دانشگاه صنعتی اصفهان

دانشکده مکانیک

The J integral

HRR theory

- **H**utchinson [1968] and **R**ice and **R**osengren [1968] independently evaluated the character of crack tip stress field in the case of power-law hardening materials.
- J characterizes the crack-tip field in a non-linear elastic material.

Assumptions:

- Stress & strain fields near the tip of a stationary crack within plastic zone.
- Consider 2D plane strain / plane stress & Mode I loading.
- Material is characterized by small strain J_2 deformation theory of plasticity.

HRR theory

- For uniaxial deformation:

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha\left(\frac{\sigma}{\sigma_0}\right)^n$$ Ramberg-Osgood equation

σ_0 = yield strength

$\varepsilon_0 = \sigma_0 / E$

α : dimensionless constant

n : strain-hardening exponent

} material properties

Power law relationship assumed between plastic strain and stress.

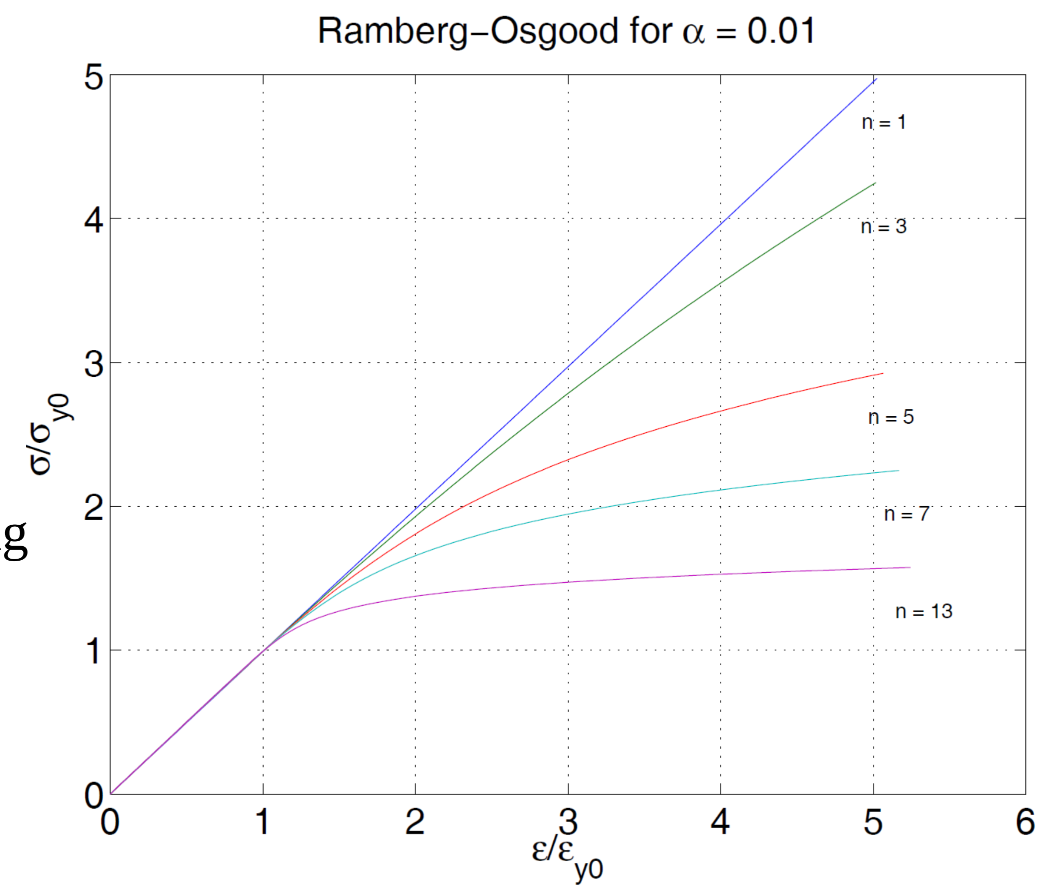
For a linear elastic material $n = 1$.

Ramberg−Osgood model

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha\left(\frac{\sigma}{\sigma_0}\right)^n$$

$$\varepsilon = \varepsilon^{el} + \varepsilon^{pl}$$

- Elastic model: Unlike plasticity unloading in on the same line
- Higher *n* closer to elastic perfectly plastic



Stress-strain relation according to the Ramberg-Osgood material law

HRR solution

Hutchinson, Rice and Rosengren(HRR) solution

- Near crack tip "plastic" strains dominate:

$$\frac{\varepsilon}{\varepsilon_0} = \alpha\left(\frac{\sigma}{\sigma_0}\right)^n \qquad (*)$$

- Assume the following r dependence for σ and ε

$$\sigma = \frac{c_1}{r^x}$$

$$\varepsilon = \frac{c_2}{r^y}$$

1. Bounded energy:

$$\sigma\varepsilon \propto \frac{1}{r} \quad \Rightarrow \quad x + y = 1$$

2. $\varepsilon - \sigma$ relation (*)

$$y = nx$$

$$\Longrightarrow \quad x = \frac{1}{1+n}, \qquad y = \frac{n}{1+n}$$

HRR solution

- Asymptotic field derived by **H**utchinson **R**ice and **R**osengren:

$$\varepsilon_{ij} = A_2 \left(\frac{J}{r} \right)^{n/(n+1)} \qquad \sigma_{ij} = A_1 \left(\frac{J}{r} \right)^{1/(n+1)} \qquad u_i = A_3\, J^{n/(n+1)}\, r^{1/(n+1)}$$

A_i are regular functions that depend on θ and the previous parameters.

The $1/\sqrt{r}$ singularity is recovered when $n = 1$.

Path independence of J $\Rightarrow$ The product $\sigma_{ij}\,\varepsilon_{ij}$ varies as $1/r$:

From $$J = r \int_{-\pi}^{\pi} \left[w(r,\theta)\cos\theta - T_i(r,\theta)\, \frac{\partial u_i(r,\theta)}{\partial x} \right] d\theta$$

$$\sigma_{ij}\,\varepsilon_{ij} \rightarrow \frac{f(\theta)}{r} \quad as \quad r \rightarrow 0$$

J defines the amplitude of the HRR field as K does in the linear case.

HRR solution

- Hutchinson, Rice & Rosengren proposed following form for plastic crack tip fields:

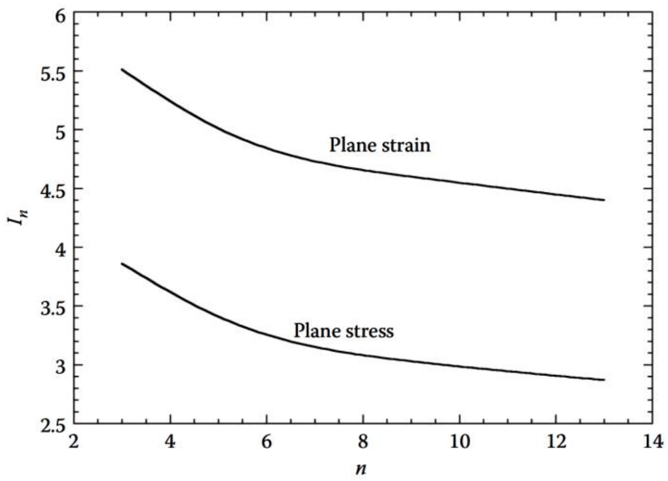
$$\sigma_{ij} = \sigma_0 \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{1/(n+1)} \tilde{\sigma}_{ij}(n,\theta) \qquad \varepsilon_{ij} = \frac{\alpha \sigma_0}{E} \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{n/(n+1)} \tilde{\varepsilon}_{ij}(n,\theta)$$

(see: Appendix 3A.4)

where I_n is an integration constant that depends on n, and $\tilde{\sigma}_{ij}$ and $\tilde{\varepsilon}_{ij}$ are dimensionless functions of n and θ.

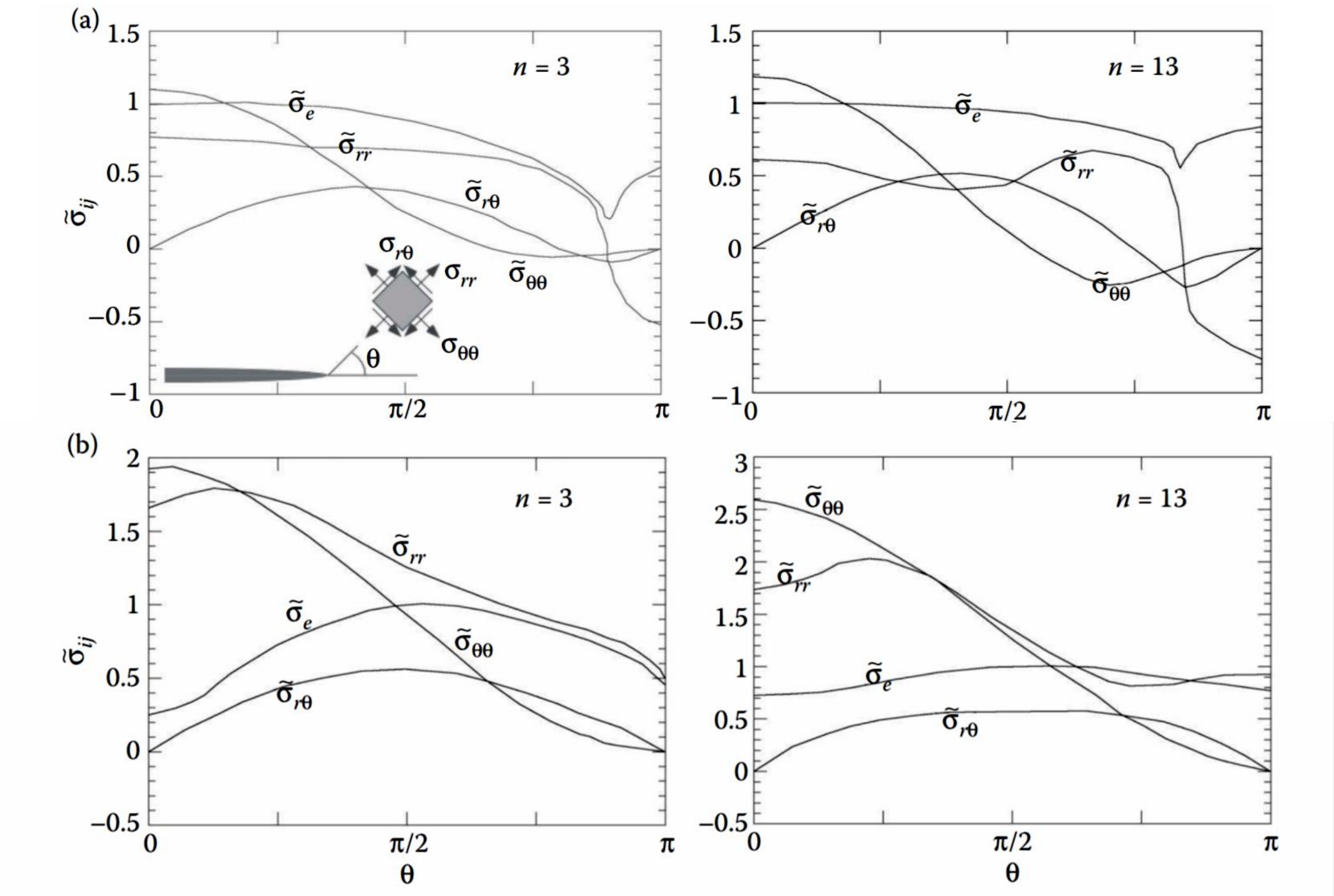
- J defines the amplitude of the HRR field as K does in the linear case.

- The equations are called the HRR singularity, named after Hutchinson, Rice, and Rosengren.



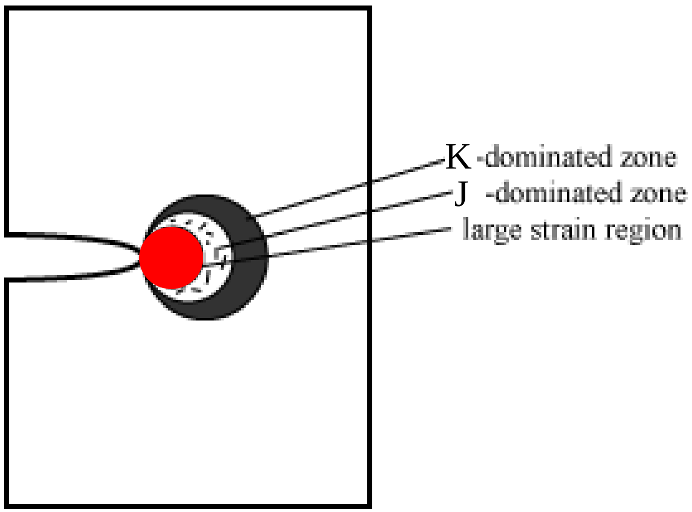
Effect of the strain hardening exponent on the HRR integration constant

HRR solution

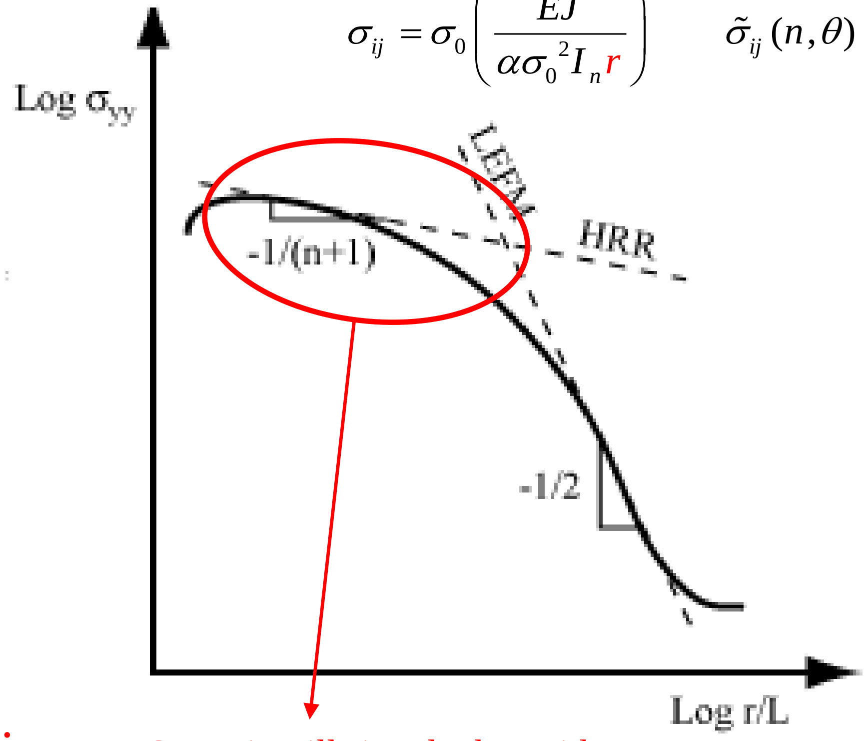


Angular variation of dimensionless stress for n = 3 and 13 (a) plane stress and (b) plane strain.

Two singular zones can be identified:



$$\sigma_{ij} = \sigma_0 \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{1/(n+1)} \tilde{\sigma}_{ij}(n,\theta)$$



Small region where crack blunting occurs.

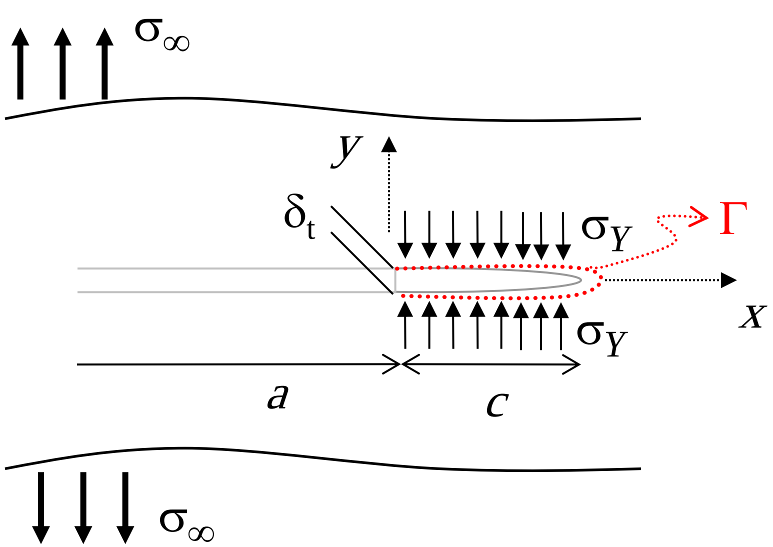
Stress is still singular but with a weaker power of singularity!

Large deformation

HRR based upon small displacements non applicable.

- Relationship between J and CTOD

Consider again the strip-yield problem,



The first term in the J integral vanishes because *dy= 0* (slender zone)

$$J = -\int_{\Gamma} \sigma_{ij} n_j \frac{\partial u_i}{\partial x} ds$$

$$\text{but} \quad \sigma_{ij} n_j \frac{\partial u_i}{\partial x} ds = \sigma_{yy} n_y \frac{\partial u_y}{\partial x} ds = -\sigma_Y \frac{\partial u_y}{\partial x} dx$$

$$J = \int_{\Gamma} \sigma_Y \frac{\partial u_y}{\partial x} dx = \int_{-\delta_t/2}^{\delta_t/2} \sigma_Y du_y = \sigma_Y \delta_t$$

Relationship between J and CTOD

General unique relationship between J and CTOD:

$$J = m\,\sigma_Y\,\delta_t$$

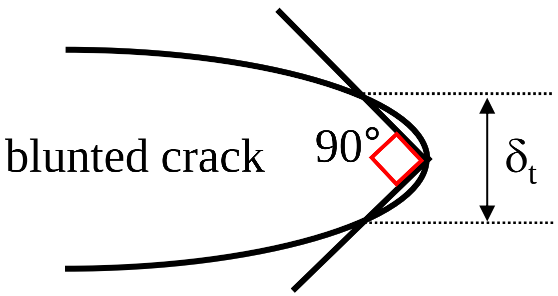
m: dimensionless parameter depending on the stress state and materials properties

• The strip-yield model predicts that m=1 (non-hardening material, plane stress condition)

• This relation is more generally derived for *hardening* materials (n >1) using the HRR displacements near the crack tip, i.e.

$$u_i = A_3\,J^{n/(n+1)}\,r^{1/(n+1)}$$

Shih proposed this definition for δ_t:



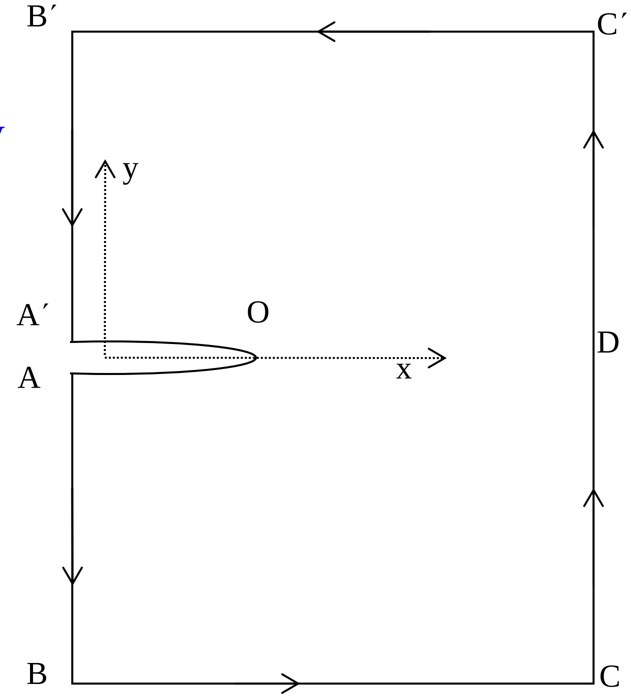
➡ m becomes a (complicated) function of n

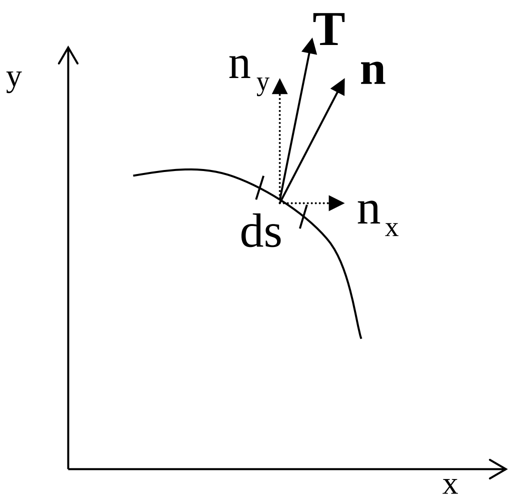
➡ The proposed definition of δ_t agrees with the one of the Irwin model

Moreover, $G = \dfrac{\pi}{4}\sigma_Y\,\delta_t$, $m = \dfrac{\pi}{4}$ in this case

Applications the J-integral

J-integral evaluated explicitly along specific contours





Loads and geometry symmetric / Ox

$$J = \int_{\Gamma} \left(w\,dy - \sigma_{ij} n_j \frac{\partial u_i}{\partial x} ds \right) \quad ?$$

$$w = \int \sigma_{ij} d\varepsilon_{ij} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + 2\sigma_{xy}\varepsilon_{xy}\right)$$

for a plane stress, linear elastic problem

From stress-strain relation,

$$w=\frac{1}{2E}\left(\sigma_{xx}^2+\sigma_{yy}^2-2\nu\sigma_{xx}\sigma_{yy}\right)+\frac{1+\nu}{E}\sigma_{xy}^2$$

Expanded form for $\sigma_{ij}n_j\frac{\partial u_i}{\partial x}ds$

$$=\sigma_{xx}n_x\frac{\partial u_x}{\partial x}ds+\sigma_{xy}n_y\frac{\partial u_x}{\partial x}ds+\sigma_{yx}n_x\frac{\partial u_y}{\partial x}ds+\sigma_{yy}n_y\frac{\partial u_y}{\partial x}ds$$ (2D problem)

Simplification :

Along AB or B´A´ $n_x = -1,\ n_y = 0$ and $ds=-dy \neq 0$

$$=\sigma_{xx}\frac{\partial u_x}{\partial x}dy+\sigma_{yx}n_x\frac{\partial u_y}{\partial x}dy$$

Along CD or DC´ $n_x = 1,\ n_y = 0$ and $ds=dy \neq 0$

$$=\sigma_{xx}\frac{\partial u_x}{\partial x}dy+\sigma_{yx}\frac{\partial u_y}{\partial x}dx$$

Applications the J-integral

Along BC or C´B´

BC : $n_x = 0, n_y = -1$ and $ds=dx \neq 0$

C'B' : $n_x = 0, n_y = 1$ and $ds=-dx \neq 0$

$$=-\sigma_{xy}\frac{\partial u_x}{\partial x}dx-\sigma_{yy}\frac{\partial u_y}{\partial x}dx$$

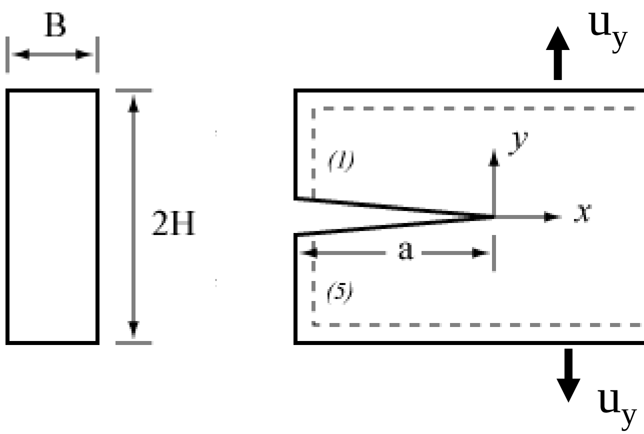
Along OA and A´O J is zero since $dy = 0$ and $T_i = 0$

Finally,

$$J=2\int_A^B\left[w-\sigma_{xx}\frac{\partial u_x}{\partial x}-\sigma_{xy}\frac{\partial u_y}{\partial x}\right]dy+2\int_B^C\left[\sigma_{xy}\frac{\partial u_x}{\partial x}+\sigma_{yy}\frac{\partial u_y}{\partial x}\right]dx+2\int_C^D\left[w-\sigma_{xx}\frac{\partial u_x}{\partial x}-\sigma_{xy}\frac{\partial u_y}{\partial x}\right]dy$$

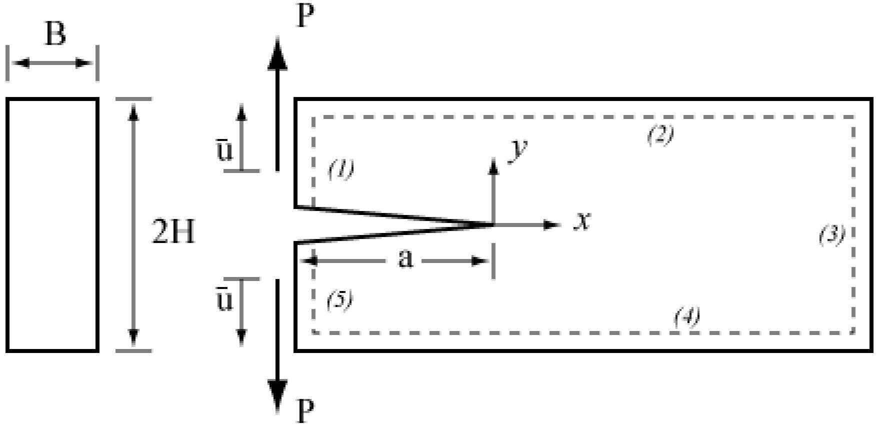
Applications the J-integral

Example 1



$$J = 2hw = \frac{(1-\nu)Eu_y^2}{(1+\nu)(1-2\nu)h}$$

Example 2

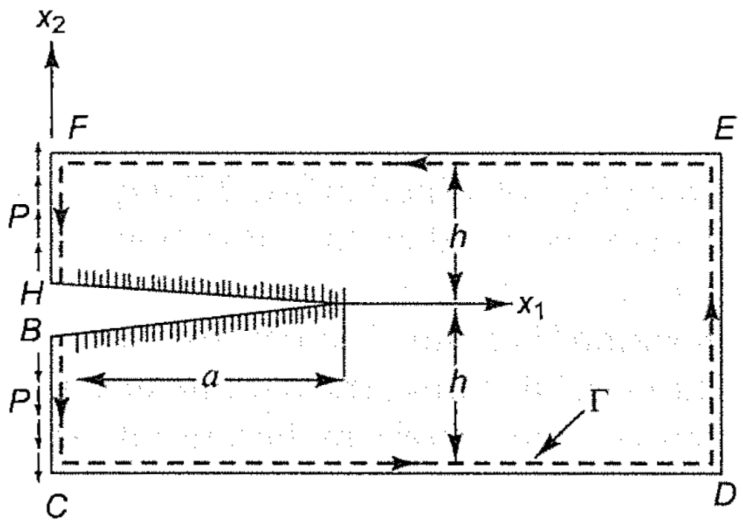


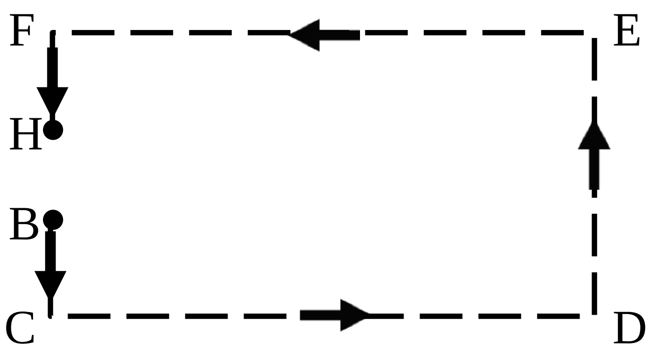
$$J = \frac{12P^2a^2}{EB^2h^3}$$

Applications the J-integral

Example 3:

J integral for double cantilever beam, if each cantilever is pulled by a distributed load P, as shown

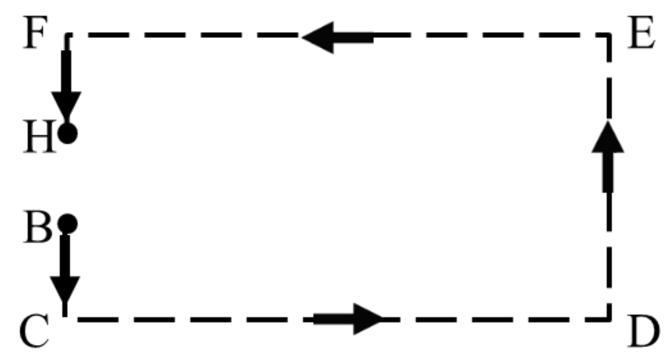




The chosen path Γ is BCDEFH and it coincides with the body contour.

Applications the J-integral

Contour of the crack; therefore, $J = J_{BC} + J_{CD} + J_{DE} + J_{EF} + J_{FH}$



J_{BC} : As bending moment is zero, bending stress is zero. So, $w=0$

$$J_{BC} = \int_{BC} \left(w\, dy - T_i \frac{\partial u_i}{\partial x} ds \right)$$

$$= \int_{BC} \left(0 - T_i \frac{\partial u_i}{\partial x} ds \right)$$

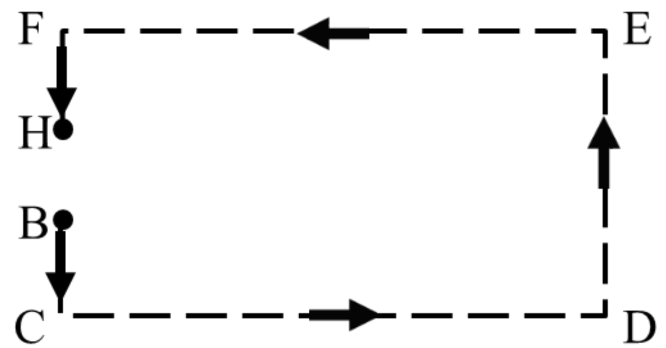
$$= -\int_{BC} T_i \frac{\partial u_i}{\partial x} ds$$

$$J_{BC} = -\int_0^h T \frac{\partial v}{\partial x} dy$$

Assuming:

- A small element of length dy on the path then $ds = dy$
- Along y-direction $u_i = v$
- Length is h so limits are 0 to h.

Applications the J-integral



Contour of the crack; therefore, $J=J_{BC}+J_{CD}+J_{DE}+J_{EF}+J_{FH}$

J_{CD} :

$$J_{CD} = \int_{CD} \left(w\, dy - T_i \frac{\partial u_i}{\partial x} ds \right)$$

dy is negligible and $T_i = 0$ ⇒ $J_{CD} = 0$

J_{EF} :

$$J_{EF} = \int_{EF} \left(w\, dy - T_i \frac{\partial u_i}{\partial x} ds \right)$$

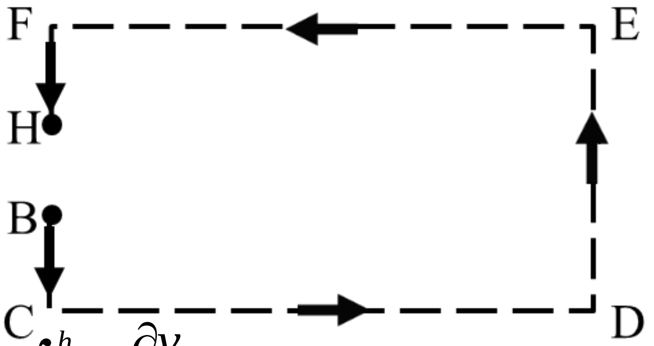
dy is negligible and $T_i = 0$ ⇒ $J_{EF} = 0$

J_{DF} :

$$J_{DE} = \int_{DE} \left(w\, dy - T_i \frac{\partial u_i}{\partial x} ds \right)$$

stresses are very small, which in turn, make w and T_i negligible. ⇒ $J_{DE} = 0$

Applications the J-integral



Contour of the crack; therefore, $J=J_{BC}+J_{CD}+J_{DE}+J_{EF}+J_{FH}$

J_{FH}:
$$J_{FH} = \int_{FH} \left(w\, dy - T_i \frac{\partial u_i}{\partial x} ds \right)$$

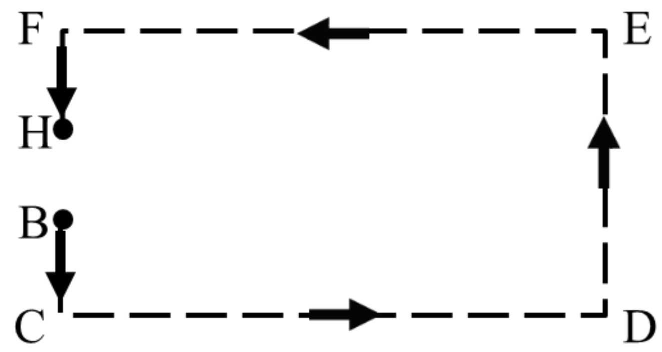
On segments BC and FH, w is negligible, ⟹ $J_{FH} = -\int_0^h T \frac{\partial v}{\partial x} dy$

Hence
$$J = J_{BC} + J_{CD} + J_{DE} + J_{EF} + J_{FH}$$
$$J = -\int_0^h T \frac{\partial v}{\partial x} dy + 0 + 0 + 0 - \int_0^h T \frac{\partial v}{\partial x} dy$$
$$= -2\int_0^h T \frac{\partial v}{\partial x} dy$$

Now we can find $\frac{\partial v}{\partial x}$ using the bending moment equation; Bending moment = P*x

$$\frac{\partial^2 v}{\partial x^2} = \frac{Px}{EI} \Rightarrow \frac{\partial v}{\partial x} = \frac{P}{EI}\frac{x^2}{2} + c \quad \left(\text{at } x=a,\ \frac{\partial v}{\partial x} = 0\right) \Rightarrow c = -\frac{P}{EI}\frac{a^2}{2} \Rightarrow \frac{\partial v}{\partial x} = \frac{P}{EI}\frac{x^2}{2} - \frac{P}{EI}\frac{a^2}{2}$$

Applications the J-integral



Contour of the crack; therefore, $J=J_{BC}+J_{CD}+J_{DE}+J_{EF}+J_{FH}$

(at $x=0$, $\frac{\partial v}{\partial x} = -\frac{P}{EI}\frac{a^2}{2}$) and $I = \frac{Bh^3}{12}$

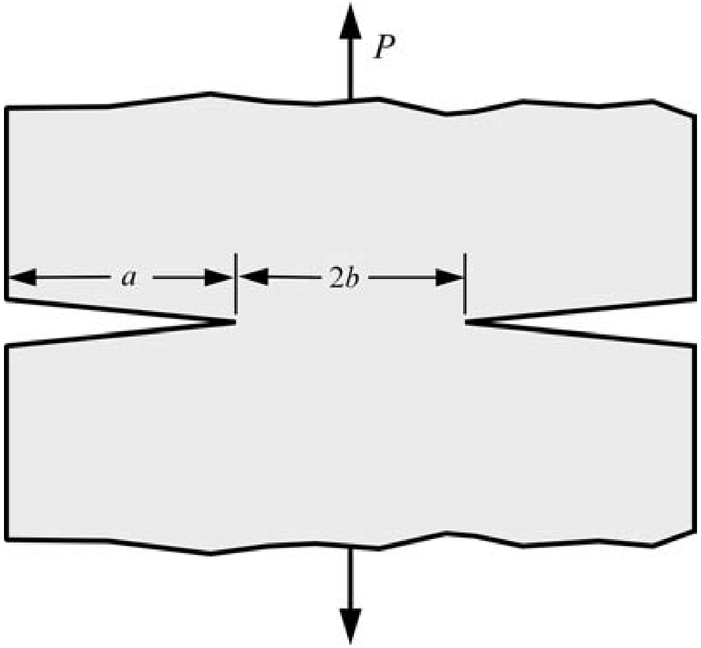
$$\Rightarrow \quad \frac{\partial v}{\partial x} = -\frac{6Pa^2}{EBh^3}$$

Hence $$J = -2\int_0^h T*(-\frac{6Pa^2}{EBh^3})dy$$

$$= \frac{12Pa^2}{EBh^3}\int_0^h Tdy$$

But on face FH: $B\int_0^h Tdy = P$ $\Rightarrow$ $J = \frac{12P^2a^2}{EB^2h^3}$

Double-edge-notched tension (DENT)



$$J=\frac{1}{2}\int_0^P\left(\frac{\partial\Delta}{\partial a}\right)_P dP=-\frac{1}{2}\int_0^P\left(\frac{\partial\Delta}{\partial b}\right)_P dP$$

$$\Delta=\Delta_{el}+\Delta_p$$

$$J=-\frac{1}{2}\int_0^P\left[\left(\frac{\partial\Delta_{el}}{\partial b}\right)_P+\left(\frac{\partial\Delta_p}{\partial b}\right)_P\right]dP$$

$$=\frac{K_I^2}{E'}-\frac{1}{2}\int_0^P\left(\frac{\partial\Delta_p}{\partial b}\right)_P dP$$

Assume that b is the only length dimension that influences Δ_p.

For plane stress

E'=E

For plane strain

E'=E/(1-v^2)

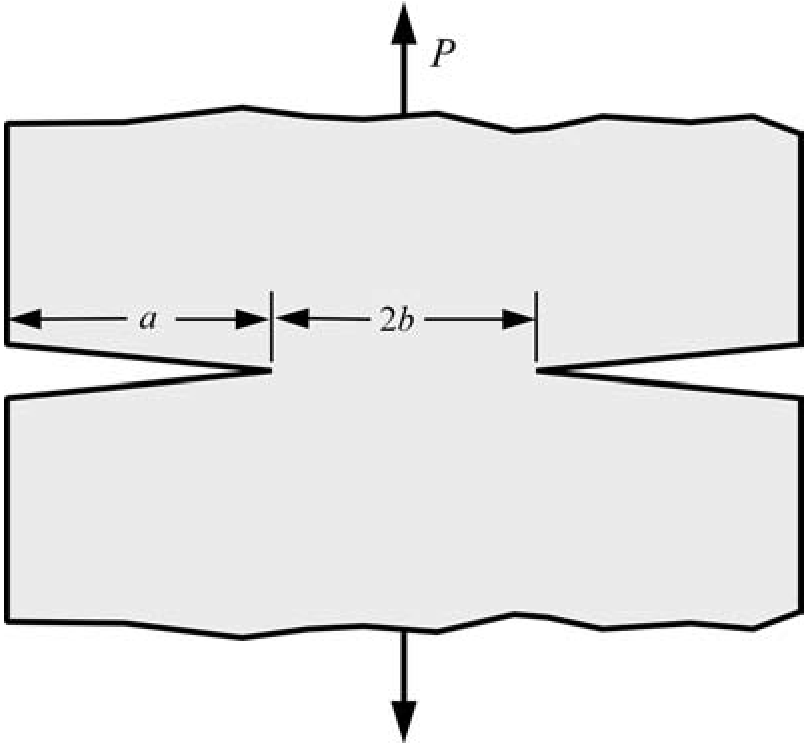
$$\Delta_p=bH\left(\frac{P}{b}\right)\Rightarrow\left(\frac{\partial\Delta_p}{\partial b}\right)_P=H\left(\frac{P}{b}\right)-H'\left(\frac{P}{b}\right)\frac{P}{b}$$

$$\left(\frac{\partial\Delta_p}{\partial P}\right)_b=H'\left(\frac{P}{b}\right)$$

$$\left(\frac{\partial\Delta_p}{\partial b}\right)_P=\frac{1}{b}\left[\Delta_p-P\left(\frac{\partial\Delta_p}{\partial P}\right)_b\right]$$

Applications the J-integral

Double-edge-notched tension (DENT)



$$\left(\frac{\partial \Delta_p}{\partial b}\right)_P = \frac{1}{b}\left[\Delta_p - P\left(\frac{\partial \Delta_p}{\partial P}\right)_b\right]$$

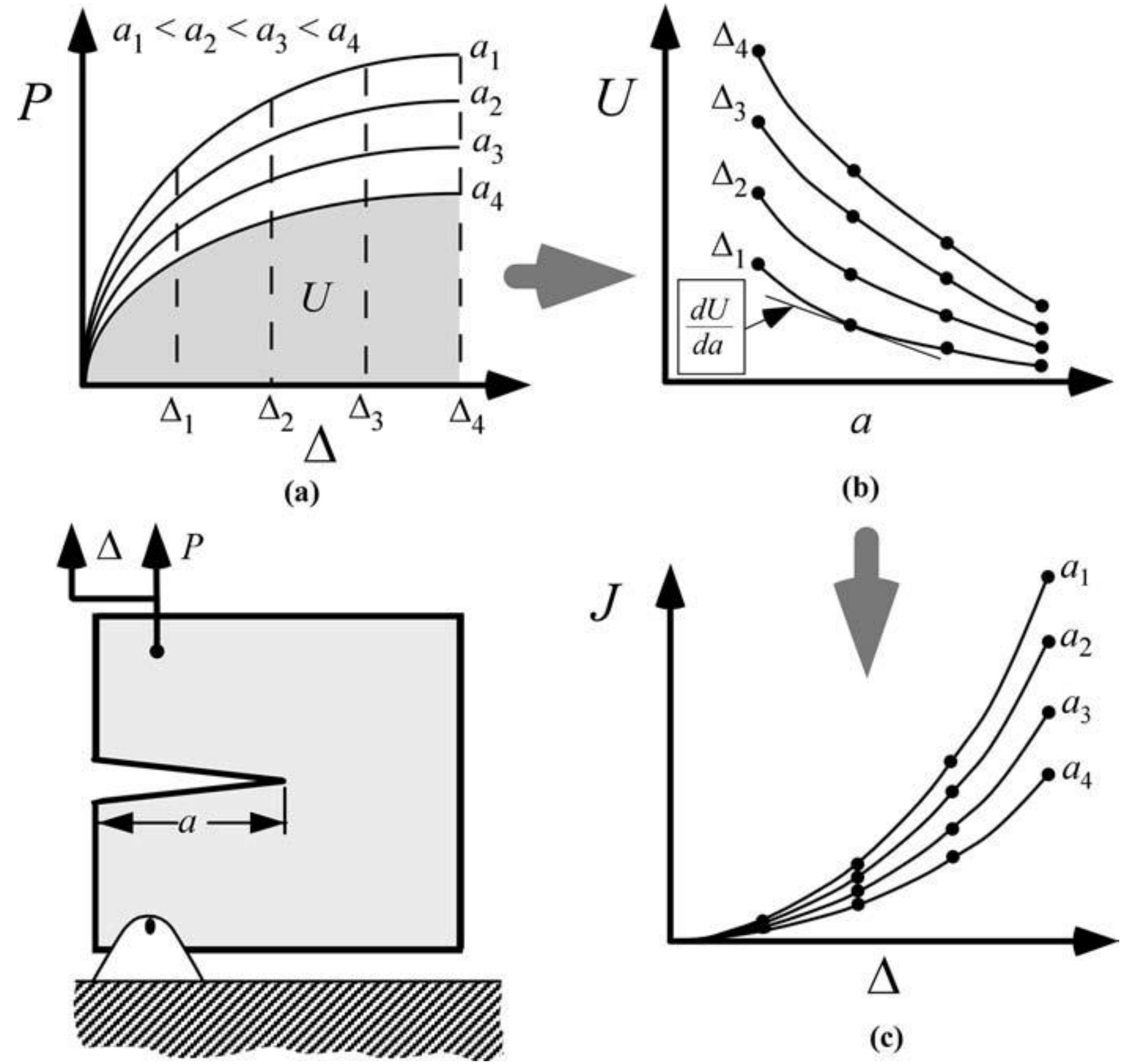
$$J = -\frac{1}{2}\int_0^P \left[\left(\frac{\partial \Delta_{el}}{\partial b}\right)_P + \left(\frac{\partial \Delta_p}{\partial b}\right)_P\right] dP$$

$$= \frac{K_I^2}{E'} - \frac{1}{2}\int_0^P \left(\frac{\partial \Delta_p}{\partial b}\right)_P dP$$

$$J = \frac{K_I^2}{E'} + \frac{1}{2b}\left[2\int_0^{\Delta_p} P d\Delta_p - P\Delta_p\right]$$

Laboratory Measurement

Laboratory Measurement



Computing the J integral is somewhat difficult when the material is nonlinear

U vs. crack length
at various fixed displacements

$$J = -\frac{1}{B}\left(\frac{\partial U}{\partial a}\right)_{\Delta}$$