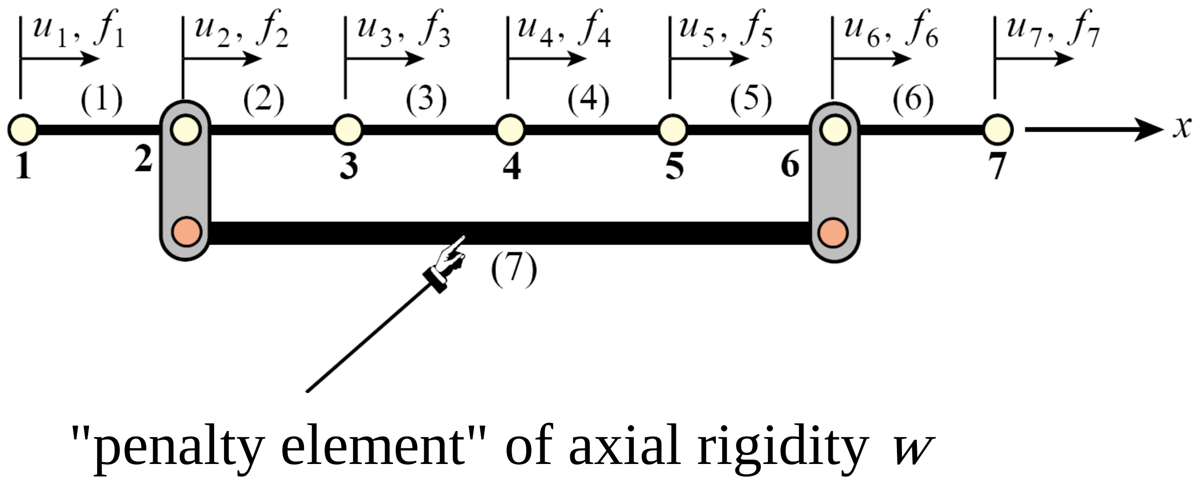


under the homogeneous MPC

$$u_2 = u_6$$

Multi-Point Constraints- Penalty Function Method

❖ Penalty Function Method, Physical Interpretation



$$w\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_2 \\ u_6 \end{bmatrix} = \begin{bmatrix} f_2^{(7)} \\ f_6^{(7)} \end{bmatrix}$$

❖ Penalty Function Method, Physical Interpretation

Upon merging the penalty element the modified stiffness equations are

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22}+w & K_{23} & 0 & 0 & -w & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & -w & 0 & 0 & K_{56} & K_{66}+w & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

This modified system is submitted to the equation solver.
Note that **u** retains the same arrangement of DOFs.

Multi-Point Constraints- Penalty Function Method

❖ Penalty Function Method - General MPCs

$$3u_3 + u_5 - 4u_6 = 1 \quad \Rightarrow \quad \begin{bmatrix} 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_3 \\ u_5 \\ u_6 \end{bmatrix} = 1$$

Premultiply both sides by $\begin{bmatrix} 3 & 1 & -4 \end{bmatrix}^T$

$$\begin{bmatrix} 9 & 3 & -12 \\ 3 & 1 & -4 \\ -12 & -4 & 16 \end{bmatrix} \begin{bmatrix} u_3 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$$

Scale by w and merge:

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33}+9w & K_{34} & 3w & -12w & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 3w & K_{45} & K_{55}+w & K_{56}-4w & 0 \\ 0 & 0 & -12w & 0 & K_{56}-4w & K_{66}+16w & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3+3w \\ f_4 \\ f_5+w \\ f_6-4w \\ f_7 \end{bmatrix}$$

Multi-Point Constraints- Penalty Function Method

❖ Theory of Penalty Function Method - General MPCs

$$\mathbf{t} = \mathbf{CU} - \mathbf{Q}$$ (Constrain equations)

$$\boldsymbol{\Pi}_p = \frac{1}{2}\mathbf{U}^T\mathbf{KU} - \mathbf{U}^T\mathbf{F} + \frac{1}{2}\mathbf{t}^T\boldsymbol{\alpha}\mathbf{t}$$

$\boldsymbol{\alpha}=[\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_m]$ is a diagonal matrix of ‘penalty numbers’

stationary condition of the modified functional requires the derivatives of Π_p with respect to the $\mathbf{U}_i$ to vanish

$$\frac{\partial \Pi_p}{d\mathbf{U}} = 0 \rightarrow \quad \mathbf{KU} - \mathbf{F} + \mathbf{C}^T\boldsymbol{\alpha}\mathbf{CU} + \mathbf{C}^T\boldsymbol{\alpha}\mathbf{Q} = \mathbf{0}$$

$$[\mathbf{K} + \mathbf{C}^T\boldsymbol{\alpha}\mathbf{C}]\mathbf{U} = \mathbf{F} + \underbrace{\mathbf{C}^T\boldsymbol{\alpha}\mathbf{Q}}_{\text{Penalty matrix}}$$

Multi-Point Constraints- Penalty Function Method

❖ Theory of Penalty Function Method - General MPCs

[Zienkiewicz *et al*., 2000] :

α = constant $(1/h)^{p+1}$

p is the order of element used

characteristic size of element

$$\alpha = 1.0 \times 10^{4-6} \times$$ max (diagonal elements in the stiffness matrix)

or

$$\alpha = 1.0 \times 10^{5-8} \times$$ Young's modulus

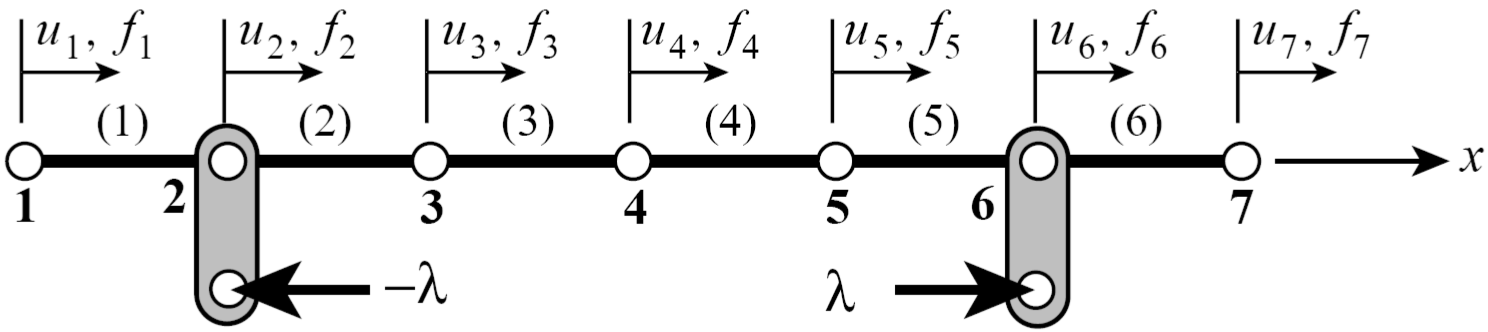
Multi-Point Constraints- Penalty Function Method

Assessment of Penalty Function Method

- ADVANTAGES
  - general application (inc' nonlinear MPCs)
  - easy to implement using FE library and standard assembler
  - no change in vector of unknowns
  - retains positive definiteness
  - insensitive to constraint dependence
- DISADVANTAGES
  - selection of weight left to user
  - accuracy limited by ill-conditioning
  - the constraint equations can only be satisfied approximately.

Multi-Point Constraints- Lagrange Multiplier Method

❖ Lagrange Multiplier Method, Physical Interpretation



force-pair that enforces MFC

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 - \lambda \\ f_3 \\ f_4 \\ f_5 \\ f_6 + \lambda \\ f_7 \end{bmatrix}$$

❖ Lagrange Multiplier Method

Because λ is unknown, it is passed to the LHS and appended to the node-displacement vector:

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 & 1 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} & -1 \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ \lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

This is now a system of 7 equations and 8 unknowns.
Needs an extra equation: the MPC.

❖ Lagrange Multiplier Method

Append MPC as additional equation:

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 & 1 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} & -1 \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ \lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ 0 \end{bmatrix}$$

This is the *multiplier-augmented system*. The new coefficient matrix is called the *bordered stiffness*.

- IMPLEMENTATION OF MPC EQUATIONS

$$\mathbf{KU} = \mathbf{F}$$ (Global system equation)

$$\mathbf{CU} - \mathbf{Q} = \mathbf{0}$$ (Matrix form of MPC equations)

Constant matrices

- Optimization problem for solution of nodal degrees of freedom:

- Find **U** to

Minimize: $\Pi_p = \frac{1}{2}\mathbf{U}^T\mathbf{K}\mathbf{U} - \mathbf{U}^T\mathbf{F}$

Subject to: $\mathbf{CU} - \mathbf{Q} = \mathbf{0}$

Multi-Point Constraints- Lagrange Multiplier Method

Lagrange multiplier method

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \end{bmatrix}^T$$ (Lagrange multipliers)

$$\boldsymbol{\lambda}^T \{\mathbf{CU} - \mathbf{Q}\} = 0$$ Multiplied to MPC equations

- Find **U** and $\boldsymbol{\lambda}$ to

Minimize: $$L = \frac{1}{2}\mathbf{U}^T\mathbf{K}\mathbf{U} - \mathbf{U}^T\mathbf{F} + \boldsymbol{\lambda}^T\{\mathbf{CU} - \mathbf{Q}\}$$ Added to functional

The stationary condition requires the derivatives of L with respect to the $\mathbf{U}_i$ and $\boldsymbol{\lambda}_i$ to vanish.

$$\frac{\partial L}{d\mathbf{U}} = 0 \rightarrow \quad \mathbf{KU} - \mathbf{F} + \mathbf{C}^T\boldsymbol{\lambda} = \mathbf{0}$$

$$\frac{\partial L}{d\boldsymbol{\lambda}} = 0 \rightarrow \quad \mathbf{CU} - \mathbf{Q} = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} \mathbf{K} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{U} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{Q} \end{Bmatrix}$$

Matrix equation is solved

Multi-Point Constraints- Lagrange Multiplier Method

❖ Lagrange Multiplier Method - Multiple MPCs

Three MPCs: $u_2 - u_6 = 0, \quad 5u_2 - 8u_7 = 3, \quad 3u_3 + u_5 - 4u_6 = 1$

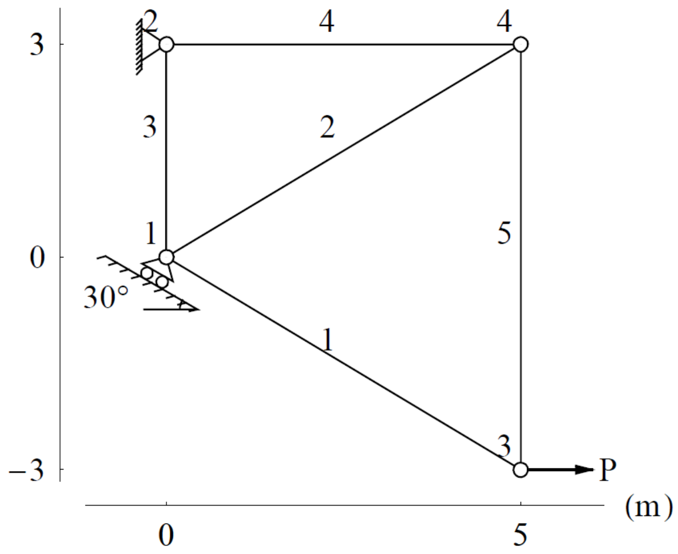
Recipe step #1: append the 3 constraints

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & -8 \\ 0 & 0 & 3 & 0 & 1 & -4 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

Recipe step #2: append multipliers, symmetrize and fill

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} & -1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} & 0 & -8 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & -4 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

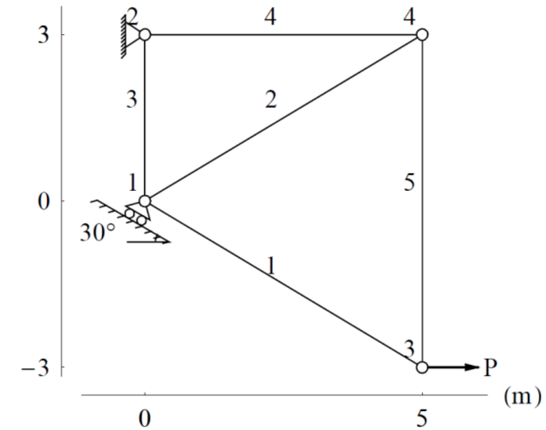
❖ Example: Five bar truss with inclined support



$E = 70$ Gpa, $A = 10^{-3}$ m^2, P = 20 kN.

❖ Example: Five bar truss with inclined support

Equations for element 1



$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	1	0	0
2	3	5000.	−3000.

$x_1 = 0 \qquad y_1 = 0 \qquad x_2 = 5000. \qquad y_2 = -3000.$

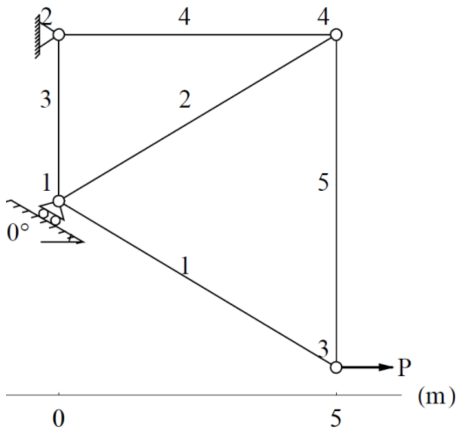
$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5830.95$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.857493 \qquad m_s = \dfrac{y_2 - y_1}{L} = -0.514496$

$$\mathbf{k}^{(1)} = \begin{pmatrix} 8827.13 & -5296.28 & -8827.13 & 5296.28 \\ -5296.28 & 3177.77 & 5296.28 & -3177.77 \\ -8827.13 & 5296.28 & 8827.13 & -5296.28 \\ 5296.28 & -3177.77 & -5296.28 & 3177.77 \end{pmatrix}$$

Multi-Point Constraints- Lagrange Multiplier Method

❖ Example: Five bar truss with inclined support



Equations for element 2

$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	1	0	0
2	4	5000.	3000.

$x_1 = 0 \qquad y_1 = 0 \qquad x_2 = 5000. \qquad y_2 = 3000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5830.95$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.857493 \qquad m_s = \dfrac{y_2 - y_1}{L} = 0.514496$

$$\mathbf{k}^{(2)} = \begin{pmatrix} 8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 5296.28 & 3177.77 & -5296.28 & -3177.77 \\ -8827.13 & -5296.28 & 8827.13 & 5296.28 \\ -5296.28 & -3177.77 & 5296.28 & 3177.77 \end{pmatrix}$$

❖ Example: Five bar truss with inclined support

Equations for element 3

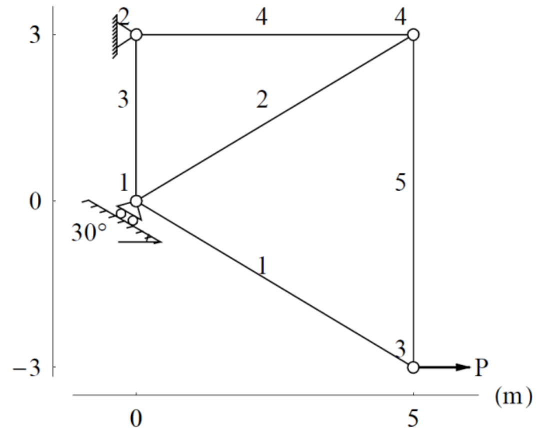
$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	1	0	0
2	2	0	3000.

$x_1 = 0 \qquad y_1 = 0 \qquad x_2 = 0 \qquad y_2 = 3000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 3000.$$

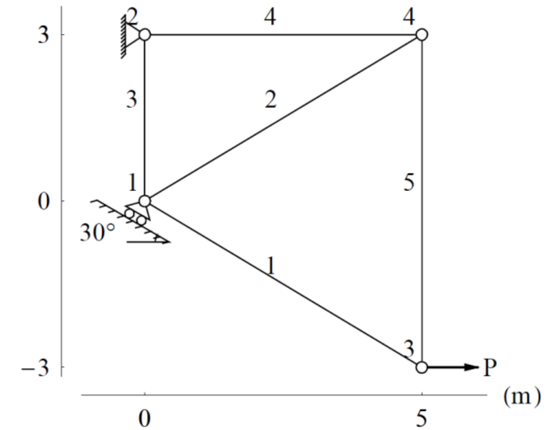
Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0 \qquad m_s = \dfrac{y_2 - y_1}{L} = 1.$



$$\mathbf{k}^{(3)} = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 23333.3 & 0. & -23333.3 \\ 0. & 0. & 0. & 0. \\ 0. & -23333.3 & 0. & 23333.3 \end{pmatrix}$$

❖ Example: Five bar truss with inclined support

Equations for element 4



$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	2	0	3000.
2	4	5000.	3000.

$x_1 = 0 \qquad y_1 = 3000. \qquad x_2 = 5000. \qquad y_2 = 3000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5000.$$

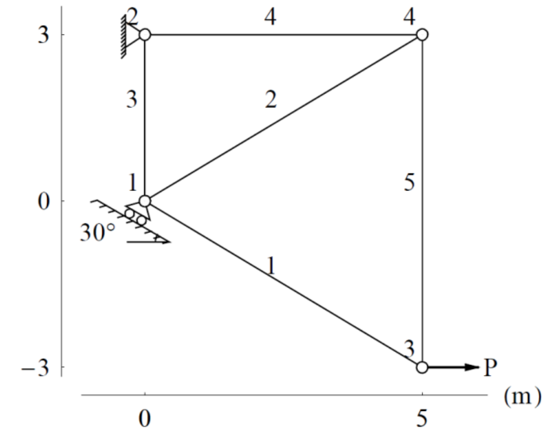
Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 1. \qquad m_s = \dfrac{y_2 - y_1}{L} = 0.$

$$\mathbf{k}^{(4)} = \begin{pmatrix} 14000. & 0. & -14000. & 0. \\ 0. & 0. & 0. & 0. \\ -14000. & 0. & 14000. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}$$

Multi-Point Constraints- Lagrange Multiplier Method

❖ Example: Five bar truss with inclined support

Equations for element 4



$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	2	0	3000.
2	4	5000.	3000.

$x_1 = 0 \qquad y_1 = 3000. \qquad x_2 = 5000. \qquad y_2 = 3000.$

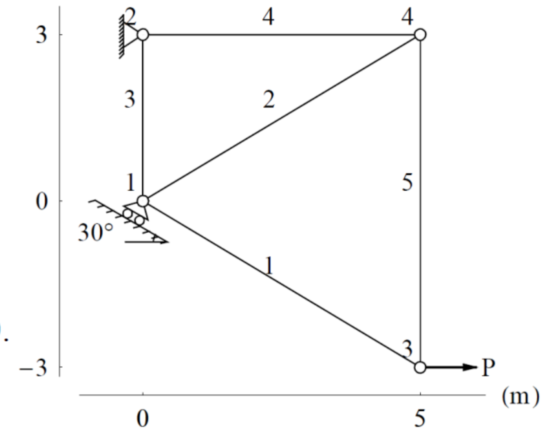
$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5000.$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 1. \qquad m_s = \dfrac{y_2 - y_1}{L} = 0.$

$$\mathbf{k}^{(4)} = \begin{pmatrix} 14000. & 0. & -14000. & 0. \\ 0. & 0. & 0. & 0. \\ -14000. & 0. & 14000. & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}$$

❖ Example: Five bar truss with inclined support

Equations for element 5



$E = 70000 \qquad A = 1000$

Element node	Global node number	x	y
1	3	5000.	−3000.
2	4	5000.	3000.

$x_1 = 5000. \qquad y_1 = -3000. \qquad x_2 = 5000. \qquad y_2 = 3000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 6000.$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0. \qquad m_s = \dfrac{y_2 - y_1}{L} = 1.$

$$\mathbf{k}^{(5)} = \begin{pmatrix} 0. & 0. & 0. & 0. \\ 0. & 11666.7 & 0. & -11666.7 \\ 0. & 0. & 0. & 0. \\ 0. & -11666.7 & 0. & 11666.7 \end{pmatrix}$$

Multi-Point Constraints- Lagrange Multiplier Method

❖ Example: Five bar truss with inclined support

$$\begin{pmatrix} 17654.3 & 0 & 0 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 0 & 29688.9 & 0 & -23333.3 & 5296.28 & -3177.77 & -5296.28 & -3177.77 \\ 0 & 0 & 14000. & 0 & 0 & 0 & -14000. & 0 \\ 0 & -23333.3 & 0 & 23333.3 & 0 & 0 & 0 & 0 \\ -8827.13 & 5296.28 & 0 & 0 & 8827.13 & -5296.28 & 0 & 0 \\ 5296.28 & -3177.77 & 0 & 0 & -5296.28 & 14844.4 & 0 & -11666.7 \\ -8827.13 & -5296.28 & -14000. & 0 & 0 & 0 & 22827.1 & 5296.28 \\ -5296.28 & -3177.77 & 0 & 0 & 0 & -11666.7 & 5296.28 & 14844.4 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

Node	dof	Value
2	u_2	0
	v_2	0

After adjusting for essential boundary conditions

$$\begin{pmatrix} 17654.3 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 \\ 0 & 29688.9 & 5296.28 & -3177.77 & -5296.28 & -3177.77 \\ -8827.13 & 5296.28 & 8827.13 & -5296.28 & 0 & 0 \\ 5296.28 & -3177.77 & -5296.28 & 14844.4 & 0 & -11666.7 \\ -8827.13 & -5296.28 & 0 & 0 & 22827.1 & 5296.28 \\ -5296.28 & -3177.77 & 0 & -11666.7 & 5296.28 & 14844.4 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Multipoint constraint due to inclined support at node 1: $u_1 \sin(\pi/6) + v_1 \cos(\pi/6) = 0$

The augmented global equations with the Lagrange multiplier are as follows.

$$\begin{pmatrix} 17654.3 & 0 & -8827.13 & 5296.28 & -8827.13 & -5296.28 & 1/2 \\ 0 & 29688.9 & 5296.28 & -3177.77 & -5296.28 & -3177.77 & \frac{\sqrt{3}}{2} \\ -8827.13 & 5296.28 & 8827.13 & -5296.28 & 0 & 0 & 0 \\ 5296.28 & -3177.77 & -5296.28 & 14844.4 & 0 & -11666.7 & 0 \\ -8827.13 & -5296.28 & 0 & 0 & 22827.1 & 5296.28 & 0 \\ -5296.28 & -3177.77 & 0 & -11666.7 & 5296.28 & 14844.4 & 0 \\ 1/2 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 20000. \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the final system of global equations we get

$$\{u_1 = 5.14286,\ v_1 = -2.96923,\ u_3 = 16.8629,\ v_3 = 12.788,\ u_4 = -1.42857,\ v_4 = 11.7594,\ \lambda = 80000.\}$$

Solution for element 1

Nodal coordinates

Element node	Global node number	x	y
1	1	0	0
2	3	5000.	−3000.

$x_1 = 0 \qquad y_1 = 0 \qquad x_2 = 5000. \qquad y_2 = -3000.$

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 5830.95$$

Direction cosines: $\ell_s = \dfrac{x_2 - x_1}{L} = 0.857493 \qquad m_s = \dfrac{y_2 - y_1}{L} = -0.514496$

Global to local transformation matrix

$$\boldsymbol{T} = \begin{pmatrix} 0.857493 & -0.514496 & 0 & 0 \\ 0 & 0 & 0.857493 & -0.514496 \end{pmatrix}$$

Element nodal displacements in global coordinates

$$\boldsymbol{d} = \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 5.14286 \\ -2.96923 \\ 16.8629 \\ 12.788 \end{pmatrix}$$

Element nodal displacements in local coordinates

$$\mathbf{u}_l = \boldsymbol{T}\,\boldsymbol{d} = \begin{pmatrix} 5.93762 \\ 7.88048 \end{pmatrix}$$

$E = 70000$ $A = 1000$

Axial strain, $\epsilon = (d_2 - d_1)/L = 0.000333197$

Axial stress, $\sigma = E\epsilon = 23.3238$

Axial force $= \sigma A = 23323.8$

	Stress	Axial force
1	23.3238	23323.8
2	23.3238	23323.8
3	69.282	69282.
4	−20.	−20000.
5	−12.	−12000.

Assessment of Lagrange Multiplier Method

- ADVANTAGES
  - General application
  - Constraint equations are satisfied exactly
- DISADVANTAGES
  - Difficult implementation
  - Total number of unknowns is increased
  - Expanded stiffness matrix is non-positive definite due to the presence of zero diagonal terms
  - Efficiency of solving the system equations is lower
  - sensitive to constraint dependence

❖ MPC Application Methods: Assessment Summary

	Master-Slave Elimination	Penalty Function	Lagrange Multiplier
Generality	fair	excellent	excellent
Ease of implementation	poor to fair	good	fair
Sensitivity to user decisions	high	high	small to none
Accuracy	variable	mediocre	excellent
Sensitivity as regards constraint dependence	high	none	high
Retains positive definiteness	yes	yes	no

References

1- Finite Element Method: A Practical Course by: S. S. Quek, G.R. Liu, 2003.

2- Introduction to Finite Element Methods, by: Carlos Felippa, University of Colorado at Boulder . http://www.colorado.edu/engineering/cas/courses.d/IFEM.d/