دانشگاه صنعتی اصفهان
دانشکده مکانیك

The J integral

The J contour integral as yield criterion

Properties of the J-integral

Closed contour around A

1) J is *zero* for any closed contour containing *no crack tip*.

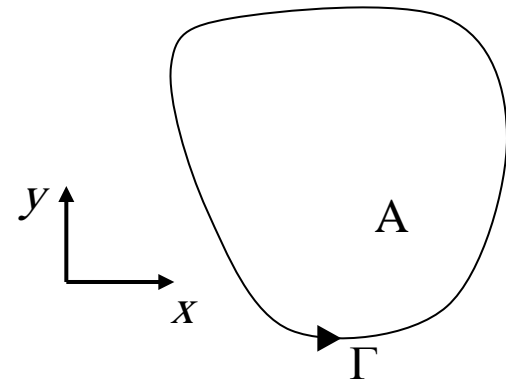Consider $\left. J \right|_\Gamma = \oint_\Gamma \left(w\, dy - T_i \frac{\partial u_i}{\partial x}\, ds \right)$

Using the Green Theorem, i.e. $\oint_\Gamma P(x,y)\,dx + Q(x,y)\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$

We have $\left. J \right|_\Gamma = \int_A \frac{\partial w}{\partial x}\, dx\,dy - \oint_\Gamma T_i \frac{\partial u_i}{\partial x}\, ds \quad = \int_A \frac{\partial w}{\partial x}\, dx\,dy - \oint_\Gamma \sigma_{ij} \frac{\partial u_i}{\partial x}\, n_j\, ds$

From the divergence theorem,

$\oint_\Gamma \sigma_{ij} \frac{\partial u_i}{\partial x}\, n_j\, ds \quad = \int_A \frac{\partial}{\partial x_j} \left(\sigma_{ij} \frac{\partial u_i}{\partial x} \right) dx\,dy$

The integral becomes,

$$J\big|_\Gamma = \int_A \left[\frac{\partial w}{\partial x} - \frac{\partial}{\partial x_j}\left(\sigma_{ij} \frac{\partial u_i}{\partial x} \right) \right] dxdy$$

However,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial x} = \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial x} = \sigma_{ij} \frac{\partial}{\partial x}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] \qquad \text{since} \quad \sigma_{ij} = \sigma_{ji}$$

$$= \sigma_{ij} \frac{\partial}{\partial x}\frac{\partial u_i}{\partial x_j} = \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x} \right)$$

Invoking the equilibrium equation, $\quad \dfrac{\partial \sigma_{ij}}{\partial x_j} = 0$

$$\frac{\partial}{\partial x_j}\left(\sigma_{ij} \frac{\partial u_i}{\partial x} \right) = \overset{0}{\frac{\partial \sigma_{ij}}{\partial x_j}} \frac{\partial u_i}{\partial x} + \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x} \right) = \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x} \right)$$

Replacing in the integral, $\quad J\big|_\Gamma = 0$

2) J is path-independent

Consider the *closed* contour:
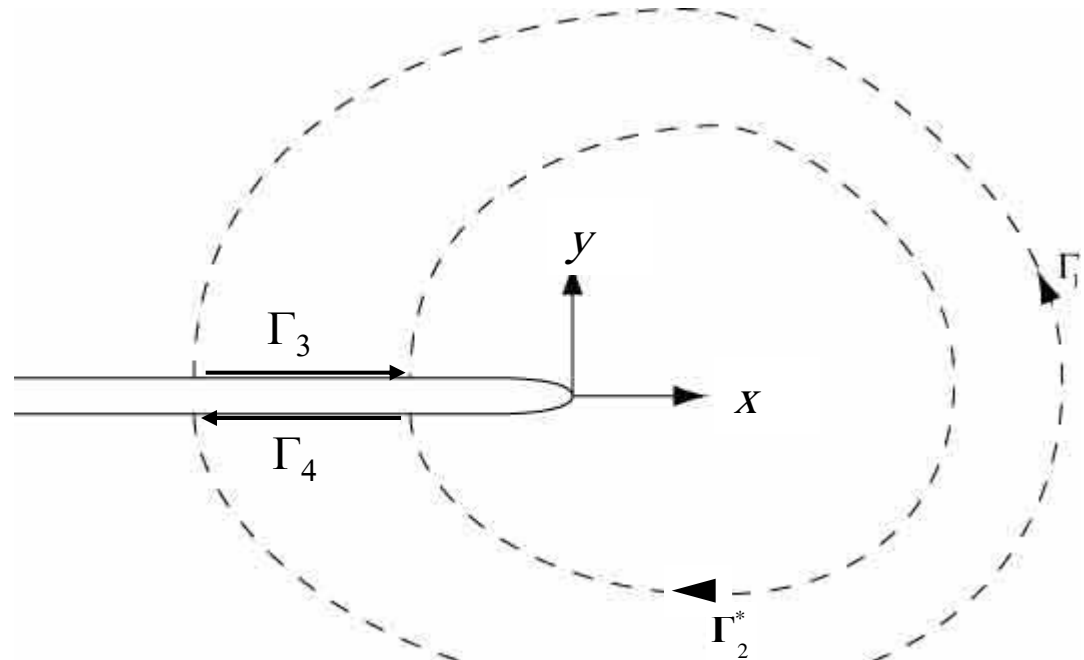
$$\Gamma = \Gamma_1 + \Gamma_3 + \Gamma_2^* + \Gamma_4$$

We have $\quad J\big|_\Gamma = J\big|_{\Gamma_1} + J\big|_{\Gamma_2^*} + J\big|_{\Gamma_3} + J\big|_{\Gamma_4} \quad$ and $\quad J\big|_\Gamma = 0$

The crack faces are traction free :

$T_i = \sigma_{ij}\, n_j = 0 \quad$ on $\ \Gamma_3 \ $ and $\ \Gamma_4$

$dy = 0$ along these contours

$\Longrightarrow \quad J\big|_{\Gamma_3} = J\big|_{\Gamma_4} = 0$

2) J is path-independent

Note that,

$$J\big|_{\Gamma_2^*} = -J\big|_{\Gamma_2}$$

and $\quad J\big|_{\Gamma} = J\big|_{\Gamma_1} - J\big|_{\Gamma_2} = 0$

$\Longrightarrow \quad J\big|_{\Gamma_1} = J\big|_{\Gamma_2}$
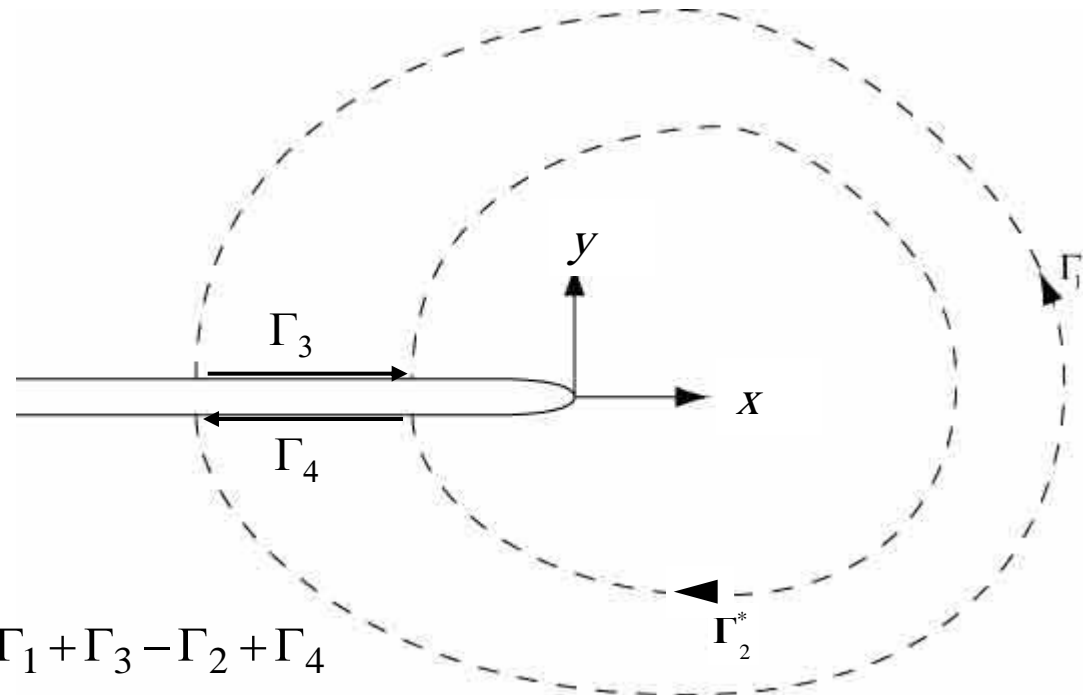
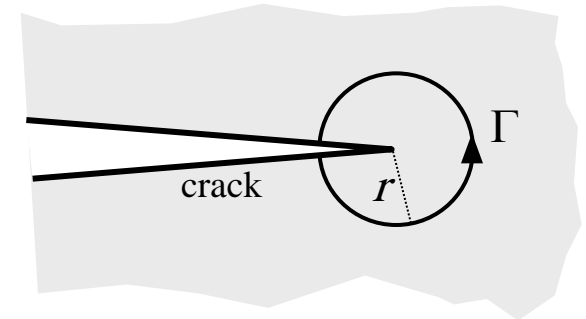$\Gamma = \Gamma_1 + \Gamma_3 - \Gamma_2 + \Gamma_4$

Γ_2 followed in the *counter-clockwise* direction.

Any arbitrary (counterclockwise) path around a crack gives the same value of *J*

$\Longrightarrow \quad$ J is *path*-independent

J can be evaluated when the path is a circle of radius r around the crack tip

Γ is followed from $\theta = -\pi$ to $\theta = \pi$

We have,
$$ds = rd\theta$$
$$dy = r\cos\theta \, d\theta$$

J integral becomes,

$$J = \int_{-\pi}^{\pi} \left[w(r,\theta)\cos\theta - T_i(r,\theta)\frac{\partial u_i(r,\theta)}{\partial x} \right] r \, d\theta$$

When $r \to 0$ only the singular terms remain

For LEFM , we can obtain : $\quad J = G = \dfrac{K_I^2}{E^{'}}$ \quad (if mode I loading)

HRR theory

➢ **H**utchinson [1968] and **R**ice and **R**osengren [1968] independently evaluated the character of crack tip stress field in the case of power-law hardening materials.

➢ J characterizes the crack-tip field in a non-linear elastic material.

Assumptions:

○ Stress & strain fields near the tip of a stationary crack within plastic zone.

○ Consider 2D plane strain / plane stress & Mode I loading.

○ Material is characterized by small strain J_2 deformation theory of plasticity.

HRR theory

- For uniaxial deformation:

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha \left(\frac{\sigma}{\sigma_0} \right)^n \quad \text{Ramberg-Osgood equation}$$

σ_0 = yield strength

$\varepsilon_0 = \sigma_0 / E$

α : dimensionless constant

n : strain-hardening exponent

} material properties

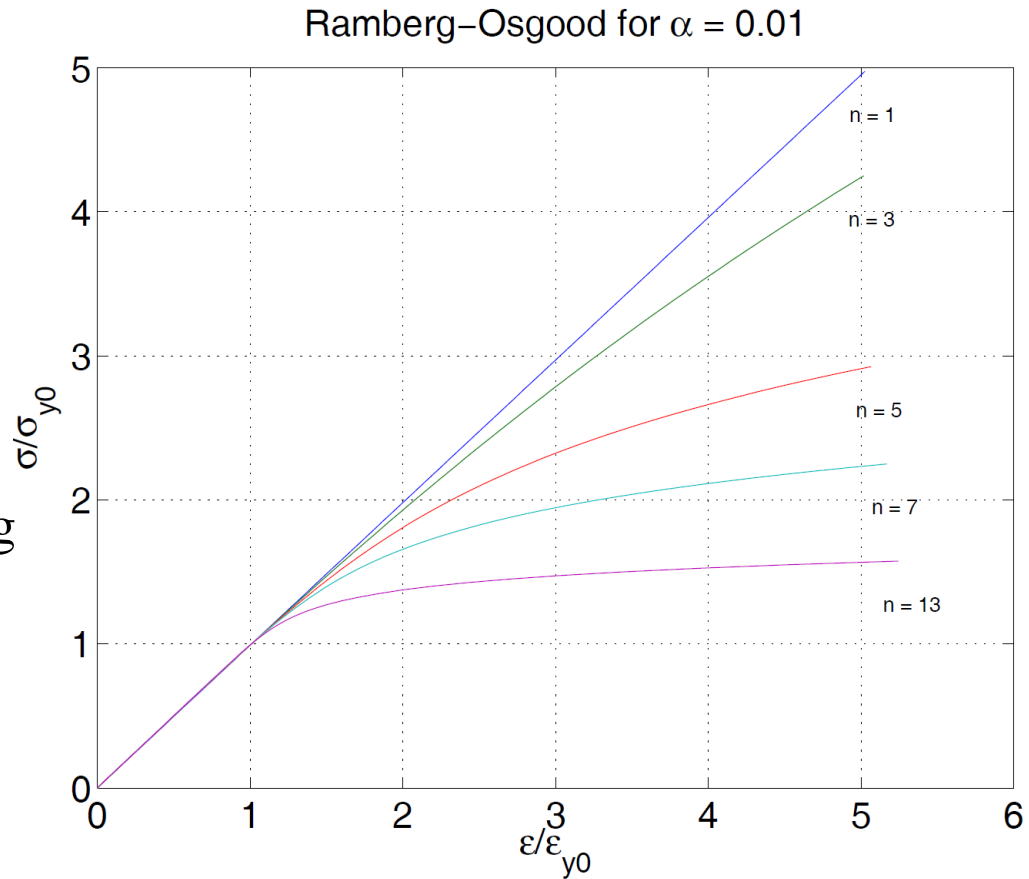Power law relationship assumed between plastic strain and stress.

For a linear elastic material $n = 1$.

Ramberg–Osgood model

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha \left(\frac{\sigma}{\sigma_0} \right)^n$$

$$\varepsilon = \varepsilon^{el} + \varepsilon^{pl}$$

- Elastic model:
Unlike plasticity unloading in on the same line
- Higher *n* closer to elastic perfectly plastic

Stress-strain relation according to the Ramberg-Osgood material law

Hutchinson, Rice and Rosengren(HRR) solution

- Near crack tip "plastic" strains dominate:

$$\frac{\varepsilon}{\varepsilon_0} = \alpha \left(\frac{\sigma}{\sigma_0} \right)^n \qquad (*)$$

  - Assume the following r dependence for σ and ε

$$\sigma = \frac{c_1}{r^x}$$

$$\varepsilon = \frac{c_2}{r^y}$$

1. Bounded energy:

$$\sigma\varepsilon \propto \frac{1}{r} \quad \Rightarrow \quad x + y = 1$$

2. $\varepsilon - \sigma$ relation (*)

$$x = \frac{1}{1+n}$$

$$y = \frac{n}{1+n}$$

- Asymptotic field derived by **H**utchinson **R**ice and **R**osengren:

$$\varepsilon_{ij} = A_2 \left(\frac{J}{r} \right)^{n/(n+1)} \qquad \sigma_{ij} = A_1 \left(\frac{J}{r} \right)^{1/(n+1)} \qquad u_i = A_3 \, J^{n/(n+1)} \, r^{1/(n+1)}$$

A_i are regular functions that depend on θ and the previous parameters.

The $1/\sqrt{r}$ singularity is recovered when $n = 1$.

Path independence of J $\implies$ The product $\sigma_{ij}\,\varepsilon_{ij}$ varies as $1/r$:

$$\text{From} \quad J = r \int_{-\pi}^{\pi} \left[w(r,\theta)\cos\theta - T_i(r,\theta) \frac{\partial u_i(r,\theta)}{\partial x} \right] d\theta$$

$$\sigma_{ij}\,\varepsilon_{ij} \to \frac{f(\theta)}{r} \quad as \quad r \to 0$$

J defines the amplitude of the HRR field as *K* does in the linear case.
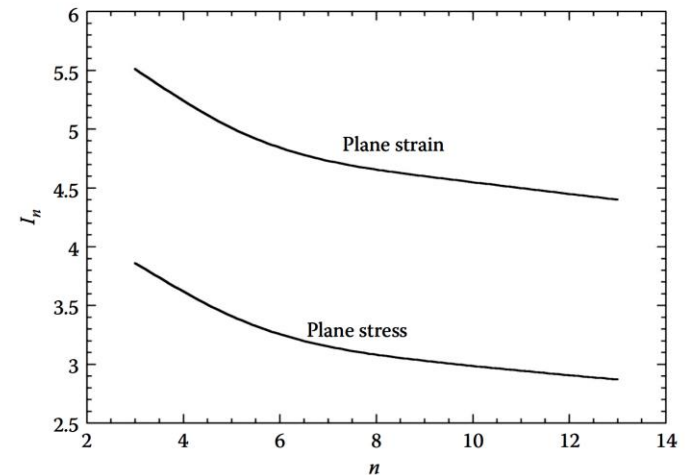
HRR solution

➤ Hutchinson, Rice & Rosengren proposed following form for plastic crack tip fields:

$$\sigma_{ij} = \sigma_0 \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{1/(n+1)} \tilde{\sigma}_{ij}(n,\theta) \qquad \varepsilon_{ij} = \frac{\alpha \sigma_0}{E} \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{n/(n+1)} \tilde{\varepsilon}_{ij}(n,\theta)$$

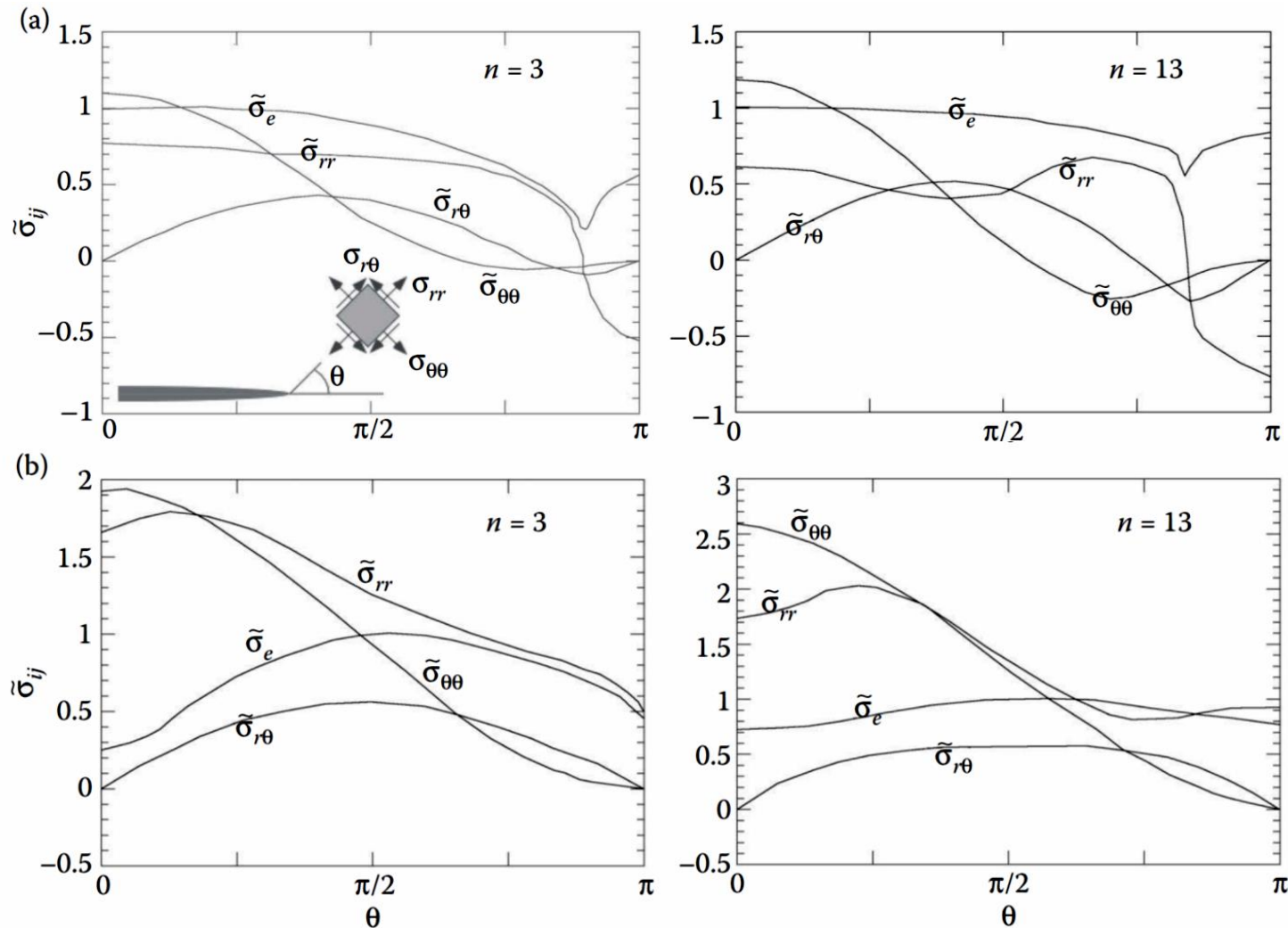(see: Appendix 3A.4)

where I_n is an integration constant that depends on n, and $\tilde{\sigma}_{ij}$ and $\tilde{\varepsilon}_{ij}$ are dimensionless functions of n and θ.

➤ *J* defines the amplitude of the HRR field as *K* does in the linear case.

➤ The equations are called the HRR singularity, named after Hutchinson, Rice, and Rosengren.

Effect of the strain hardening exponent on the HRR integration constant
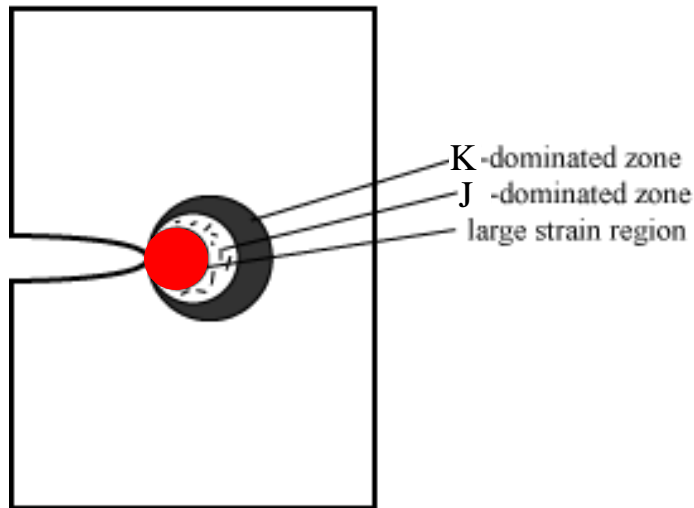
Angular variation of dimensionless stress for n = 3 and 13 (a) plane stress and (b) plane strain.

Two singular zones can be identified:
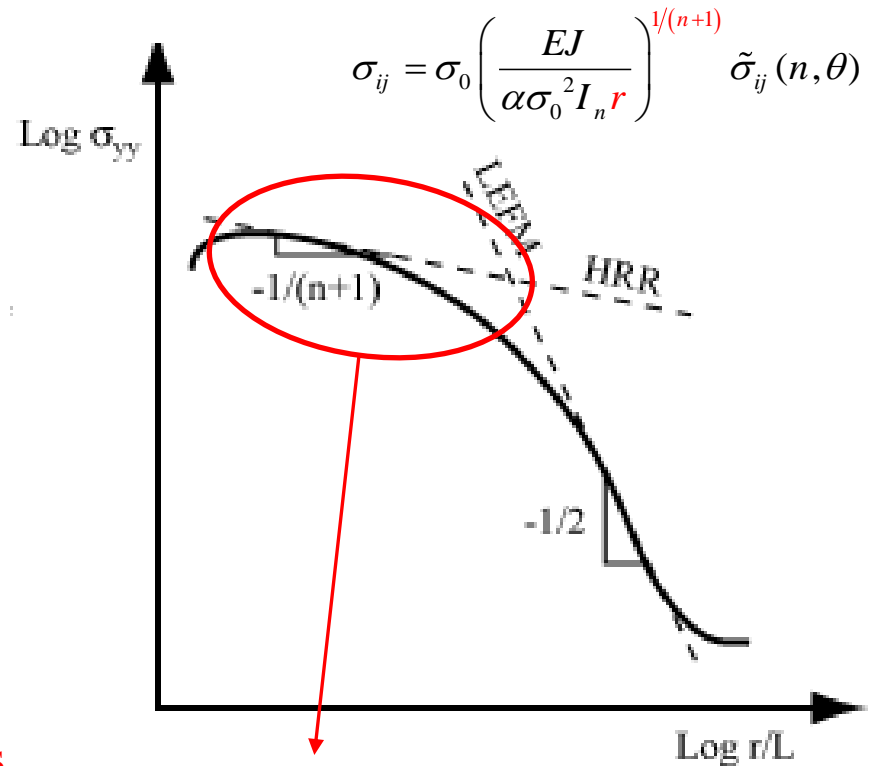
$$\sigma_{ij} = \sigma_0 \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{1/(n+1)} \tilde{\sigma}_{ij}(n, \theta)$$

K -dominated zone
J -dominated zone
large strain region

Log σ_{yy}

LEFM

HRR

-1/(n+1)

-1/2

Log r/L

Small region where crack blunting occurs.

➥ Large deformation

HRR based upon small displacements non applicable.
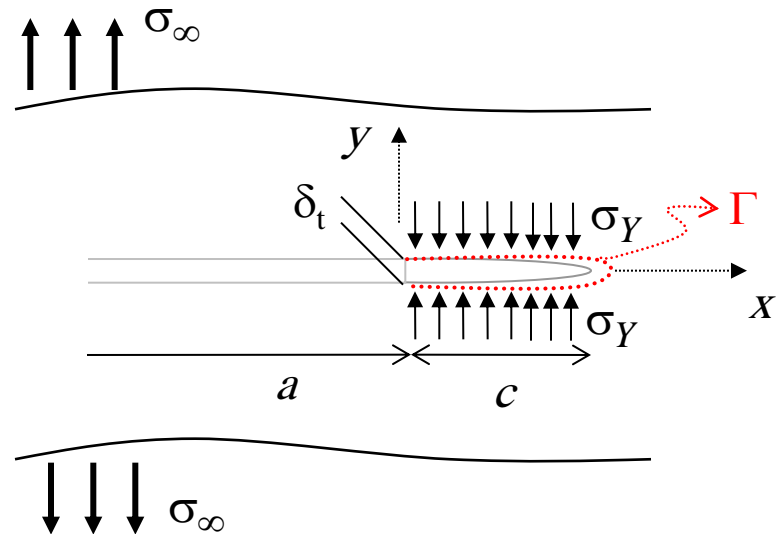
Stress is still singular but with
a weaker power of singularity!

▪ Relationship between J and CTOD

Consider again the strip-yield problem,

The first term in the J integral vanishes because $dy = 0$ (slender zone)

$$J = -\int_\Gamma \sigma_{ij} n_j \frac{\partial u_i}{\partial x} ds$$

$$\text{but} \quad \sigma_{ij} n_j \frac{\partial u_i}{\partial x} ds = \sigma_{yy} n_y \frac{\partial u_y}{\partial x} ds = -\sigma_Y \frac{\partial u_y}{\partial x} dx$$

$$J = \int_\Gamma \sigma_Y \frac{\partial u_y}{\partial x} dx = \int_{-\delta_t/2}^{\delta_t/2} \sigma_Y du_y = \sigma_Y \delta_t$$

General unique relationship between J and CTOD:

$$J = m \, \sigma_Y \, \delta_t$$

m : dimensionless parameter depending on the stress state and materials properties

• The strip-yield model predicts that $m=1$ (non-hardening material, plane stress condition)

• This relation is more generally derived for *hardening* materials ($n>1$) using the HRR displacements near the crack tip, i.e.

$$u_i = A_3 \, J^{n/(n+1)} \, r^{1/(n+1)}$$

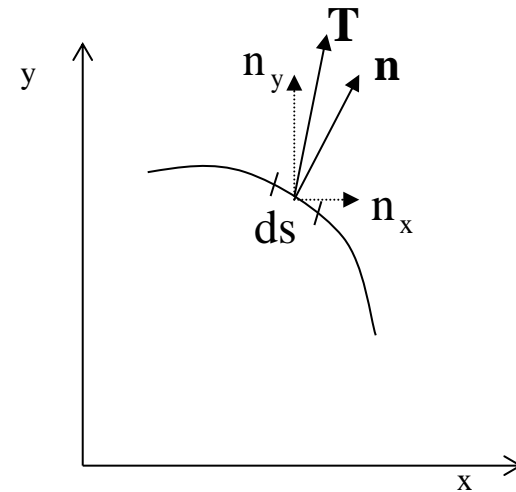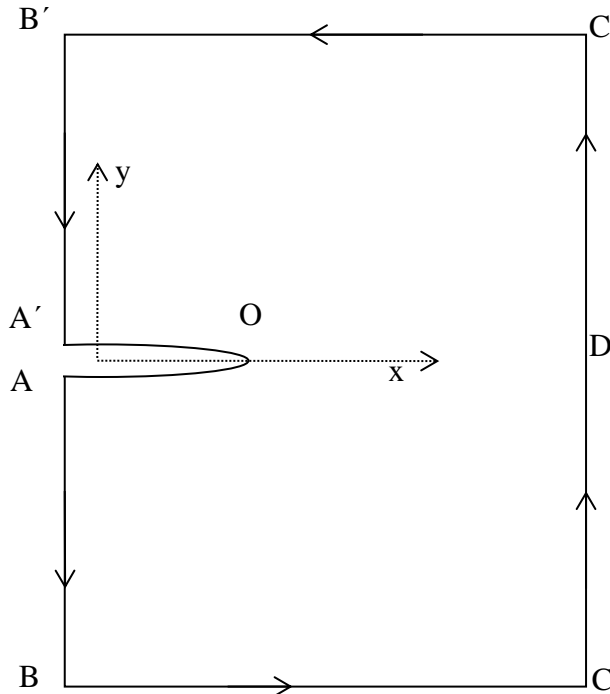Shih proposed this definition for δ_t :

blunted crack   90° δ_t
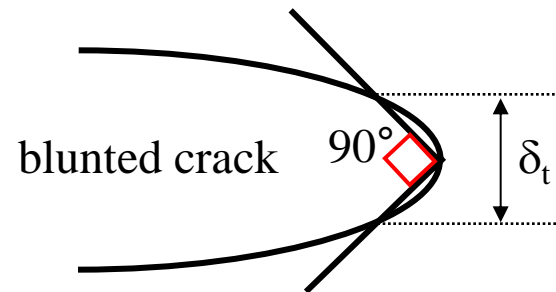
➡ m becomes a (complicated) function of n

➡ The proposed definition of δ_t agrees with the one of the Irwin model

Moreover, $G = \dfrac{\pi}{4}\sigma_Y \, \delta_t$, $m = \dfrac{\pi}{4}$ in this case

Applications the J-integral

J-integral evaluated explicitly along specific contours

Loads and geometry symmetric / Ox

$$J = \int_{\Gamma} \left(w\, dy - \sigma_{ij} n_j \frac{\partial u_i}{\partial x}\, ds \right) \quad ?$$

$$w = \int \sigma_{ij} d\varepsilon_{ij} = \frac{1}{2} \sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + 2\sigma_{xy}\varepsilon_{xy} \right)$$

for a plane stress, linear elastic problem

From stress-strain relation,

$$w = \frac{1}{2E}\left(\sigma_{xx}^2 + \sigma_{yy}^2 - 2\nu\sigma_{xx}\sigma_{yy}\right) + \frac{1+\nu}{E}\sigma_{xy}^2$$

Expanded form for $\sigma_{ij}n_j\dfrac{\partial u_i}{\partial x}ds$

$$= \sigma_{xx}n_x\frac{\partial u_x}{\partial x}ds + \sigma_{xy}n_y\frac{\partial u_x}{\partial x}ds + \sigma_{yx}n_x\frac{\partial u_y}{\partial x}ds + \sigma_{yy}n_y\frac{\partial u_y}{\partial x}ds \qquad \text{(2D problem)}$$

Simplification :

Along AB or B´ A´ $n_x = -1, n_y = 0$ and $ds = -dy \neq 0$

$$= \sigma_{xx}\frac{\partial u_x}{\partial x}dy + \sigma_{yx}n_x\frac{\partial u_y}{\partial x}dy$$

Along CD or DC´ $n_x = 1, n_y = 0$ and $ds = dy \neq 0$

$$= \sigma_{xx}\frac{\partial u_x}{\partial x}dy + \sigma_{yx}\frac{\partial u_y}{\partial x}dx$$

Along   BC or C′B′   BC : $n_x = 0$, $n_y = -1$ and ds=dx $\neq$ 0

$$=-\sigma_{xy}\frac{\partial u_x}{\partial x}dx-\sigma_{yy}\frac{\partial u_y}{\partial x}dx$$

C'B' : $n_x = 0$, $n_y = 1$ and ds=-dx $\neq$ 0

Along OA and A′O J is zero since $dy = 0$ and $T_i = 0$

Finally,

$$J=2\int_A^B\left[w-\sigma_{xx}\frac{\partial u_x}{\partial x}-\sigma_{xy}\frac{\partial u_y}{\partial x}\right]dy+2\int_B^C\left[\sigma_{xy}\frac{\partial u_x}{\partial x}+\sigma_{yy}\frac{\partial u_y}{\partial x}\right]dx+2\int_C^D\left[w-\sigma_{xx}\frac{\partial u_x}{\partial x}-\sigma_{xy}\frac{\partial u_y}{\partial x}\right]dy$$