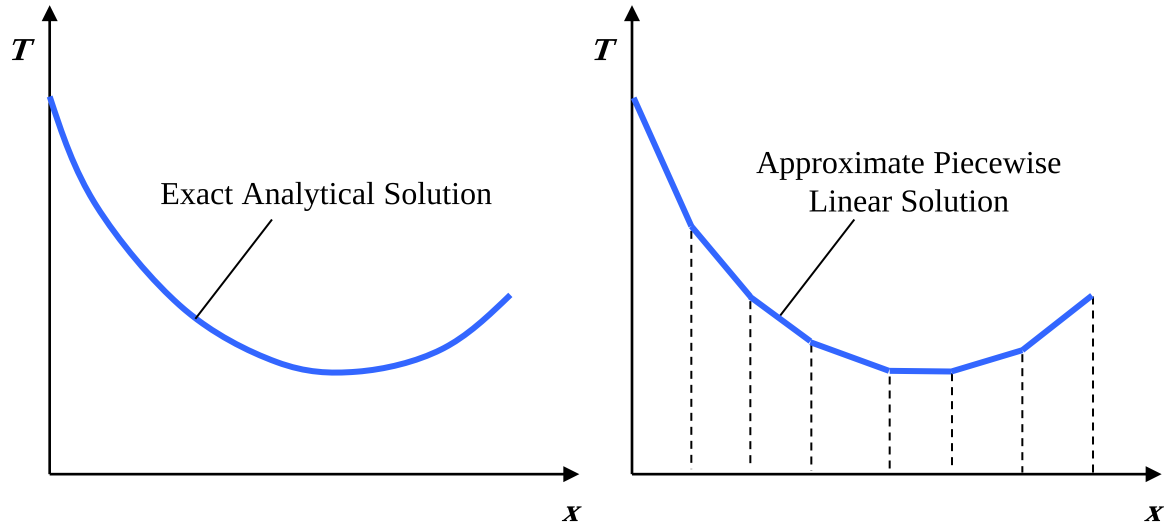
Shape Functions generation, requirements

Basic Concept of the Finite Element Method

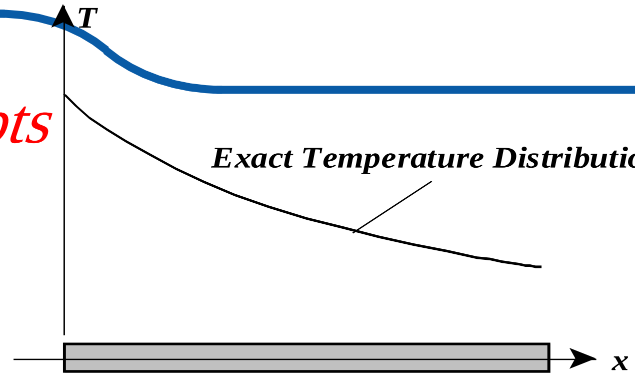
Any continuous solution field such as stress, displacement, temperature, pressure, etc. can be approximated by a discrete model composed of a set of piecewise continuous functions defined over a finite number of subdomains.

One-Dimensional Temperature Distribution

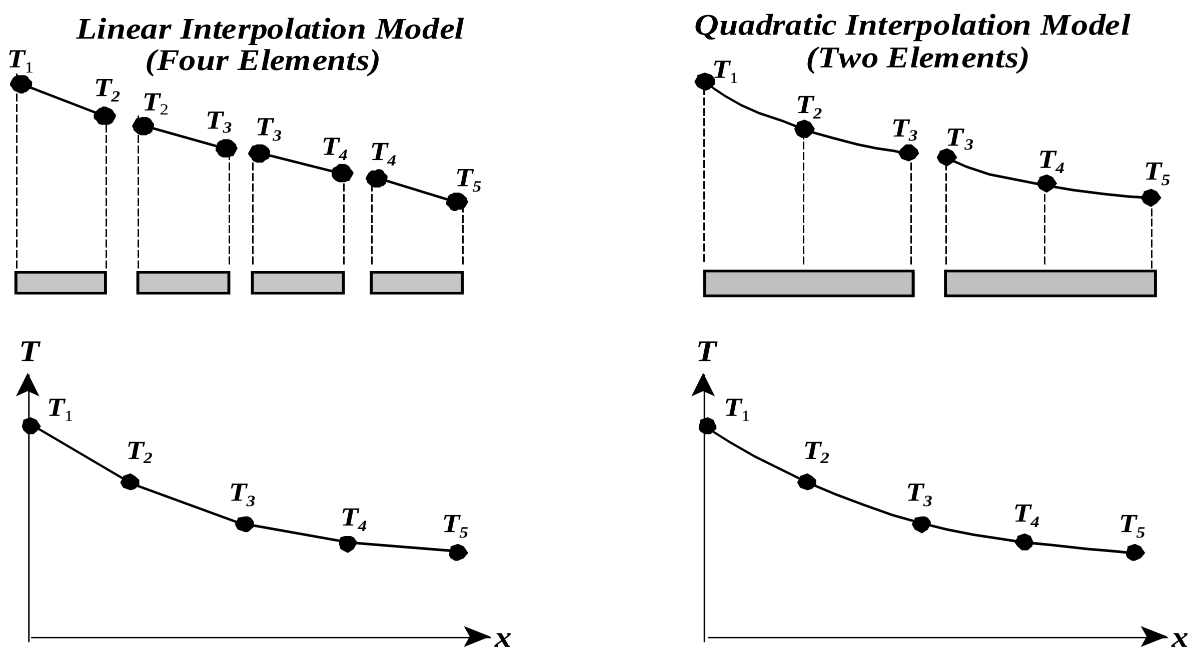


Basic Concept of the Finite Element Method

Discretization Concepts



Finite Element Discretization



Piecewise Linear Approximation

Temperature Continuous but with Discontinuous Temperature Gradients

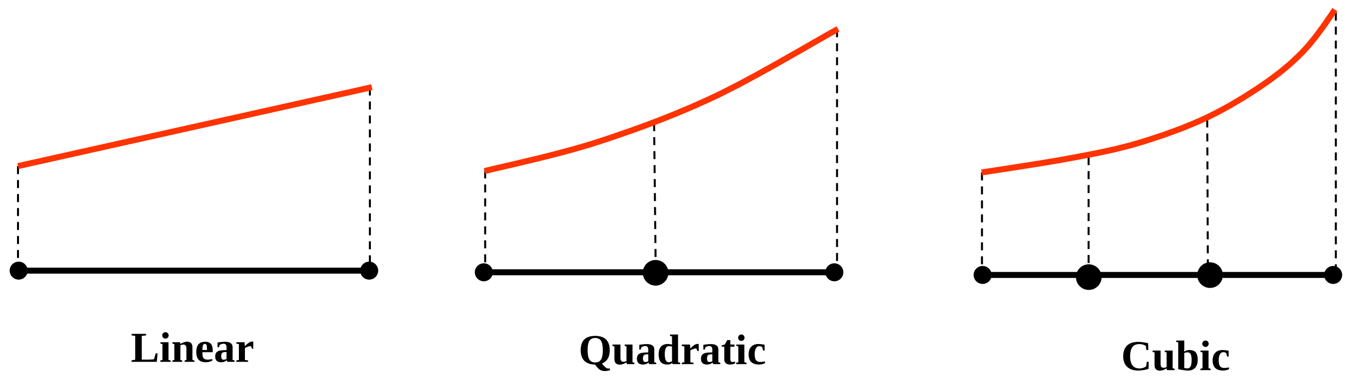
Piecewise Quadratic Approximation

Temperature and Temperature Gradients Continuous

Common Approximation Schemes
One-Dimensional Examples

Polynomial Approximation

Most often polynomials are used to construct approximation functions for each element. Depending on the order of approximation, different numbers of element parameters are needed to construct the appropriate function.



Special Approximation

For some cases (e.g. infinite elements, crack or other singular elements) the approximation function is chosen to have special properties as determined from theoretical considerations

Requirements for Shape Functions

Requirements for shape functions are motivated by **convergence**: as the mesh is refined the FEM solution should approach the analytical solution of the mathematical model.

1. The requirement for compatibility: The interpolation has to be such that field of displacements is :

1. continual and derivable inside the element
2. continual across the element border

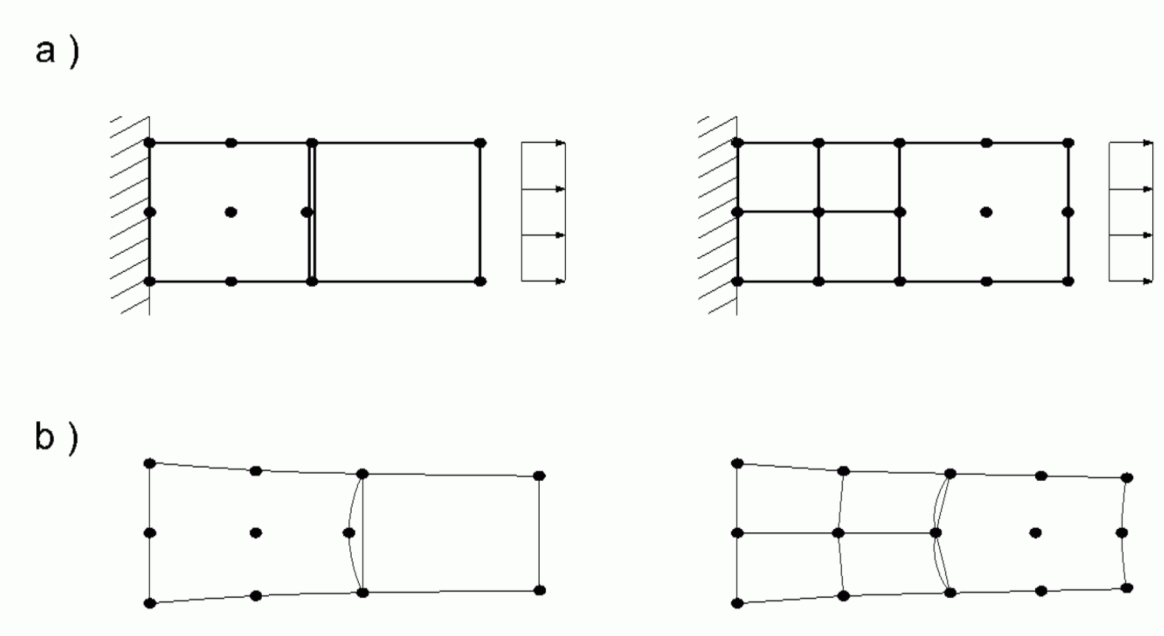
The finite elements that satisfy this property are called **conforming**, or **compatible**. (The use of elements that violate this property, **nonconforming** or **incompatible elements** is however common)

2. The requirement for completeness: The interpolation has to be able to represent:

1. the rigid body displacement
2. constant strain state

Requirement for Compatibility:

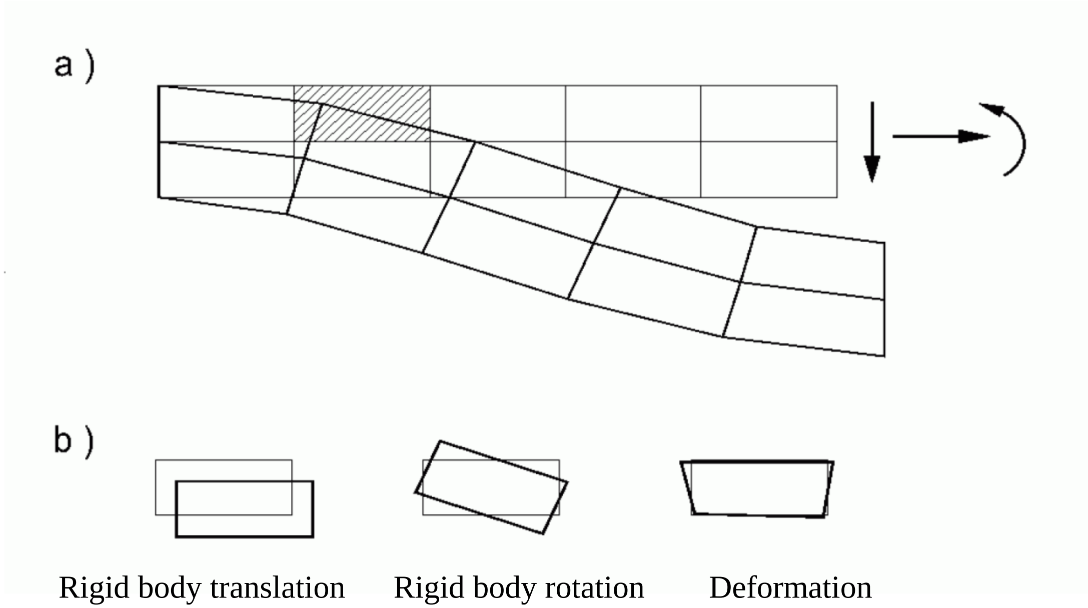
The shape functions should provide *displacement continuity* between elements. Physically this insure that no material gaps appear as the elements deform. As the mesh is refined, such gaps would multiply and may absorb or release spurious energy.



Compatibility violation by using different types of elements.
a) Discretization and load; b) Deformed shape (left gap, right overlapping)

Requirement for Completeness: The interpolation has to be able to represent:

1. The rigid body displacement
2. Constant strain state



a) Deformation of cantilever beam. b) Rigid body displacement and deformation of hatched element

Requirements for Shape Functions

If the stiffness integrands involve derivatives of order m, then requirements for shape functions can be formulated as follows:

1. The requirement for compatibility: The shape functions must be $C^{(m-1)}$ continuous between elements, and C^m piecewise differentiable inside each element.

2. The requirement for completeness: The element shape functions must represent exactly all polynomial terms of order $\leq m$ in the Cartesian coordinates. A set of shape functions that satisfies this condition is called m-complete.

Requirements for Shape Functions

Differential operator $D_{\varepsilon u}$ for different types of physical or mechanical problems

Physical or mechanical problem	**Heat conduction**	Truss (N=1) **Two-dimensional problem (N=2)** Three-dimensional continuum (N=3)	Euler-Bernoulli beam (N=1) **Thin plate (N=2) (Kirchhoff plate)**	**Thick plates (N=2) (Reissner-Mindlin plate)**
Independent primary variables	Temperature 1	Displacements N	Transverse displacement 1	Transverse displacement 1 Rotation N
Operator $\mathbf{D}_{\varepsilon u}$ (for N=2)	$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix}$	$\begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix}$	$\begin{bmatrix} \partial^2/\partial x^2 \\ \partial^2/\partial y^2 \\ 2\partial^2/\partial x \partial y \end{bmatrix}$	$\begin{bmatrix} 0 & \partial/\partial x & 0 \\ 0 & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \\ \partial/\partial x & 1 & 0 \\ \partial/\partial y & 0 & 1 \end{bmatrix}$
Required continuity	C^0	C^0	C^1	C^0

PROPERTIES OF THE SHAPE FUNCTIONS

1. Kronecker delta property: The shape function at any node has a value of *1* at that node and a value of *zero* at ALL other nodes.

2. Compatibility: The displacement approximation is continuous across element boundaries

3. Completeness

Rigid body mode

Constant strain states

Compatibility + Completeness ⇒ Convergence

Ensure that the solution gets better as more elements are introduced and, in the limit, approaches the exact answer.

Lagrangian Shape Functions:

- Can perform this for any number of points at any designated locations.

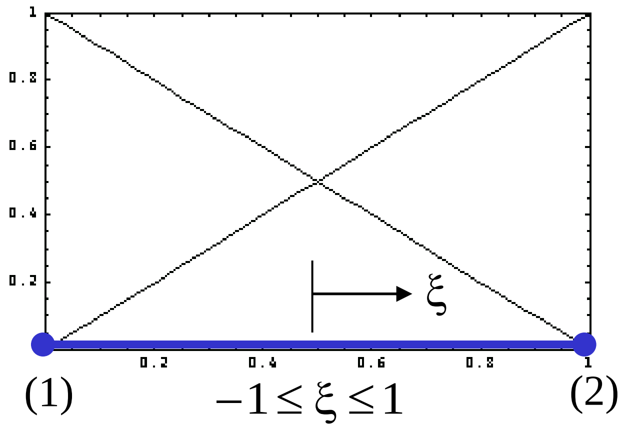
$$L_k^{(m)}(\xi)=\frac{(\xi-\xi_0)(\xi-\xi_1)\cdots(\xi-\xi_{k-1})(\xi-\xi_{k+1})\cdots(\xi-\xi_m)}{(\xi_k-\xi_0)(\xi_k-\xi_1)\cdots(\xi_k-\xi_{k-1})(\xi_k-\xi_{k+1})\cdots(\xi_k-\xi_m)}=\prod_{\substack{i=0\\ i\neq k}}^{m}\frac{(\xi-\xi_i)}{(\xi_k-\xi_i)}.$$

No ξ-ξ_k term!

Lagrange polynomial of order *m* at node *k*

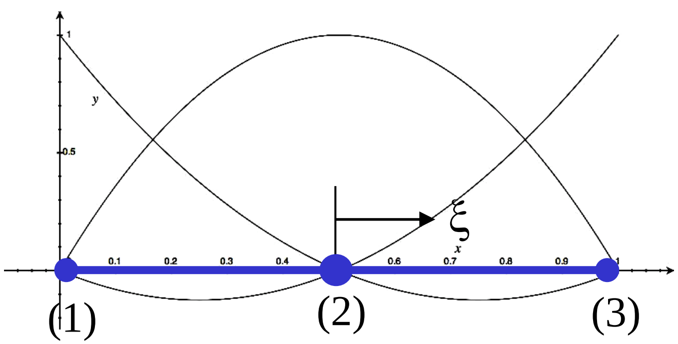
Lagrange Interpolation Functions

$$N_i(\xi_j) = \begin{cases} 1\ ,\ i = j \\ 0\ ,\ i \neq j \end{cases}$$



$$N_1 = \frac{1}{2}(1-\xi)$$

$$N_2 = \frac{1}{2}(1+\xi)$$

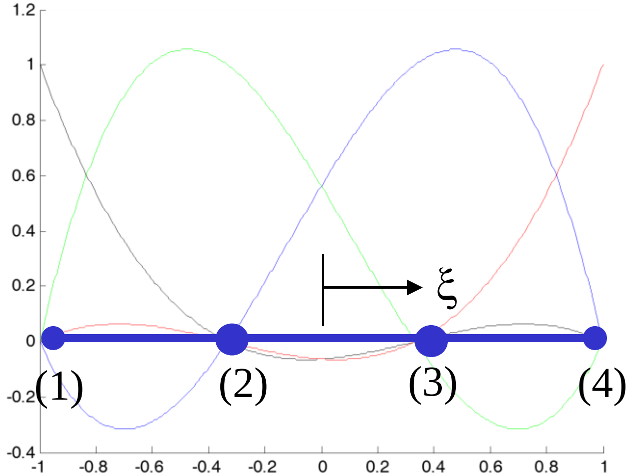


$$N_1 = -\frac{1}{2}\xi(1-\xi)$$

$$N_2 = (1-\xi)(1+\xi)$$

$$N_3 = \frac{1}{2}\xi(1+\xi)$$

Lagrange Interpolation Functions



$$N_1 = -\frac{9}{16}(1-\xi)(\frac{1}{3}+\xi)(\frac{1}{3}-\xi)$$

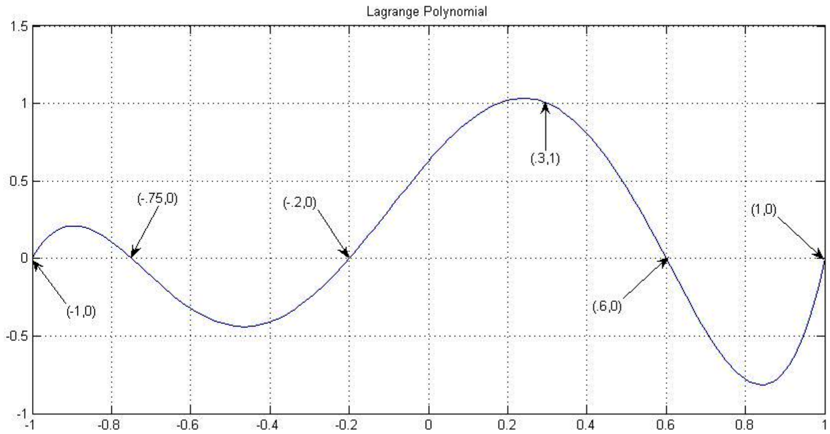
$$N_2 = \frac{27}{16}(1-\xi)(1+\xi)(\frac{1}{3}-\xi)$$

$$N_3 = \frac{27}{16}(1-\xi)(1+\xi)(\frac{1}{3}+\xi)$$

$$N_4 = -\frac{9}{16}(\frac{1}{3}+\xi)(\frac{1}{3}-\xi)(1+\xi)$$

Lagrangian Shape Functions:

- Uses a procedure that <u>automatically</u> satisfies the Kronecker delta property for shape functions.
  - Consider 1D example of 6 points; want function = 1 at $\xi_3 = 0.3$ and function = 0 at other designated points:



$$\xi_0 = -1; \quad \xi_1 = -.75; \quad \xi_2 = -.2; \quad \xi_3 = .3; \quad \xi_4 = .6; \quad \xi_5 = 1.$$

$$L_3^{(5)}(\xi) = \frac{(\xi-\xi_0)(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_4)(\xi-\xi_5)}{(\xi_3-\xi_0)(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)(\xi_3-\xi_5)}.$$

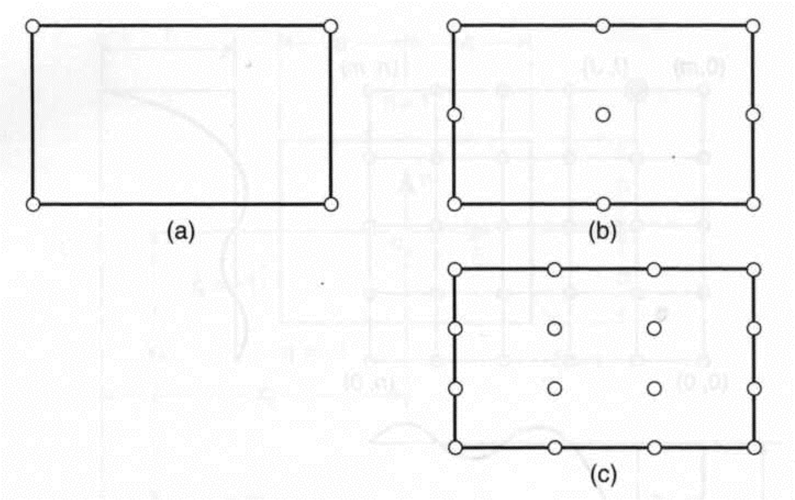
Shape Functions of Plane Elements

Classification of shape functions according to:

- the element form:
  - triangular elements,
  - rectangular elements.
- polynomial degree of the shape functions:
  - linear
  - quadratic
  - cubic
  - …
- type of the shape functions
  - Lagrange shape functions
  - serendipity shape functions

Lagrangian Elements:

- Order *n* element has $(n+1)^2$ nodes arranged in square-symmetric pattern – requires internal nodes.



- Shape functions are products of *n*th order polynomials in each direction. ("biquadratic", "bicubic", …)
- Bilinear quad is a Lagrangian element of order $n = 1$.

Rectangular elements – Lagrange family

Lagrange interpolation polynomial
in one direction :

$$l_k^n(\xi_1) = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{\xi_1 - \xi_1^i}{\xi_1^k - \xi_1^i}$$

An easy and systematic method of generating shape functions of any order now can be achieved by simple products of Lagrange polynomials in the two coordinates :

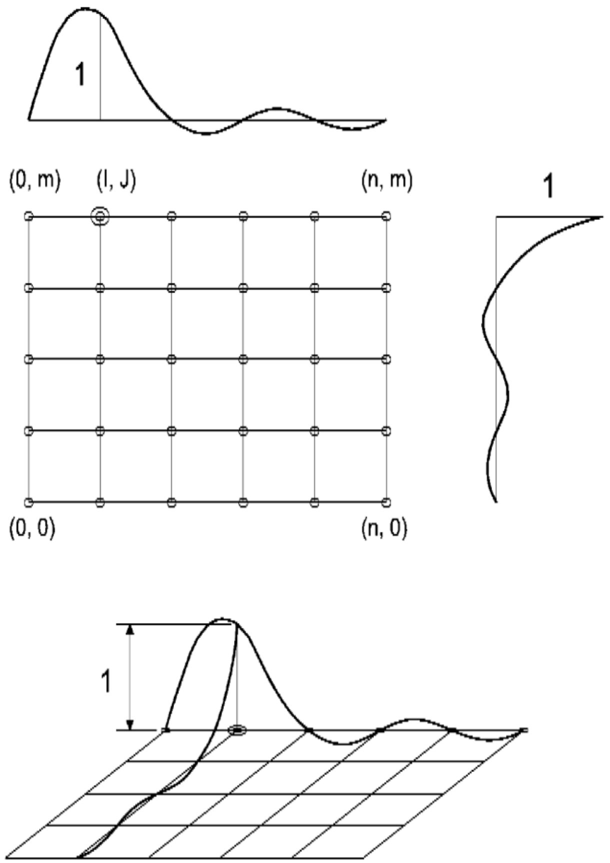
$$N_a = N_{IJ} = l_I^n(\xi_1) l_J^m(\xi_2)$$

where :

$$\xi_1 = \frac{2(x - x_e)}{a_e} \quad \xi_2 = \frac{2(y - y_e)}{b_e}$$

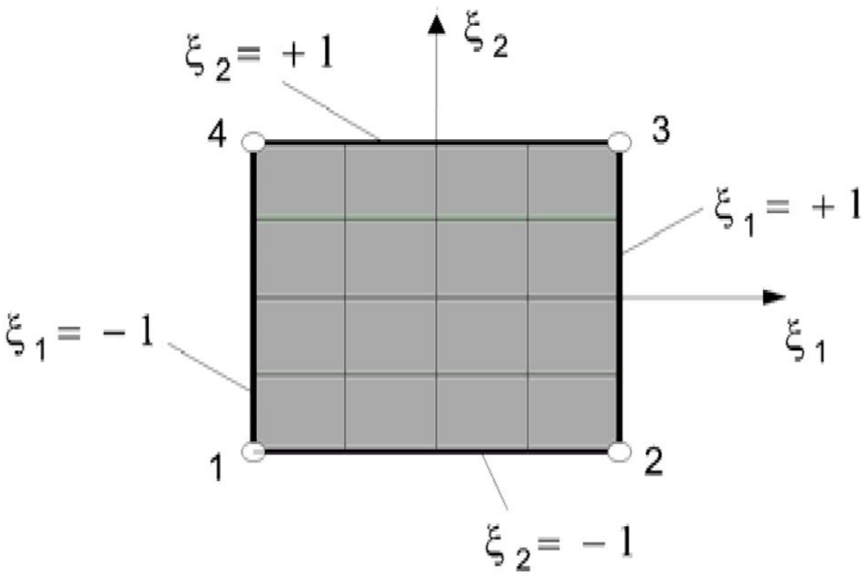
x_e, y_e are coordinates of the center of the element

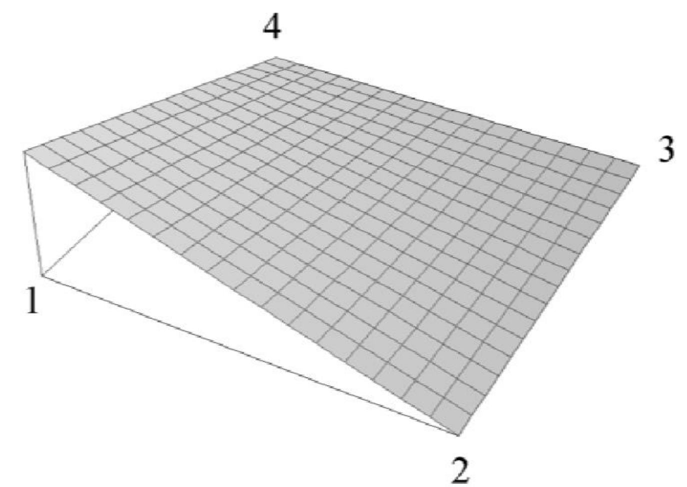
a_e, b_e are dimensions of the element



Rectangular elements – Lagrange family

The Four-Node Bilinear Quadrilateral

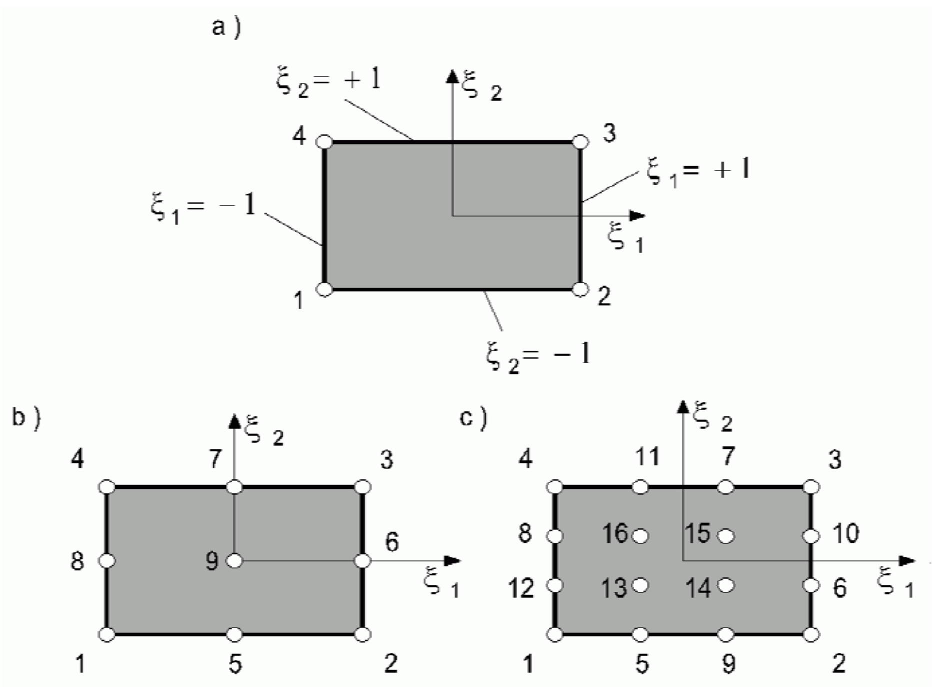


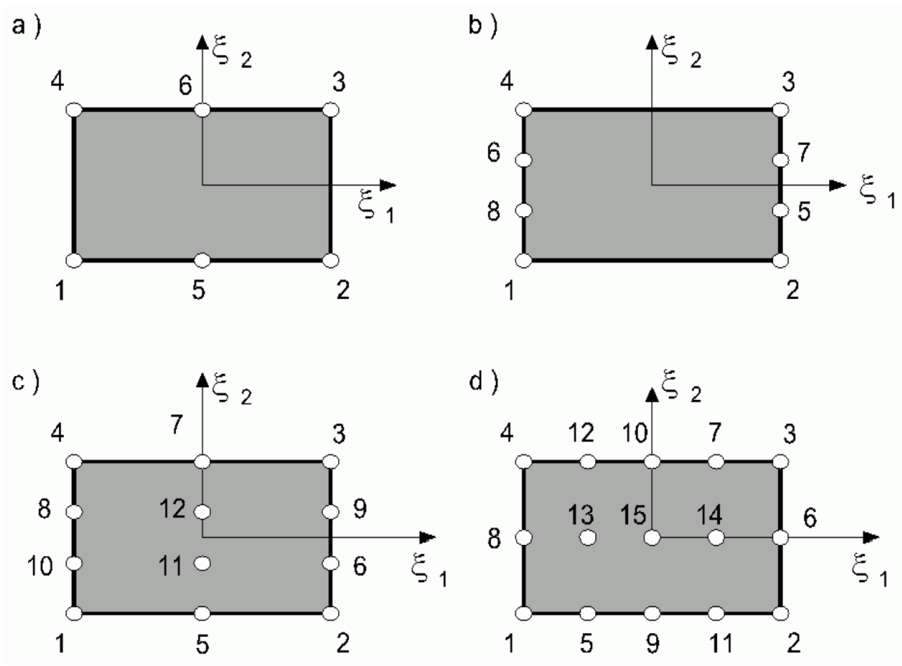


$$N^1(\xi) = \frac{1}{4}(1-\xi_1)(1-\xi_2) \qquad N^3(\xi) = \frac{1}{4}(1+\xi_1)(1+\xi_2)$$

$$N^2(\xi) = \frac{1}{4}(1+\xi_1)(1-\xi_2) \qquad N^4(\xi) = \frac{1}{4}(1-\xi_1)(1+\xi_2)$$

Rectangular elements – Lagrange family

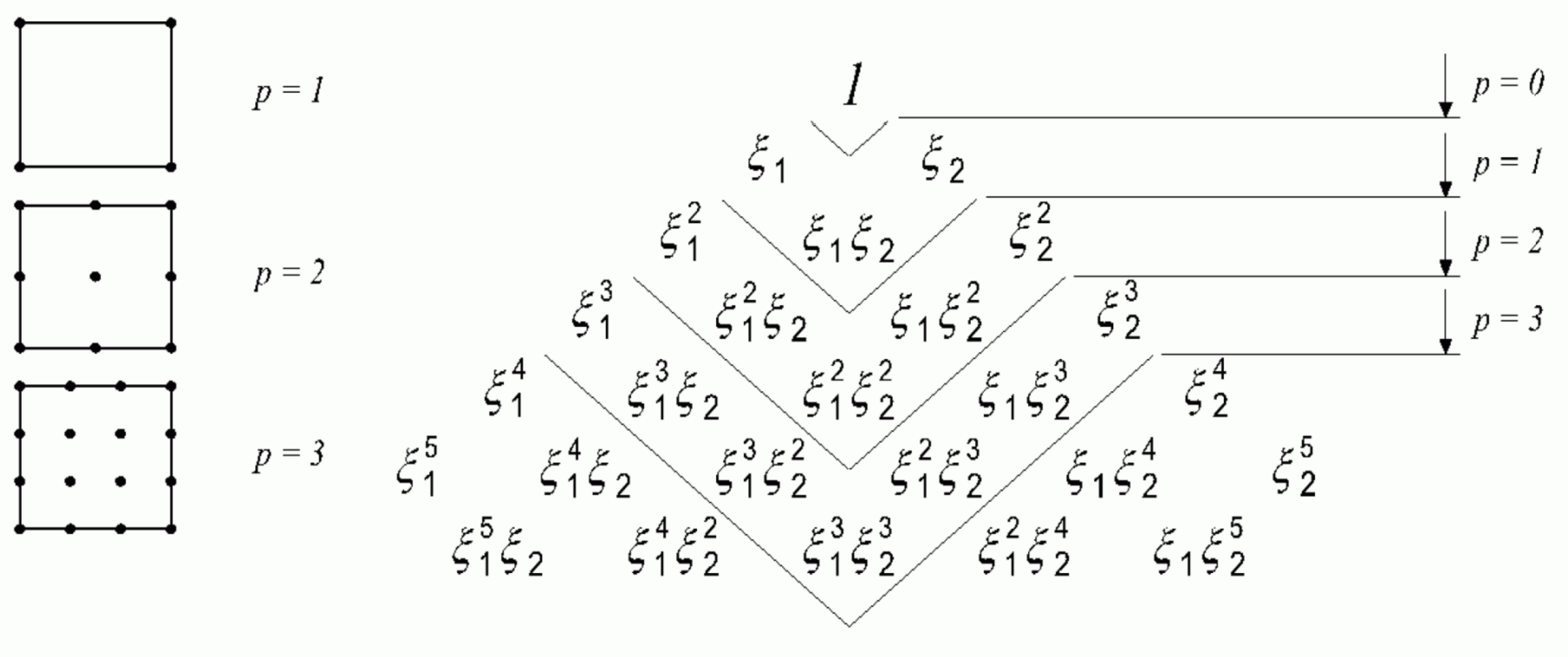




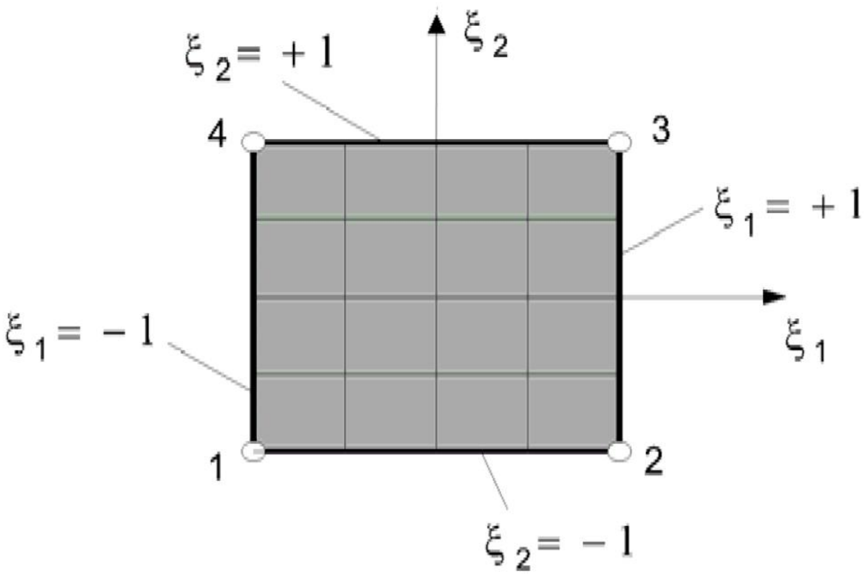
The Quadrilateral Lagrangian elements: a) bilinear, b) biquadratic c) bicubic

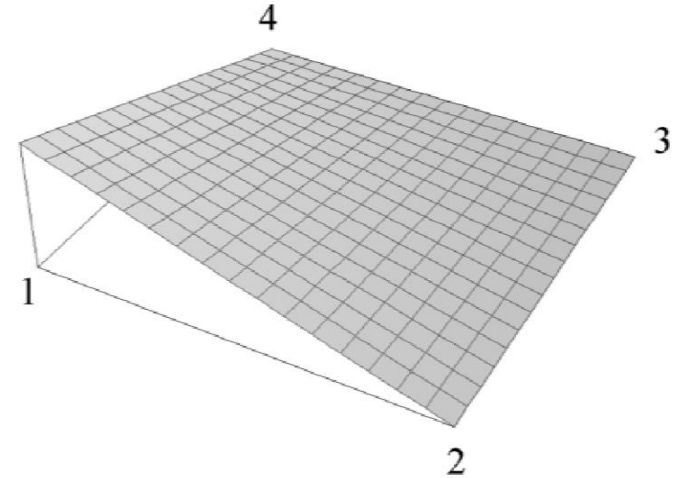
The Quadrilateral Lagrangian elements: a) quadratic-linear, b) linear-cubic c) quadratic-cubic, d) quartic-quadratic

Complete two-dimensional Lagrange polynomials in the Pascal triangle



The Four-Node Bilinear Quadrilateral

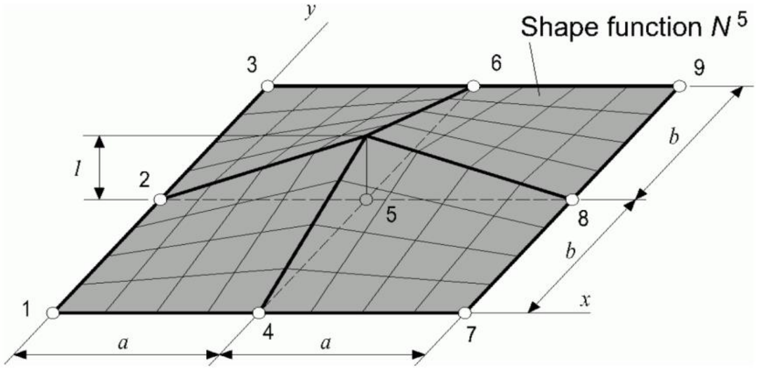




$$N^1(\xi) = \frac{1}{4}(1-\xi_1)(1-\xi_2) \qquad N^3(\xi) = \frac{1}{4}(1+\xi_1)(1+\xi_2)$$

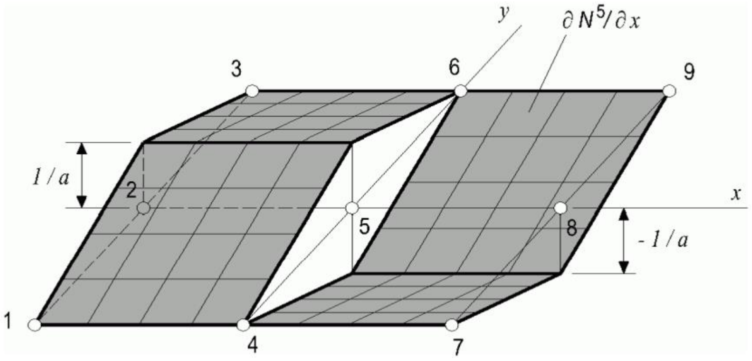
$$N^2(\xi) = \frac{1}{4}(1+\xi_1)(1-\xi_2) \qquad N^4(\xi) = \frac{1}{4}(1-\xi_1)(1+\xi_2)$$

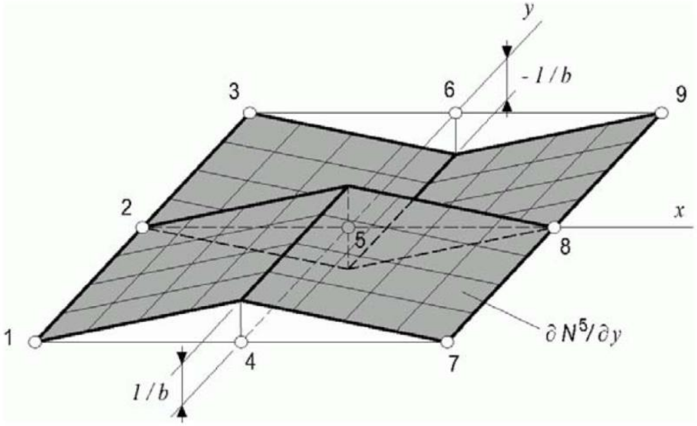
The Four-Node Bilinear Quadrilateral
Check of compatibility



Assemblage of four bilinear quadrilateral elements

Change of N^5 along the edge is linear and it is uniquely defined by two nodes





Partial derivatives with respect to x and y of the shape functions N^5

Derivative inside element exists, and on the boundary has finite discontinuity

Check of completeness

A set of shape functions is complete for a continuum element if they can represent exactly any linear displacement motions such as :

$$u_x = \alpha_0 + \alpha_1 x + \alpha_2 y, \qquad u_y = \beta_0 + \beta_1 x + \beta_2 y \qquad (1)$$

The nodal point displacements corresponding to this displacement field are :

$$u_{xi} = \alpha_0 + \alpha_1 x_i + \alpha_2 y_i, \quad u_{yi} = \beta_0 + \beta_1 x_i + \beta_2 y_i \qquad (2)$$

The displacements (1) have to be obtained within the element when the element nodal point displacements are given by (2).
In the isoparametric formulation we have the displacement interpolation :

$$u_x = \sum_{i=1}^{n} u_{xi} N_i^e, \quad u_y = \sum_{i=1}^{n} u_{yi} N_i^e$$

Computation for the displacement u_x:

$$u_x = \sum_{i=1}^{e} (\alpha_0 + \alpha_1 x_i + \alpha_2 y_i) N_i^e = \alpha_0 \sum_{i=1}^{n} N_i^e + \alpha_1 \sum_{i=1}^{n} x_i N_i^e + \alpha_2 \sum_{i=1}^{n} y_i N_i^e$$

Rectangular elements – Lagrange family

Since in the isoparametric formulation the coordinates are interpolated in the same way as the displacements, we can use :

$$x = \sum_{i=1}^{n} x_i N_i^e \,, \qquad y = \sum_{i=1}^{n} y_i N_i^e$$

to obtain :

$$u_x = \alpha_0 \sum_{i=1}^{n} N_i^e + \alpha_1 x + \alpha_2 y \qquad (3)$$

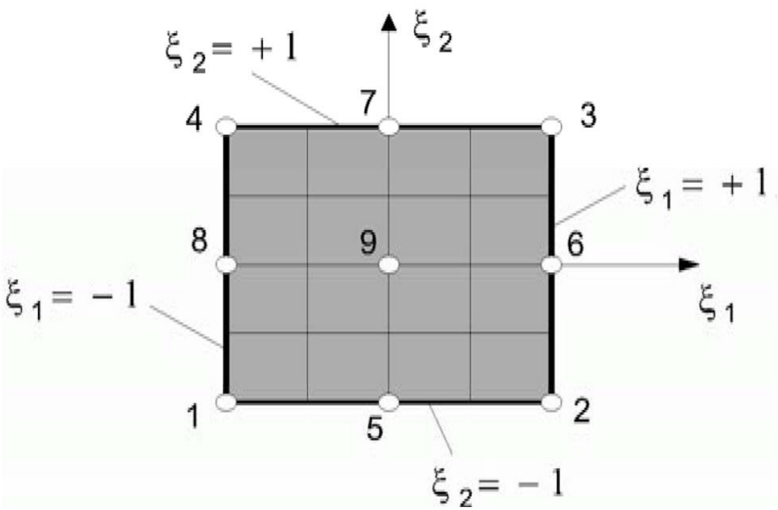
The displacements defined in (3) are the same as those given (1), provided that for any point in the element :

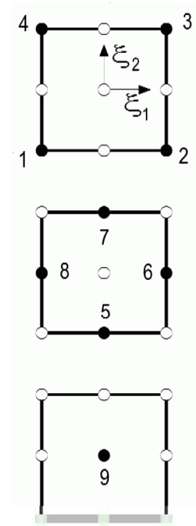
$$\sum_{i=0}^{n} N_i = 1 \qquad (4)$$

The relation (4) is the condition on the interpolation functions for the completeness requirements to be satisfied.

Rectangular elements – Lagrange family

The Nine-Node Biquadratic Quadrilateral



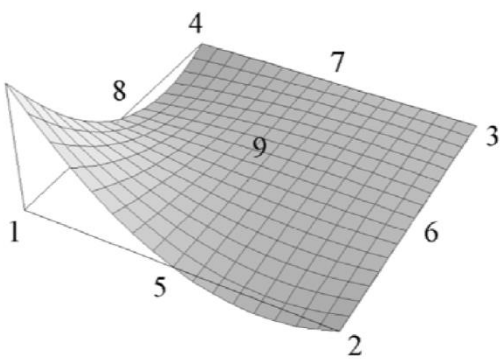


$$N^i = \tfrac{1}{4}(1+\xi_1^i\xi_1)\xi_1^i\xi_1(1+\xi_2^i\xi_2)\xi_2^i\xi_2$$

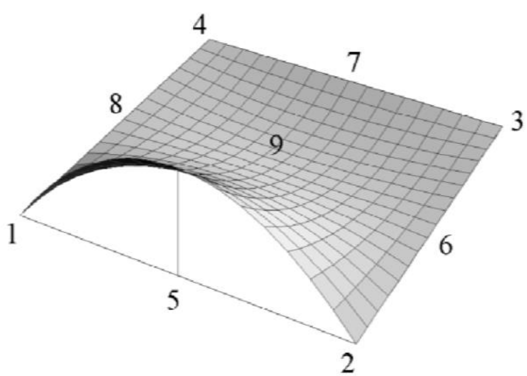
$$N^i = \tfrac{1}{2}(1-\xi_2^2)(1+\xi_1^i\xi_1)\xi_1^i\xi_1 \,,\; \xi_2^i = 0$$

$$N^i = \tfrac{1}{2}(1-\xi_1^2)(1+\xi_2^i\xi_2)\xi_2^i\xi_2 \,,\; \xi_1^i = 0$$

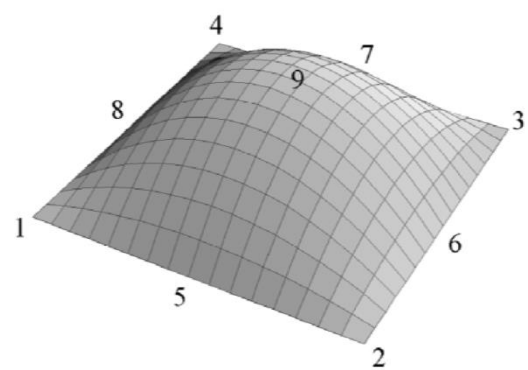
$$N^9 = (1-\xi_1^2)(1-\xi_2^2)$$



$$N^1(\xi) = \frac{1}{4}(1-\xi_1)(1-\xi_2)\,\xi_1\,\xi_2$$

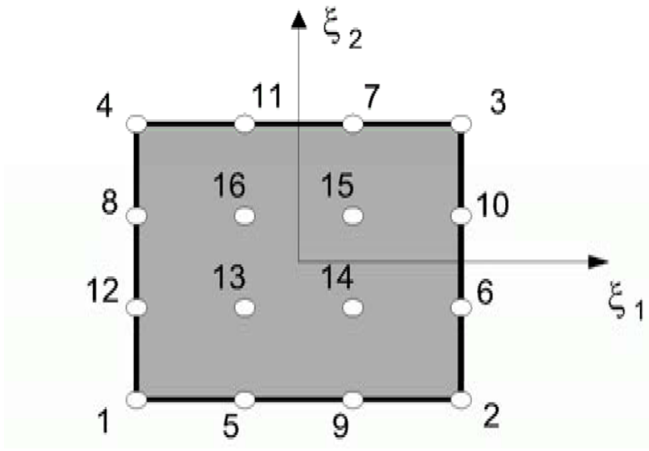


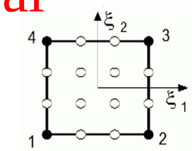
$$N^5(\xi) = \frac{1}{2}(1-\xi_1^2)\,(\xi_2-1)\xi_2$$



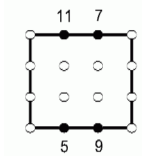
$$N^9(\xi) = (1-\xi_1^2)(1-\xi_2^2)$$

The 16-Node Bicubic Quadrilateral

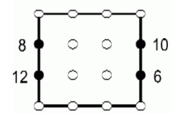




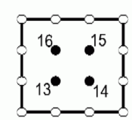
$$N^i = \tfrac{81}{256}(1+\xi_1^i\xi_1)(1+\xi_2^i\xi_2)(\tfrac{1}{9}-\xi_1^2)(\tfrac{1}{9}-\xi_2^2)$$



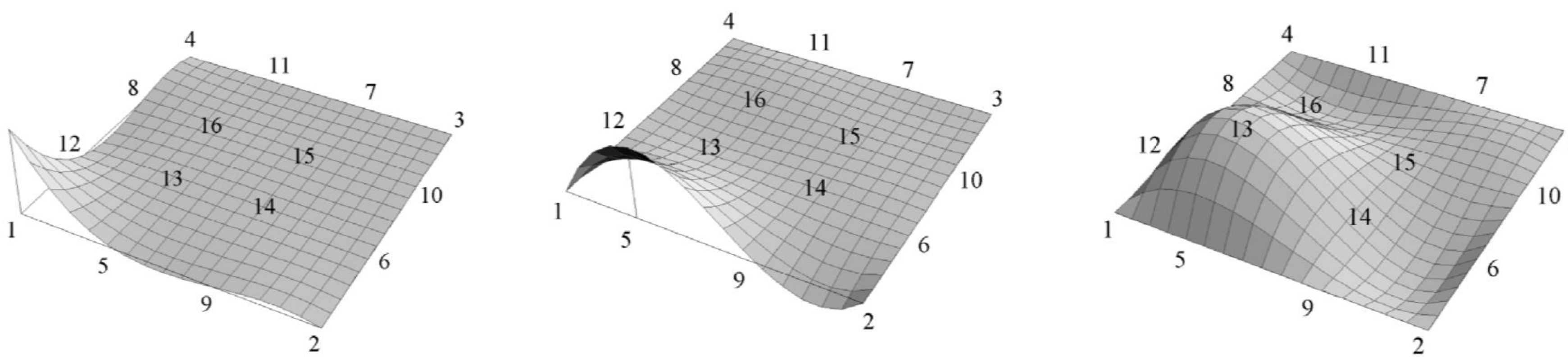
$$N^i = \tfrac{243}{256}(1-\xi_1^2)(\xi_2^2-\tfrac{1}{9})(\tfrac{1}{3}+3\xi_1^i\xi_1)(1+\xi_2^i\xi_2)$$



$$N^i = \tfrac{243}{256}(1-\xi_2^2)(\xi_1^2-\tfrac{1}{9})(\tfrac{1}{3}+3\xi_2^i\xi_2)(1+\xi_1^i\xi_1)$$

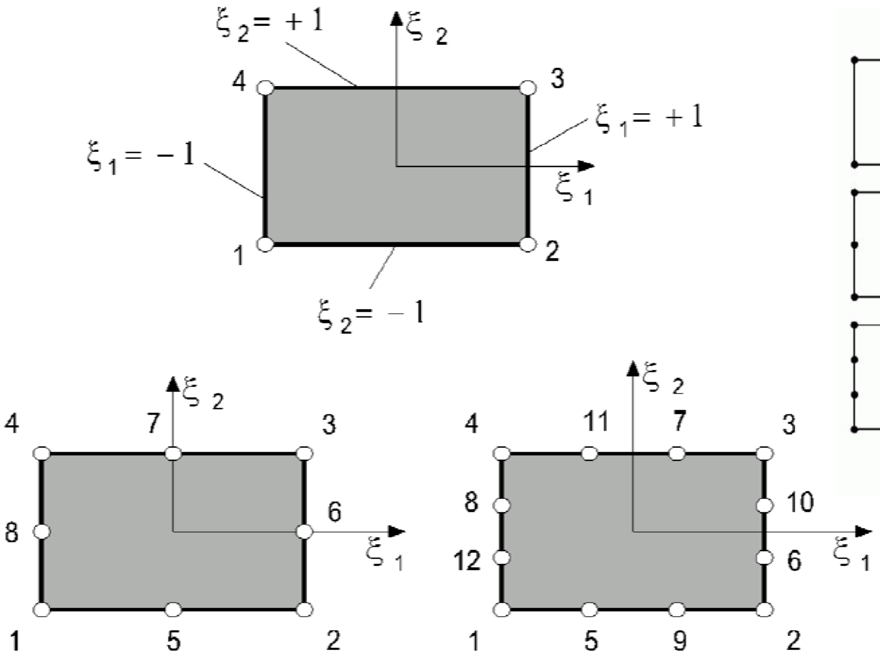


$$N^i = \tfrac{729}{256}(1-\xi_1^2)(1-\xi_2^2)(\tfrac{1}{3}+3\xi_1^i\xi_1)(\tfrac{1}{3}+3\xi_2^i\xi_2)$$

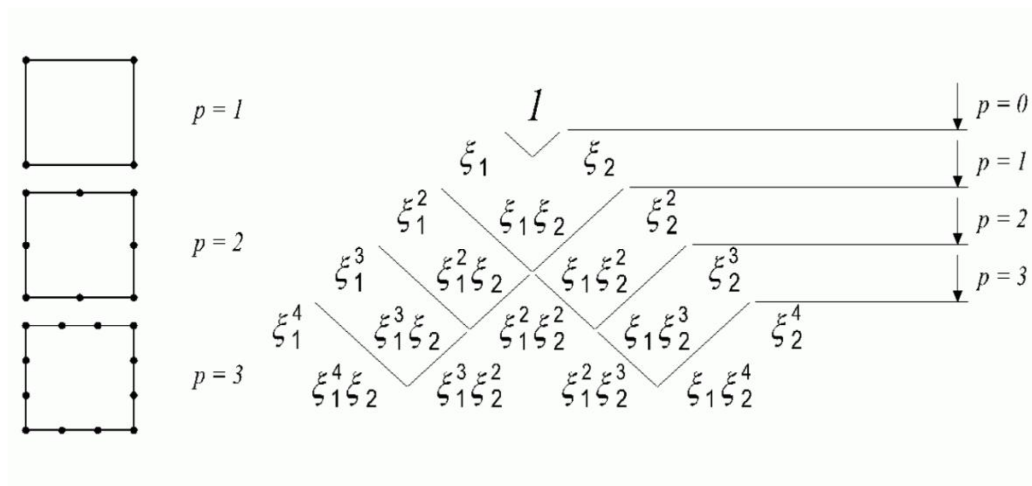


Rectangular elements – Serendipity elements

Serendipity elements are constructed with nodes only on the element boundary

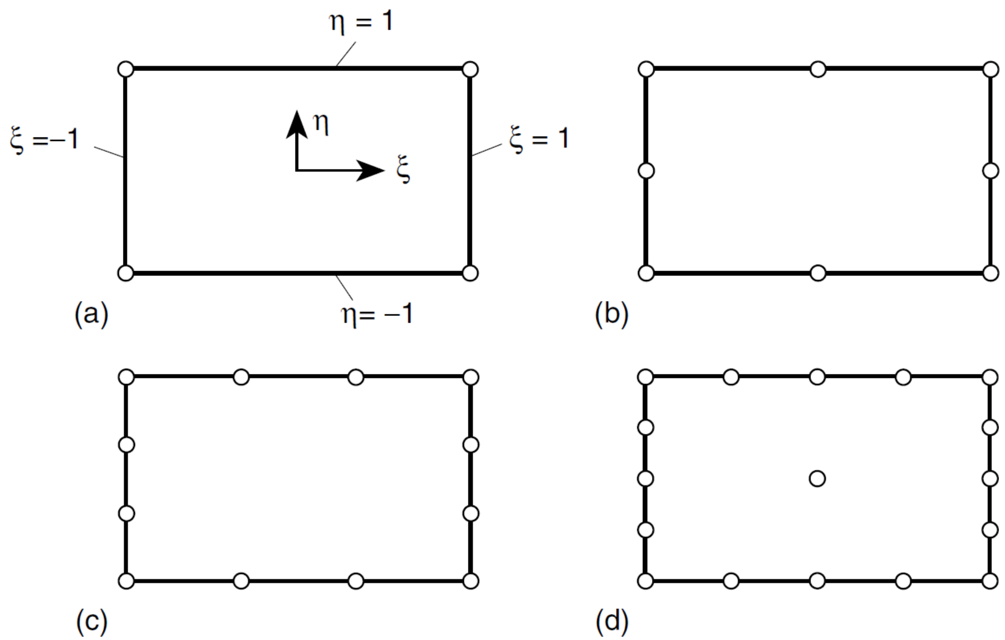


Serendipity quadrilateral elements:
a) bilinear , b) biquadratique, c) bicubic



Two dimensional serendipity polynomials of quadrilateral elements in Pascal triangle

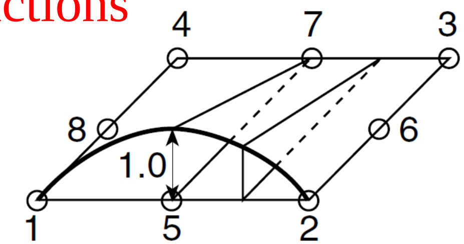
Rectangular elements – Serendipity elements



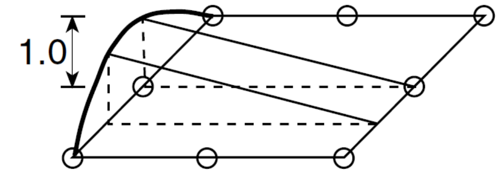
Rectangles of boundary node (serendipity) family: (a) linear, (b) quadratic, (c) cubic, (d) quartic.

Serendipity Biquadratic Shape functions

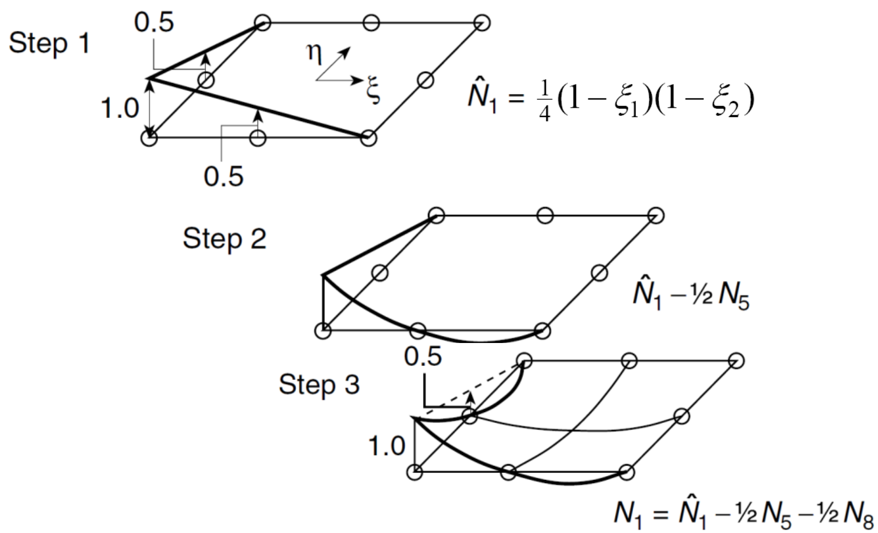
For mid-side nodes a lagrangian interpolation of quadratic x linear type suffices to determine N^i at nodes 5 to 8. For corner nodes start with bilinear lagragian family (step 1), and successive subtraction (step 2, step 3) ensures zero value at nodes 5, 8



$$N^5 = \tfrac{1}{2}(1-\xi_1^2)(1-\xi_2)$$



$$N^8 = \tfrac{1}{2}(1-\xi_1)(1-\xi_2^2)$$

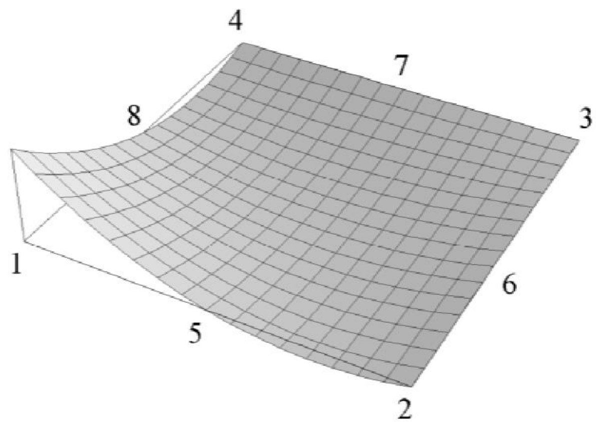


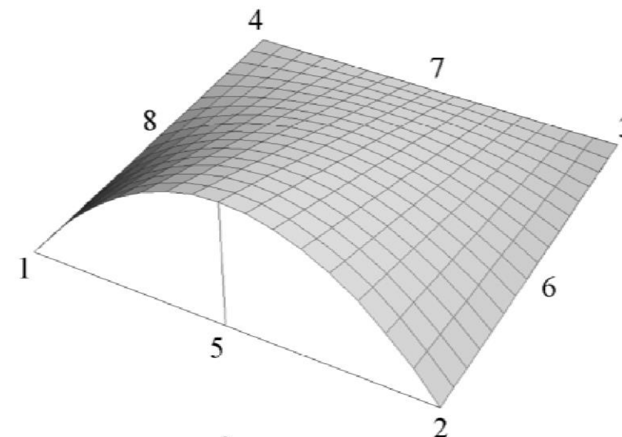
$$N^1 = \tfrac{1}{4}(1-\xi_1)(1-\xi_2) - \tfrac{1}{2}\cdot\tfrac{1}{2}(1-\xi_1^2)(1-\xi_2) - \tfrac{1}{2}\cdot\tfrac{1}{2}(1-\xi_1)(1-\xi_2^2) =$$

$$\tfrac{1}{4}(1-\xi_1)(1-\xi_2)[1-(1-\xi_1)-(1-\xi_2)]$$

$$N^1 = \tfrac{1}{4}(1-\xi_1)(1-\xi_2)(-1-\xi_1-\xi_2)$$

Serendipity Biquadratic Shape functions





$$N^1(\boldsymbol{\xi}) = \tfrac{1}{4}(1-\xi_1)(1-\xi_2)(-1-\xi_1-\xi_2)$$

$$N^i(\boldsymbol{\xi}) = \tfrac{1}{4}(1+\xi_1^i\xi_1)(1+\xi_2^i\xi_2)(-1+\xi_1^i\xi_1+\xi_2^i\xi_2)$$

$$N^5(\boldsymbol{\xi}) = \tfrac{1}{2}(1-\xi_1^2)(1-\xi_2)$$

$$N^i(\boldsymbol{\xi}) = \tfrac{1}{2}(1-\xi_1^2)(1+\xi_2^i\xi_2)\,, \text{ for } \xi_1^i = 0$$

$$N^i(\boldsymbol{\xi}) = \tfrac{1}{2}(1+\xi_1^i\xi_1)(1-\xi_2^2)\,, \text{ for } \xi_2^i = 0$$

Serendipity Shape functions

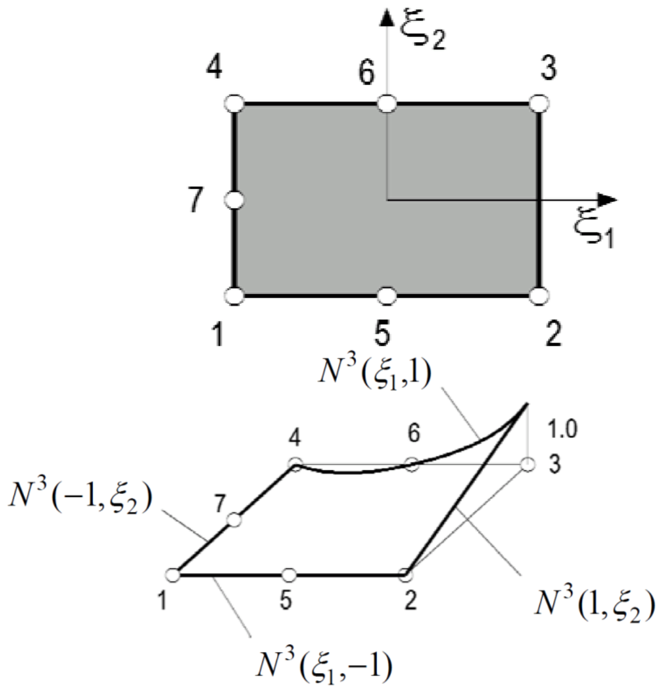
In general serendipity shape functions can be obtained with the following expression:

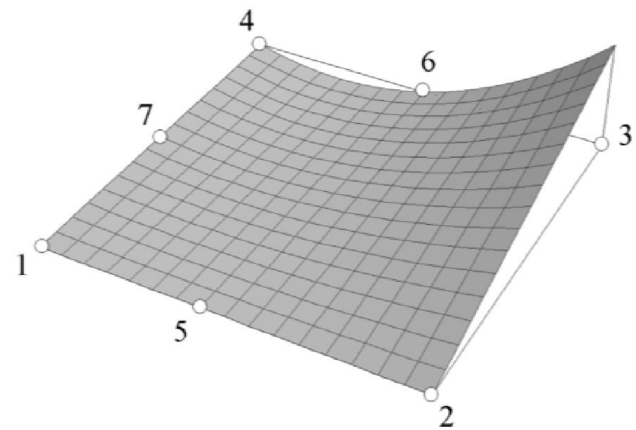
$$
\begin{aligned}
N^i(\xi_1,\xi_2) = {} & \tfrac{1}{2}(1-\xi_2)N^i(\xi_1,-1)+\tfrac{1}{2}(1+\xi_1)N^i(1,\xi_2) \\
& +\tfrac{1}{2}(1+\xi_2)N^i(\xi_1,1)+\tfrac{1}{2}(1-\xi_1)N^i(-1,\xi_2) \\
& -\tfrac{1}{4}(1-\xi_1)(1-\xi_2)N^i(-1,-1)-\tfrac{1}{4}(1+\xi_1)(1-\xi_2)N^i(1,-1) \\
& -\tfrac{1}{4}(1+\xi_1)(1+\xi_2)N^i(1,1)-\tfrac{1}{4}(1-\xi_1)(1+\xi_2)N^i(-1,1)
\end{aligned}
$$

where functions $N^i(\xi_1,-1)$, $N^i(1,\xi_2)$, $N^i(\xi_1,1)$, $N^i(-1,\xi_2)$ are lagrangian interpolations along the corresponding boundary and values $N^i(-1,-1)$, $N^i(1,-1)$, $N^i(1,1)$, $N^i(-1,1)$ have values 0 or 1 and represent values of interpolation on corners

Rectangular elements – Serendipity elements

Example: Find shape function of the node N^3





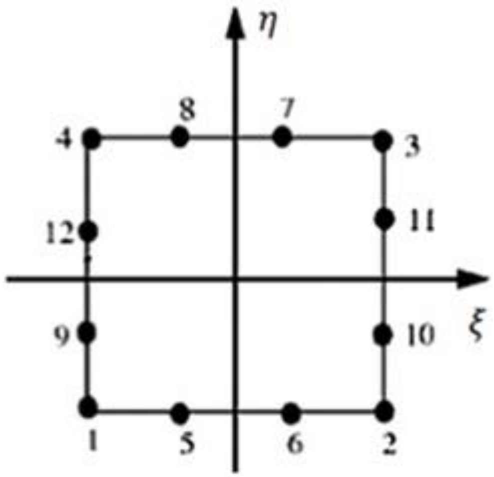
$$N^3(\xi_1,\xi_2)=\tfrac{1}{4}\xi_1(1+\xi_1)(1+\xi_2)$$

$$N^3(\xi_1,\xi_2)=\tfrac{1}{2}(1+\xi_1)N^3(1,\xi_2)+\tfrac{1}{2}(1+\xi_2)N^3(\xi_1,1)-\tfrac{1}{4}(1+\xi_1)(1+\xi_2)N^3(1,1)=$$

$$\tfrac{1}{2}(1+\xi_1)\cdot\tfrac{1}{2}(1+\xi_2)+\tfrac{1}{2}(1+\xi_2)\cdot\tfrac{1}{2}\xi_1(1+\xi_1)-\tfrac{1}{4}(1+\xi_1)(1+\xi_2)\cdot 1=$$

$$\tfrac{1}{4}\xi_1(1+\xi_1)(1+\xi_2)$$

Example: Find cubic serendipity shape function



$$N_5 = \frac{27}{32}(1-\xi^2)(\frac{1}{3}-\xi)(1-\eta)$$

$$N_6 = \frac{27}{32}(1-\xi^2)(\frac{1}{3}+\xi)(1-\eta)$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)-\frac{2}{3}N_5-\frac{1}{3}N_6-\frac{2}{3}N_9-\frac{1}{3}N_{12}$$

$$N_i(\xi,\eta) = \frac{1}{32}(1+\xi\xi_i)(1+\eta\eta_i)\left[9(\xi^2+\eta^2)-10\right], \qquad i=1,2,3,4$$

$$N_i(\xi,\eta) = \frac{9}{32}(1+\xi^2)(1+\eta\eta_i)(1+9\xi\xi_i), \qquad i=5,6,7,8$$

$$N_i(\xi,\eta) = \frac{9}{32}(1+\xi\xi_i)(1-\eta^2)(1+9\eta\eta_i), \qquad i=9,10,11,12$$

Reference:

Zienkiewicz O.C. , Taylor R.L. , Zhu J.Z. : The Finite Element Method: Its Basis and Fundamentals, 6. Edition, Elsevier Butterworth-Heinemann, 2005.