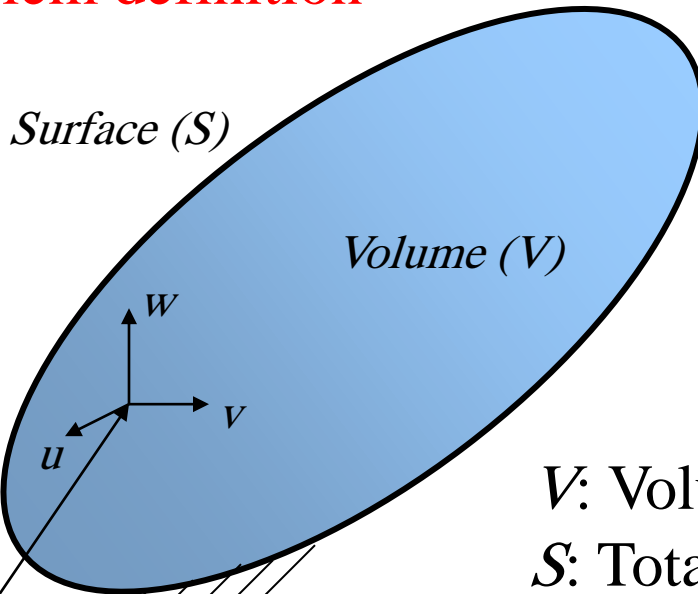




Solid Elements for 3-D Problems

Problem definition



V : Volume of body

S : Total surface of the body

The deformation at point $x = [x, y, z]^T$ is given by the 3 components of its displacement

$$\mathbf{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

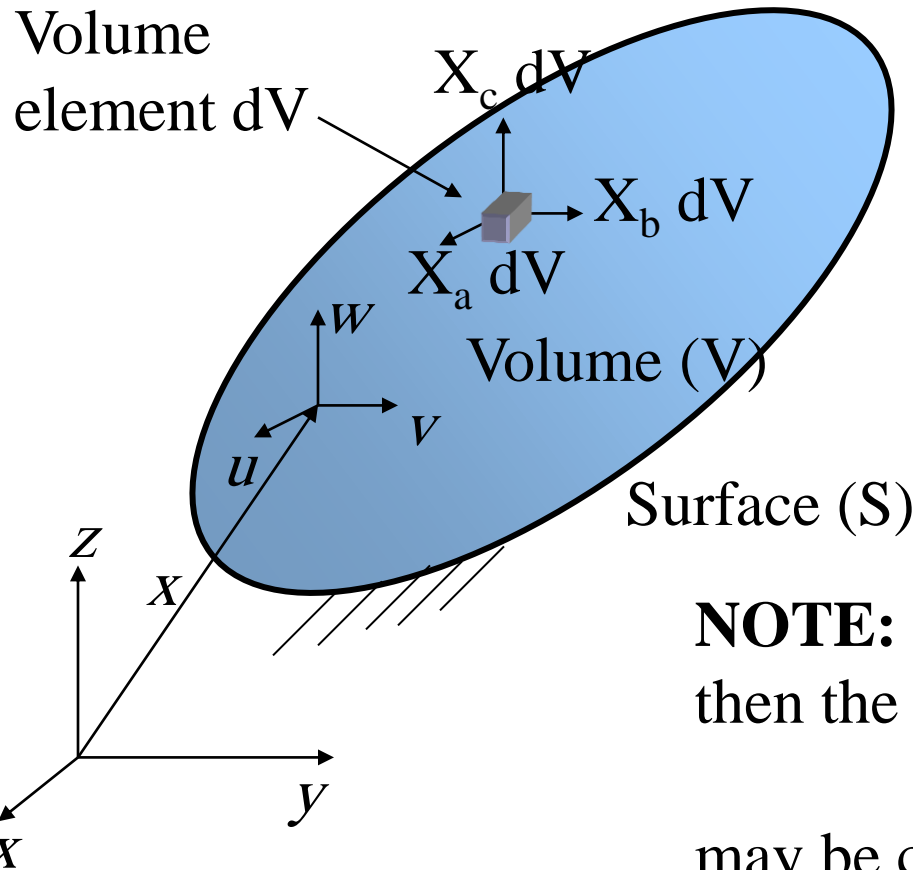


External forces acting on the body

Two basic types of **external forces** act on a body

1. **Body force** (force per unit **volume**) e.g., weight, inertia, etc
2. **Surface traction** (force per unit **surface area**) e.g., friction

Body force



Body force: distributed force per unit volume (e.g., weight, inertia, etc)

$$\mathbf{X} = \begin{Bmatrix} X_a \\ X_b \\ X_c \end{Bmatrix}$$

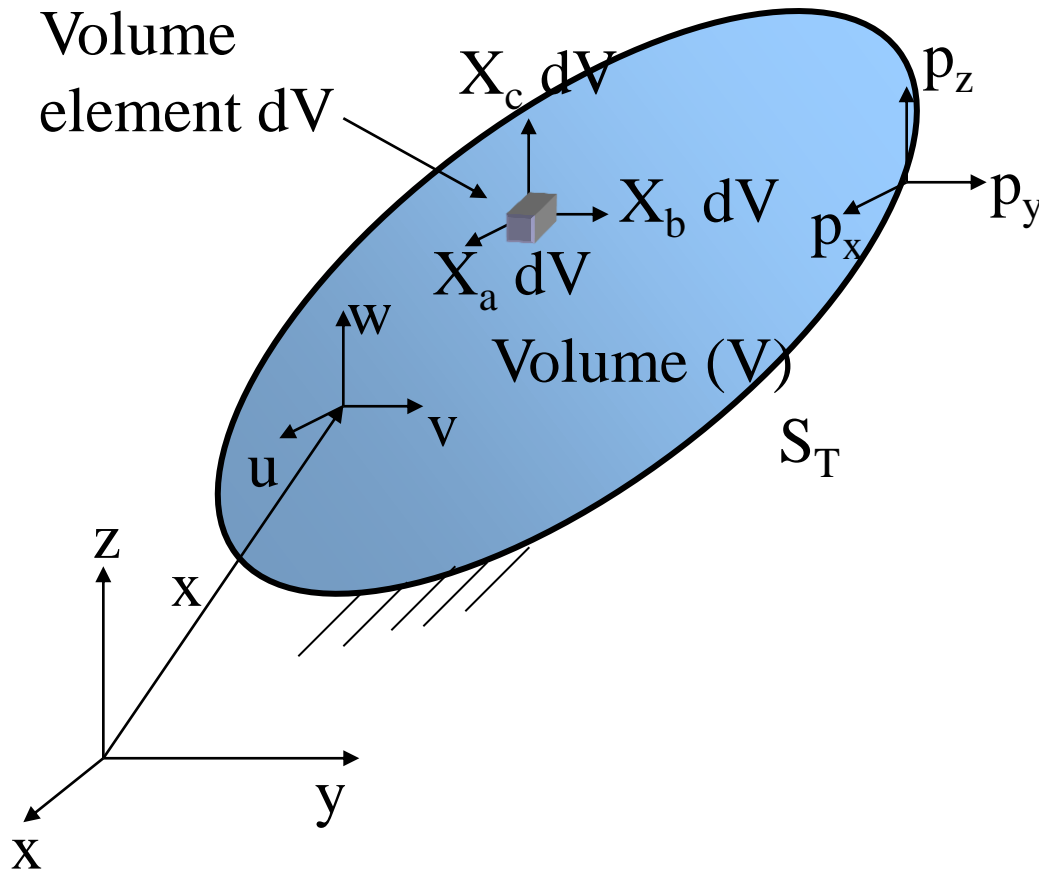
NOTE: If the body is accelerating, then the **inertia force**

$$\rho \ddot{\mathbf{u}} = \begin{Bmatrix} \rho \ddot{u} \\ \rho \ddot{v} \\ \rho \ddot{w} \end{Bmatrix}$$

may be considered as part of \mathbf{X}

$$\mathbf{X} = \tilde{\mathbf{X}} - \rho \ddot{\mathbf{u}}$$

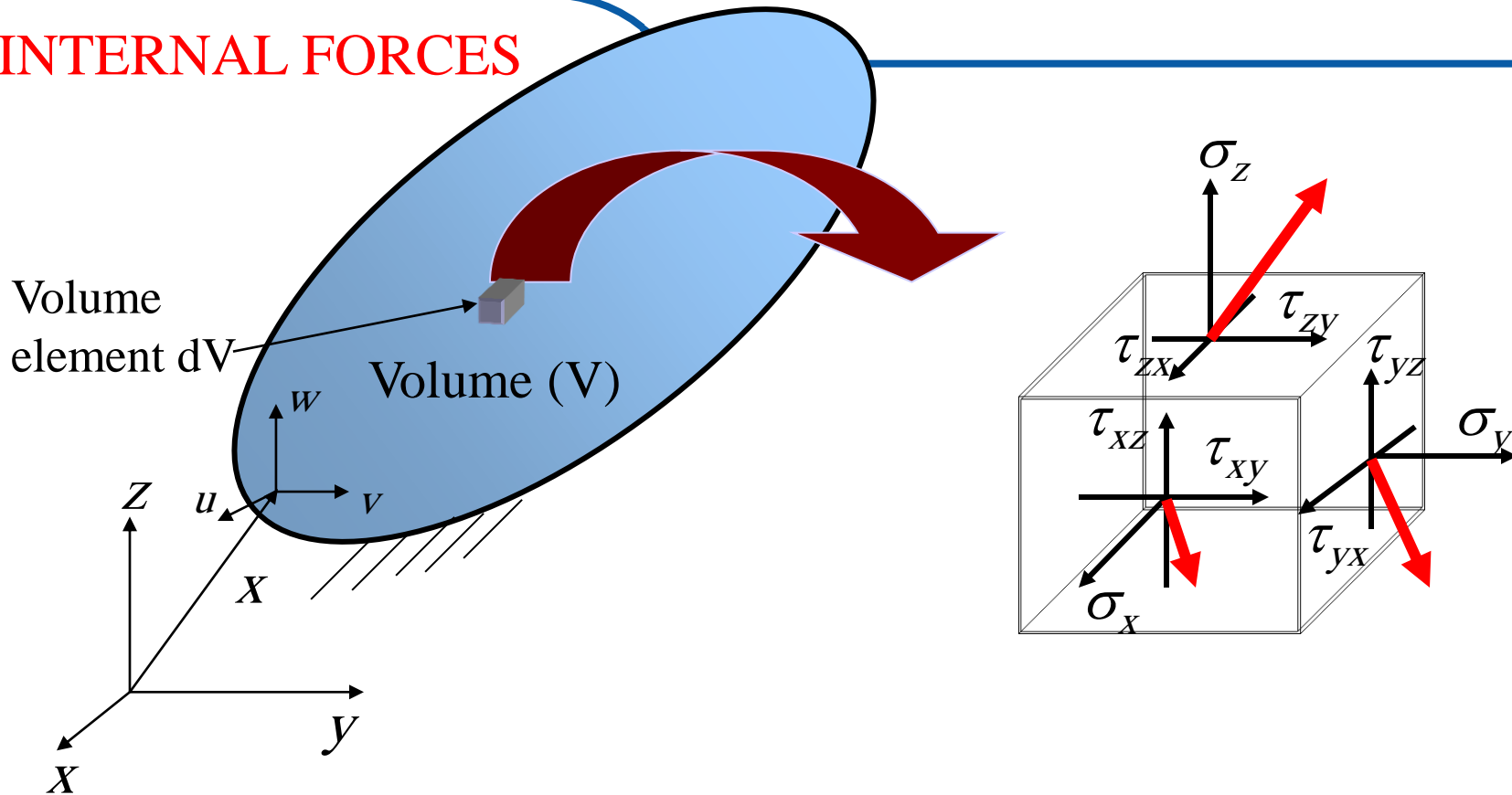
Surface Traction



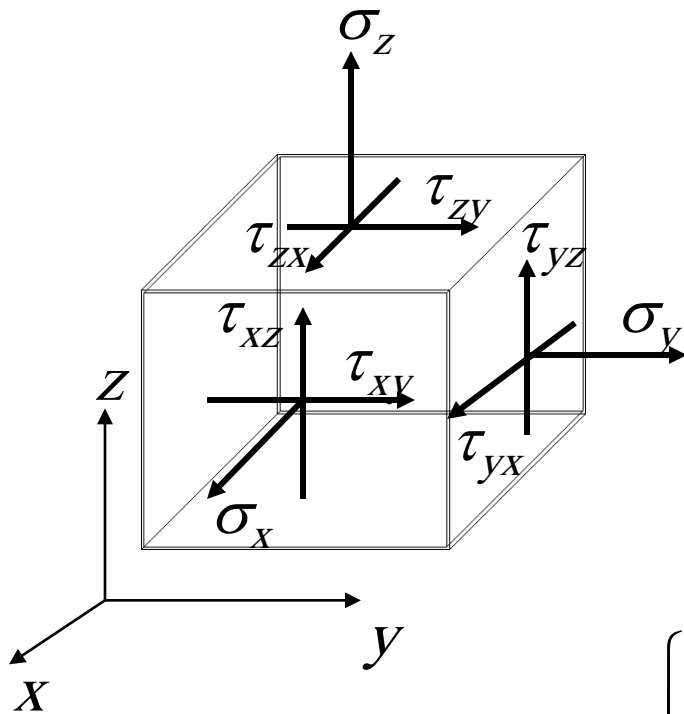
Traction: Distributed force per unit surface area

$$\mathbf{T}_S = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

INTERNAL FORCES



If I take out a chunk of material from the body, I will see that, due to the external forces applied to it, there are reaction forces (e.g., due to the loads applied to a truss structure, internal forces develop in each truss member). For the cube in the figure, the **internal reaction forces per unit area (red arrows)**, on each surface, may be decomposed into three orthogonal components.



Strains: 6 independent **strain components**

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

σ_x , σ_y and σ_z are **normal stresses**.

The rest 6 are the **shear stresses**.
Convention τ_{xy} is the stress on the face perpendicular to the x-axis and points in the +ve y direction. Total of 9 stress components of which only 6 are independent since

The **stress vector** is therefore

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{yz} = \tau_{zy}$$

$$\tau_{zx} = \tau_{xz}$$

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$



Consider the equilibrium of a differential volume element to obtain the **3 equilibrium equations** of elasticity

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X_a = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + X_b = 0$$

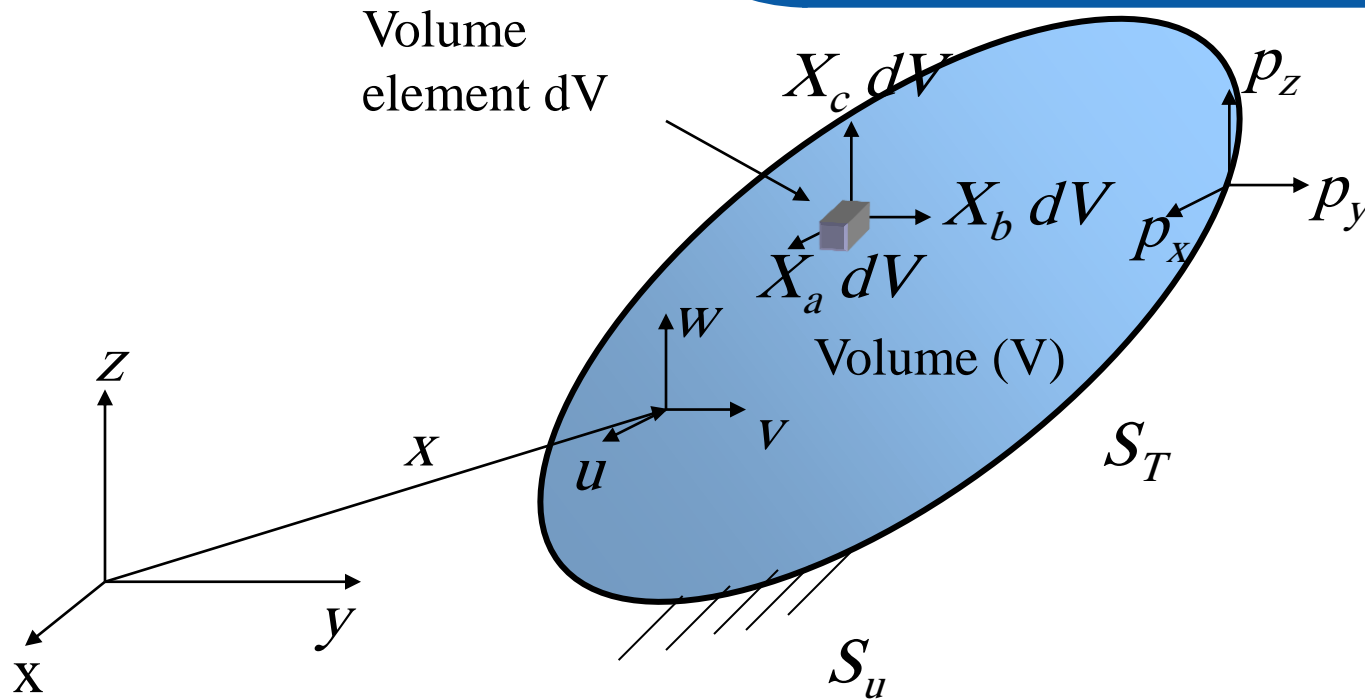
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + X_c = 0$$

$$\sigma_{ij,j} + X_i = 0$$



3D elasticity problem is completely defined once we understand the following three concepts

- Strong formulation (governing differential equation + boundary conditions)
- Strain-displacement relationship
- Stress-strain relationship



1. Strong formulation of the 3D elasticity problem: “Given the externally applied loads (on S_T and in V) and the specified displacements (on S_u) we want to solve for the resultant displacements, strains and stresses required to maintain equilibrium of the body.”



Equilibrium equations:

$$\sigma_{ij,j} + X_i = 0$$

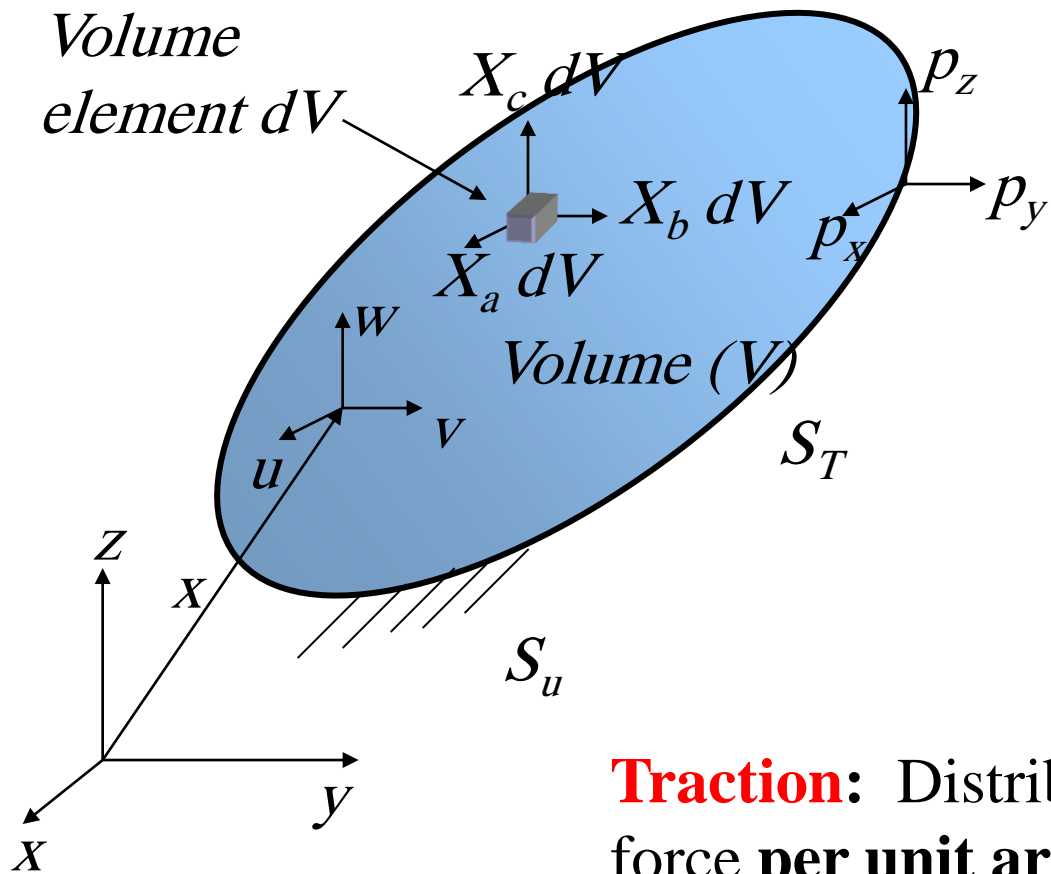
Boundary conditions:

1. Displacement boundary conditions: **Displacements** are specified on portion S_u of the boundary

$$\mathbf{u} = \mathbf{u}^{specified} \quad \text{on } S_u$$

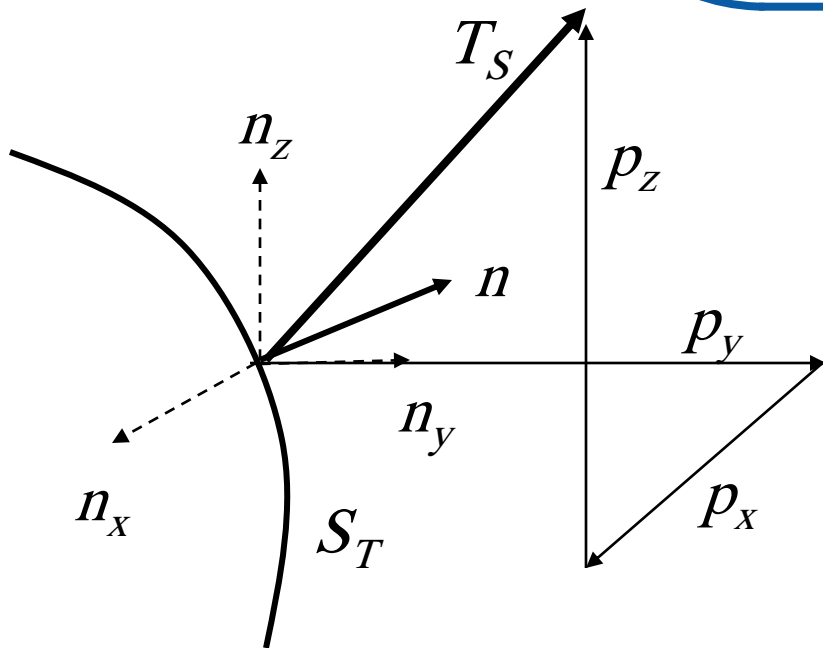
2. Traction (force) boundary conditions: **Tractions** are specified on portion S_T of the boundary

Now, how do I express this mathematically?



Traction: Distributed force per unit area

$$\mathbf{T}_S = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$



$$\mathbf{T}_S = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

If the unit outward normal to S_T :

$$\mathbf{n} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

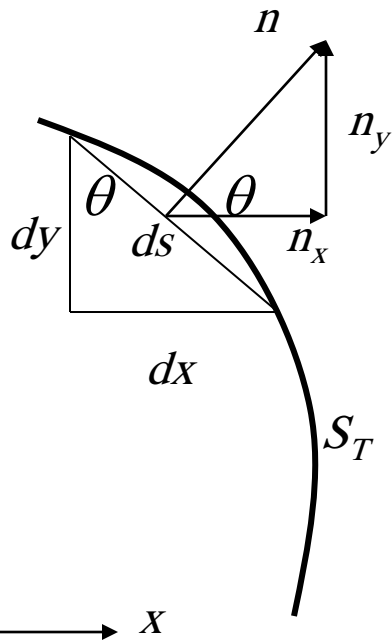
Then

$$p_x = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z$$

$$p_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z$$

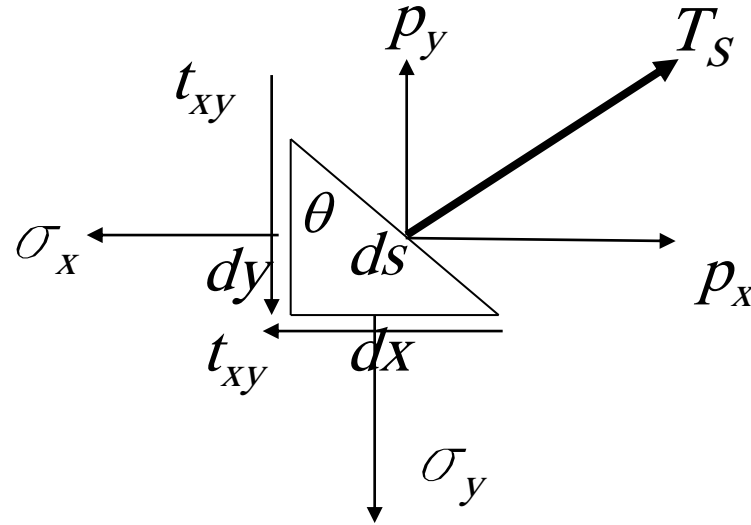
$$p_z = \tau_{xz} n_x + \tau_{zy} n_y + \sigma_z n_z$$

In 2D



$$\sin \theta = \frac{dx}{ds} = n_y$$

$$\cos \theta = \frac{dy}{ds} = n_x$$



Consider the equilibrium of the wedge in x-direction

$$p_x ds = \sigma_x dy + \tau_{xy} dx$$

$$\Rightarrow p_x = \sigma_x \frac{dy}{ds} + \tau_{xy} \frac{dx}{ds}$$

$$\Rightarrow p_x = \sigma_x n_x + \tau_{xy} n_y$$

Similarly

$$p_y = \tau_{xy} n_x + \sigma_y n_y$$



3D elasticity problem is completely defined once we understand the following three concepts

- Strong formulation (governing differential equation + boundary conditions)
- Strain-displacement relationship
- Stress-strain relationship



2. Strain-displacement relationships:

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$



3D elasticity problem is completely defined once we understand the following three concepts

- Strong formulation (governing differential equation + boundary conditions)
- Strain-displacement relationship
- Stress-strain relationship

3. Stress-Strain relationship:

Linear elastic material (Hooke's Law)

$$\sigma = \mathbf{E}\varepsilon$$

Linear elastic isotropic material:

$$\mathbf{E} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$



Displacement Field:

$$u = \sum_{i=1}^N N_i u_i$$

$$v = \sum_{i=1}^N N_i v_i$$

$$w = \sum_{i=1}^N N_i w_i$$

In matrix form:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}_{(3 \times 1)} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \dots \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \dots \end{bmatrix}_{(3 \times 3N)} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \end{Bmatrix}_{(3N \times 1)}$$

or $\mathbf{u} = \mathbf{N} \mathbf{d}$



Using the previous relations, we can derive the strain vector

$$\boldsymbol{\varepsilon} = \mathbf{B} \mathbf{d} \qquad \textit{Stiffness Matrix: } \mathbf{k} = \int_v \mathbf{B}^T \mathbf{E} \mathbf{B} dv$$

$(6 \times 1) \quad (6 \times 3N) \times (3N \times 1)$ $(3N \times 3N) \quad (3N \times 6) \times (6 \times 6) \times (6 \times 3N)$

Numerical quadratures are often needed to evaluate the above integration.

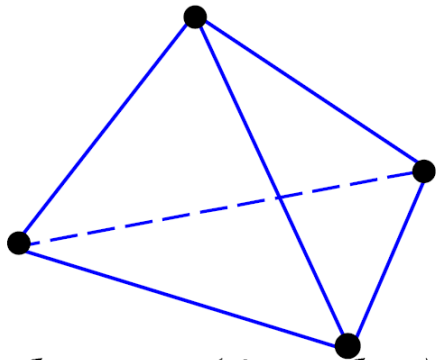
Rigid-body motions for 3-D bodies (6 components): *3 translations*,

3 rotations.

These rigid-body motions (singularity of the system of equations) must be removed from the FEA model to ensure the quality of the analysis.

المان چهار وجهی

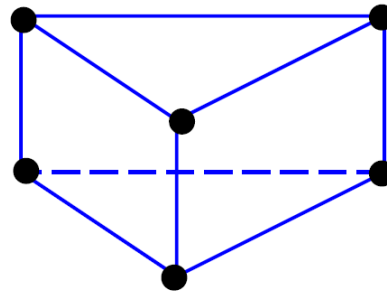
Tetrahedron:



linear (4 nodes)

المان پنج وجهی

Penta:

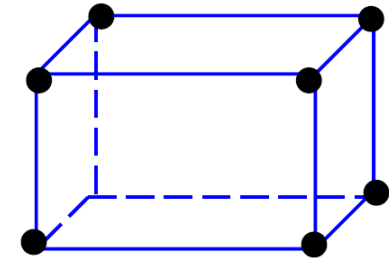


linear (6 nodes)

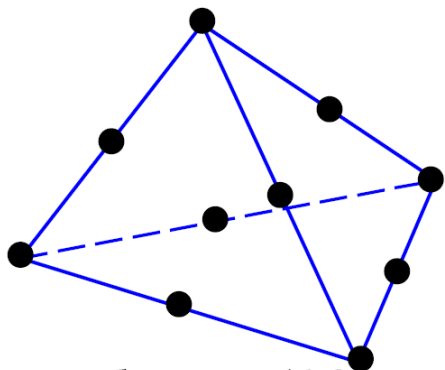
المان شش وجهی

Hexahedron

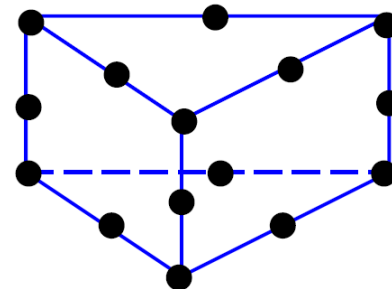
(brick):



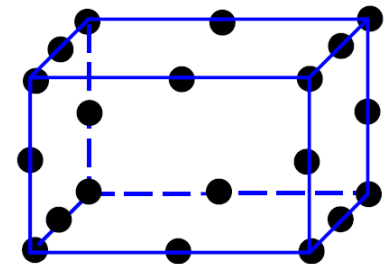
linear (8 nodes)



quadratic (10 nodes)

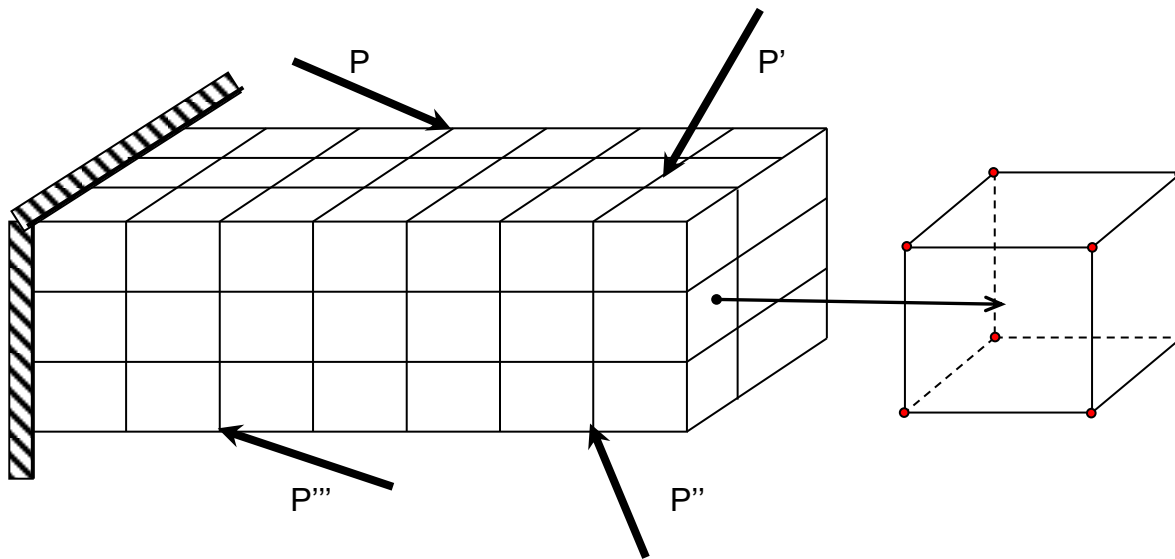


quadratic (15 nodes)



quadratic (20 nodes)

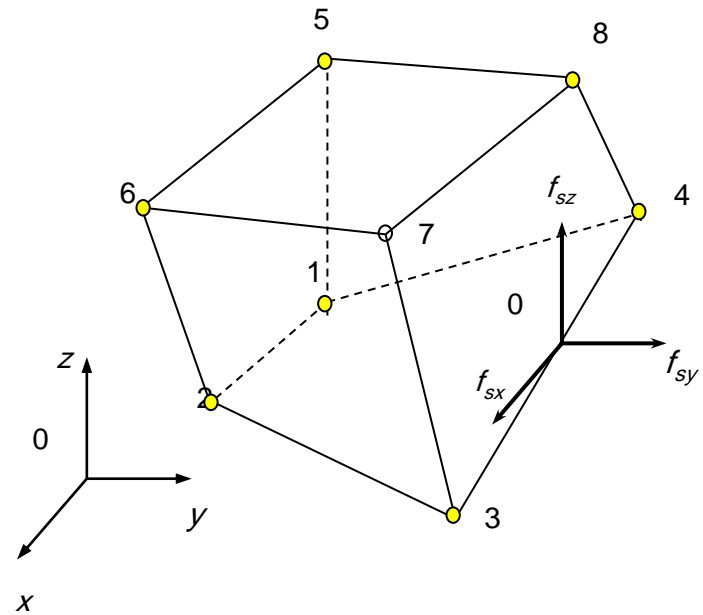
- 3D solid meshed with hexahedron elements



Shape functions

$$\mathbf{U} = \mathbf{N}\mathbf{d}$$

$$\mathbf{d} = \begin{cases} \mathbf{d}_1 & \text{displacement components at node 1} \\ \mathbf{d}_2 & \text{displacement components at node 2} \\ \mathbf{d}_3 & \text{displacement components at node 3} \\ \mathbf{d}_4 & \text{displacement components at node 4} \\ \mathbf{d}_5 & \text{displacement components at node 5} \\ \mathbf{d}_6 & \text{displacement components at node 6} \\ \mathbf{d}_7 & \text{displacement components at node 7} \\ \mathbf{d}_8 & \text{displacement components at node 8} \end{cases}$$

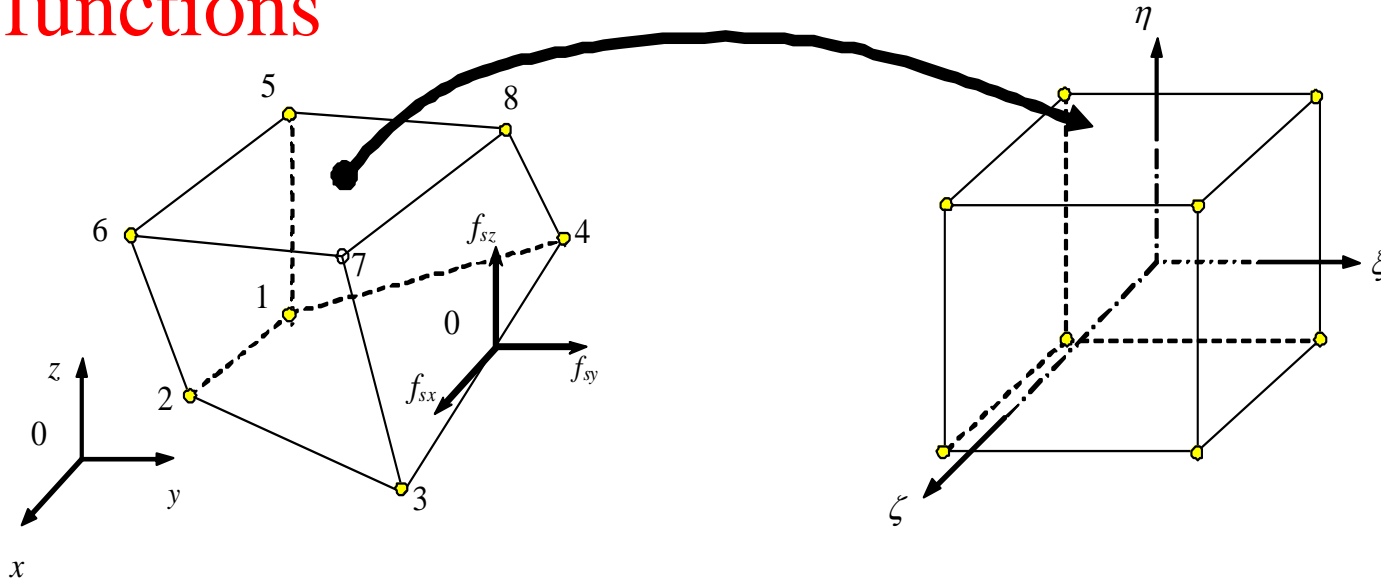


$$\mathbf{d}_i = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix} \quad (i = 1, 2, \dots, 8)$$

$$\mathbf{N}_i = \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \quad (i = 1, 2, \dots, 8)$$

$$\mathbf{N} = [\mathbf{N}_1 \quad \mathbf{N}_2 \quad \mathbf{N}_3 \quad \mathbf{N}_4 \quad \mathbf{N}_5 \quad \mathbf{N}_6 \quad \mathbf{N}_7 \quad \mathbf{N}_8]$$

Shape functions



$$x = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) x_i$$

$$y = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) y_i$$

$$z = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) z_i$$

$$N_i = \frac{1}{8} (1 + \xi \xi_i)(1 + \eta \eta_i)(1 + \zeta \zeta_i)$$

(Tri-linear functions)

Shape functions:

$$N_1(\xi, \eta, \zeta) = \frac{1}{8} (1 - \xi) (1 - \eta) (1 - \zeta) ,$$

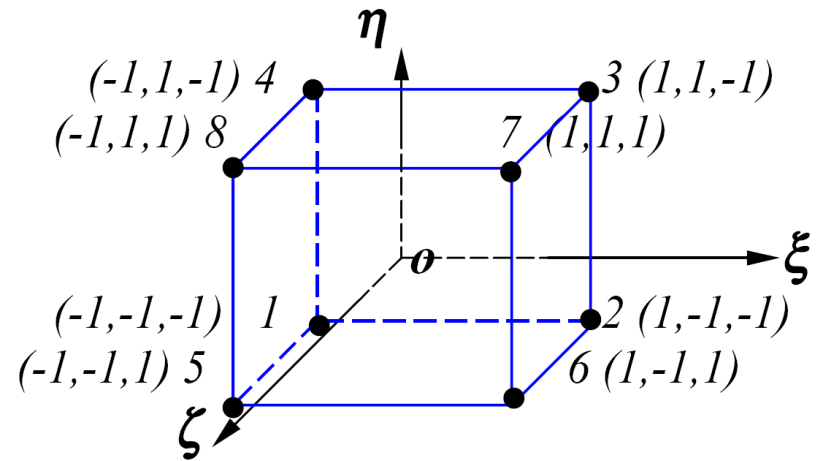
$$N_2(\xi, \eta, \zeta) = \frac{1}{8} (1 + \xi) (1 - \eta) (1 - \zeta) ,$$

$$N_3(\xi, \eta, \zeta) = \frac{1}{8} (1 + \xi) (1 + \eta) (1 - \zeta) ,$$

⋮

⋮

$$N_8(\xi, \eta, \zeta) = \frac{1}{8} (1 - \xi) (1 + \eta) (1 + \zeta) .$$



$$N_i = \frac{1}{8} (1 + \xi_i \xi) (1 + \eta_i \eta) (1 + \zeta_i \zeta)$$

Note that we have the following relations for the shape functions:

$$N_i(\xi_j, \eta_j, \zeta_j) = \delta_{ij} , \quad i, j = 1, 2, \dots, 8. \quad \text{and} \quad \sum_{i=1}^8 N_i(\xi, \eta, \zeta) = 1.$$



Strain matrix

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial x}{\partial x} + \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{Bmatrix} = \dots \text{use (15)} = \mathbf{B} \mathbf{d} \quad \boldsymbol{\varepsilon} = \mathbf{B} \mathbf{d}$$

(6×1) (6×24)×(24×1)



Strain matrix

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d}$$

$$\mathbf{B} = [\mathbf{B}_1 \quad \mathbf{B}_2 \quad \mathbf{B}_3 \quad \mathbf{B}_4 \quad \mathbf{B}_5 \quad \mathbf{B}_6 \quad \mathbf{B}_7 \quad \mathbf{B}_8]$$

$$\mathbf{B}_i = \mathbf{D}\mathbf{N}_i = \begin{bmatrix} \partial N_i / \partial x & 0 & 0 \\ 0 & \partial N_i / \partial y & 0 \\ 0 & 0 & \partial N_i / \partial z \\ 0 & \partial N_i / \partial z & \partial N_i / \partial y \\ \partial N_i / \partial z & 0 & \partial N_i / \partial x \\ \partial N_i / \partial y & \partial N_i / \partial x & 0 \end{bmatrix}$$

Note: Shape functions are expressed in natural coordinates – chain rule of differentiation



Strain matrix

$$\frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \xi} + \frac{\partial N_i}{\partial z} \frac{\partial z}{\partial \xi}$$

$$\frac{\partial N_i}{\partial \eta} = \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \eta} + \frac{\partial N_i}{\partial z} \frac{\partial z}{\partial \eta}$$

$$\frac{\partial N_i}{\partial \zeta} = \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \zeta} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \zeta} + \frac{\partial N_i}{\partial z} \frac{\partial z}{\partial \zeta}$$

Chain rule of differentiation

$$\Rightarrow \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix} \quad \text{where} \quad \mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}$$



Strain matrix

The same shape functions are used as for the displacement field (***Isoparametric element***).

Coordinate Transformation: $x = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) x_i$, $y = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) y_i$, $z = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) z_i$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & \frac{\partial N_5}{\partial \xi} & \frac{\partial N_6}{\partial \xi} & \frac{\partial N_7}{\partial \xi} & \frac{\partial N_8}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} & \frac{\partial N_5}{\partial \eta} & \frac{\partial N_6}{\partial \eta} & \frac{\partial N_7}{\partial \eta} & \frac{\partial N_8}{\partial \eta} \\ \frac{\partial N_1}{\partial \zeta} & \frac{\partial N_2}{\partial \zeta} & \frac{\partial N_3}{\partial \zeta} & \frac{\partial N_4}{\partial \zeta} & \frac{\partial N_5}{\partial \zeta} & \frac{\partial N_6}{\partial \zeta} & \frac{\partial N_7}{\partial \zeta} & \frac{\partial N_8}{\partial \zeta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \\ x_7 & y_7 & z_7 \\ x_8 & y_8 & z_8 \end{bmatrix}$$

or
$$\mathbf{J} = \begin{bmatrix} \sum_{i=1}^8 x_i \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^8 y_i \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^8 z_i \frac{\partial N_i}{\partial \xi} \\ \sum_{i=1}^8 x_i \frac{\partial N_i}{\partial \eta} & \sum_{i=1}^8 y_i \frac{\partial N_i}{\partial \eta} & \sum_{i=1}^8 z_i \frac{\partial N_i}{\partial \eta} \\ \sum_{i=1}^8 x_i \frac{\partial N_i}{\partial \zeta} & \sum_{i=1}^8 y_i \frac{\partial N_i}{\partial \zeta} & \sum_{i=1}^8 z_i \frac{\partial N_i}{\partial \zeta} \end{bmatrix}$$



Strain matrix

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{Bmatrix}$$

Used to replace derivatives w.r.t. x, y, z with derivatives w.r.t. ξ, η, ζ

$$\mathbf{B}_i = \mathbf{D}\mathbf{N}_i = \begin{bmatrix} \partial N_i / \partial x & 0 & 0 \\ 0 & \partial N_i / \partial y & 0 \\ 0 & 0 & \partial N_i / \partial z \\ 0 & \partial N_i / \partial z & \partial N_i / \partial y \\ \partial N_i / \partial z & 0 & \partial N_i / \partial x \\ \partial N_i / \partial y & \partial N_i / \partial x & 0 \end{bmatrix}$$



Strain energy:

$$U = \frac{1}{2} \int_V \boldsymbol{\sigma}^T \boldsymbol{\varepsilon} dV = \frac{1}{2} \int_V (\mathbf{E}\boldsymbol{\varepsilon})^T \boldsymbol{\varepsilon} dV = \frac{1}{2} \int_V \boldsymbol{\varepsilon}^T \mathbf{E} \boldsymbol{\varepsilon} dV$$

$$= \frac{1}{2} \mathbf{d}^T \left[\int_V \mathbf{B}^T \mathbf{E} \mathbf{B} dV \right] \mathbf{d}$$

Element stiffness matrix:

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} dV$$

(24×24) (24×6)×(6×6)×(6×24)

In $\xi\eta\zeta$ coordinates:

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} dV = \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} \mathbf{B}^T \mathbf{E} \mathbf{B} \det[\mathbf{J}] d\xi d\eta d\zeta$$

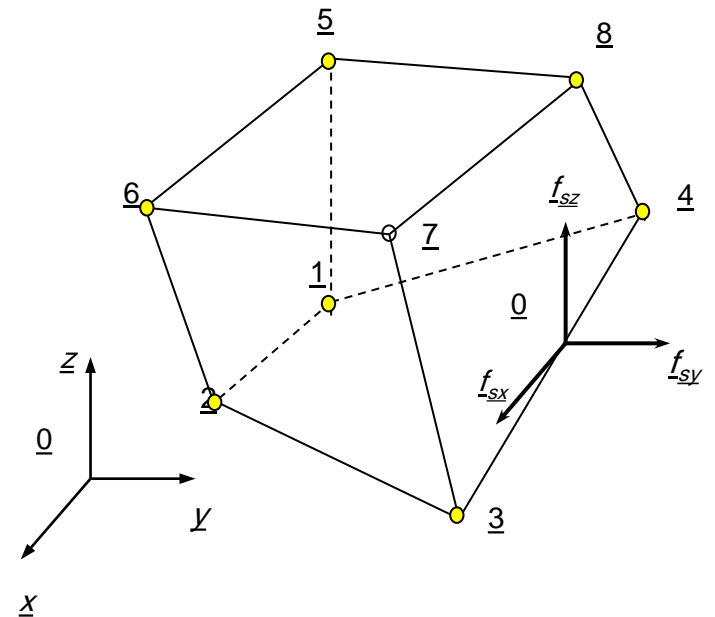
Gauss integration: $I = \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta, \zeta) d\xi d\eta d\zeta = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^l w_i w_j w_k f(\xi_i, \eta_j, \zeta_k)$

Note: 3-D elements usually do not use rotational DOFs.

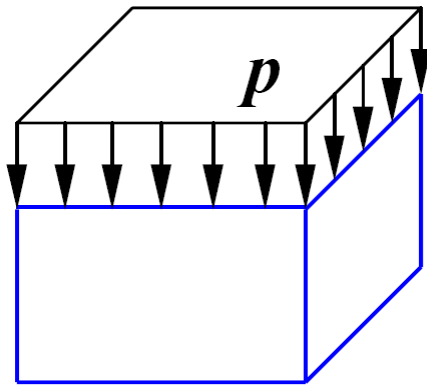
Loads: Distributed loads \rightarrow Nodal forces

$$\mathbf{f}_e = \int_l [\mathbf{N}]^T \Big|_{3-4} \begin{Bmatrix} f_{sx} \\ f_{sy} \\ f_{sz} \end{Bmatrix} dl$$

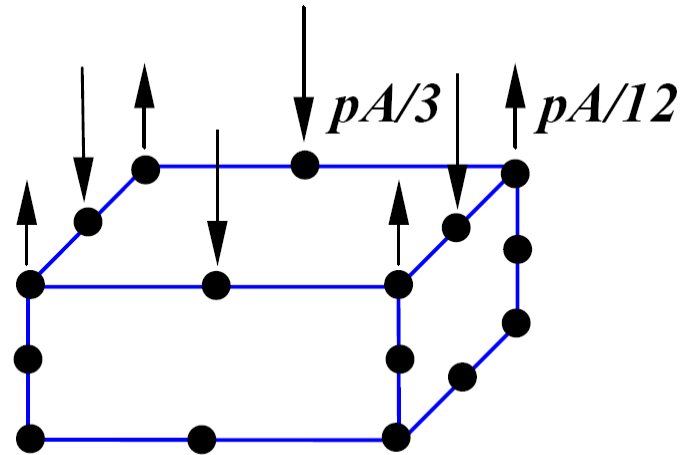
For uniformly distributed load: $\mathbf{f}_e = \frac{1}{2} l_{3-4} \begin{Bmatrix} \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{3 \times 1} \\ f_{sx} \\ f_{sy} \\ f_{sz} \\ f_{sx} \\ f_{sy} \\ f_{sz} \\ \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{3 \times 1} \end{Bmatrix}$



Loads: Distributed loads \rightarrow Nodal forces



$$Area = A$$





Stresses:

$$\boldsymbol{\sigma} = \mathbf{E} \boldsymbol{\varepsilon} = \mathbf{E} \mathbf{B} \mathbf{d}$$

Principal stresses: $\sigma_1, \sigma_2, \sigma_3$.

von Mises stress:
$$\sigma_e = \sigma_{VM} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}.$$

Stresses are evaluated at selected points (including nodes) on each element. Averaging (around a node, for example) may be employed to smooth the field.



Example:

For the rectangular solid element, find the stiffness term k_{11} , assuming isotropic material.

$$[\mathbf{k}_{11}] = abc \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [\mathbf{B}]_{1,1}^T [\mathbf{E}] [\mathbf{B}]_{1,1} d\xi d\eta d\zeta$$

If the hexahedron is rectangular with dimensions of $a \times b \times c$, the determinate of the Jacobian matrix is simply given by

$$\det[\mathbf{J}] = abc = V_e$$

$$[\mathbf{B}]_{1,1} = \begin{Bmatrix} \partial N_1 / \partial x \\ 0 \\ 0 \\ \partial N_1 / \partial y \\ 0 \\ \partial N_1 / \partial z \end{Bmatrix} = -\frac{1}{8} \begin{Bmatrix} \frac{1}{a}(1-\eta)(1-\zeta) \\ 0 \\ 0 \\ \frac{1}{b}(1-\xi)(1-\zeta) \\ 0 \\ \frac{1}{c}(1-\xi)(1-\eta) \end{Bmatrix}$$



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$$[\mathbf{k}_{11}] = abc \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \left(-\frac{1}{8} \right) \left[\begin{array}{cccccc} \frac{1}{a}(1-\eta)(1-\zeta) & 0 & 0 & \frac{1}{b}(1-\xi)(1-\zeta) & 0 & \frac{1}{c}(1-\xi)(1-\eta) \end{array} \right]$$

$$\frac{E}{(1+\nu)(1-2\nu)} \left[\begin{array}{cccccc} f & \nu & \nu & 0 & 0 & 0 \\ \nu & f & \nu & 0 & 0 & 0 \\ \nu & \nu & f & 0 & 0 & 0 \\ 0 & 0 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & 0 & g \end{array} \right] \left(-\frac{1}{8} \right) \left\{ \begin{array}{c} \frac{1}{a}(1-\eta)(1-\zeta) \\ 0 \\ 0 \\ \frac{1}{b}(1-\xi)(1-\zeta) \\ 0 \\ \frac{1}{c}(1-\xi)(1-\eta) \end{array} \right\} d\xi d\eta d\zeta$$

where: $f = 1 - \nu$
 $g = 1 - 2\nu$



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$$[\mathbf{k}_{11}] = \frac{Eabc}{64(1+\nu)(1-2\nu)} \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \left[\begin{array}{ccccccc} \frac{1}{a}(1-\eta)(1-\zeta) & 0 & 0 & \frac{1}{b}(1-\xi)(1-\zeta) & 0 & \frac{1}{c}(1-\xi)(1-\eta) \\ \frac{1}{a}(1-\eta)(1-\zeta)f \\ \frac{1}{a}(1-\eta)(1-\zeta)\nu \\ \frac{1}{a}(1-\eta)(1-\zeta)\nu \\ \frac{1}{b}(1-\xi)(1-\zeta)g \\ 0 \\ \frac{1}{c}(1-\xi)(1-\eta)g \end{array} \right] d\xi d\eta d\zeta$$



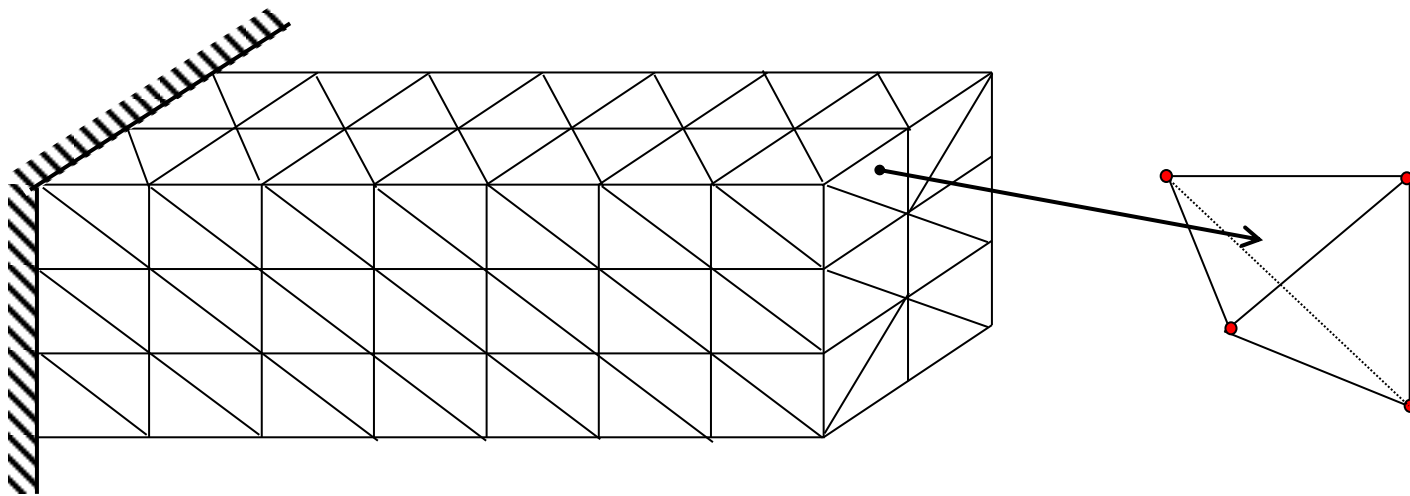
محاسبه ماتریس سختی المان شش وجهی

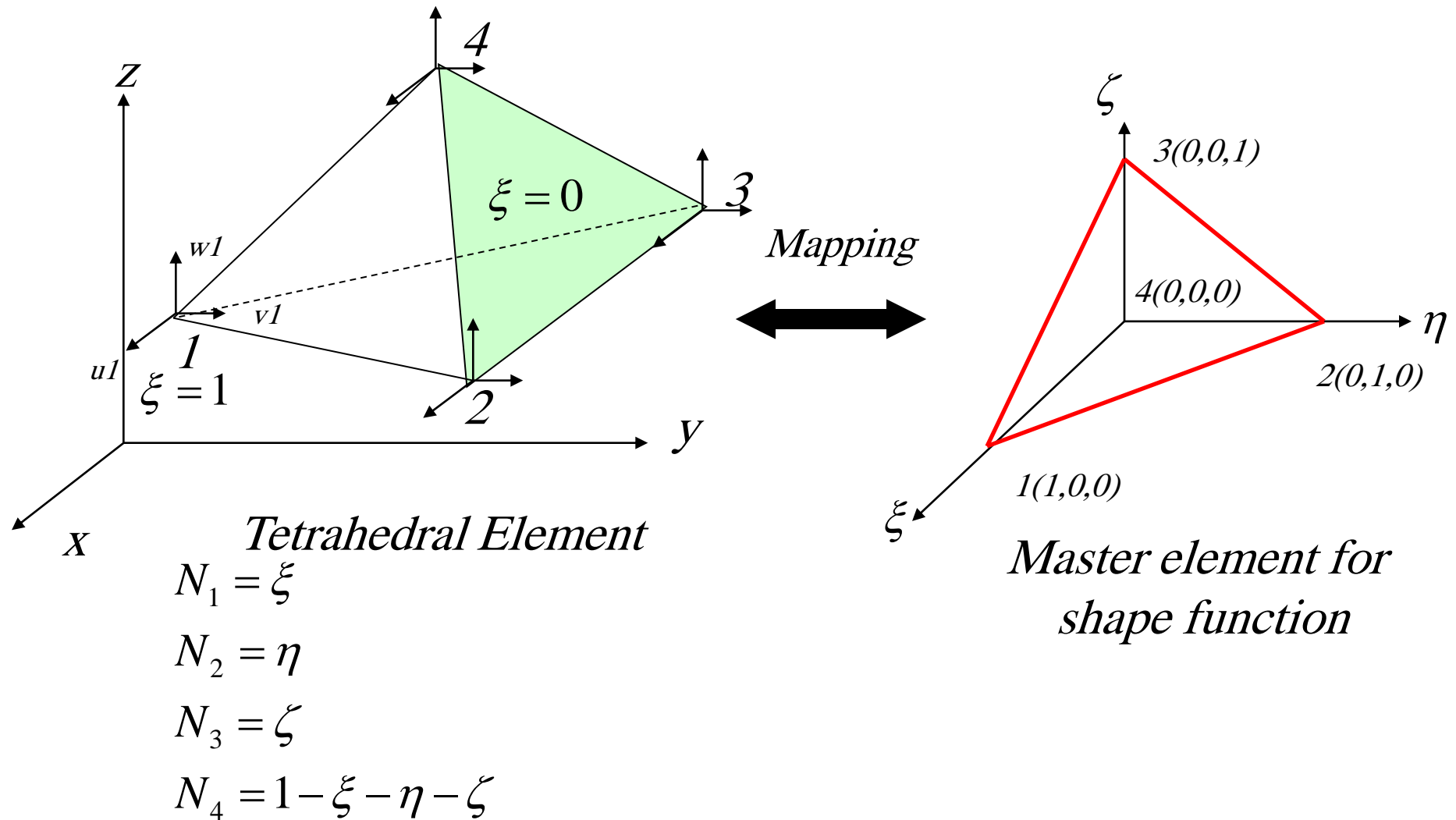
$$[\mathbf{k}_{11}] = \frac{Eabc}{64(1+\nu)(1-2\nu)} \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \left[\frac{1}{a^2} (1-\eta)^2 (1-\zeta)^2 f + \frac{1}{b^2} (1-\xi)^2 (1-\zeta)^2 g + \frac{1}{c^2} (1-\xi)^2 (1-\eta)^2 g \right] d\xi d\eta d\zeta$$

$$\mathbf{k}_{11} = \frac{2abcE}{9(1+\nu)(1-2\nu)} \left(\frac{f}{a^2} + \frac{g}{b^2} + \frac{g}{2c^2} \right)$$

$$\mathbf{k}_{11} = \frac{2abcE}{9(1+\nu)(1-2\nu)} \left(\frac{1-\nu}{a^2} + \frac{1-2\nu}{b^2} + \frac{1-2\nu}{2c^2} \right)$$

- 3D solid meshed with tetrahedron elements





$$\mathbf{u} = \mathbf{N}\mathbf{d}$$

$$\mathbf{N} = \begin{bmatrix} \underbrace{N_1 & 0 & 0}_{\text{node 1}} & \underbrace{N_2 & 0 & 0}_{\text{node 2}} & \underbrace{N_3 & 0 & 0}_{\text{node 3}} & \underbrace{N_4 & 0 & 0}_{\text{node 4}} \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix}$$

$$\mathbf{d} = \{u_1 \ v_1 \ w_1 \ \dots \ u_4 \ v_4 \ w_4\}^T$$

For iso-parametric formulation we have:

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 \quad \text{Or}$$

$$z = N_1 z_1 + N_2 z_2 + N_3 z_3 + N_4 z_4$$

$$x = x_4 + x_{14}\xi + x_{24}\eta + x_{34}\zeta$$

$$y = y_4 + y_{14}\xi + y_{24}\eta + y_{34}\zeta$$

$$z = z_4 + z_{14}\xi + z_{24}\eta + z_{34}\zeta$$



ماتریس سختی المان چهار وجهی خطی

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{Bmatrix} \quad \mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} x_{14} & y_{14} & z_{14} \\ x_{24} & y_{24} & z_{24} \\ x_{34} & y_{34} & z_{34} \end{bmatrix}$$

or

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} \end{Bmatrix} \quad \mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{24}z_{34} - y_{34}z_{24} & y_{34}z_{14} - y_{14}z_{34} & y_{14}z_{24} - y_{24}z_{14} \\ z_{24}x_{34} - z_{34}x_{24} & z_{34}x_{14} - z_{14}x_{34} & z_{14}x_{24} - z_{24}x_{14} \\ x_{24}y_{34} - x_{34}y_{24} & x_{34}y_{14} - x_{14}y_{34} & x_{14}y_{24} - x_{24}y_{14} \end{bmatrix} = \mathbf{A}$$



ماتریس سختی المان چهار وجهی خطی

$$\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{U} = \mathbf{D}\mathbf{N}\mathbf{d} = \mathbf{B}\mathbf{d}$$

$$\mathbf{B} = \mathbf{D}\mathbf{N} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \\ \partial/\partial y & \partial/\partial x & 0 \end{bmatrix} \mathbf{N}$$

where

$$\tilde{A}_1 = A_{11} + A_{12} + A_{13}$$

$$\tilde{A}_2 = A_{21} + A_{22} + A_{23}$$

$$\tilde{A}_3 = A_{31} + A_{32} + A_{33}$$

$$\mathbf{B} = \begin{bmatrix} A_{11} & 0 & 0 & A_{12} & 0 & 0 & A_{13} & 0 & 0 & -\tilde{A}_1 & 0 & 0 \\ 0 & A_{21} & 0 & 0 & A_{22} & 0 & 0 & A_{23} & 0 & 0 & -\tilde{A}_2 & 0 \\ 0 & 0 & A_{31} & 0 & 0 & A_{32} & 0 & 0 & A_{33} & 0 & 0 & -\tilde{A}_3 \\ 0 & A_{31} & A_{21} & 0 & A_{32} & A_{22} & 0 & A_{33} & A_{23} & 0 & -\tilde{A}_3 & -\tilde{A}_2 \\ A_{31} & 0 & A_{11} & A_{32} & 0 & A_{12} & A_{33} & 0 & A_{13} & -\tilde{A}_3 & 0 & -\tilde{A}_1 \\ A_{21} & A_{11} & 0 & A_{22} & A_{12} & 0 & A_{23} & A_{13} & 0 & -\tilde{A}_2 & -\tilde{A}_1 & 0 \end{bmatrix}$$