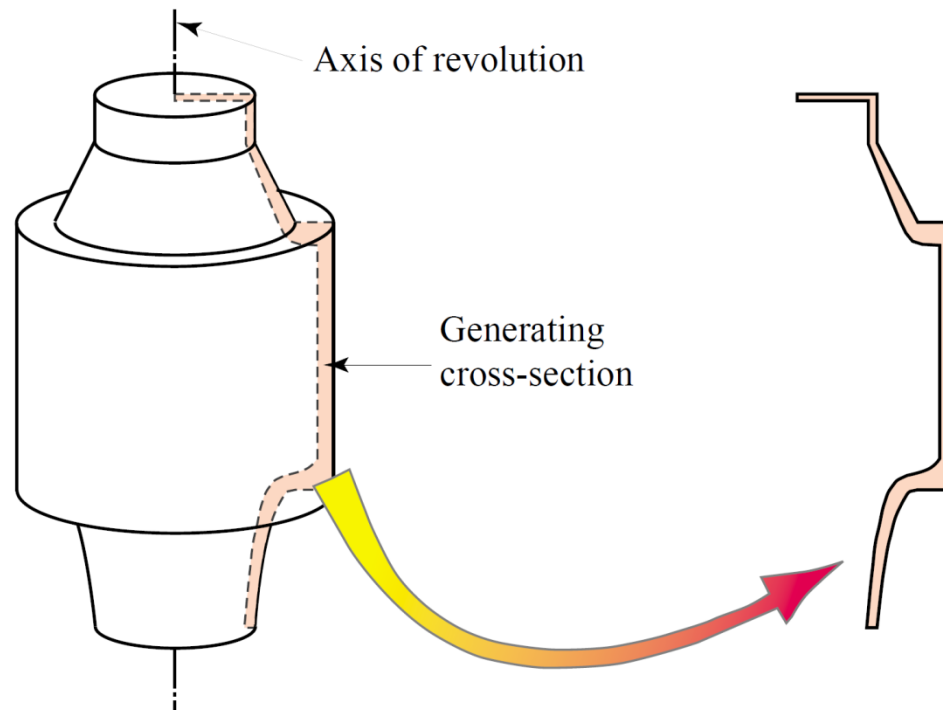




Solids of Revolution

(Axisymmetric Solids)

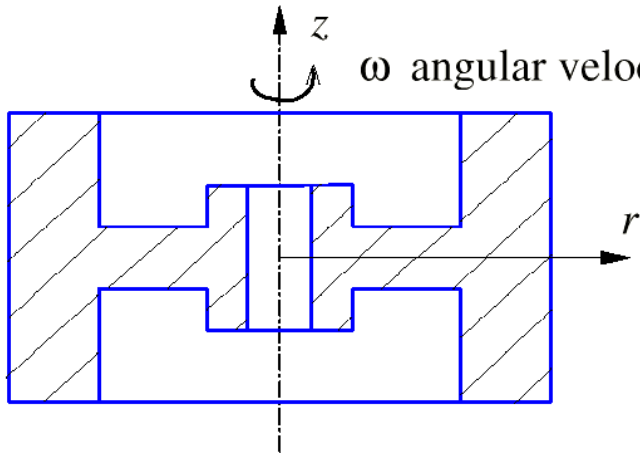
تعریف:



A structure of revolution is generated by rotating a generating cross section about an axis of revolution.

Rotating Flywheel:

مثال و کاربرد:

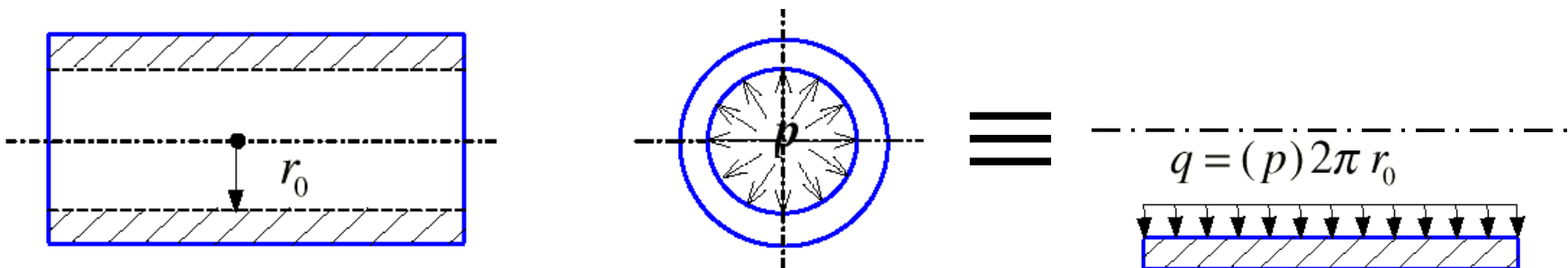


Body forces:

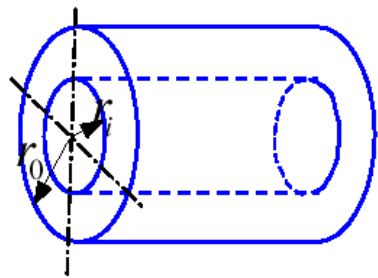
$$f_r = \rho r \omega^2 \text{ (equivalent radial centrifugal or inertial force)}$$

$$f_z = -\rho g \text{ (gravitational force)}$$

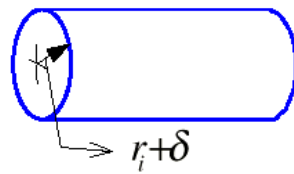
Cylinder Subject to Internal Pressure:



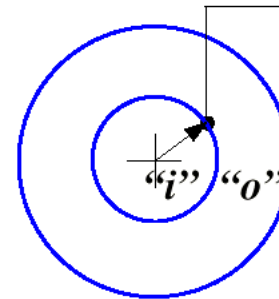
Press Fit:



ring (Sleeve)



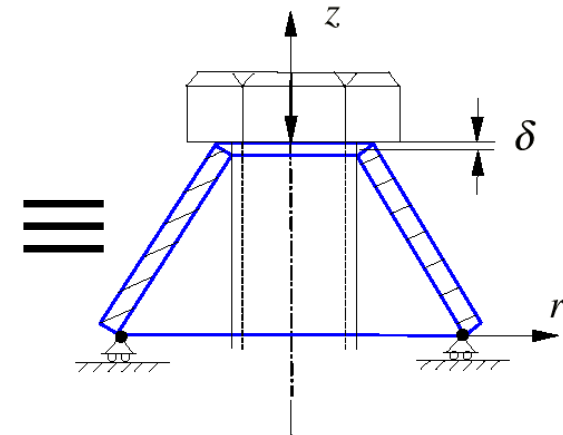
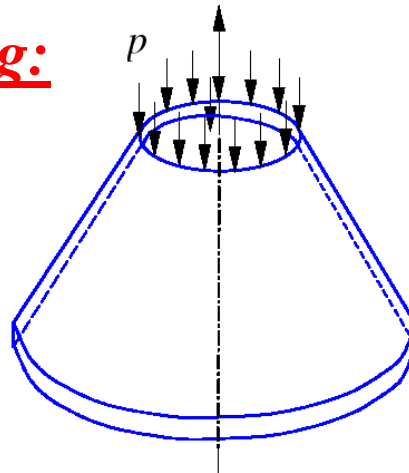
shaft

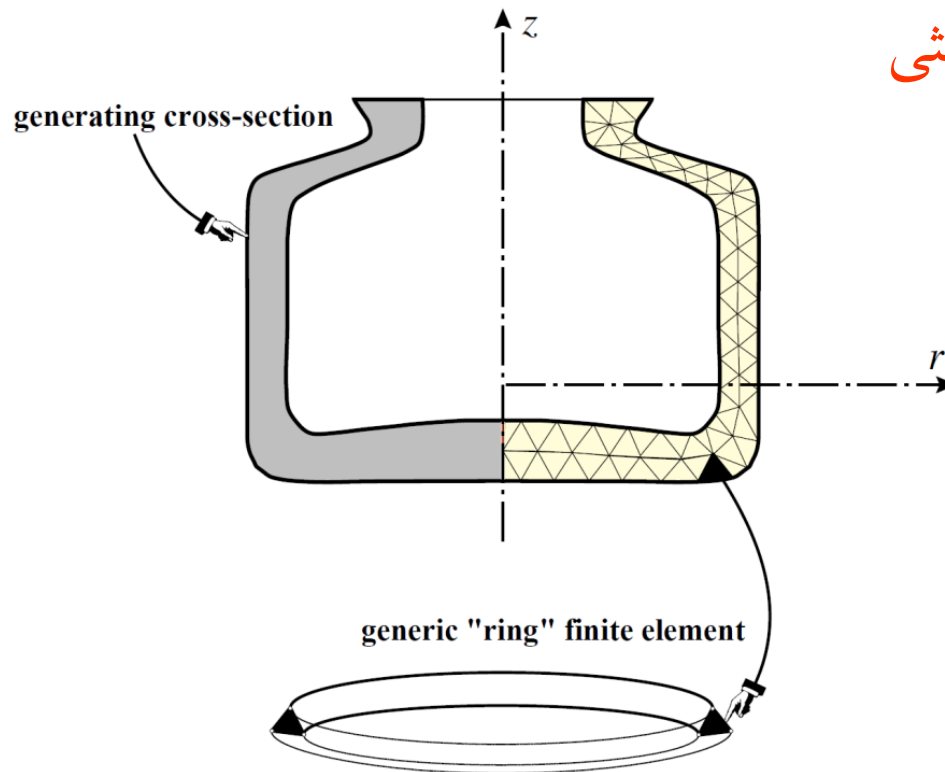


مثال و کاربرد:

$$\begin{aligned} \text{at } r = r_i: \\ u_o - u_i = \delta \\ \Rightarrow MPC \end{aligned}$$

Belleville (Conical) Spring:

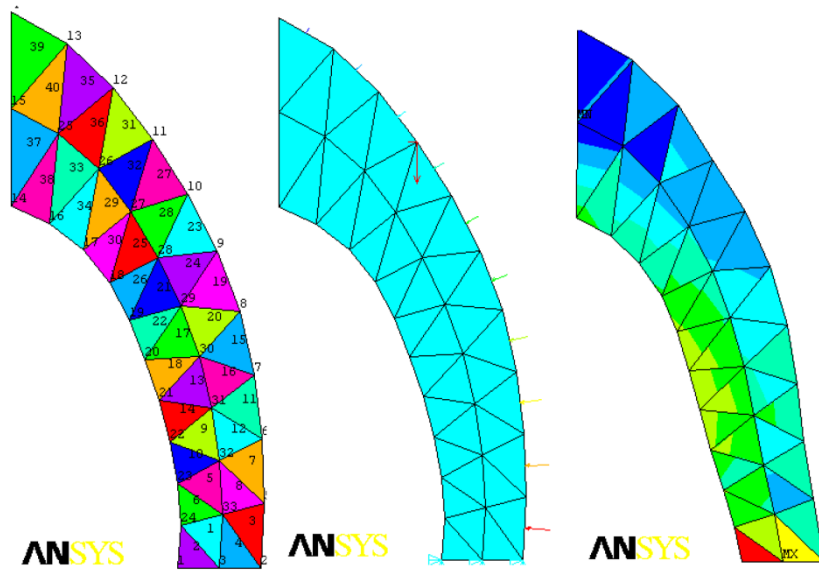
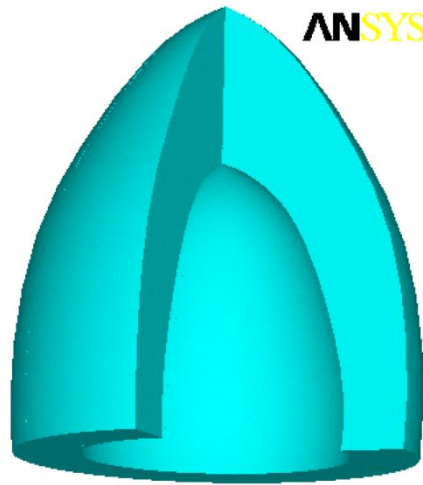




Axisymmetric solid “ring” element



(a)



(b)

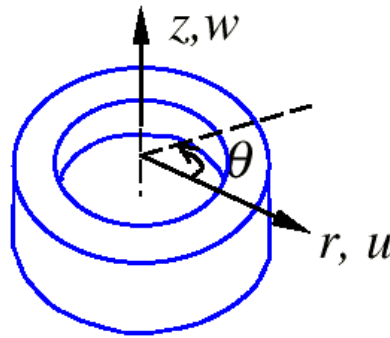
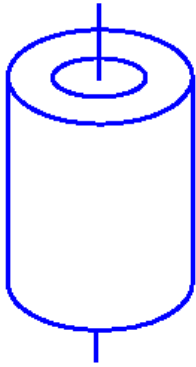
(c)

(d)

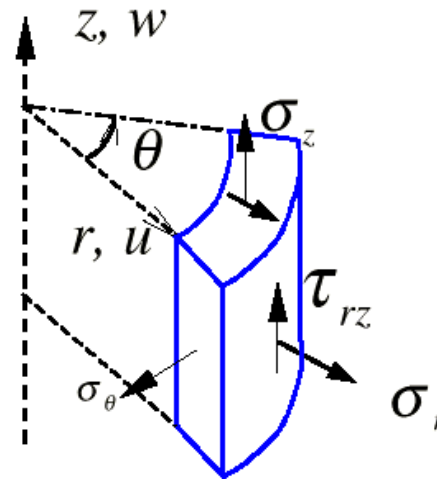
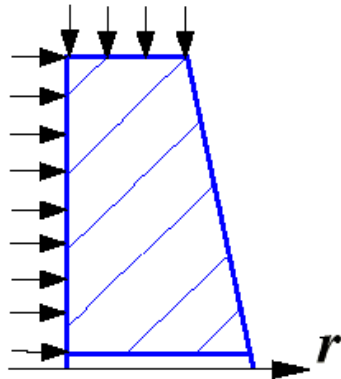
(a) An axisymmetric solid, (b) FE mesh, (c) applied loads, (d) stress distribution.

استخراج ماتریس سختی المان تقارن محوری

$$(x, y, z) \Rightarrow (r, \theta, z)$$



با توجه به ماهیت مسئله، لازم است از دستگاه مختصات استوانه‌ای سود برد.



میدان تغییر مکان:

$$u = u(r, z)$$

$$w = w(r, z)$$

$$v = 0$$

استخراج ماتریس سختی المان تقارن محوری

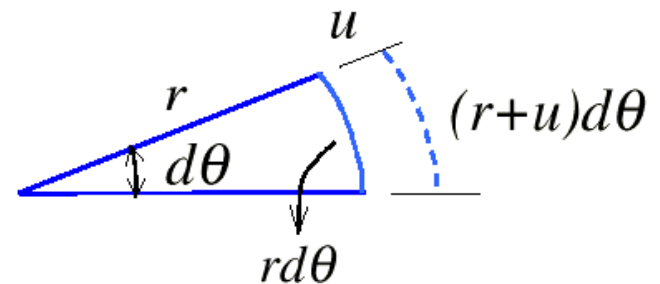
میدان تغییر مکان:

$$u = u(r, z), \quad w = w(r, z) \quad (\text{No } v\text{-circumferential component})$$

کرنش:

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}, \quad (\gamma_{r\theta} = \gamma_{z\theta} = 0)$$

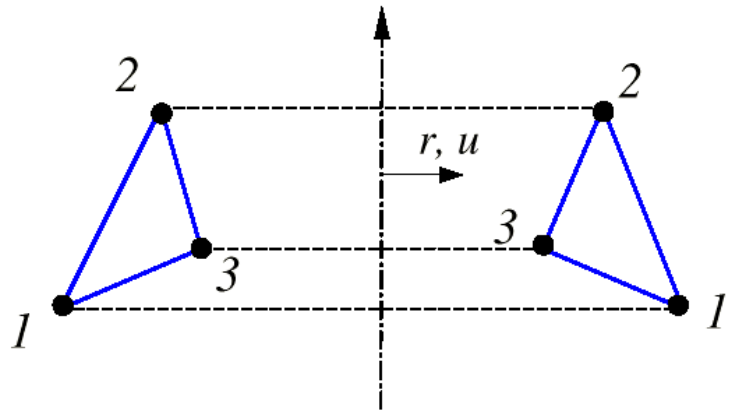


تنش:

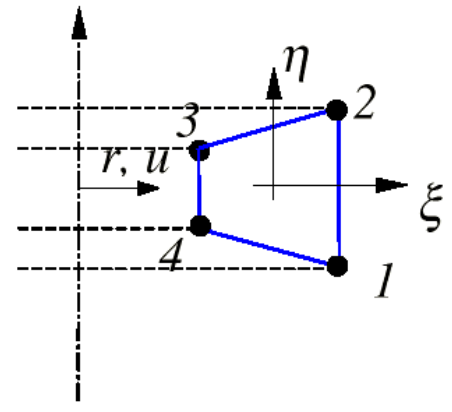
$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \tau_{rz} \\ \sigma_\theta \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 & \nu \\ \nu & 1-\nu & 0 & \nu \\ 0 & 0 & \frac{1-2\nu}{2} & 0 \\ \nu & \nu & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \gamma_{rz} \\ \varepsilon_\theta \end{Bmatrix}$$

استخراج ماتریس سختی المان تقارن محوری

Axisymmetric Elements:



3-node element (ring)



4-node element (ring)

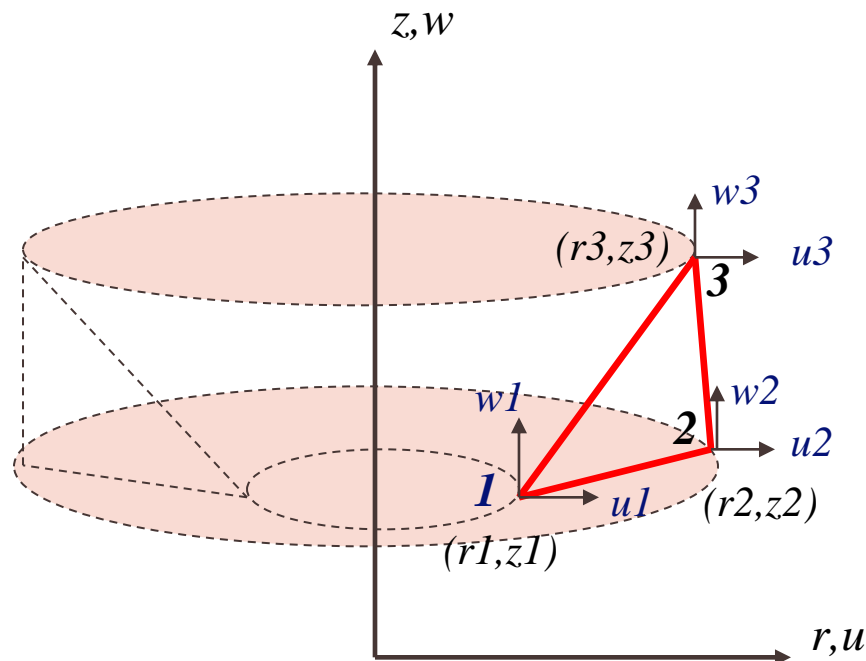
المان تقارن محوری

مشابه المان تنش صفحه‌ای برای ماتریس سختی می‌توان نوشت:

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} r dr d\theta dz$$

$$\mathbf{k} = \int_0^{2\pi} \int_{-1}^1 \int_{-1}^1 \mathbf{B}^T \mathbf{E} \mathbf{B} r (\det \mathbf{J}) d\xi d\eta d\theta = 2\pi \int_{-1}^1 \int_{-1}^1 \mathbf{B}^T \mathbf{E} \mathbf{B} r (\det \mathbf{J}) d\xi d\eta$$

The elements are defined in r - z plane. The definition of connectivity of elements and the nodal coordinates are the same as CST element. Note that r and z are replaced respectively by x and y . Using 3 shape Function N_1 , N_2 and N_3 .





استخراج ماتریس سختی المان تقارن محوری مثلثی

$$\begin{aligned}
 u &= \xi u_1 + \eta u_2 + (1 - \xi - \eta) u_3 \\
 w &= \xi w_1 + \eta w_2 + (1 - \xi - \eta) w_3
 \end{aligned}
 \quad \& \quad
 \begin{aligned}
 \mathbf{u} &= \mathbf{N} \mathbf{d} \\
 \mathbf{d} &= \{u_1 \ w_1 \ u_2 \ w_2 \ u_3 \ w_3\}^T \\
 \mathbf{N} &= \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}
 \end{aligned}$$

Using iso-parametric representation:

$$r = \xi r_1 + \eta r_2 + (1 - \xi - \eta) r_3$$

$$z = \xi z_1 + \eta z_2 + (1 - \xi - \eta) z_3$$

The chain rule of differentiation gives:

$$\begin{aligned}
 \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} &= \mathbf{J} \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial z} \end{Bmatrix} \quad \& \quad \begin{Bmatrix} \frac{\partial w}{\partial \xi} \\ \frac{\partial w}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \frac{\partial w}{\partial r} \\ \frac{\partial w}{\partial z} \end{Bmatrix}, \quad \mathbf{J} = \begin{bmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{bmatrix} \quad \text{where} \quad \begin{aligned} r_{ij} &= r_i - r_j \\ z_{ij} &= z_i - z_j \end{aligned}
 \end{aligned}$$



استخراج ماتریس سختی المان تقارن محوری مثلثی

$$\det \mathbf{J} = r_{13}z_{23} - z_{13}r_{23} = 2A_e$$

$$\begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \frac{\partial w}{\partial r} \\ \frac{\partial w}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial w}{\partial \xi} \\ \frac{\partial w}{\partial \eta} \end{Bmatrix} \quad \text{where} \quad \mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & r_{13} \end{bmatrix}$$

Thus strain vector would be in the form of:

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial w}{\partial z} \\ \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \\ \frac{u}{r} \end{Bmatrix} = \begin{Bmatrix} \frac{z_{23}(u_1 - u_3) - z_{13}(u_2 - u_3)}{\det \mathbf{J}} \\ \frac{-r_{23}(w_1 - w_3) + r_{13}(w_2 - w_3)}{\det \mathbf{J}} \\ \frac{-r_{23}(u_1 - u_3) + r_{13}(u_2 - u_3) + z_{23}(w_1 - w_3) - z_{13}(w_2 - w_3)}{\det \mathbf{J}} \\ \frac{N_1 u_1 + N_2 u_2 + N_2 u_2}{r} \end{Bmatrix}$$



استخراج ماتریس سختی المان تقارن محوری مثلثی

or $\boldsymbol{\varepsilon} = \mathbf{Bd}$

Thus

$$\mathbf{B} = \begin{bmatrix} \frac{z_{23}}{\det \mathbf{J}} & 0 & \frac{z_{31}}{\det \mathbf{J}} & 0 & \frac{z_{12}}{\det \mathbf{J}} & 0 \\ 0 & \frac{r_{32}}{\det \mathbf{J}} & 0 & \frac{r_{13}}{\det \mathbf{J}} & 0 & \frac{r_{21}}{\det \mathbf{J}} \\ \frac{r_{32}}{\det \mathbf{J}} & \frac{z_{23}}{\det \mathbf{J}} & \frac{r_{13}}{\det \mathbf{J}} & \frac{z_{31}}{\det \mathbf{J}} & \frac{r_{21}}{\det \mathbf{J}} & \frac{z_{12}}{\det \mathbf{J}} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \end{bmatrix}$$

$$\mathbf{k} = 2\pi \int_{A_e} \mathbf{B}^T \mathbf{E} \mathbf{B} r dA$$

The integral is logarithmic, numerical integration is preferred.

On the other hand, as a simple approximation, \mathbf{B} and r can be evaluated at the centroid of the triangle and used as representative values for the triangle. At the centroid of the triangle,.

$$\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{N}_3 = \frac{1}{3}, \quad \bar{r} = \frac{r_1 + r_2 + r_3}{3}$$

where \bar{r} is the radius of the centroid. Denoting $\bar{\mathbf{B}}$ as the element strain-displacement matrix \mathbf{B} evaluated at the centroid, we get

$$\mathbf{k} = 2\pi \bar{\mathbf{B}}^T \mathbf{E} \bar{\mathbf{B}} \bar{r} \int_{A_e} dA = 2\pi \bar{r} A_e \bar{\mathbf{B}}^T \mathbf{E} \bar{\mathbf{B}} \quad A_e = \frac{\det(\mathbf{J})}{2}$$



Body Force Term

$$2\pi \int_A \mathbf{u}^T \mathbf{f} r dA = \mathbf{d}^T \mathbf{f}^e$$

$$2\pi \int_A \mathbf{u}^T \mathbf{f} r dA = 2\pi \int_A (u f_r + w f_z) r dA =$$

$$2\pi \int_A [(N_1 u_1 + N_2 u_2 + N_3 u_3) f_r + (N_1 w_1 + N_2 w_2 + N_3 w_3) f_z] r dA$$

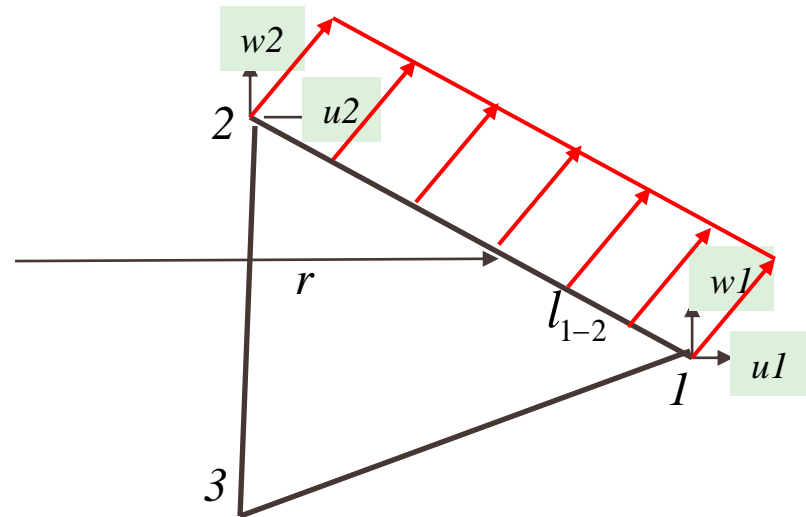
Surface Traction

$$2\pi \int_e \mathbf{u}^T \mathbf{T} r dl = \mathbf{d}^T \mathbf{T}^e$$

$$r = N_1 r_1 + N_2 r_2$$

For a uniformly distributed load on edge 1-2, we have:

$$\mathbf{d} = \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \end{Bmatrix}, \quad \mathbf{T}^e = 2\pi l_{1-2} \begin{Bmatrix} aT_r \\ aT_z \\ bT_r \\ bT_z \end{Bmatrix}$$

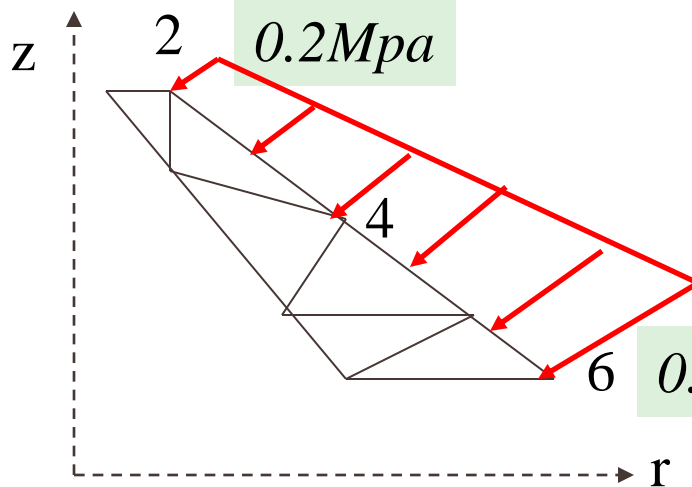


$$a = \frac{2r_1 + r_2}{6}, \quad b = \frac{r_1 + 2r_2}{6}$$

$$l_{1-2} = \sqrt{(r_2 - r_1)^2 + (z_2 - z_1)^2}$$

Problem 1: Calculate the equivalent body force applied on A triangular element under gravity force in z direction and An inertia force due to rotation around z axis.

Problem 2: An axisymmetric body with a linearly distributed Load on the conical surface is shown in Fig. 1. Determine the Equivalent point load at nodes 2,4 and 6.

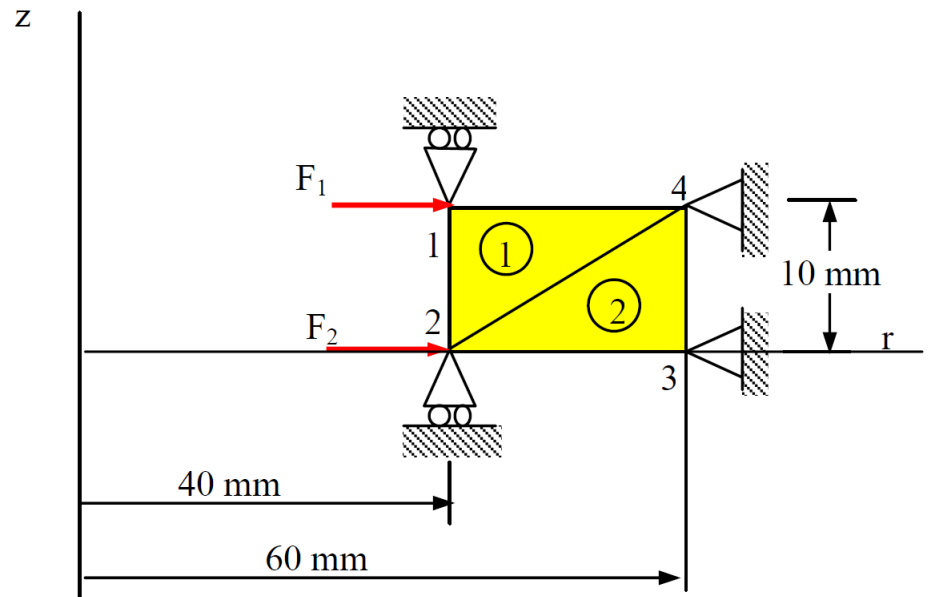
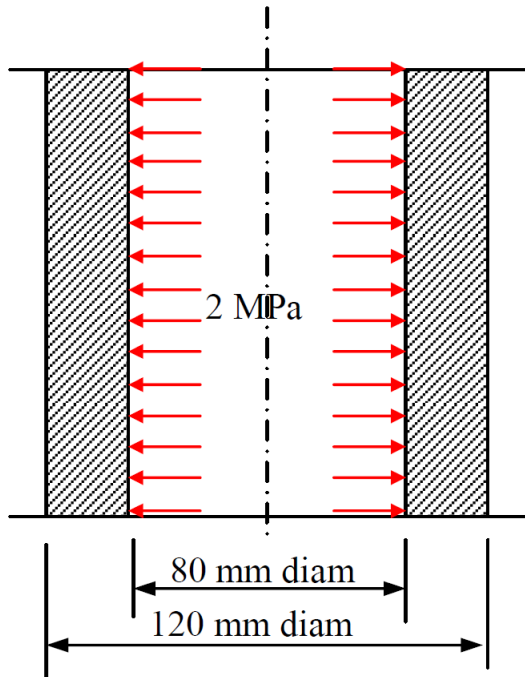


$$r_2 = 20\text{mm}, \quad z_2 = 70\text{mm}$$

$$r_4 = 40\text{mm}, \quad z_4 = 55\text{mm}$$

$$r_6 = 60\text{mm}, \quad z_6 = 40\text{mm}$$

Example: The cylinder is then subjected to an internal pressure of 2 MPa . Using two elements on the 10 mm length shown, find the displacements at the inner radius. $E = 200 \text{ GPa}$, $\nu = 0.3$.



Element connectivity:

Element	Node numbers		
	1	2	3
1	1	2	4
2	2	3	4

nodal coordinates:

Node	Coordinates	
	r	z
1	40	10
2	40	0
3	60	0
4	60	10

$$\mathbf{E} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 & \nu \\ \nu & 1-\nu & 0 & \nu \\ 0 & 0 & \frac{1-2\nu}{2} & 0 \\ \nu & \nu & 0 & 1-\nu \end{bmatrix}$$

$$\mathbf{E} = 10^5 \begin{bmatrix} 2.69 & 1.15 & 0 & 1.15 \\ 1.15 & 2.69 & 0 & 1.15 \\ 0 & 0 & 0.77 & 0 \\ 1.15 & 1.15 & 0 & 2.69 \end{bmatrix}$$

$$E = 200000 \text{ MPa},$$

$$\nu = 0.3.$$

$$F_1 = F_3 = \frac{2\pi r_1 l_{1-2} p_i}{2} = \frac{2\pi(40)(10)(2)}{2} = 2514 \text{ N}$$

For element 1: $\bar{r} = \frac{1}{3}(40 + 40 + 60) = 46.67 \text{ mm},$

$$\bar{\mathbf{B}}^I = \begin{bmatrix} -0.05 & 0 & 0 & 0 & 0.05 & 0 \\ 0 & 0.1 & 0 & -0.1 & 0 & 0 \\ 0.1 & -0.05 & -0.1 & 0 & 0 & 0.05 \\ 0.0071 & 0 & 0.0071 & 0 & 0.0071 & 0 \end{bmatrix}$$

For element 2: $\bar{r} = \frac{1}{3}(40 + 60 + 60) = 53.53 \text{ mm},$

$$\bar{\mathbf{B}}^I = \begin{bmatrix} -0.05 & 0 & 0.05 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 0 & 0.1 \\ 0 & -0.05 & -0.1 & 0.05 & 0.1 & 0 \\ 0.00625 & 0 & 0.00625 & 0 & 0.00625 & 0 \end{bmatrix}$$

$$\mathbf{k} = 2\pi\bar{r}A_e [\bar{\mathbf{B}}]^T [\mathbf{E}][\bar{\mathbf{B}}]$$

	1	2	3	4	7	8	→ Global dof
$[\mathbf{k}^1] = 10^7$	4.03	-2.58	-2.34	1.45	-1.932	1.13	
		8.45	1.37	-7.89	1.93	-0.565	
			2.30	-0.24	0.16	-1.13	
				7.89	-1.93	0	
					2.25	0	
							0.565

Symmetric



$$\mathbf{k} = 2\pi\bar{r}A_e [\bar{\mathbf{B}}]^T [\mathbf{E}][\bar{\mathbf{B}}]$$

	3	4	5	6	7	8	→ Global dof
$[\mathbf{k}^2] = 10^7$	2.05	0	-2.22	1.69	-0.085	-1.69	
		0.645	1.29	-0.645	-1.29	0	
			5.11	-3.46	-2.42	2.17	
				9.66	1.05	-9.01	
					2.62	0.241	
						9.01	

Symmetric

Using the elimination approach, on assembling the matrices with reference to the degrees of freedom 1 and 3, we get

$$10^7 \begin{bmatrix} 4.03 & -2.34 \\ -2.34 & 4.35 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 2514 \\ 2514 \end{Bmatrix} \quad \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.014 \times 10^{-2} \\ 0.0133 \times 10^{-2} \end{Bmatrix} \text{ mm}$$

The element stresses can be calculated from

$$\boldsymbol{\sigma} = \mathbf{E} \bar{\mathbf{B}} \mathbf{d}$$

$$\mathbf{d}^{(1)} = \begin{Bmatrix} 0.014 \times 10^{-2} \\ 0 \\ 0.0133 \times 10^{-2} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \mathbf{d}^{(2)} = \begin{Bmatrix} 0.0133 \times 10^{-2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\boldsymbol{\sigma}^{(1)} = \begin{Bmatrix} -166 \\ -58.2 \\ 5.4 \\ -28.4 \end{Bmatrix} \times 10^{-2} \text{ MPa}$$

$$\boldsymbol{\sigma}^{(2)} = \begin{Bmatrix} -169.3 \\ -66.9 \\ 0 \\ -54.1 \end{Bmatrix} \times 10^{-2} \text{ MPa}$$