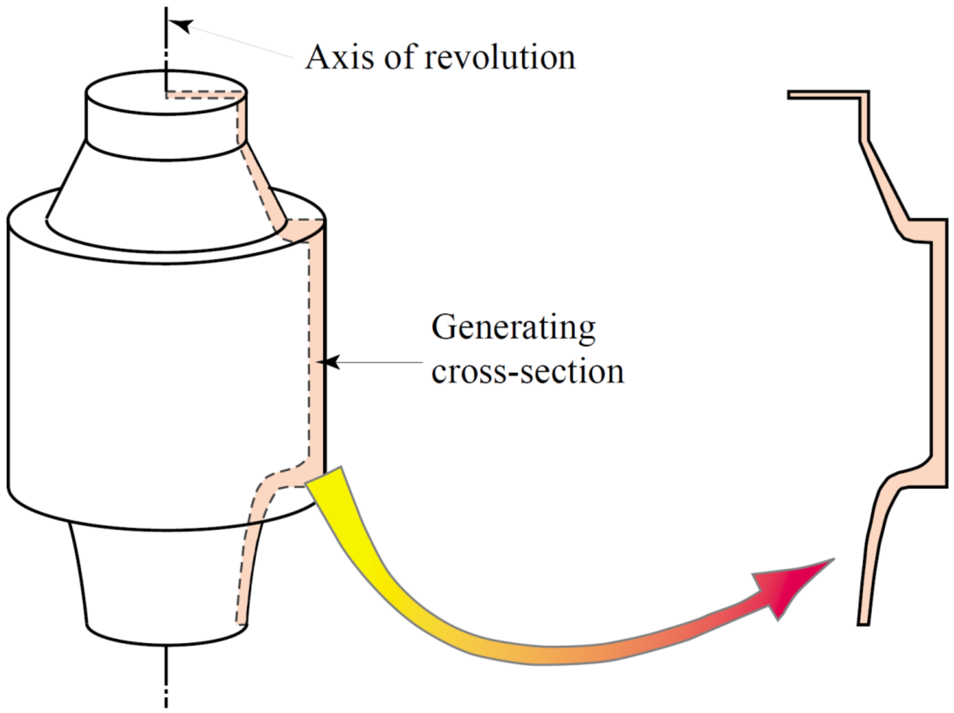
Solids of Revolution
(Axisymmetric Solids)

المان تقارن محوری

تعریف:



A structure of revolution is generated by rotating a generating cross section about an axis of revolution.

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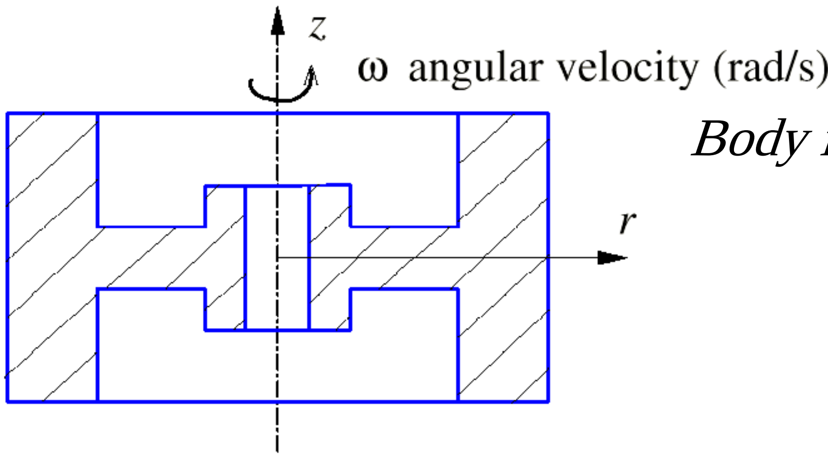
مثال و کاربرد:

Rotating Flywheel:

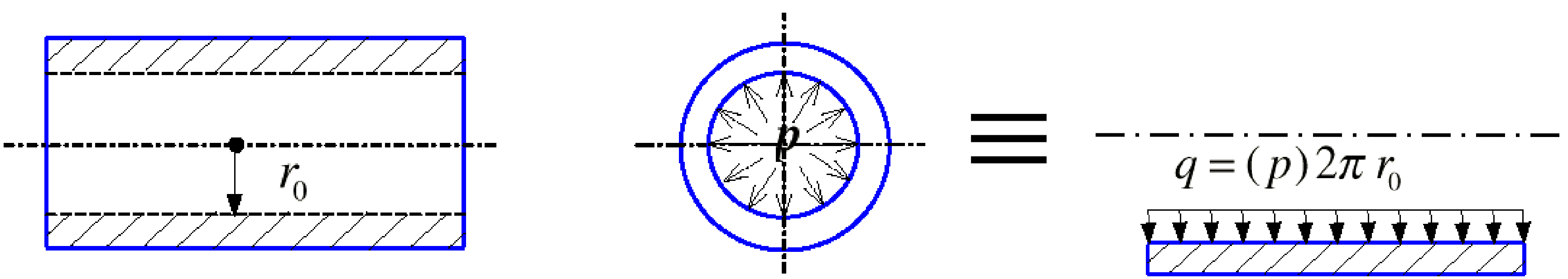


Body forces:

$$f_r = \rho r \omega^2$$ (equivalent radial centrifugal or inertial force)

$$f_z = -\rho g$$ (gravitational force)

Cylinder Subject to Internal Pressure:



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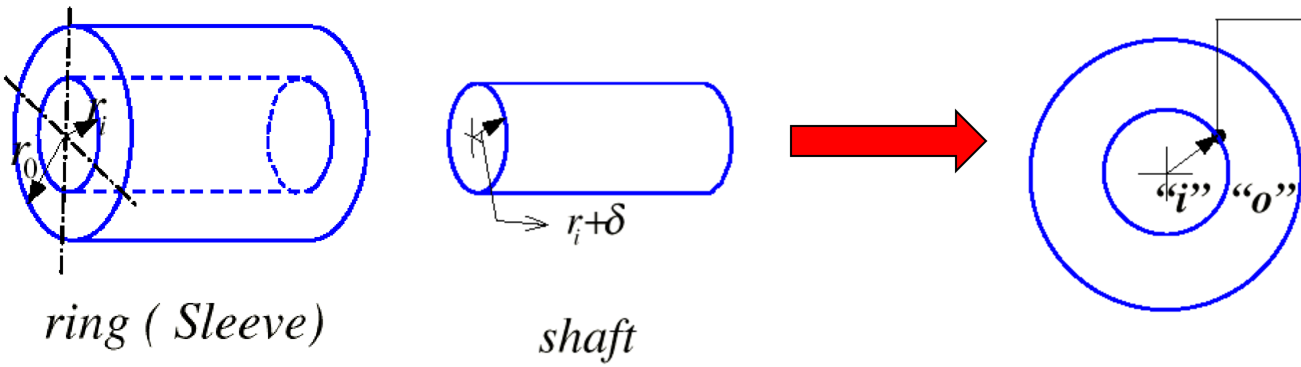
Press Fit:

مثال و کاربرد:

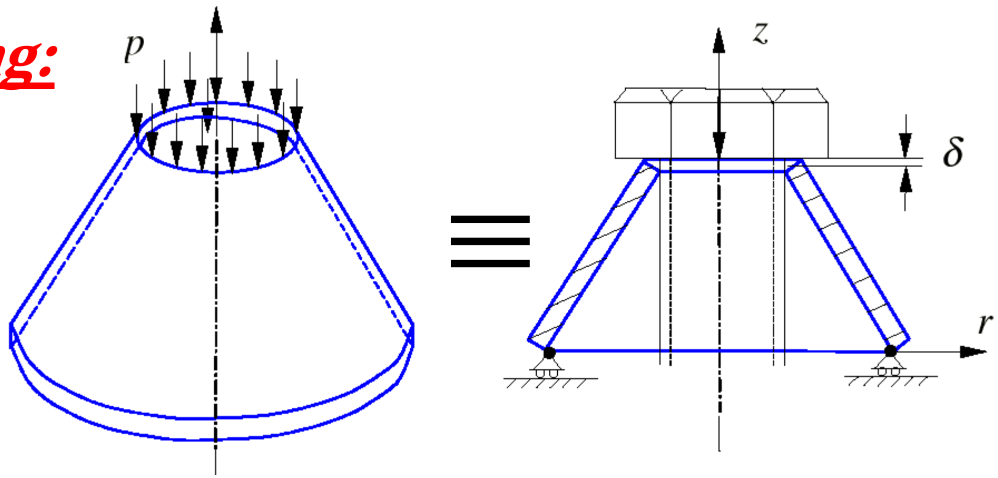


$at\ r = r_i:$

$u_o - u_i = \delta$

$\Rightarrow\ MPC$

Belleville (Conical) Spring:



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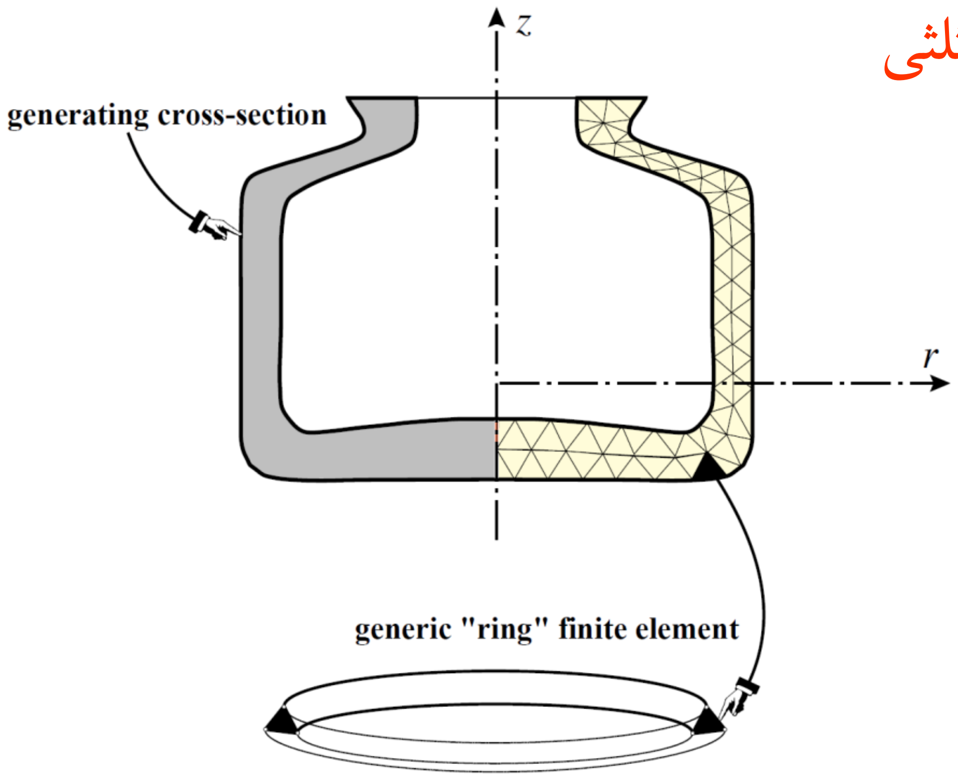
المان تقارن محوری مثلثی



Axisymmetric solid “ring” element

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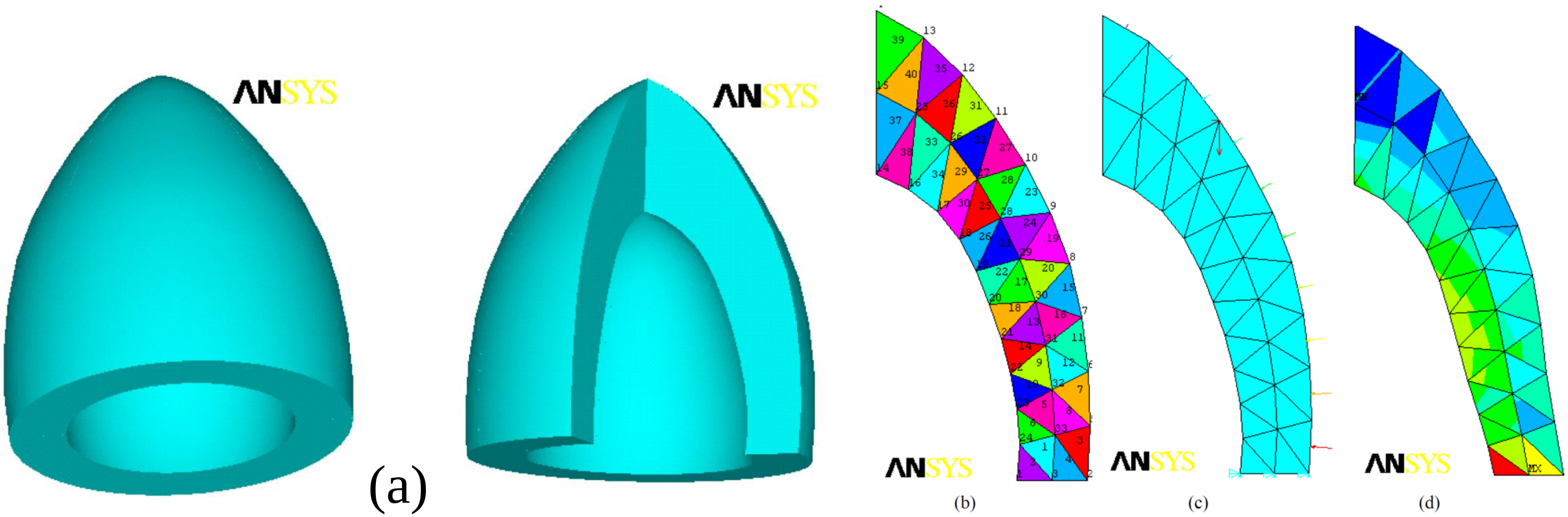
المان تقارن محوری مثلثی

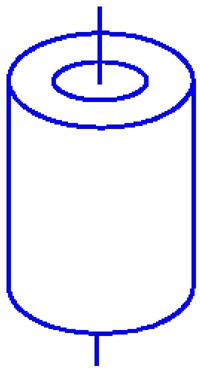


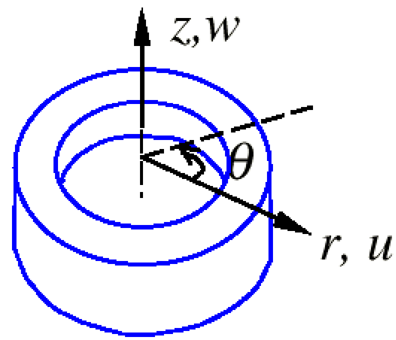
(a) An axisymmetric solid, (b) FE mesh, (c) applied loads, (d) stress distribution.

استخراج ماتریس سختی المان تقارن محوری

$$(x, y, z) \Rightarrow (r, \theta, z)$$

با توجه به ماهیت مسئله، لازم است از دستگاه مختصات استوانه‌ای سود برد.



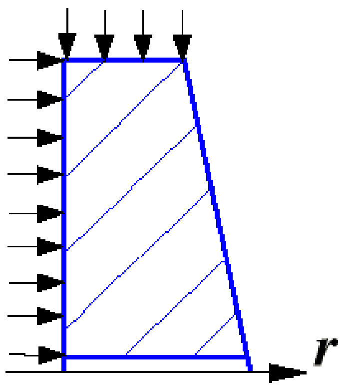


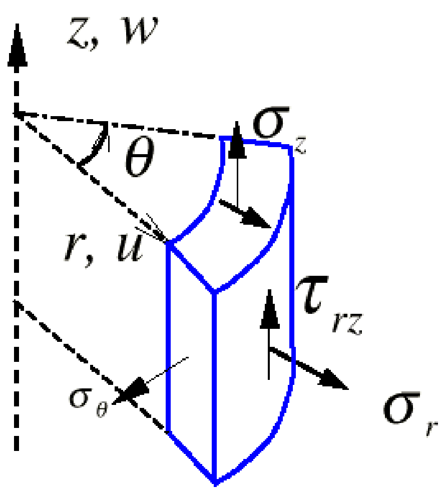
میدان تغییرمکان:

$$u = u(r, z)$$

$$w = w(r, z)$$

$$v = 0$$





استخراج ماتریس سختی المان تقارن محوری

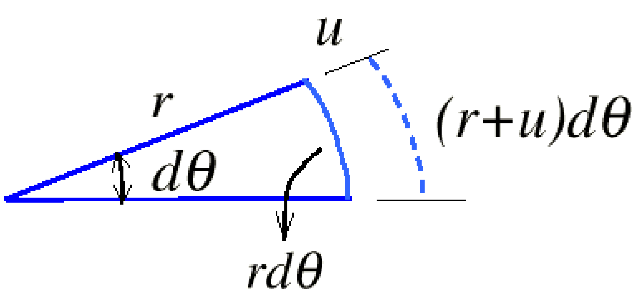
میدان تغییر مکان:

$$u = u(r,z)\ ,\quad w = w(r,z)\quad (\text{No } v - \text{circumferential component})$$

کرنش:

$$\varepsilon_r = \frac{\partial u}{\partial r}\ ,\qquad \varepsilon_\theta = \frac{u}{r}\ ,\qquad \varepsilon_z = \frac{\partial w}{\partial z}\ ,$$

$$\gamma_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\ ,\ (\gamma_{r\theta} = \gamma_{z\theta} = 0)$$



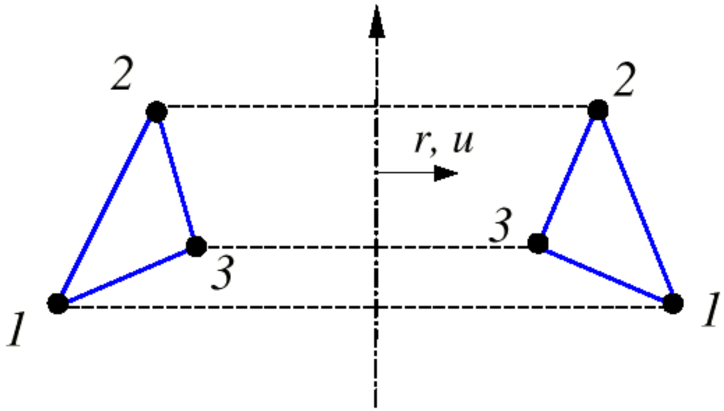
تنش:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \tau_{rz} \\ \sigma_\theta \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 & v \\ v & 1-v & 0 & v \\ 0 & 0 & \dfrac{1-2v}{2} & 0 \\ v & v & 0 & 1-v \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \gamma_{rz} \\ \varepsilon_\theta \end{Bmatrix}$$

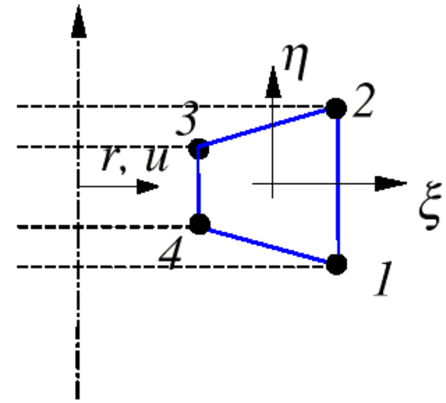
استخراج ماتریس سختی المان تقارن محوری

Axisymmetric Elements:

المان تقارن محوری



3-node element (ring)



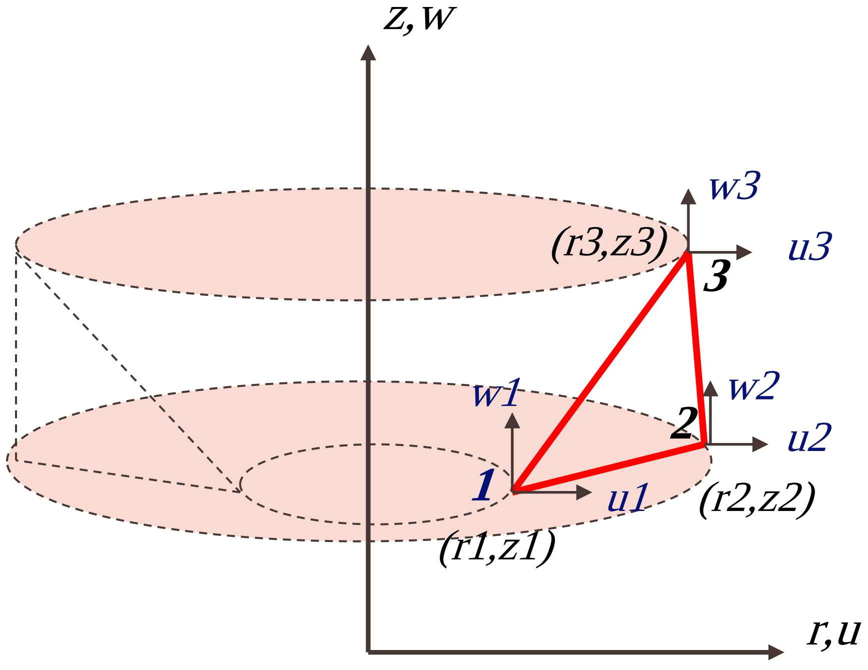
4-node element (ring)

مشابه المان تنش صفحه‌ای برای ماتریس سختی می‌توان نوشت:

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} \, r dr \, d\theta \, dz$$

$$\mathbf{k} = \int_0^{2\pi} \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}^T \mathbf{E} \mathbf{B} \, r (\det \mathbf{J}) \, d\xi \, d\eta \, d\theta \;=\; 2\pi \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}^T \mathbf{E} \mathbf{B} \, r (\det \mathbf{J}) \, d\xi \, d\eta$$

The elements are defined in *r-z* plane. The definition of connectivity of elements and the nodal coordinates are the same as CST element. Note that *r* and *z* are replaced respectively by *x* and *y*. Using *3* shape Function N_1, N_2 and N_3.



استخراج ماتریس سختی المان تقارن محوری مثلثی

$$u = \xi u_1 + \eta u_2 + (1-\xi-\eta) u_3$$
$$w = \xi w_1 + \eta w_2 + (1-\xi-\eta) w_3$$

&

$$\mathbf{u} = \mathbf{N}\mathbf{d}$$
$$\mathbf{d} = \{u_1 \;\; w_1 \;\; u_2 \;\; w_2 \;\; u_3 \;\; w_3\}^T$$
$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

Using iso-parametric representation:

$$r = \xi r_1 + \eta r_2 + (1-\xi-\eta) r_3$$
$$z = \xi z_1 + \eta z_2 + (1-\xi-\eta) z_3$$

The chain rule of differentiation gives:

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} \quad \& \quad \begin{Bmatrix} \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial w}{\partial r} \\ \dfrac{\partial w}{\partial z} \end{Bmatrix} \;, \quad \mathbf{J} = \begin{bmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{bmatrix} \quad \textit{where} \quad \begin{aligned} r_{ij} &= r_i - r_j \\ z_{ij} &= z_i - z_j \end{aligned}$$

استخراج ماتریس سختی المان تقارن محوری مثلثی

$$\det \mathbf{J} = r_{13}z_{23} - z_{13}r_{23} = 2A_e$$

$$\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \quad and \quad \begin{Bmatrix} \dfrac{\partial w}{\partial r} \\ \dfrac{\partial w}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \end{Bmatrix} \quad \text{where} \quad \mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & r_{13} \end{bmatrix}$$

Thus strain vector would be in the form of:

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial w}{\partial r} + \dfrac{\partial u}{\partial z} \\ \dfrac{u}{r} \end{Bmatrix} = \begin{Bmatrix} \dfrac{z_{23}(u_1 - u_3) - z_{13}(u_2 - u_3)}{\det \mathbf{J}} \\ \dfrac{-r_{23}(w_1 - w_3) + r_{13}(w_2 - w_3)}{\det \mathbf{J}} \\ \dfrac{-r_{23}(u_1 - u_3) + r_{13}(u_2 - u_3) + z_{23}(w_1 - w_3) - z_{13}(w_2 - w_3)}{\det \mathbf{J}} \\ \dfrac{N_1u_1 + N_2u_2 + N_2u_2}{r} \end{Bmatrix}$$

استخراج ماتریس سختی المان تقارن محوری مثلثی

or $\boldsymbol{\varepsilon} = \mathbf{Bd}$

Thus

$$\mathbf{B} = \begin{bmatrix} \frac{z_{23}}{\det \mathbf{J}} & 0 & \frac{z_{31}}{\det \mathbf{J}} & 0 & \frac{z_{12}}{\det \mathbf{J}} & 0 \\ 0 & \frac{r_{32}}{\det \mathbf{J}} & 0 & \frac{r_{13}}{\det \mathbf{J}} & 0 & \frac{r_{21}}{\det \mathbf{J}} \\ \frac{r_{32}}{\det \mathbf{J}} & \frac{z_{23}}{\det \mathbf{J}} & \frac{r_{13}}{\det \mathbf{J}} & \frac{z_{31}}{\det \mathbf{J}} & \frac{r_{21}}{\det \mathbf{J}} & \frac{z_{12}}{\det \mathbf{J}} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \end{bmatrix}$$

$$\mathbf{k} = 2\pi \int_{A_e} \mathbf{B}^T \mathbf{E} \mathbf{B} \, r dA$$

The integral is logarithmic, numerical integration is preferred.

On the other hand, as a simple approximation, **B** and r can be evaluated at the centroid of the triangle and used as representative values for the triangle. At the centroid of the triangle,.

$$\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{N}_3 = \frac{1}{3}, \qquad \bar{r} = \frac{r_1 + r_2 + r_3}{3}$$

where $\bar{r}$ is the radius of the centroid. Denoting $\bar{\mathbf{B}}$ as the element strain-displacement matrix **B** evaluated at the centroid, we get

$$\mathbf{k} = 2\pi \bar{\mathbf{B}}^T \mathbf{E} \bar{\mathbf{B}} \bar{r} \int_{A_e} dA = 2\pi \bar{r} A_e \bar{\mathbf{B}}^T \mathbf{E} \bar{\mathbf{B}} \qquad A_e = \frac{\det(\mathbf{J})}{2}$$

Body Force Term

$$2\pi\int_A \mathbf{u}^T\,\mathbf{f}rdA=\mathbf{d}^T\mathbf{f}^e$$

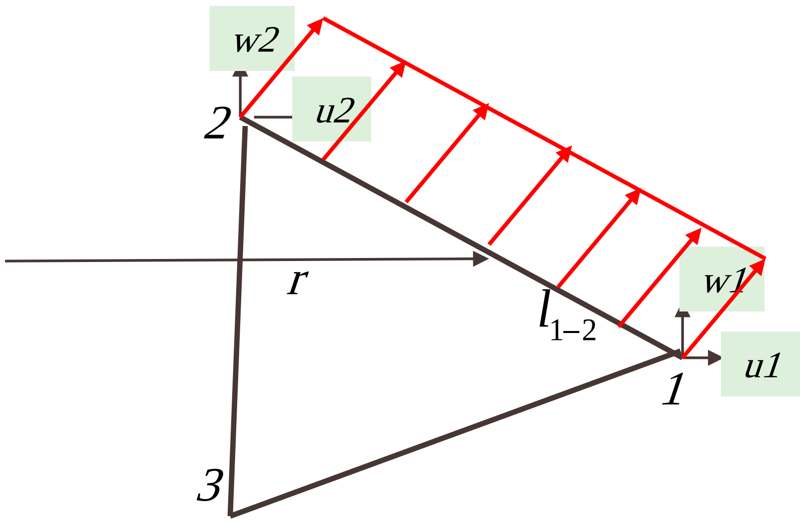
$$2\pi\int_A \mathbf{u}^T\,\mathbf{f}rdA=2\pi\int_A (uf_r+wf_z)rdA=$$

$$2\pi\int_A\left[(N_1u_1+N_2u_2+N_3u_3)f_r+(N_1w_1+N_2w_2+N_3w_3)f_z\right]rdA$$

Surface Traction

$$2\pi \int_e \mathbf{u}^T T r dl = \mathbf{d}^T \mathbf{T}^e$$

$$r = N_1 r_1 + N_2 r_2$$



For a uniformly distributed load on edge 1-2, we have:

$$\mathbf{d} = \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \end{Bmatrix}, \qquad \mathbf{T}^e = 2\pi l_{1-2} \begin{Bmatrix} aT_r \\ aT_z \\ bT_r \\ bT_z \end{Bmatrix}$$

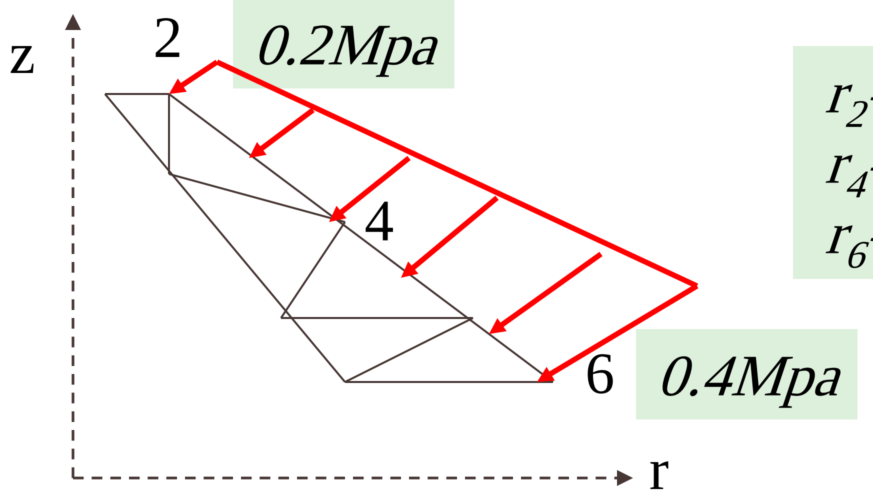
$$a = \frac{2r_1 + r_2}{6}, \quad b = \frac{r_1 + 2r_2}{6}$$

$$l_{1-2} = \sqrt{(r_2 - r_1)^2 + (z_2 - z_1)^2}$$

استخراج ماتریس سختی المان تقارن محوری مثلثی

Problem 1: Calculate the equivalent body force applied on A triangular element under gravity force in z direction and An inertia force due to rotation around z axis.

Problem 2: An axisymmetric body with a linearly distributed Load on the conical surface is shown in Fig. 1. Determine the Equivalent point load at nodes 2,4 and 6.

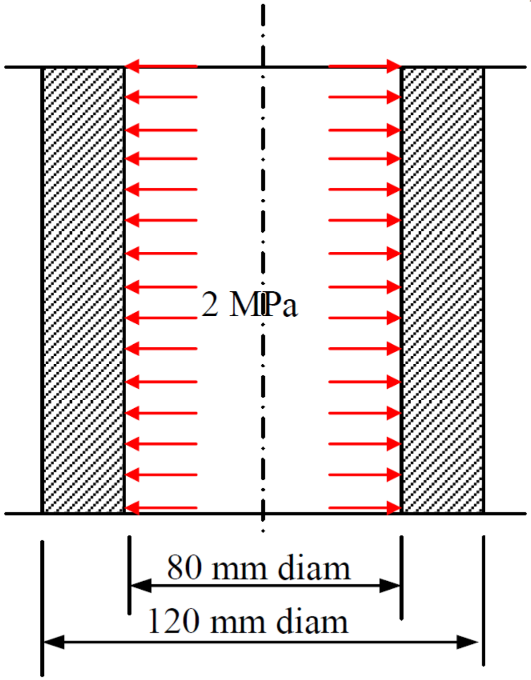


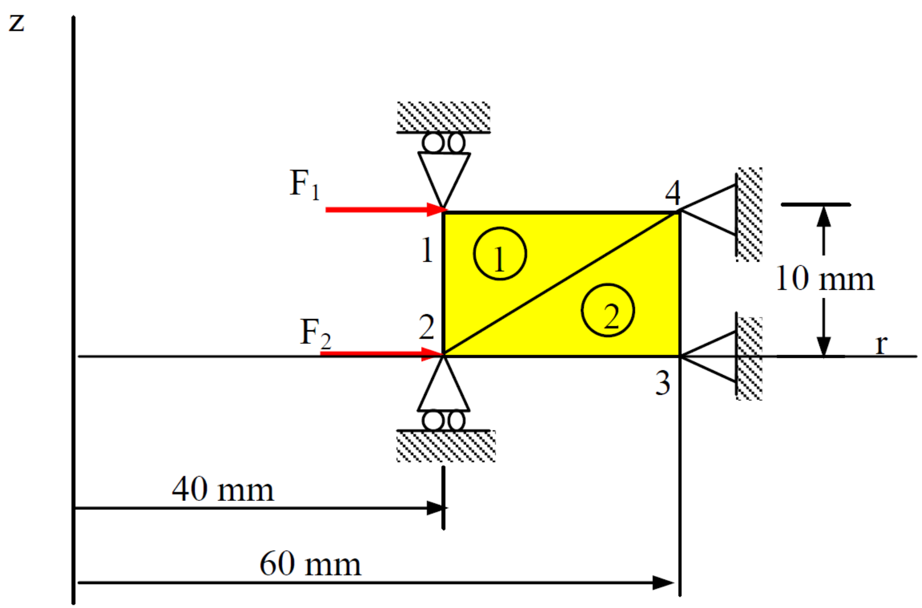
$r_2=20mm,\ \ z_2=70mm$

$r_4=40mm,\ \ z_4=55mm$

$r_6=60mm,\ \ z_6=40mm$

Example: The cylinder is then subjected to an internal pressure of *2 Mpa*. Using two elements on the *10 mm* length shown, find the displacements at the inner radius. $E = 200$ GPa, $\nu = 0.3$.





Element connectivity:

Element	Node numbers		
	1	2	3
1	1	2	4
2	2	3	4

nodal coordinates:

Node	Coordinates	
	r	z
1	40	10
2	40	0
3	60	0
4	60	10

$$\mathbf{E} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 & v \\ v & 1-v & 0 & v \\ 0 & 0 & \frac{1-2v}{2} & 0 \\ v & v & 0 & 1-v \end{bmatrix}$$

$$\mathbf{E} = 10^5 \begin{bmatrix} 2.69 & 1.15 & 0 & 1.15 \\ 1.15 & 2.69 & 0 & 1.15 \\ 0 & 0 & 0.77 & 0 \\ 1.15 & 1.15 & 0 & 2.69 \end{bmatrix}$$

$E = 200000\ MPa,$
$v = 0.3.$

$$F_1 = F_3 = \frac{2\pi r_1 l_{1-2} p_i}{2} = \frac{2\pi(40)(10)(2)}{2} = 2514 \quad N$$

For element 1: $\bar{r} = \frac{1}{3}\left(40+40+60\right) = 46.67 \text{ mm},$

$$\bar{\mathbf{B}}^{1} = \begin{bmatrix} -0.05 & 0 & 0 & 0 & 0.05 & 0 \\ 0 & 0.1 & 0 & -0.1 & 0 & 0 \\ 0.1 & -0.05 & -0.1 & 0 & 0 & 0.05 \\ 0.0071 & 0 & 0.0071 & 0 & 0.0071 & 0 \end{bmatrix}$$

For element 2: $\bar{r} = \frac{1}{3}\left(40+60+60\right) = 53.53 \text{ mm},$

$$\bar{\mathbf{B}}^{1} = \begin{bmatrix} -0.05 & 0 & 0.05 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 0 & 0.1 \\ 0 & -0.05 & -0.1 & 0.05 & 0.1 & 0 \\ 0.00625 & 0 & 0.00625 & 0 & 0.00625 & 0 \end{bmatrix}$$

$$\mathbf{k} = 2\pi\bar{r}A_e\left[\bar{B}\right]^T[E]\left[\bar{B}\right]$$

$$\begin{matrix} 1 & 2 & 3 & 4 & 7 & 8 & \rightarrow \text{Global dof} \end{matrix}$$

$$\left[k^1\right] = 10^7 \begin{bmatrix} 4.03 & -2.58 & -2.34 & 1.45 & -1.932 & 1.13 \\ & 8.45 & 1.37 & -7.89 & 1.93 & -0.565 \\ & & 2.30 & -0.24 & 0.16 & -1.13 \\ & & & 7.89 & -1.93 & 0 \\ & \text{Symmetric} & & & 2.25 & 0 \\ & & & & & 0.565 \end{bmatrix}$$

المان تقارن محوری مثلثی

$$\mathbf{k} = 2\pi\bar{r}A_e\left[\bar{B}\right]^T\left[E\right]\left[\bar{B}\right]$$

$$\begin{matrix} 3 & 4 & 5 & 6 & 7 & 8 & \rightarrow \text{Global dof} \end{matrix}$$

$$\left[k^2\right] = 10^7 \begin{bmatrix} 2.05 & 0 & -2.22 & 1.69 & -0.085 & -1.69 \\ & 0.645 & 1.29 & -0.645 & -1.29 & 0 \\ & & 5.11 & -3.46 & -2.42 & 2.17 \\ & & & 9.66 & 1.05 & -9.01 \\ & \text{Symmetric} & & & 2.62 & 0.241 \\ & & & & & 9.01 \end{bmatrix}$$

Using the elimination approach, on assembling the matrices with reference to the degrees of freedom 1 and 3, we get

$$10^7 \begin{bmatrix} 4.03 & -2.34 \\ -2.34 & 4.35 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 2514 \\ 2514 \end{Bmatrix} \qquad \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.014 \times 10^{-2} \\ 0.0133 \times 10^{-2} \end{Bmatrix} mm$$

The element stresses can be calculated from $\boldsymbol{\sigma} = \mathbf{E}\bar{\mathbf{B}}\mathbf{d}$

$$\mathbf{d}^{(1)} = \begin{Bmatrix} 0.014 \times 10^{-2} \\ 0 \\ 0.0133 \times 10^{-2} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \mathbf{d}^{(2)} = \begin{Bmatrix} 0.0133 \times 10^{-2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\boldsymbol{\sigma}^{(1)} = \begin{Bmatrix} -166 \\ -58.2 \\ 5.4 \\ -28.4 \end{Bmatrix} \times 10^{-2} \quad MPa$$

$$\boldsymbol{\sigma}^{(2)} = \begin{Bmatrix} -169.3 \\ -66.9 \\ 0 \\ -54.1 \end{Bmatrix} \times 10^{-2} \quad MPa$$