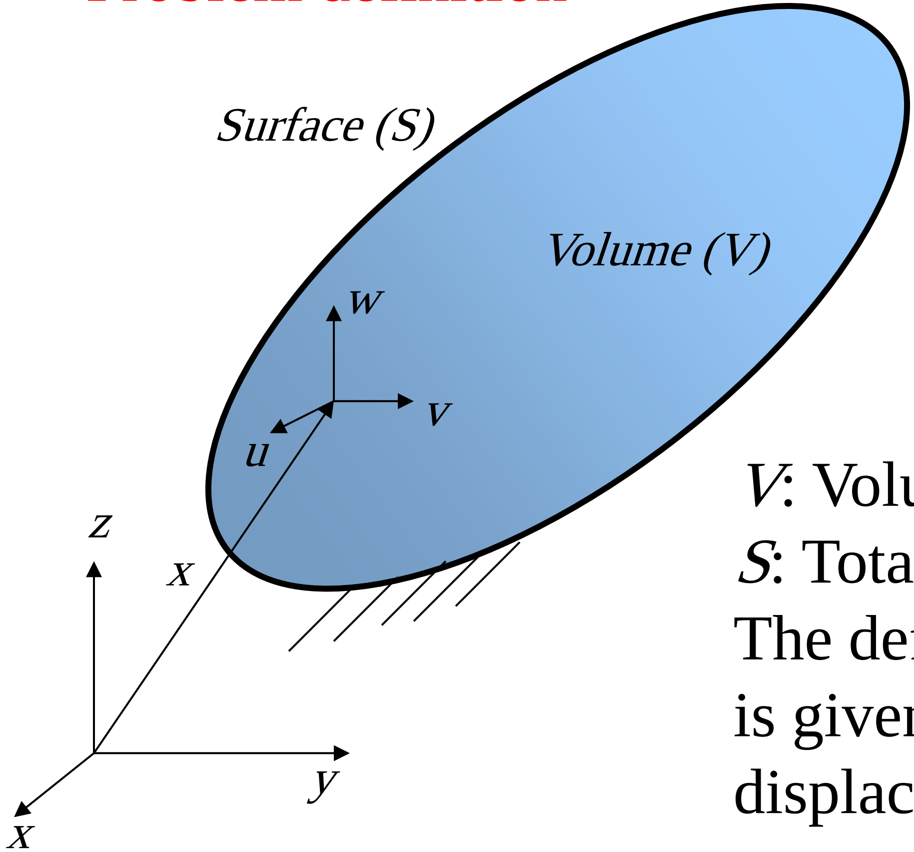
Problem definition



V: Volume of body

S: Total surface of the body

The deformation at point $x = [x,y,z]^T$ is given by the 3 components of its displacement

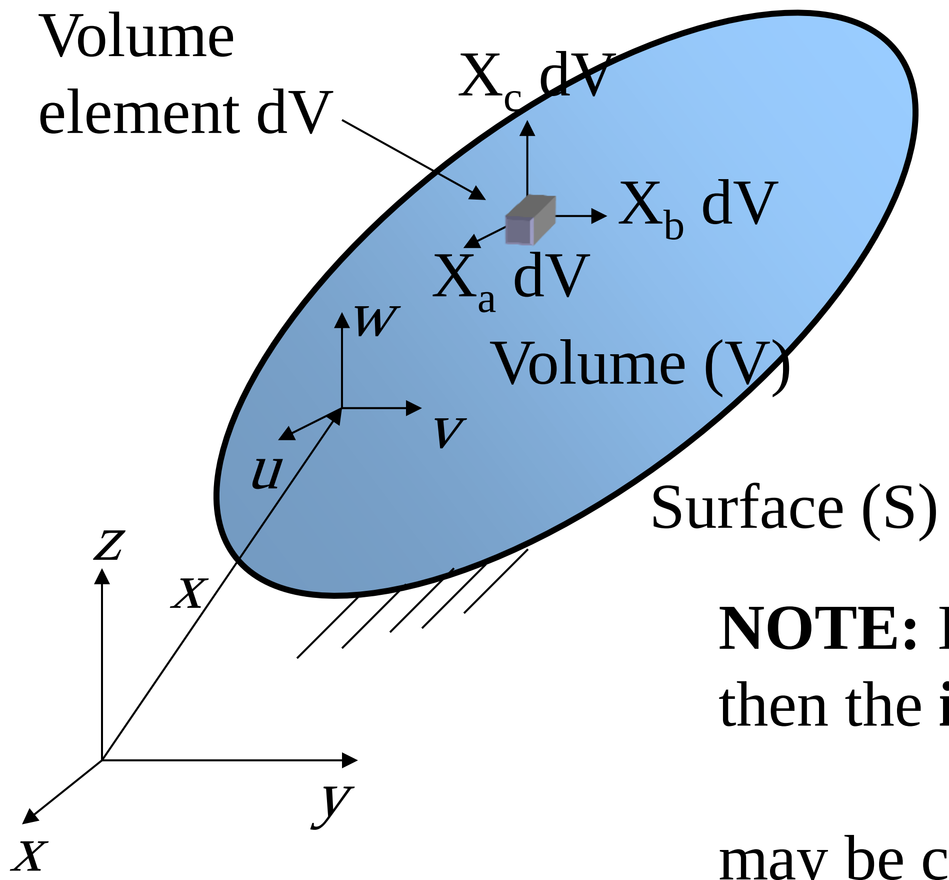
$$\mathbf{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

External forces acting on the body

Two basic types of **external forces** act on a body

1. Body force (force per unit **volume**) e.g., weight, inertia, etc
2. Surface traction (force per unit **surface area**) e.g., friction

Body force



Body force: distributed force per unit volume (e.g., weight, inertia, etc)

$$\mathbf{X} = \begin{Bmatrix} X_a \\ X_b \\ X_c \end{Bmatrix}$$

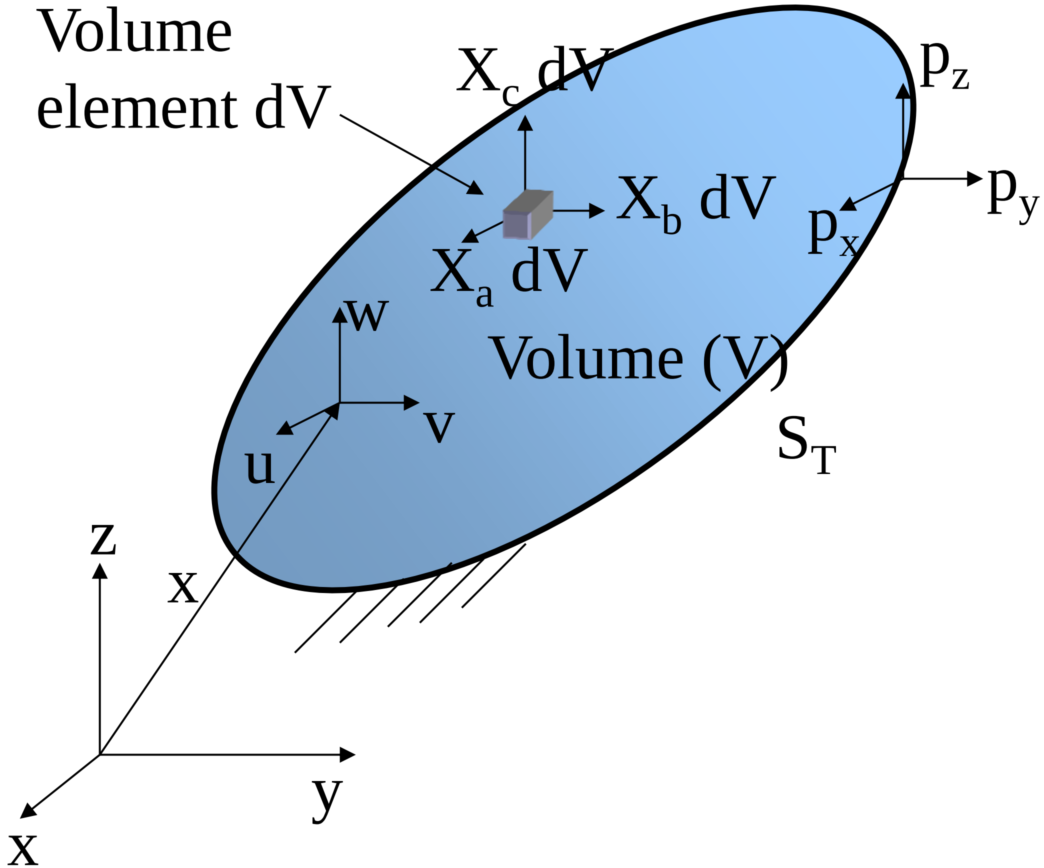
NOTE: If the body is accelerating, then the **inertia force**

$$\rho\,\ddot{\mathbf{u}} = \begin{Bmatrix} \rho\ddot{u} \\ \rho\ddot{v} \\ \rho\ddot{w} \end{Bmatrix}$$

may be considered as part of **X**

$$\mathbf{X} = \tilde{\mathbf{X}} - \rho\,\ddot{\mathbf{u}}$$

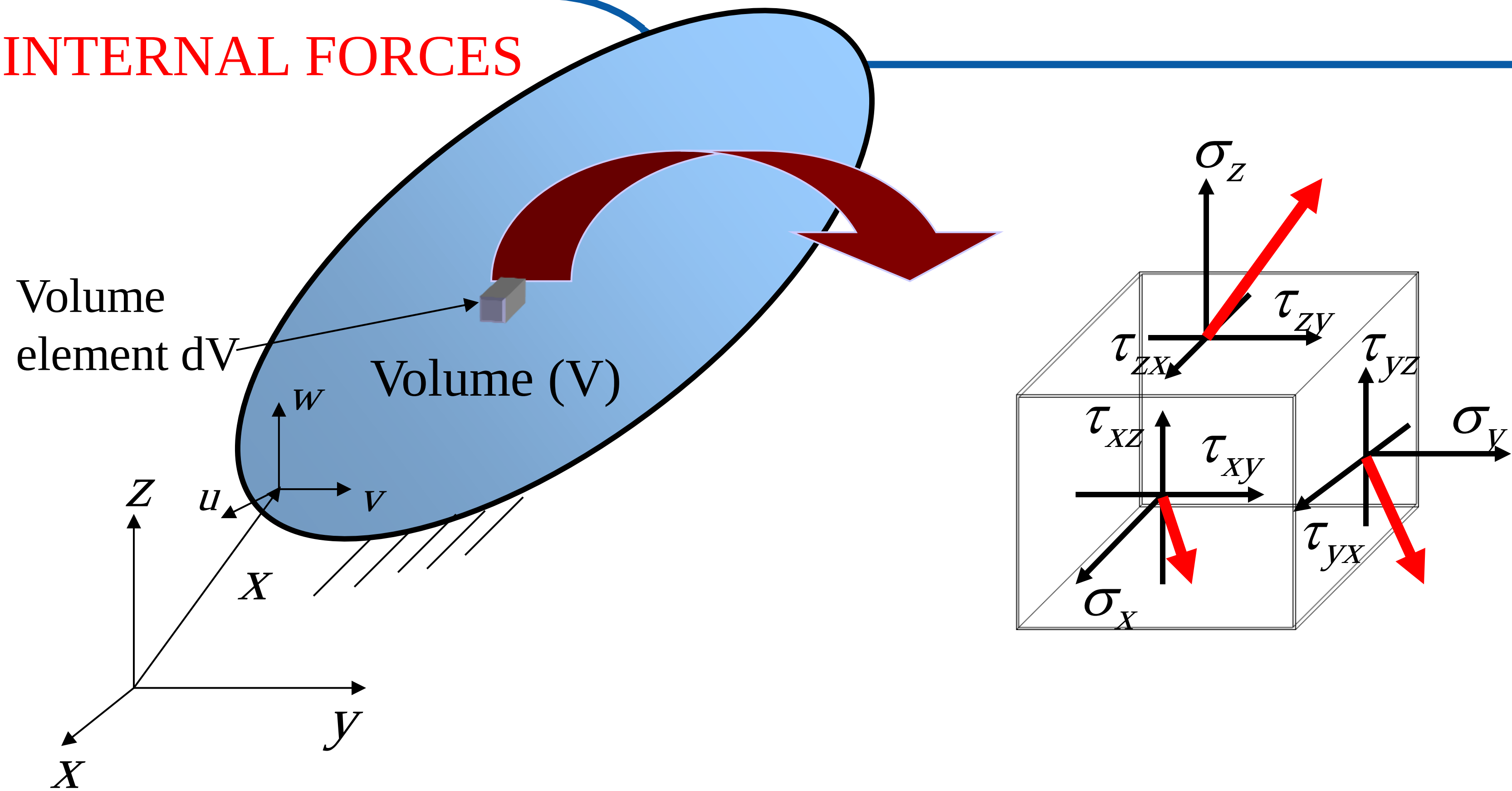
Surface Traction



Traction: Distributed force per unit surface area

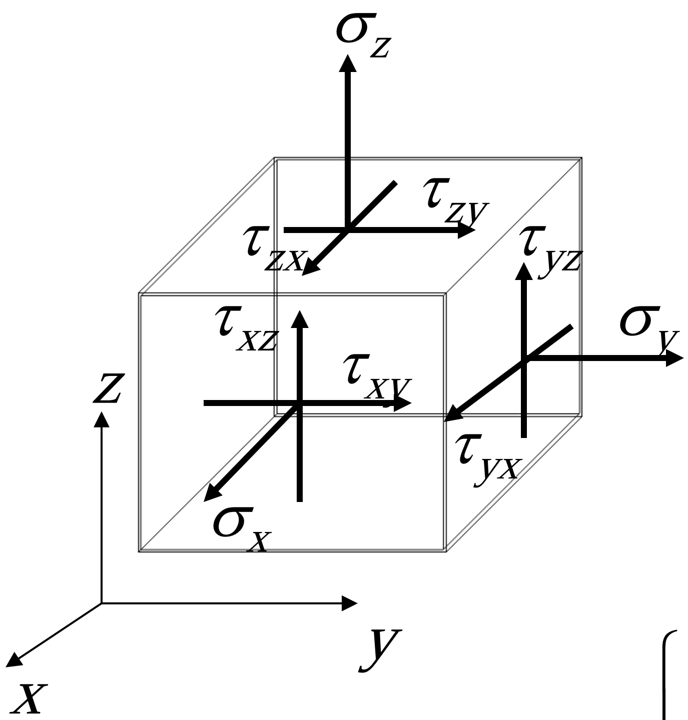
$$\mathbf{T}_S = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

INTERNAL FORCES



If I take out a chunk of material from the body, I will see that, due to the external forces applied to it, there are reaction forces (e.g., due to the loads applied to a truss structure, internal forces develop in each truss member). For the cube in the figure, the **internal reaction forces per unit area**(**red arrows**) , on each surface, may be decomposed into three orthogonal components.

الاستیسیته سه بعدی



σ_x, σ_y and σ_z are **normal stresses**.
The rest 6 are the **shear stresses**
Convention τ_{xy} is the stress on the face perpendicular to the x-axis and points in the +ve y direction. Total of 9 stress components of which only 6 are independent since

$$\tau_{xy} = \tau_{yx}$$
$$\tau_{yz} = \tau_{zy}$$
$$\tau_{zx} = \tau_{xz}$$

The **stress vector** is therefore

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

Strains: 6 independent **strain components**

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

الاستیسیته سه بعدی

Consider the equilibrium of a differential volume element to obtain the 3 **equilibrium equations** of elasticity

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X_a = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + X_b = 0$$

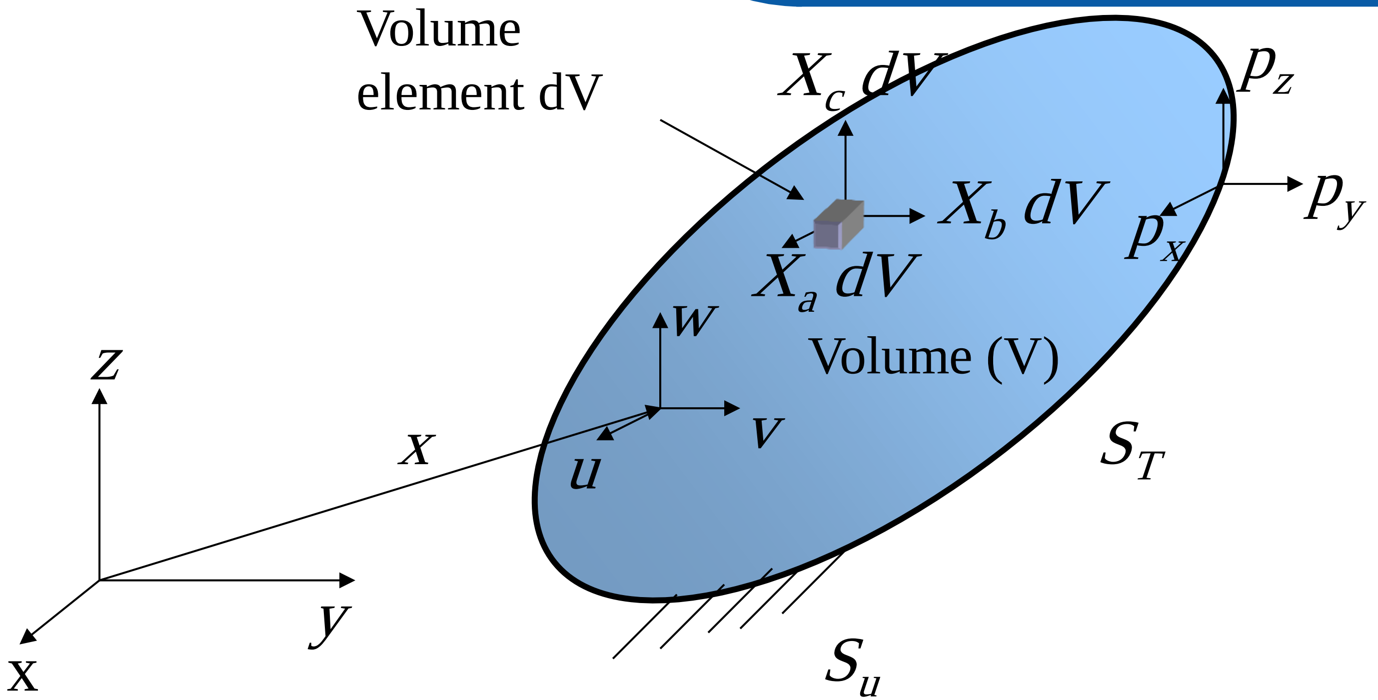
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + X_c = 0$$

$$\sigma_{ij,j} + X_i = 0$$

الاستیسیته سه بعدی

3D elasticity problem is completely defined once we understand the following three concepts

- Strong formulation (governing differential equation + boundary conditions)
- Strain-displacement relationship
- Stress-strain relationship



1. Strong formulation of the 3D elasticity problem: "Given the externally applied loads (on S_T and in V) and the specified displacements (on S_u) we want to solve for the resultant displacements, strains and stresses required to maintain equilibrium of the body."

Equilibrium equations: $$\sigma_{ij,j} + X_i = 0$$

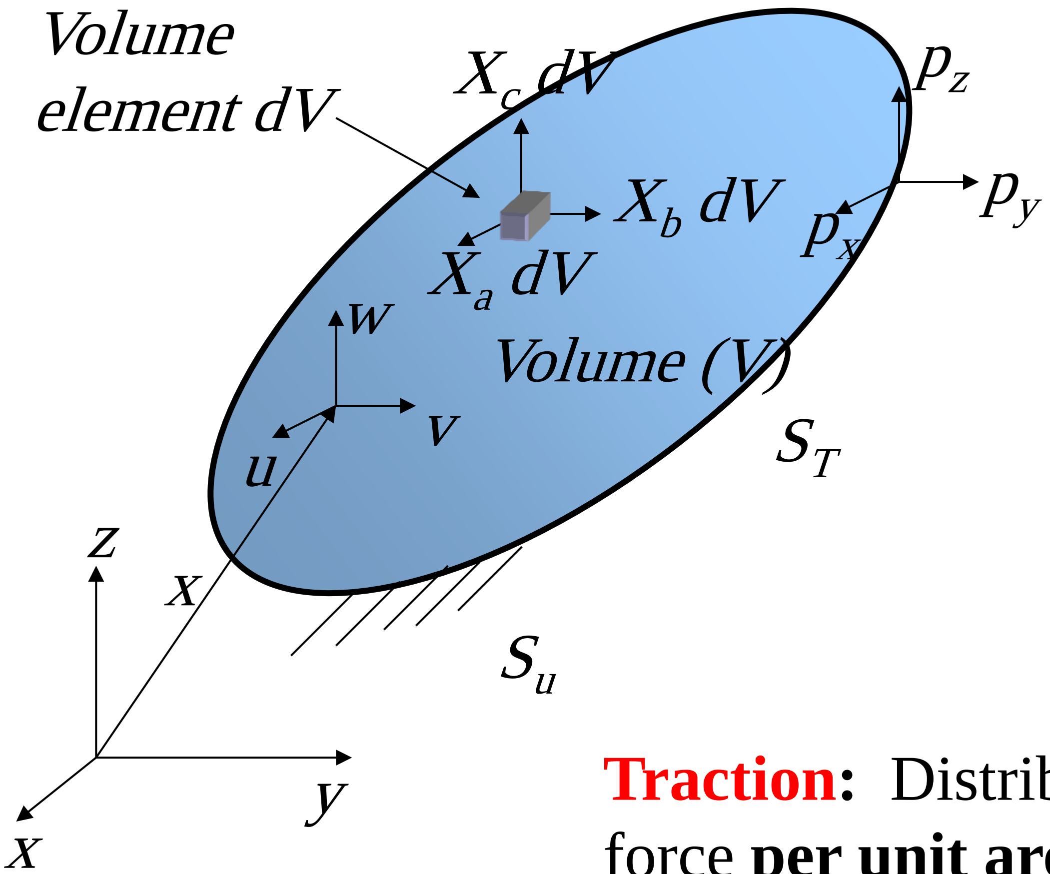
Boundary conditions:

1. Displacement boundary conditions: **Displacements** are specified on portion S_u of the boundary

$$\boldsymbol{u} = \boldsymbol{u}^{specified} \quad on \quad S_u$$

2. Traction (force) boundary conditions: **Tractions** are specified on portion S_T of the boundary

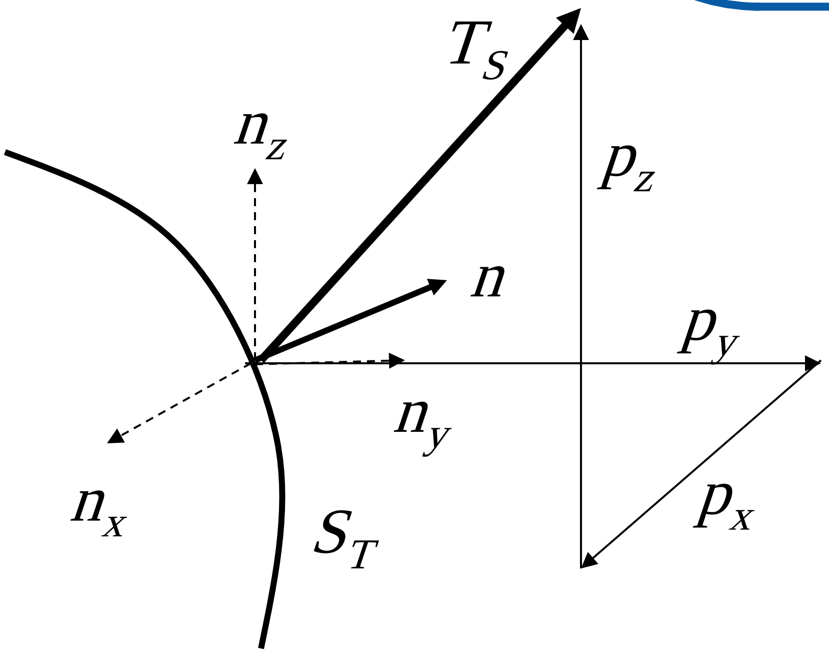
Now, how do I express this mathematically?



Traction: Distributed force **per unit area**

$$\boldsymbol{T}_S = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

الاستیسیته سه بعدی



$$\boldsymbol{T}_S = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

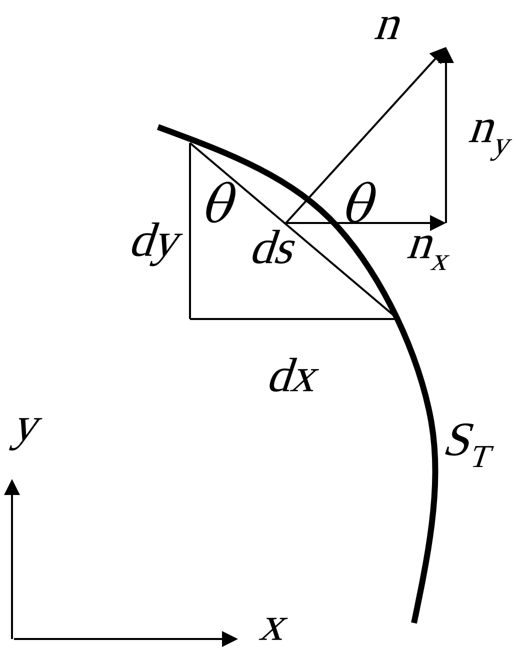
If the unit outward normal to S_T:

$$\boldsymbol{n} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

Then

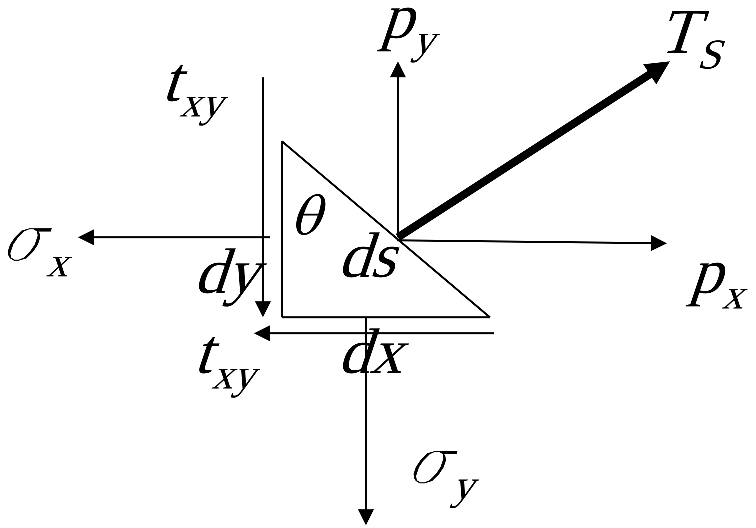
$$p_x = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z$$
$$p_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z$$
$$p_z = \tau_{xz} n_x + \tau_{zy} n_y + \sigma_z n_z$$

In 2D



$$\sin\theta = \frac{dx}{ds} = n_y$$

$$\cos\theta = \frac{dy}{ds} = n_x$$



Consider the equilibrium of the wedge in x-direction

$$p_x ds = \sigma_x dy + \tau_{xy} dx$$

$$\Rightarrow p_x = \sigma_x \frac{dy}{ds} + \tau_{xy} \frac{dx}{ds}$$

$$\Rightarrow p_x = \sigma_x n_x + \tau_{xy} n_y$$

Similarly $$p_y = \tau_{xy} n_x + \sigma_y n_y$$

الاستیسیته سه بعدی

3D elasticity problem is completely defined once we understand the following three concepts

- Strong formulation (governing differential equation + boundary conditions)
- Strain-displacement relationship
- Stress-strain relationship

2. Strain-displacement relationships:

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

3D elasticity problem is completely defined once we understand the following three concepts

- Strong formulation (governing differential equation + boundary conditions)
- Strain-displacement relationship
- Stress-strain relationship

3. Stress-Strain relationship:

Linear elastic material (Hooke's Law) $\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}$

Linear elastic isotropic material:

$$\mathbf{E} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Displacement Field:

$$u = \sum_{i=1}^{N} N_i u_i \qquad v = \sum_{i=1}^{N} N_i v_i \qquad w = \sum_{i=1}^{N} N_i w_i$$

In matrix form:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}_{(3\times1)} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \cdots \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \cdots \end{bmatrix}_{(3\times3N)} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \end{Bmatrix}_{(3N\times1)}$$

or $\mathbf{u} = \mathbf{N}\,\mathbf{d}$

Using the previous relations, we can derive the strain vector

$$\boldsymbol{\varepsilon} = \mathbf{B}\,\mathbf{d}$$

$(6\times1)\quad(6\times3N)\times(3N\times1)$

Stiffness Matrix: $$\mathbf{k} = \int_v \mathbf{B}^T \mathbf{E}\,\mathbf{B}\,dv$$

$(3N\times3N)\quad(3N\times6)\times(6\times6)\times(6\times3N)$

Numerical quadratures are often needed to evaluate the above integration.

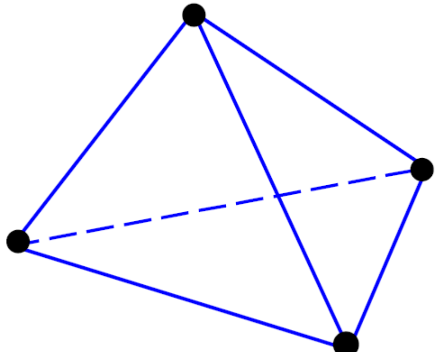
Rigid-body motions for 3-D bodies (6 components): *3 translations*, *3 rotations*.

These rigid-body motions (singularity of the system of equations) must be removed from the FEA model to ensure the quality of the analysis.

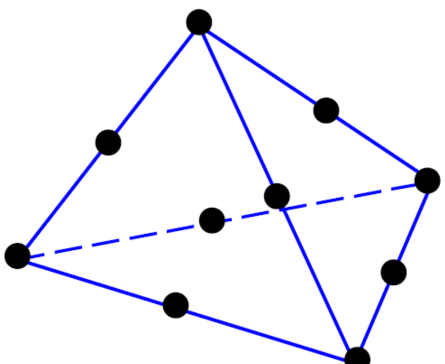
المان سه بعدی

المان چهار وجهی

Tetrahedron:



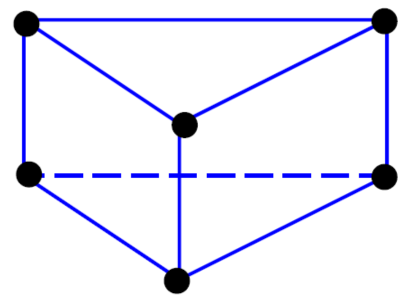
linear (4 nodes)



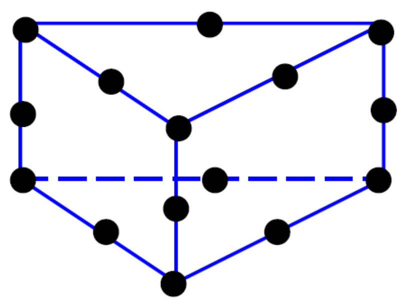
quadratic (10 nodes)

المان پنج وجهی

pentahedron:



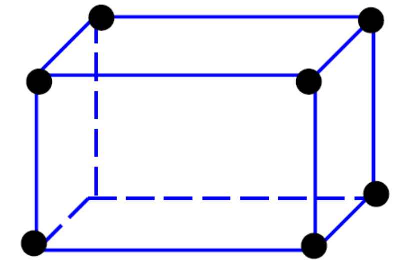
linear (6 nodes)



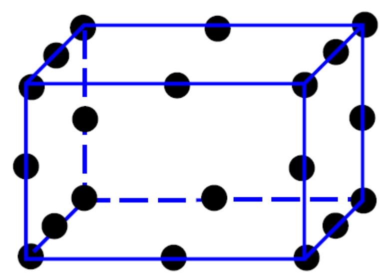
quadratic (15 nodes)

المان شش وجهی

Hexahedron (brick):

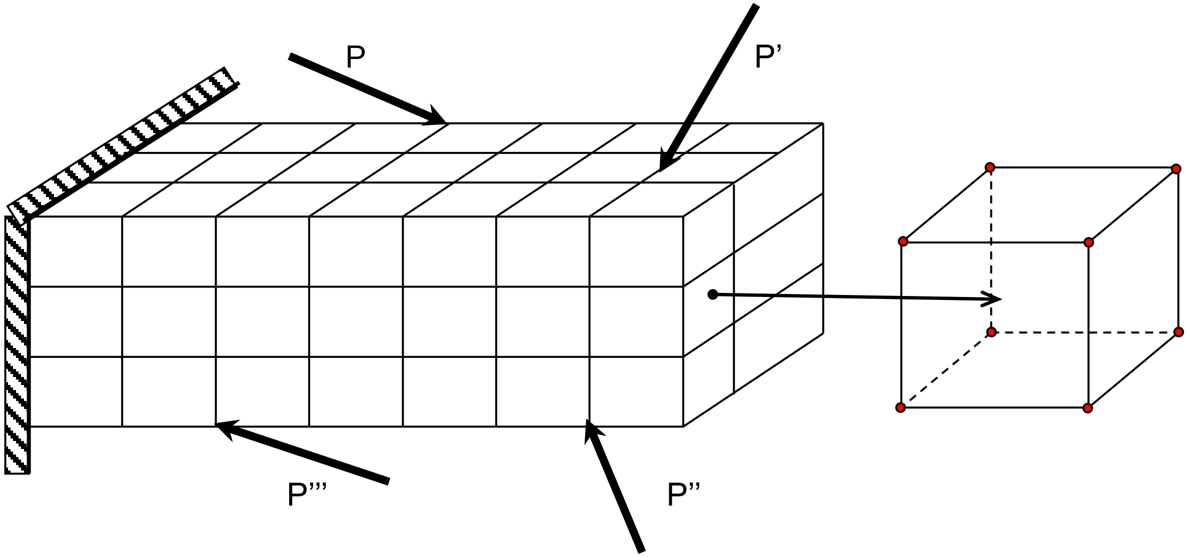


linear (8 nodes)



quadratic (20 nodes)

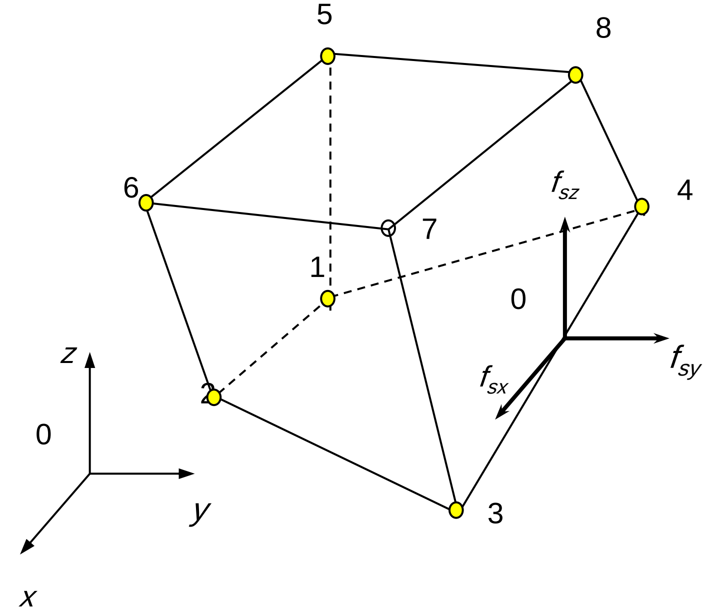
- 3D solid meshed with hexahedron elements



Shape functions

$$\mathbf{U} = \mathbf{N}\mathbf{d}$$

$$\mathbf{d} = \begin{Bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \\ \mathbf{d}_4 \\ \mathbf{d}_5 \\ \mathbf{d}_6 \\ \mathbf{d}_7 \\ \mathbf{d}_8 \end{Bmatrix} \begin{matrix} \text{displacement components at node 1} \\ \text{displacement components at node 2} \\ \text{displacement components at node 3} \\ \text{displacement components at node 4} \\ \text{displacement components at node 5} \\ \text{displacement components at node 6} \\ \text{displacement components at node 7} \\ \text{displacement components at node 8} \end{matrix}$$

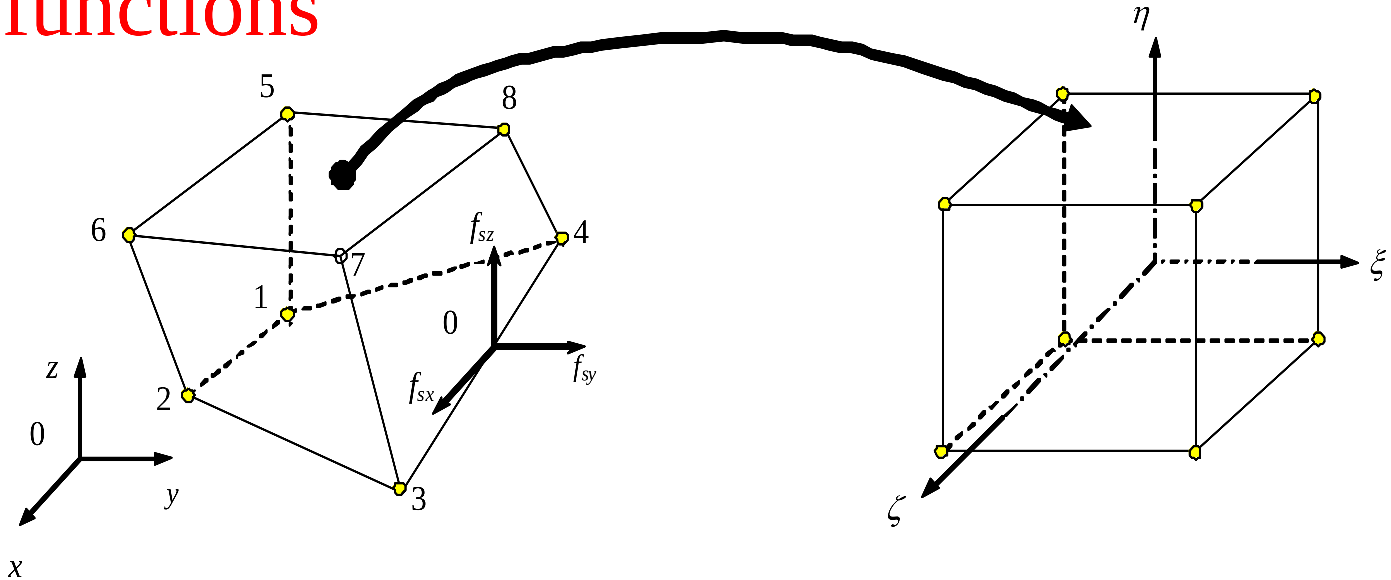


$$\mathbf{d}_i = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix} \quad (i = 1,\ 2, \cdots, 8)$$

$$\mathbf{N}_i = \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \quad (i = 1, 2, \cdots, 8)$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{N}_4 & \mathbf{N}_5 & \mathbf{N}_6 & \mathbf{N}_7 & \mathbf{N}_8 \end{bmatrix}$$

Shape functions



$$x = \sum_{i=1}^{8} N_i(\xi,\eta,\zeta)x_i$$

$$y = \sum_{i=1}^{8} N_i(\xi,\eta,\zeta)y_i$$

$$z = \sum_{i=1}^{8} N_i(\xi,\eta,\zeta)z_i$$

$$N_i = \frac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)$$

(Tri-linear functions)

ماتریس سختی المان شش وجهی خطی

Shape functions:

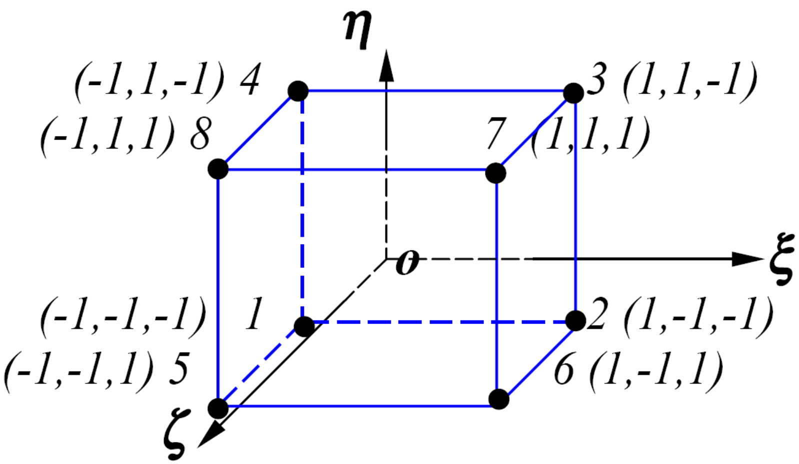
$$N_1(\xi,\eta,\zeta)=\frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)\quad ,$$

$$N_2(\xi,\eta,\zeta)=\frac{1}{8}(1+\xi)(1-\eta)(1-\zeta)\quad ,$$

$$N_3(\xi,\eta,\zeta)=\frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)\quad ,$$

$$\vdots \qquad\qquad \vdots$$

$$N_8(\xi,\eta,\zeta)=\frac{1}{8}(1-\xi)(1+\eta)(1+\zeta)\quad .$$



$$N_i=\tfrac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta)$$

Note that we have the following relations for the shape functions:

$$N_i(\xi_j,\eta_j,\zeta_j)=\delta_{ij}\quad ,\quad i,j=1,2,\cdots,8. \qquad \text{and} \qquad \sum_{i=1}^{8}N_i(\xi,\eta,\zeta)=1.$$

ماتریس سختی المان شش وجهی خطی

Strain matrix

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial x}{\partial x} + \dfrac{\partial u}{\partial y} \\ \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \end{Bmatrix} = \cdots_{use\,(15)} = \mathbf{B}\,\mathbf{d}$$

$$\boldsymbol{\varepsilon} = \mathbf{B}\,\mathbf{d}$$

$$(6\times 1) \quad (6\times 24)\times(24\times 1)$$

Strain matrix

$$\boldsymbol{\varepsilon} = \mathbf{Bd}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 & \mathbf{B}_5 & \mathbf{B}_6 & \mathbf{B}_7 & \mathbf{B}_8 \end{bmatrix}$$

$$\mathbf{B}_i = \mathbf{DN}_i = \begin{bmatrix} \partial N_i/\partial x & 0 & 0 \\ 0 & \partial N_i/\partial y & 0 \\ 0 & 0 & \partial N_i/\partial z \\ 0 & \partial N_i/\partial z & \partial N_i/\partial y \\ \partial N_i/\partial z & 0 & \partial N_i/\partial x \\ \partial N_i/\partial y & \partial N_i/\partial x & 0 \end{bmatrix}$$

Note: Shape functions are expressed in natural coordinates – chain rule of differentiation

Strain matrix

$$\left.\begin{aligned}
\frac{\partial N_i}{\partial \xi} &= \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial \xi} \\
\frac{\partial N_i}{\partial \eta} &= \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial \eta} \\
\frac{\partial N_i}{\partial \zeta} &= \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \zeta} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial \zeta}
\end{aligned}\right\}$$

Chain rule of differentiation

$$\Rightarrow \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix} \quad \text{where} \quad \mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}$$

Strain matrix

The same shape functions are used as for the displacement field (***Isoparametric element***).

Coordinate Transformation: $x=\sum_{i=1}^{8} N_i(\xi,\eta,\zeta)x_i,\quad y=\sum_{i=1}^{8} N_i(\xi,\eta,\zeta)y_i,\quad z=\sum_{i=1}^{8} N_i(\xi,\eta,\zeta)z_i$

$$\mathbf{J}=\begin{bmatrix} \frac{\partial N_1}{\partial\xi} & \frac{\partial N_2}{\partial\xi} & \frac{\partial N_3}{\partial\xi} & \frac{\partial N_4}{\partial\xi} & \frac{\partial N_5}{\partial\xi} & \frac{\partial N_6}{\partial\xi} & \frac{\partial N_7}{\partial\xi} & \frac{\partial N_8}{\partial\xi} \\ \frac{\partial N_1}{\partial\eta} & \frac{\partial N_2}{\partial\eta} & \frac{\partial N_3}{\partial\eta} & \frac{\partial N_4}{\partial\eta} & \frac{\partial N_5}{\partial\eta} & \frac{\partial N_6}{\partial\eta} & \frac{\partial N_7}{\partial\eta} & \frac{\partial N_8}{\partial\eta} \\ \frac{\partial N_1}{\partial\zeta} & \frac{\partial N_2}{\partial\zeta} & \frac{\partial N_3}{\partial\zeta} & \frac{\partial N_4}{\partial\zeta} & \frac{\partial N_5}{\partial\zeta} & \frac{\partial N_6}{\partial\zeta} & \frac{\partial N_7}{\partial\zeta} & \frac{\partial N_8}{\partial\zeta} \end{bmatrix}\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \\ x_7 & y_7 & z_7 \\ x_8 & y_8 & z_8 \end{bmatrix}$$

or

$$\mathbf{J}=\begin{bmatrix} \sum_{i=1}^{8} x_i\frac{\partial N_i}{\partial\xi} & \sum_{i=1}^{8} y_i\frac{\partial N_i}{\partial\xi} & \sum_{i=1}^{8} z_i\frac{\partial N_i}{\partial\xi} \\ \sum_{i=1}^{8} x_i\frac{\partial N_i}{\partial\eta} & \sum_{i=1}^{8} y_i\frac{\partial N_i}{\partial\eta} & \sum_{i=1}^{8} z_i\frac{\partial N_i}{\partial\eta} \\ \sum_{i=1}^{8} x_i\frac{\partial N_i}{\partial\zeta} & \sum_{i=1}^{8} y_i\frac{\partial N_i}{\partial\zeta} & \sum_{i=1}^{8} z_i\frac{\partial N_i}{\partial\zeta} \end{bmatrix}$$

Strain matrix

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \\ \dfrac{\partial N_i}{\partial \zeta} \end{Bmatrix}$$

Used to replace derivatives w.r.t. x, y, z with derivatives w.r.t. ξ, η, ζ

$$\mathbf{B}_i = \mathbf{D}\mathbf{N}_i = \begin{bmatrix} \partial N_i/\partial x & 0 & 0 \\ 0 & \partial N_i/\partial y & 0 \\ 0 & 0 & \partial N_i/\partial z \\ 0 & \partial N_i/\partial z & \partial N_i/\partial y \\ \partial N_i/\partial z & 0 & \partial N_i/\partial x \\ \partial N_i/\partial y & \partial N_i/\partial x & 0 \end{bmatrix}$$

Strain energy:

$$U = \frac{1}{2}\int_V \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}\, dV = \frac{1}{2}\int_V (\mathbf{E}\boldsymbol{\varepsilon})^T \boldsymbol{\varepsilon}\, dV = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^T \mathbf{E}\,\boldsymbol{\varepsilon}\, dV$$

$$= \frac{1}{2}\mathbf{d}^T \left[\int_V \mathbf{B}^T \mathbf{E}\,\mathbf{B}\, dV\right]\mathbf{d}$$

Element stiffness matrix:

$$\underset{(24\times24)}{\mathbf{k}} = \int_V \underset{(24\times6)\times(6\times6)\times(6\times24)}{\mathbf{B}^T \mathbf{E}\,\mathbf{B}}\, dV$$

In $\xi\eta\zeta$ coordinates: $\mathbf{k} = \int_V \mathbf{B}^T\mathbf{E}\mathbf{B}\,\mathrm{d}V = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} \mathbf{B}^T\mathbf{E}\mathbf{B}\,\det[\mathbf{J}]\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$

Gauss integration: $I = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} w_i w_j w_k\, f(\xi_i,\eta_j,\zeta_j)$

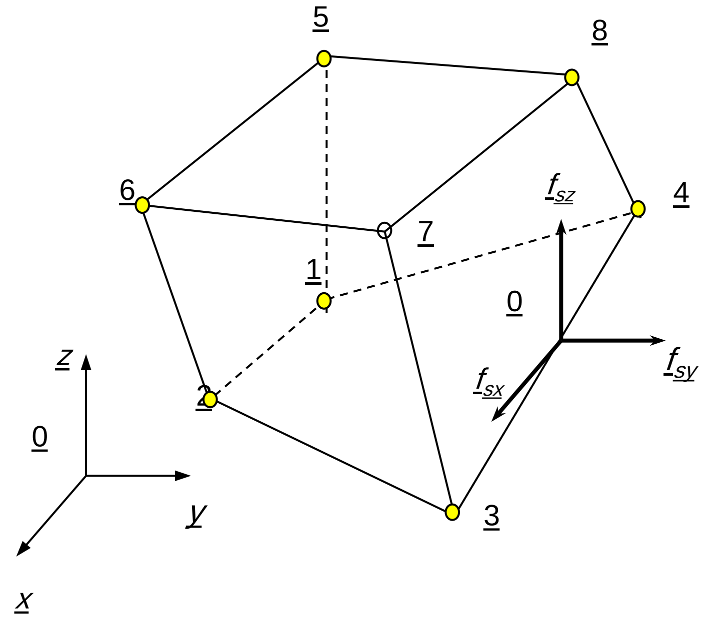
Note: 3-D elements usually do not use rotational DOFs.

Loads: Distributed loads → Nodal forces

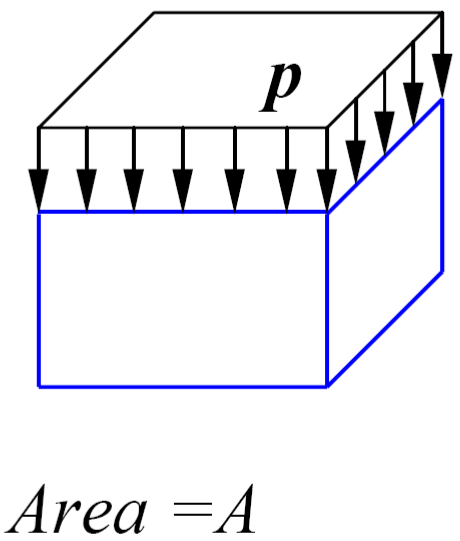
$$\mathbf{f}_e = \int_l [\mathbf{N}]^{\mathrm{T}}\Big|_{3-4} \begin{Bmatrix} f_{sx} \\ f_{sy} \\ f_{sz} \end{Bmatrix} \mathrm{d}l$$

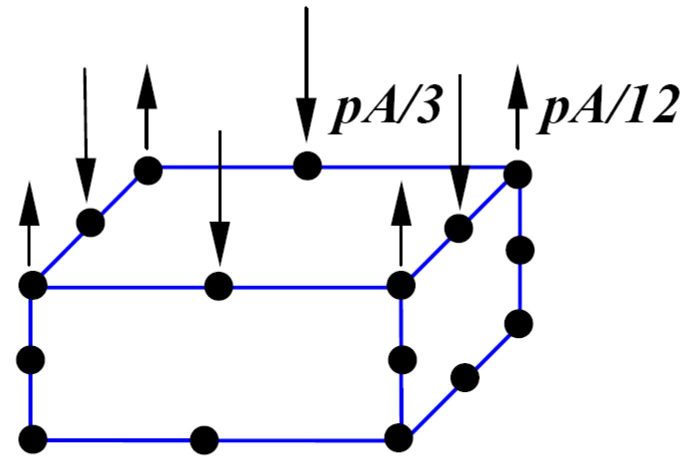
For uniformly distributed load:

$$\mathbf{f}_e = \frac{1}{2} l_{3-4} \begin{Bmatrix} \{\mathbf{0}\}_{3\times1} \\ \{\mathbf{0}\}_{3\times1} \\ \begin{Bmatrix} f_{sx} \\ f_{sy} \\ f_{sz} \end{Bmatrix} \\ \begin{Bmatrix} f_{sx} \\ f_{sy} \\ f_{sz} \end{Bmatrix} \\ \{\mathbf{0}\}_{3\times1} \\ \{\mathbf{0}\}_{3\times1} \\ \{\mathbf{0}\}_{3\times1} \\ \{\mathbf{0}\}_{3\times1} \end{Bmatrix}$$



Loads: Distributed loads → Nodal forces





Stresses:

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon} = \mathbf{E}\,\mathbf{B}\,\mathbf{d}$$

Principal stresses: $\sigma_1, \sigma_2, \sigma_3$.

von Mises stress: $\sigma_e = \sigma_{VM} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$.

Stresses are evaluated at selected points (including nodes) on each element. Averaging (around a node, for example) may be employed to smooth the field.

Example:

For the rectangular solid element, find the stiffness term k_{11}, assuming isotropic material.

$$[\mathbf{k}_{11}] = abc \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [\mathbf{B}]_{1,1}^{T} [\mathbf{E}][\mathbf{B}]_{1,1}\, d\xi d\eta d\zeta$$

If the hexahedron is rectangular with dimensions of a ×b ×c, the determinate of the Jacobian matrix is simply given by $\det[\mathbf{J}] = abc = V_e$

$$[\mathbf{B}]_{1,1} = \begin{Bmatrix} \partial N_1/\partial x \\ 0 \\ 0 \\ \partial N_1/\partial y \\ 0 \\ \partial N_1/\partial z \end{Bmatrix} = -\frac{1}{8}\begin{Bmatrix} \frac{1}{a}(1-\eta)(1-\zeta) \\ 0 \\ 0 \\ \frac{1}{b}(1-\xi)(1-\zeta) \\ 0 \\ \frac{1}{c}(1-\xi)(1-\eta) \end{Bmatrix}$$

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$$[\mathbf{k}_{11}] = abc\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left(-\frac{1}{8}\right)\left[\frac{1}{a}(1-\eta)(1-\zeta) \quad 0 \quad 0 \quad \frac{1}{b}(1-\xi)(1-\zeta) \quad 0 \quad \frac{1}{c}(1-\xi)(1-\eta)\right]$$

$$\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} f & \nu & \nu & 0 & 0 & 0 \\ \nu & f & \nu & 0 & 0 & 0 \\ \nu & \nu & f & 0 & 0 & 0 \\ 0 & 0 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & 0 & g \end{bmatrix}\left(-\frac{1}{8}\right)\begin{Bmatrix} \frac{1}{a}(1-\eta)(1-\zeta) \\ 0 \\ 0 \\ \frac{1}{b}(1-\xi)(1-\zeta) \\ 0 \\ \frac{1}{c}(1-\xi)(1-\eta) \end{Bmatrix} d\xi d\eta d\zeta$$

where: $f = 1-\nu$

$g = 1-2\nu$

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$$[\mathbf{k}_{11}] = \frac{Eabc}{64(1+\nu)(1-2\nu)} \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \left[\frac{1}{a}(1-\eta)(1-\zeta) \quad 0 \quad 0 \quad \frac{1}{b}(1-\xi)(1-\zeta) \quad 0 \quad \frac{1}{c}(1-\xi)(1-\eta) \right]$$

$$\begin{Bmatrix} \frac{1}{a}(1-\eta)(1-\zeta)f \\ \frac{1}{a}(1-\eta)(1-\zeta)\nu \\ \frac{1}{a}(1-\eta)(1-\zeta)\nu \\ \frac{1}{b}(1-\xi)(1-\zeta)g \\ 0 \\ \frac{1}{c}(1-\xi)(1-\eta)g \end{Bmatrix} d\xi d\eta d\zeta$$

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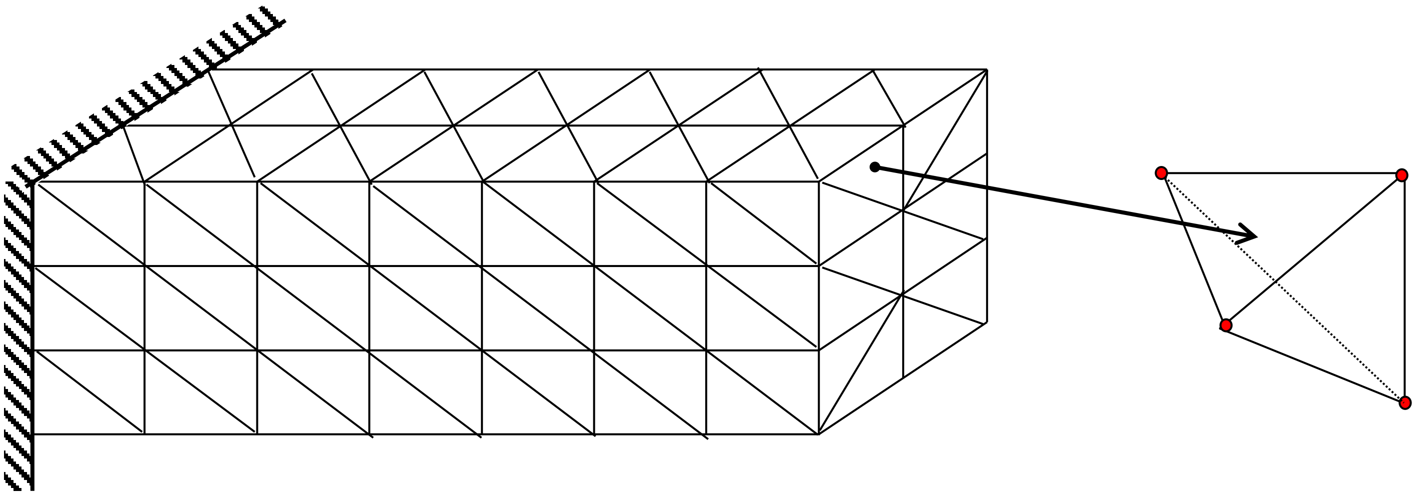
$$[\mathbf{k}_{11}] = \frac{Eabc}{64(1+\nu)(1-2\nu)} \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \left[\frac{1}{a^2}(1-\eta)^2(1-\zeta)^2 f + \frac{1}{b^2}(1-\xi)^2(1-\zeta)^2 g + \frac{1}{c^2}(1-\xi)^2(1-\eta)^2 g \right] d\xi d\eta d\zeta$$

$$\mathbf{k}_{11} = \frac{2abcE}{9(1+\nu)(1-2\nu)} \left(\frac{f}{a^2} + \frac{g}{b^2} + \frac{g}{2c^2} \right)$$

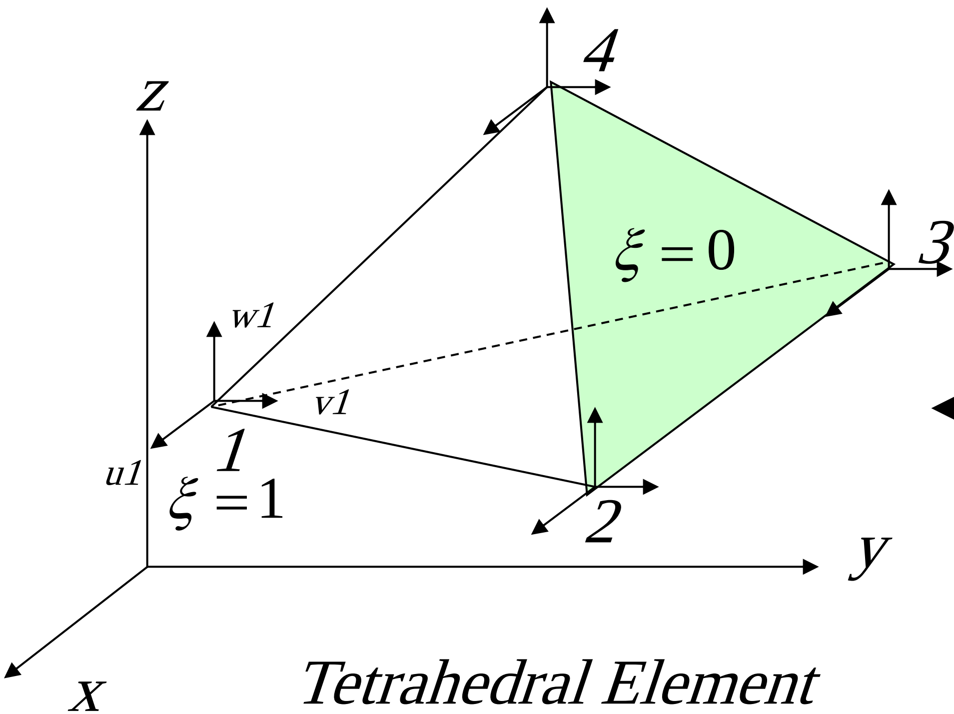
$$\mathbf{k}_{11} = \frac{2abcE}{9(1+\nu)(1-2\nu)} \left(\frac{1-\nu}{a^2} + \frac{1-2\nu}{b^2} + \frac{1-2\nu}{2c^2} \right)$$

المان چهار وجهی خطی

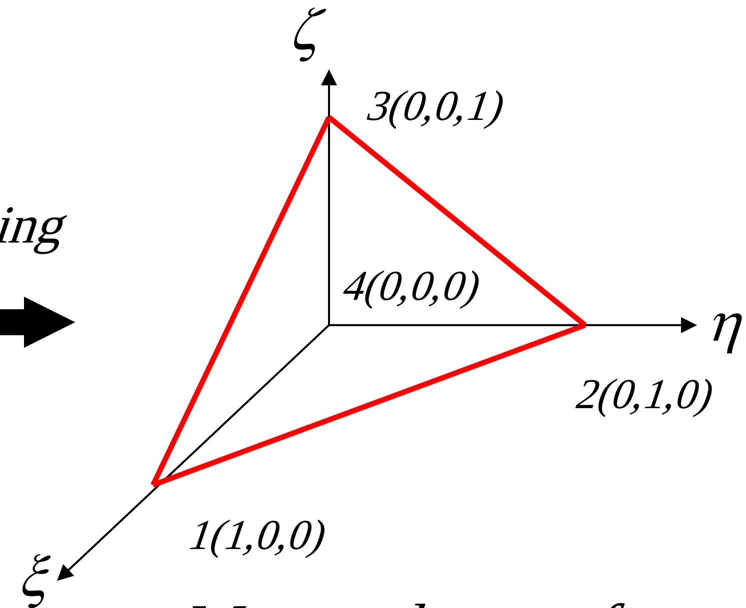
- 3D solid meshed with tetrahedron elements



المان چهار وجهی خطی



Mapping



Tetrahedral Element

$$N_1=\xi$$

$$N_2=\eta$$

$$N_3=\zeta$$

$$N_4=1-\xi-\eta-\zeta$$

Master element for shape function

ماتریس سختی المان چهار وجهی خطی

$$\mathbf{u} = \mathbf{N}\mathbf{d}$$

$$\mathbf{N} = \begin{bmatrix} \overbrace{N_1 \quad 0 \quad 0}^{\text{node 1}} & \overbrace{N_2 \quad 0 \quad 0}^{\text{node 2}} & \overbrace{N_3 \quad 0 \quad 0}^{\text{node 3}} & \overbrace{N_4 \quad 0 \quad 0}^{\text{node 4}} \\ 0 \quad N_1 \quad 0 & 0 \quad N_2 \quad 0 & 0 \quad N_3 \quad 0 & 0 \quad N_4 \quad 0 \\ 0 \quad 0 \quad N_1 & 0 \quad 0 \quad N_2 & 0 \quad 0 \quad N_3 & 0 \quad 0 \quad N_4 \end{bmatrix}$$

$$\mathbf{d} = \{u_1 \ v_1 \ w_1 \ \ldots \ u_4 \ v_4 \ w_4\}^T$$

For iso-parametric formulation we have:

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$
$$z = N_1 z_1 + N_2 z_2 + N_3 z_3 + N_4 z_4$$

Or

$$x = x_4 + x_{14}\xi + x_{24}\eta + x_{34}\zeta$$
$$y = y_4 + y_{14}\xi + y_{24}\eta + y_{34}\zeta$$
$$z = z_4 + z_{14}\xi + z_{24}\eta + z_{34}\zeta$$

ماتریس سختی المان چهار وجهی خطی

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial u}{\partial \zeta} \end{Bmatrix} = \boldsymbol{J} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} \qquad \boldsymbol{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial y}{\partial \zeta} & \dfrac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} x_{14} & y_{14} & z_{14} \\ x_{24} & y_{24} & z_{24} \\ x_{34} & y_{34} & z_{34} \end{bmatrix}$$

or

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} = \boldsymbol{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial u}{\partial \zeta} \end{Bmatrix} \qquad \boldsymbol{J}^{-1} = \frac{1}{\det \boldsymbol{J}} \begin{bmatrix} y_{24}z_{34} - y_{34}z_{24} & y_{34}z_{14} - y_{14}z_{34} & y_{14}z_{24} - y_{24}z_{14} \\ z_{24}x_{34} - z_{34}x_{24} & z_{34}x_{14} - z_{14}x_{34} & z_{14}x_{24} - z_{24}x_{14} \\ x_{24}y_{34} - x_{34}y_{24} & x_{34}y_{14} - x_{14}y_{34} & x_{14}y_{24} - x_{24}y_{14} \end{bmatrix}$$

$$= \boldsymbol{A}$$

ماتریس سختی المان چهار وجهی خطی

$$\boldsymbol{\varepsilon} = \mathbf{DU} = \mathbf{DNd} = \mathbf{Bd}$$

$$\mathbf{B} = \mathbf{DN} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \\ \partial/\partial y & \partial/\partial x & 0 \end{bmatrix} \mathbf{N}$$

where

$$\tilde{A}_1 = A_{11} + A_{12} + A_{13}$$
$$\tilde{A}_2 = A_{21} + A_{22} + A_{23}$$
$$\tilde{A}_3 = A_{31} + A_{32} + A_{33}$$

$$\mathbf{B} = \begin{bmatrix} A_{11} & 0 & 0 & A_{12} & 0 & 0 & A_{13} & 0 & 0 & -\tilde{A}_1 & 0 & 0 \\ 0 & A_{21} & 0 & 0 & A_{22} & 0 & 0 & A_{23} & 0 & 0 & -\tilde{A}_2 & 0 \\ 0 & 0 & A_{31} & 0 & 0 & A_{32} & 0 & 0 & A_{33} & 0 & 0 & -\tilde{A}_3 \\ 0 & A_{31} & A_{21} & 0 & A_{32} & A_{22} & 0 & A_{33} & A_{23} & 0 & -\tilde{A}_3 & -\tilde{A}_2 \\ A_{31} & 0 & A_{11} & A_{32} & 0 & A_{12} & A_{33} & 0 & A_{13} & -\tilde{A}_3 & 0 & -\tilde{A}_1 \\ A_{21} & A_{11} & 0 & A_{22} & A_{12} & 0 & A_{23} & A_{13} & 0 & -\tilde{A}_2 & -\tilde{A}_1 & 0 \end{bmatrix}$$