دانشگاه صنعتی اصفهان

دانشکده مکانیک

Elastic–Plastic Fracture Mechanics

- Linear elastic fracture mechanics (LEFM) is valid only as long as nonlinear material deformation is confined to a small region surrounding the crack tip. In many materials, it is virtually impossible to characterize the fracture behavior with LEFM, and an alternative fracture mechanics model is required.

- Elastic–plastic fracture mechanics applies to materials that exhibit time-independent, nonlinear behavior (i.e., plastic deformation). Two elastic–plastic parameters are introduced: the crack tip opening displacement (*CTOD*) and the *J integral*. Both parameters describe crack tip conditions in elastic–plastic materials, and each can be used as a fracture criterion. Critical values of CTOD or J give nearly size-independent measures of fracture toughness, even for relatively large amounts of crack tip plasticity.

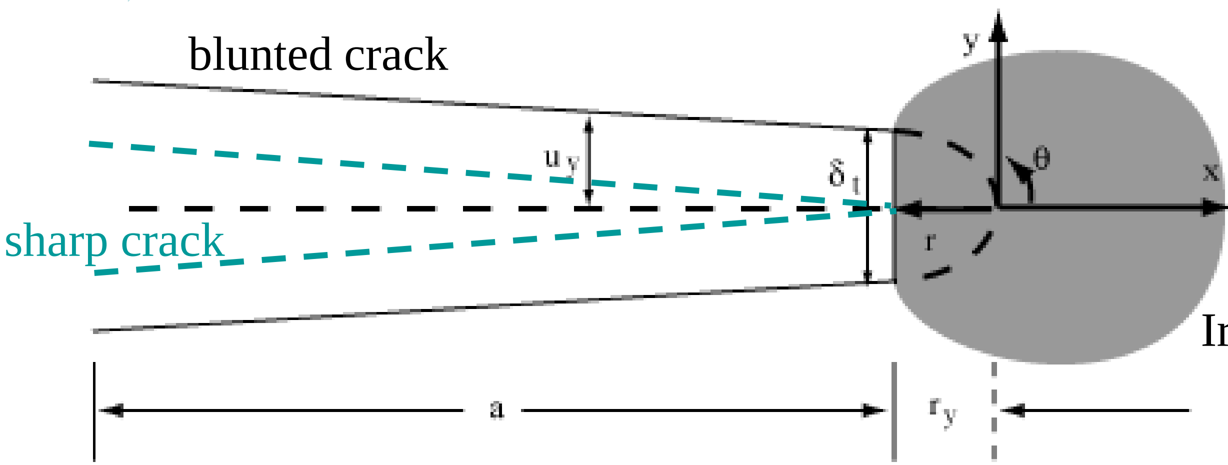
CTOD: Crack Tip Opening Displacement

Well's experimental work: attempt to measure K_{Ic} for structural steels

But

Initial sharp crack has *blunted* prior to fracture

Non-negligible plastic deformation



➡ LEFM *inaccurate* : materials too tough !!!

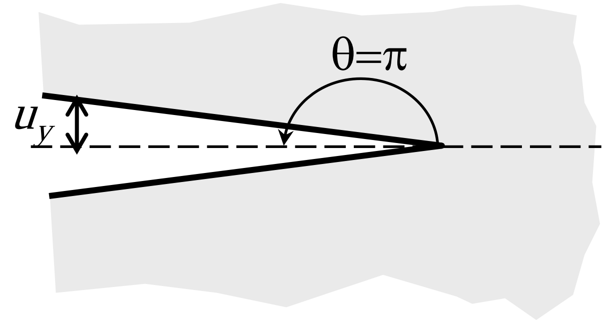
Instead, Wells proposed δ_t (CTOD) as a measure of fracture toughness.

Estimation of δ_t using Irwin model : Crack length: $a + r_y$

By definition, $\delta_t = 2u_y$ at $r = r_y$ where u_y is the crack opening

CTOD as yield criterion

Crack opening: (u_y) $$u_y = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\, sin\frac{\theta}{2}\left[\kappa+1-2cos^2\frac{\theta}{2}\right]\Bigg|_{\theta=\pi}$$



$$= \frac{K_I}{2\mu}(\kappa+1)\sqrt{\frac{r}{2\pi}}$$ (*see Table 2.2*)

We have $\mu = \frac{E}{2(1+\nu)}$ and for plane stress, $\kappa = \frac{3-\nu}{1+\nu}$

$$\Rightarrow \quad \delta_t = 2\frac{4K_I}{E}\sqrt{\frac{r_y}{2\pi}}$$

From Irwin model, the radius of the plastic zone is $r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$

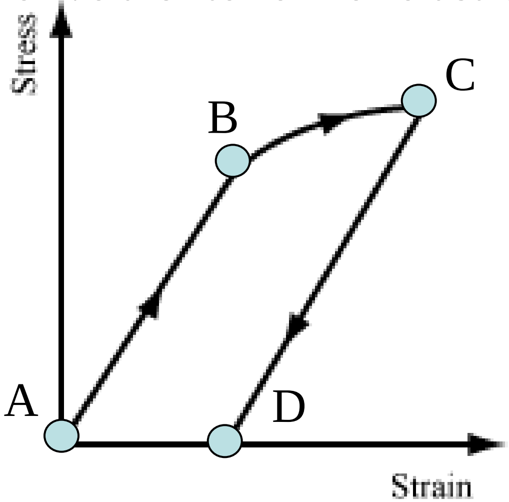
$\delta_t = \frac{4}{\pi}\frac{K_I^2}{\sigma_{YS}E}$ and also, $\delta_t = \frac{4}{\pi}\frac{G}{\sigma_{YS}}$ CTOD related uniquely to K_I and G.

➡ CTOD appropriate characterizing crack-tip-parameter when LEFM no longer valid.

Can be proved by a unique relationship between CTOD and the J integral.

The J contour integral as yield criterion

- More general criterion than K (valid for LEFM)
- Derive a criterion for elastic-plastic materials, with typical stress-strain behavior:



A→B : linear

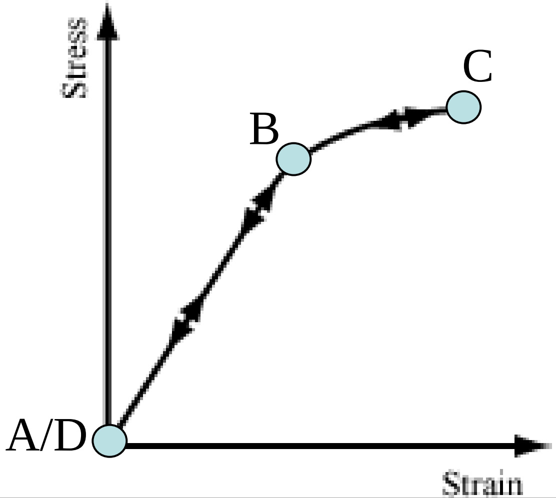
B→C : non-linear curve

C→D : non-linear, same slope as A-B

non-reversibility: A-B-C ≠ C-D-A

➡ Material behavior is strain history dependent !
Non unique solutions for stresses

- **Simplification**: non-linear elastic behavior



reversibility: A-B-C = C-D-A

➡ Correct only for a monotonic loading

= Deformation theory of Plasticity

The J contour integral as yield criterion

Definition of the J-integral

Rice defined a *path-independent* contour *integral J* for the analysis of cracked bodies

showed that its value = *energy release rate* in a *nonlinear elastic* material

J generalizes *G* to nonlinear materials :

$\rightarrow$ nonlinear elastic energy release rate

As *G* can be used as a fracture criterion J_c

reduces to G_c in the case of linear fracture

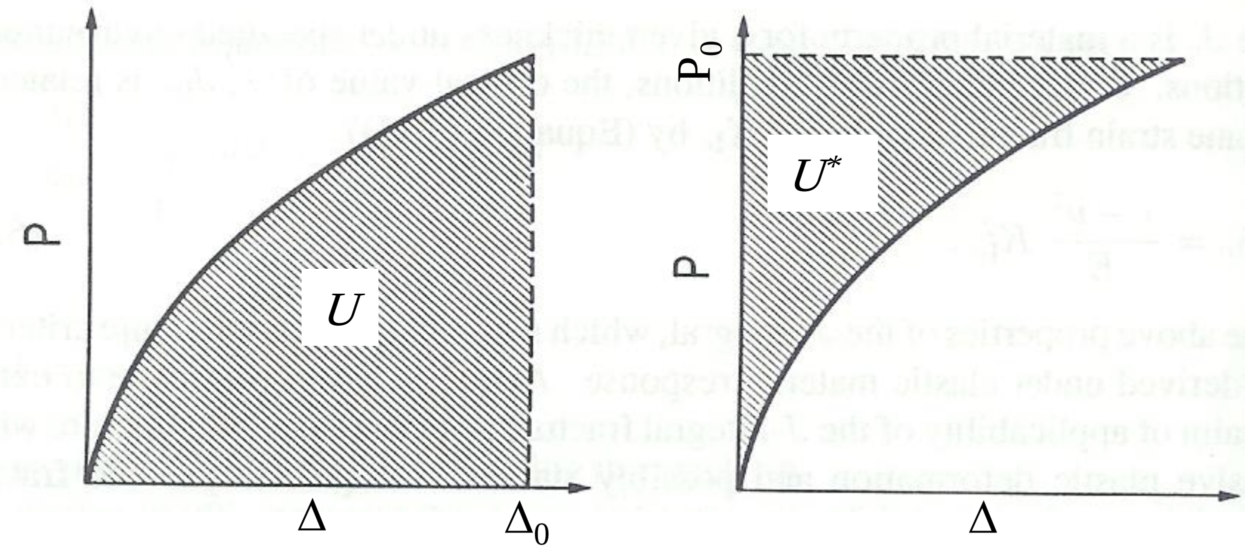
The J contour integral as yield criterion

Definition of the J-integral

- Historically,

  Rice defined a *path-independent* contour *integral J* for the analysis of crack

  showed that its value = *energy release rate* in a *nonlinear* elastic body with a crack
- J generalizes the concept of G to non-linear materials
  - For linear materials $J = G$
  - Load-displacement diagram: potential energy Π



Fixed-grips conditions:

$$\Pi = U = \int_0^{\Delta_0} P(\Delta)\, d\Delta$$

Dead-load conditions

$$-\Pi = U^* = \int_0^{P_0} \Delta(P)\, dP$$

U: Elastic strain energy

$\neq$ (in general)

U^*: Complementary energy

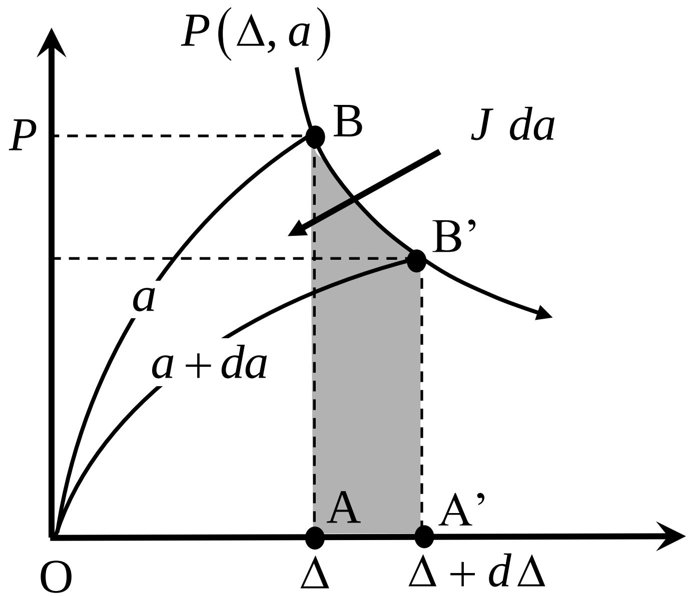
The J contour integral as yield criterion

- Definition of J using the potential energy Π :

$$J = -\frac{d\Pi}{dA}$$

$A = a\,B$: for a cracked plate with through crack

  - Geometrical interpretation:



OB and OB' :

loading/unloading for the given body with crack lengths *a* and *a+da*

$P(\Delta, a)$:

Possible relationship between the load *P* and the displacement Δ while the crack is moving.

We have $J\ dA = Pd\Delta - dU$

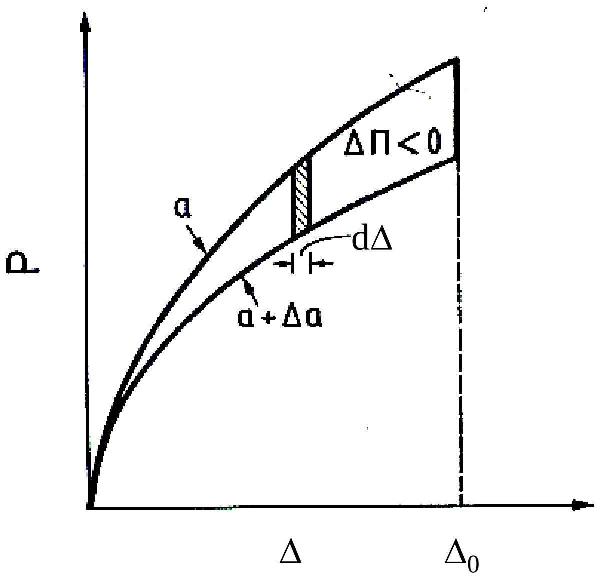
d*U* is the difference between the areas under OB' and OB : OA'B' – OAB

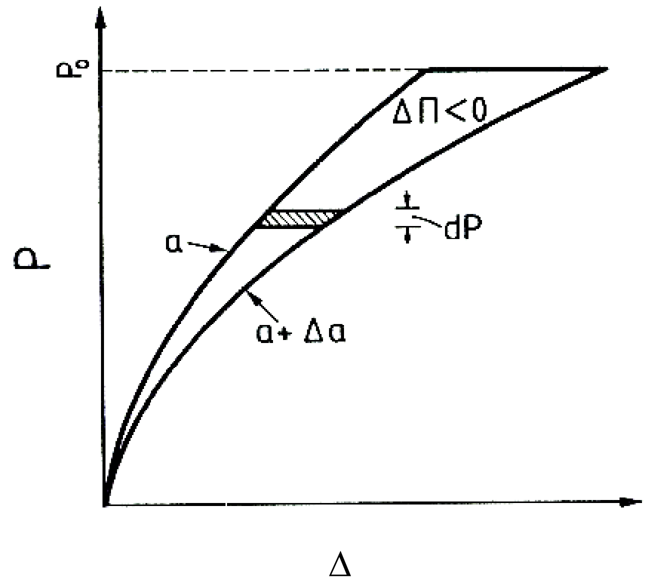
PdΔ appears as the area AA'B'B

Thus, $J\ dA = J\ B\,da$ = AA'B'B + OAB – OA'B' = OBB'

The J contour integral as yield criterion

- In particular ,





At constant displacement:

$$J = -\frac{1}{B}\left(\frac{\partial U}{\partial a}\right)_{\Delta} = -\frac{1}{B}\int_0^{\Delta_0}\left(\frac{\partial P}{\partial a}\right)_{\Delta} d\Delta$$

At constant force (dual form):

$$J = \frac{1}{B}\left(\frac{\partial U^*}{\partial a}\right)_{P} = \frac{1}{B}\int_0^{P_0}\left(\frac{\partial \Delta}{\partial a}\right)_{P} dP$$

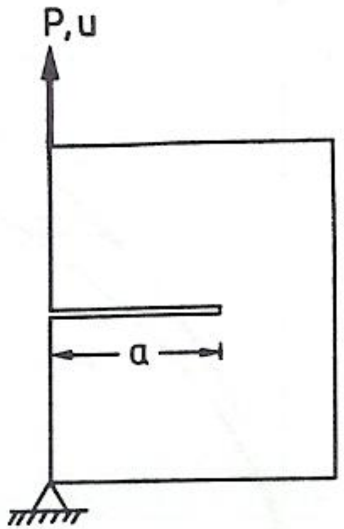
Useful expressions for the experimental determination of *J*

The J contour integral as yield criterion

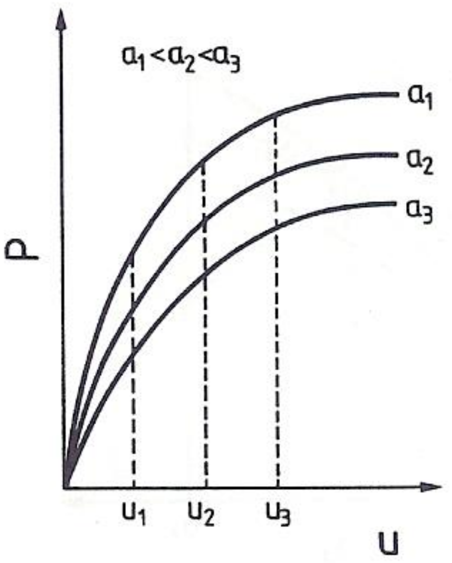
- Experimental determination of the J-integral :
  - Multiple-specimen method (Begley and Landes (1972)) :

Procedure

(1) Consider cracked specimens with different crack lengths a_i



(1)

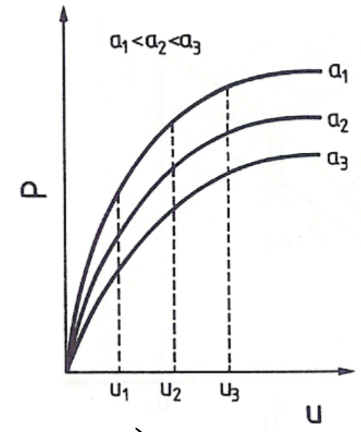


(2)

The J contour integral as yield criterion

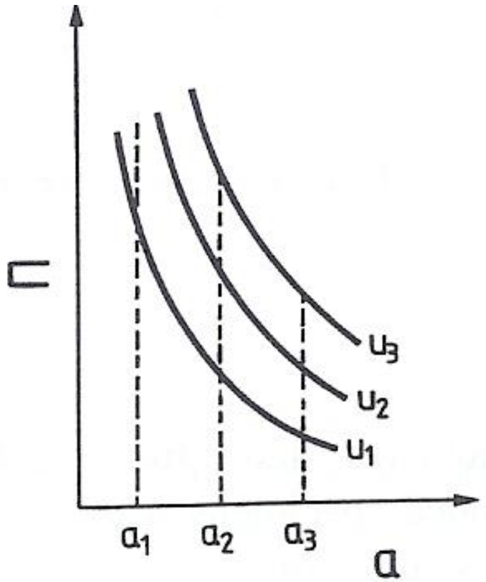
(3) Calculation of the potential energy Π for given values of displacement u

= area under the load-displacement curve

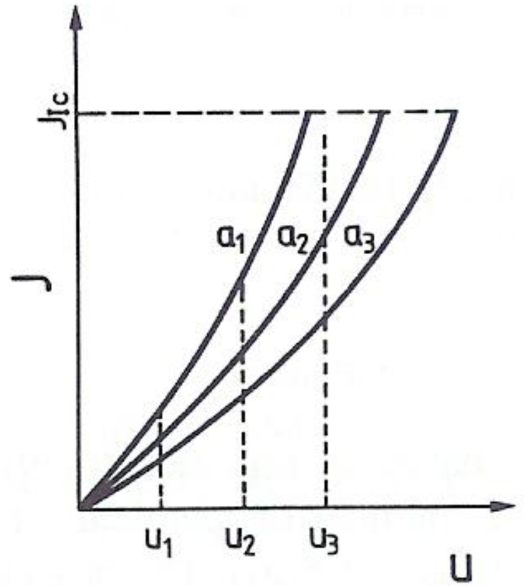


(4) Negative slopes of the P – a curves determined and plotted versus displacement for different crack lengths :

Critical value J_{Ic} of J at the onset of crack extension (material constant)



(3)



(4)

- J as a path-independent line integral

$$J = \int_{\Gamma} \left(w\,dy - T_i \frac{\partial u_i}{\partial x} ds \right) \quad \text{with} \quad w(\varepsilon_{mn}) = \int_0^{\varepsilon_{mn}} \sigma_{ij}\, d\varepsilon_{ij} \quad \text{strain energy density}$$

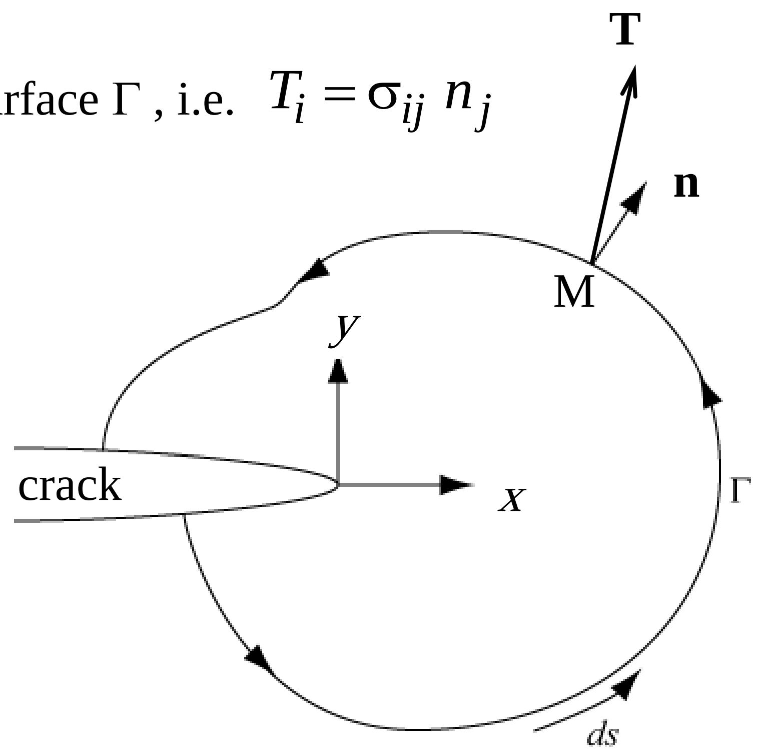
$$= \int_{\Gamma} \left(w\,dy - \mathbf{T} \cdot \frac{\partial \mathbf{u}}{\partial x} ds \right)$$

T : traction vector at a point M on the bounding surface Γ , i.e. $T_i = \sigma_{ij}\, n_j$

u : displacement vector at the same point M.

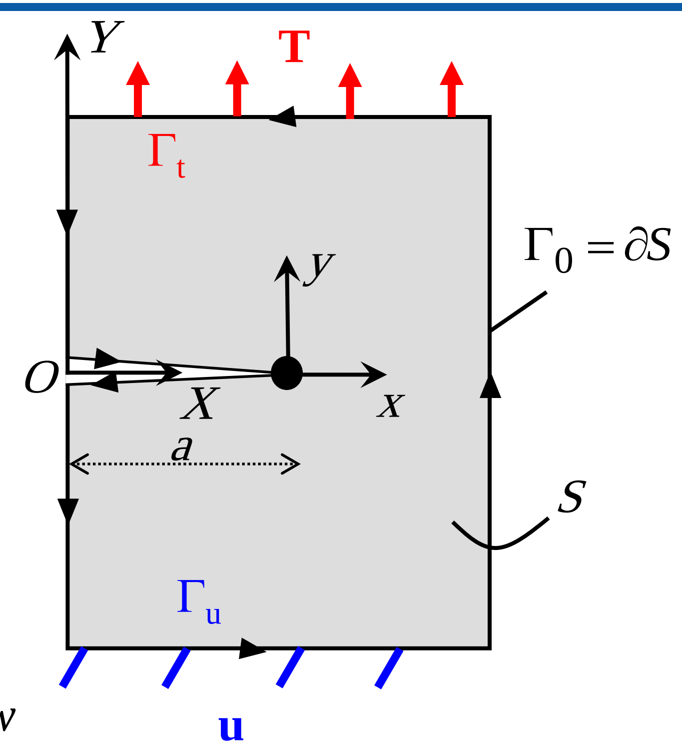
n : unit *outward* normal.

The contour Γ is followed in the *counter-clockwise* direction.



Equivalence of the two definitions

- 2D solid of unit thickness of area S , with a linear crack of length *a* along *OX* (fixed)
- Crack faces are traction-free.
- Total contour of the solid Γ_0 *including* the crack tip:
  Imposed tractions on the part of the contour Γ_t
  Displacements applied on Γ_u



Proof : Recall for the potential energy (per unit thickness),

$$\Pi(a) = \iint_S w\, dS - \int_{\Gamma_t} T_i u_i\, ds \qquad T_i = \sigma_{ij} n_j \qquad \sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}$$

The tractions and displacements imposed on Γ_t and Γ_u are independent of *a*

$$\frac{dT_i}{da} = 0, \quad on \quad \Gamma_t$$

$$\frac{du_i}{da} = 0 \quad on \quad \Gamma_u$$

$$\Rightarrow \quad \frac{d\Pi}{da} = \iint_S \frac{dw}{da} dS - \int_{\Gamma_0} T_i \frac{du_i}{da} ds$$

The J contour integral as yield criterion

Considering the moving coordinate system $x\,,\,y$ (attached to the crack tip), $x = X - a$

$\frac{d}{da}$: total derivative/crack length

$$\frac{d}{da} = \left(\frac{\partial}{\partial a}\right)_x + \left(\frac{\partial x}{\partial a}\right)_X \left(\frac{\partial}{\partial x}\right)_a \qquad = \frac{\partial}{\partial a} - \frac{\partial}{\partial x}$$

Thus,

$$\frac{d\Pi}{da} = \iint_S \left(\frac{\partial w}{\partial a} - \frac{\partial w}{\partial x}\right) dS - \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial a} - \frac{\partial u_i}{\partial x}\right) ds$$

However,

$$\frac{\partial w}{\partial a} = \frac{\partial w}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial a} \quad = \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial a} \quad = \sigma_{ij} \frac{\partial}{\partial a}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] \qquad \text{since} \quad \sigma_{ij} = \sigma_{ji}$$

$$= \sigma_{ij} \frac{\partial}{\partial a} \frac{\partial u_i}{\partial x_j} \quad = \quad \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial a}\right)$$

Thus,

$$\iint_S \frac{\partial w}{\partial a} dS = \iint_S \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial a}\right) dS$$

We have,

$$\iint_S \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial a}\right) dS = \int_{\Gamma_0} \sigma_{ij} \frac{\partial u_i}{\partial a} n_j \, ds = \int_{\Gamma_0} T_i \frac{\partial u_i}{\partial a} \, ds$$

The derivative of J reduces to,

$$\frac{d\Pi}{da} = \iint_S \left(\frac{\partial w}{\partial a} - \frac{\partial w}{\partial x}\right) dS - \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial a} - \frac{\partial u_i}{\partial x}\right) ds$$

$$= -\iint_S \left(\frac{\partial w}{\partial x}\right) dS + \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial a}\right) ds - \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial a} - \frac{\partial u_i}{\partial x}\right) ds$$

$$= -\left(\iint_S \left(\frac{\partial w}{\partial x}\right) dS - \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial x}\right) ds\right)$$

The J contour integral as yield criterion

Using the Green Theorem, i.e. $\oint_{\Gamma} P(x,y)dx + Q(x,y)dy = \iint_{A}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$

$$-\frac{d\Pi}{da} = \iint_{S}\left(\frac{\partial w}{\partial x}\right)dS - \int_{\Gamma_0} T_i\left(\frac{\partial u_i}{\partial x}\right)ds$$

$$= \int_{\Gamma_0}\left(w\,dy - T_i\left(\frac{\partial u_i}{\partial x}\right)ds\right)$$

➡ J derives from a potential