دانشگاه صنعتی اصفهان
دانشکده مکانیک

Mixed-Mode Fracture

Interaction of Multiple Cracks

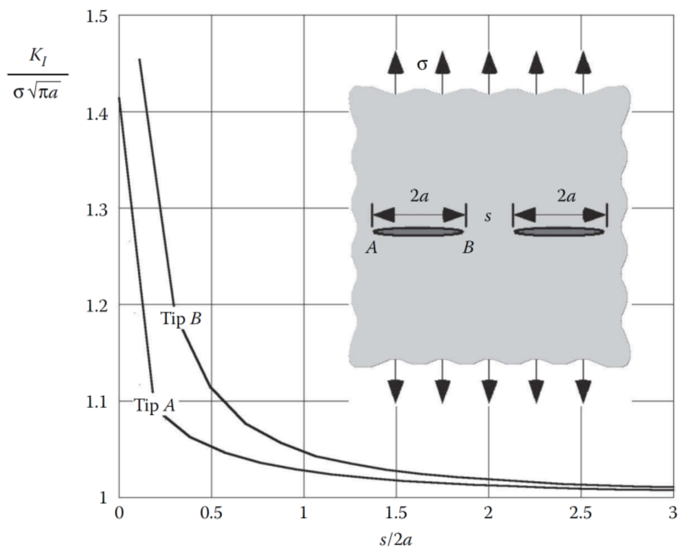
- The local stress field and crack driving force for a given flaw can be significantly affected by the presence of one or more neighboring cracks. Depending on the relative orientation of the neighboring cracks, the interaction can either magnify or diminish the stress intensity factor.

Coplanar Cracks

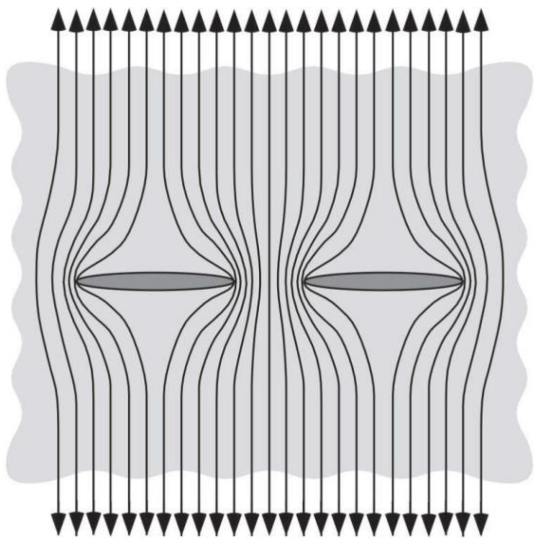
- Typical propagation from an initial crack that is not orthogonal to the applied normal stress. The loading for the initial angled crack is a combination of Modes I and II, but the crack tends to propagate normal to the applied stress, resulting in pure Mode I loading.

Coplanar Cracks

- The figure illustrates two identical coplanar cracks in an infinite plate. The lines of force represent the relative stress concentrating effect of the cracks. As the ligament between the cracks shrinks in size, the area through which the force must be transmitted decreases. Consequently, K_I is magnified for each crack as the two cracks approach one another.



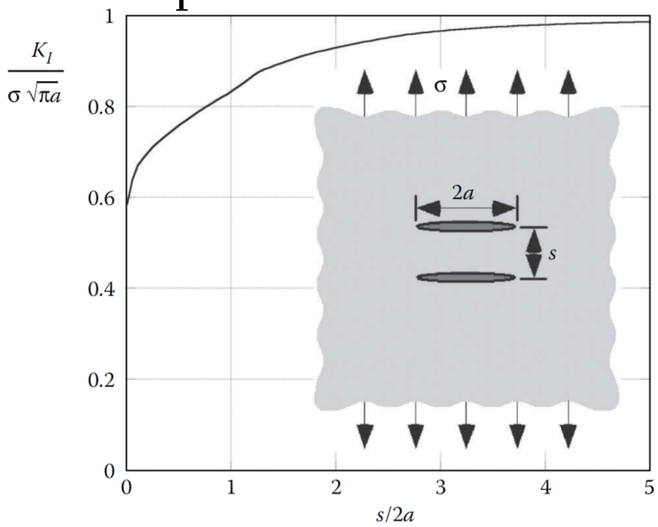
Interaction of two identical coplanar through-wall cracks in an infinite plate



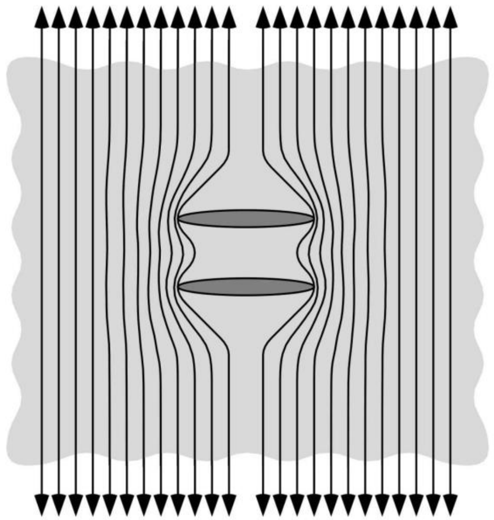
Coplanar cracks. Interaction between cracks results in a magnification of K_I

Parallel Cracks

- The figure illustrates two parallel cracks. In this case, the cracks tend to shield one another, which results in a decrease in K_I relative to the case of the single crack. This is indicative of the general case where two or more parallel cracks have a mutual shielding interaction when subject to Mode I loading. Consequently, multiple cracks that are parallel to one another are of less concern than multiple cracks in the same plane.



Interaction between two identical parallel through-wall cracks in an infinite plate



Parallel cracks. A mutual shielding effect reduces K_I in each crack.

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Elastic–Plastic Fracture Mechanics

- Linear elastic fracture mechanics (LEFM) is valid only as long as nonlinear material deformation is confined to a small region surrounding the crack tip. In many materials, it is virtually impossible to characterize the fracture behavior with LEFM, and an alternative fracture mechanics model is required.

- Elastic–plastic fracture mechanics applies to materials that exhibit time-independent, nonlinear behavior (i.e., plastic deformation). Two elastic–plastic parameters are introduced: the crack tip opening displacement (*CTOD*) and the *J integral*. Both parameters describe crack tip conditions in elastic–plastic materials, and each can be used as a fracture criterion. Critical values of CTOD or J give nearly size-independent measures of fracture toughness, even for relatively large amounts of crack tip plasticity.

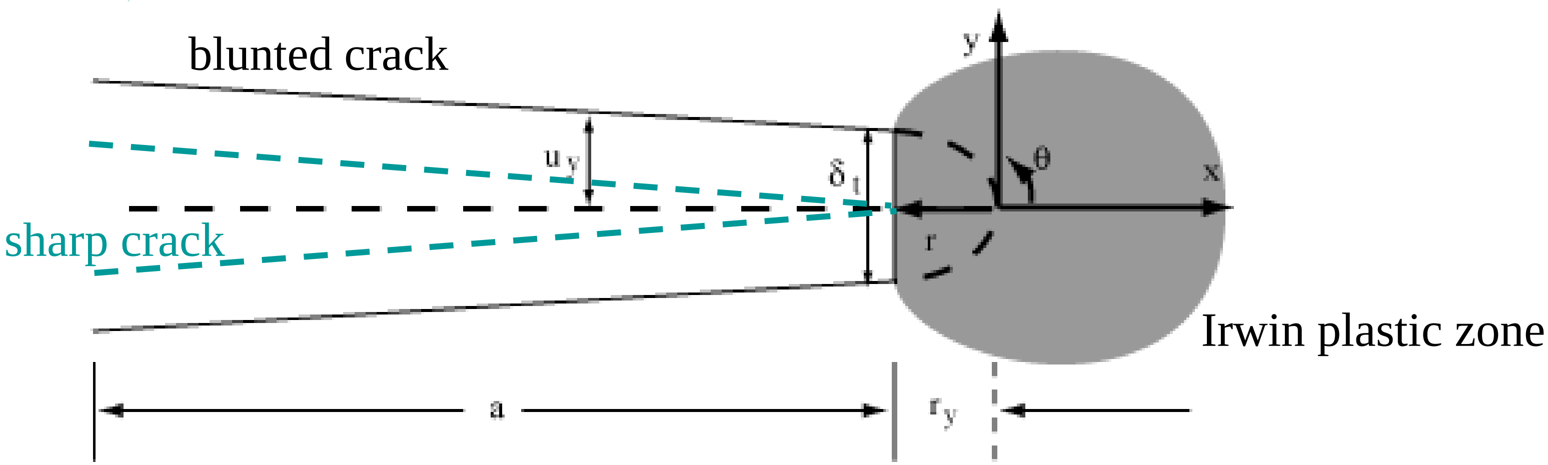
CTOD: Crack Tip Opening Displacement

Well's experimental work: attempt to measure K_{Ic} for structural steels

But

Initial sharp crack has *blunted* prior to fracture

Non-negligible plastic deformation



➡LEFM *inaccurate* : materials too tough !!!

Instead, Wells proposed δ_t (CTOD) as a measure of fracture toughness.

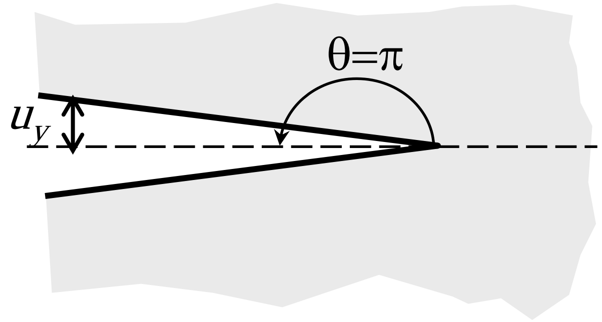
Estimation of δ_t using Irwin model : Crack length: $a + r_y$

By definition, $\delta_t = 2u_y$ at $r = r_y$ where u_y is the crack opening

CTOD as yield criterion

Crack opening: (u_y)
$$u_y = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\, sin\frac{\theta}{2}\left[\kappa+1-2cos^2\frac{\theta}{2}\right]\Bigg|_{\theta=\pi}$$

$$= \frac{K_I}{2\mu}(\kappa+1)\sqrt{\frac{r}{2\pi}} \qquad (see\ Table\ 2.2)$$



We have $\mu = \dfrac{E}{2(1+\nu)}$ and for plane stress, $\kappa = \dfrac{3-\nu}{1+\nu}$

$$\Rightarrow \quad \delta_t = 2\frac{4K_I}{E}\sqrt{\frac{r_y}{2\pi}}$$

From Irwin model, the radius of the plastic zone is $r_y = \dfrac{1}{2\pi}\left(\dfrac{K_I}{\sigma_Y}\right)^2$

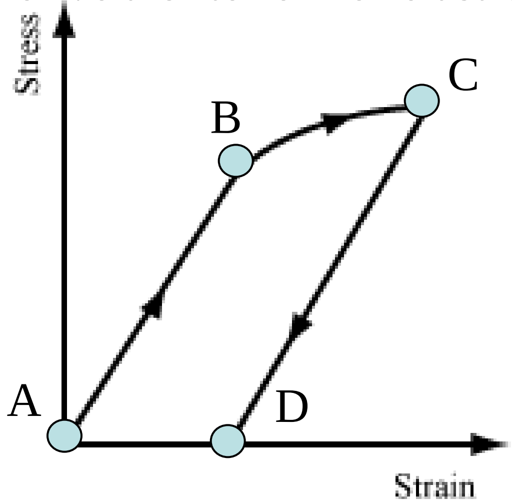
$$\delta_t = \frac{4}{\pi}\frac{K_I^2}{\sigma_Y E}$$ and also, $$\delta_t = \frac{4}{\pi}\frac{G}{\sigma_Y}$$ CTOD related uniquely to K_I and G.

➡ CTOD appropriate characterizing crack-tip-parameter when LEFM no longer valid.

Can be proved by a unique relationship between CTOD and the J integral.

The J contour integral as yield criterion

- More general criterion than K (valid for LEFM)
- Derive a criterion for elastic-plastic materials, with typical stress-strain behavior:



A→B : linear

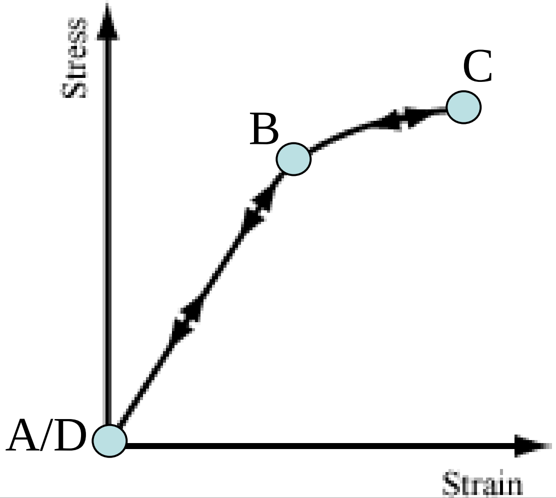
B→C : non-linear curve

C→D : non-linear, same slope as A-B

non-reversibility: A-B-C ≠ C-D-A

➡ Material behavior is strain history dependent !
Non unique solutions for stresses

- **Simplification**: non-linear elastic behavior



reversibility: A-B-C = C-D-A

➡ Correct only for a monotonic loading

= Deformation theory of Plasticity

The J contour integral as yield criterion

Definition of the J-integral

Rice defined a *path-independent* contour *integral J* for the analysis of cracked bodies

showed that its value = *energy release rate* in a *nonlinear elastic* material

J generalizes *G* to nonlinear materials :

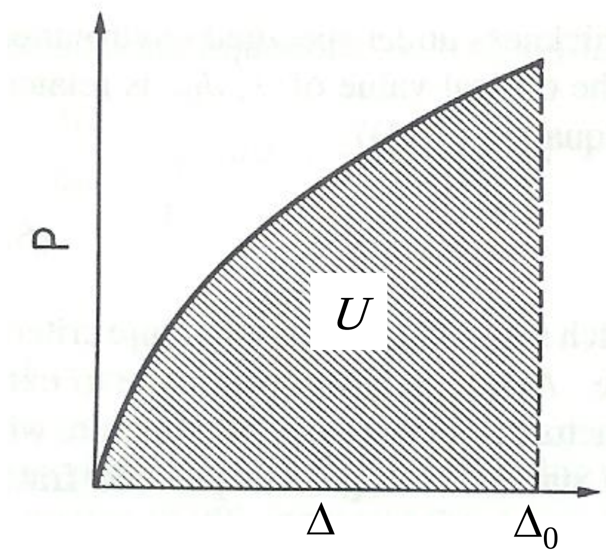
$\rightarrow$ nonlinear elastic energy release rate

As *G* can be used as a fracture criterion J_c

reduces to G_c in the case of linear fracture

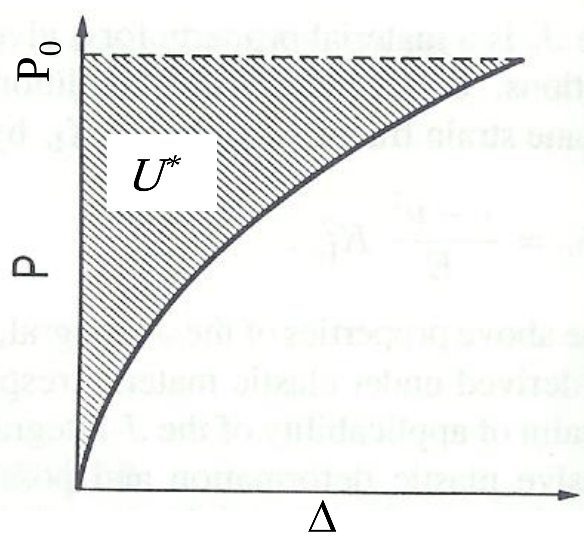
Definition of the J-integral

- Historically,
  Rice defined a *path-independent* contour *integral J* for the analysis of crack
  showed that its value = *energy release rate* in a *nonlinear* elastic body with a crack
- J generalizes the concept of G to non-linear materials
  - For linear materials $J = G$
  - Load-displacement diagram: potential energy Π



Fixed-grips conditions:

$$\Pi = U = \int_0^{\Delta_0} P(\Delta)\,d\Delta$$



Dead-load conditions

$$-\Pi = U^* = \int_0^{P_0} \Delta(P)\,dP$$

U: Elastic strain energy

$\neq$ (in general)

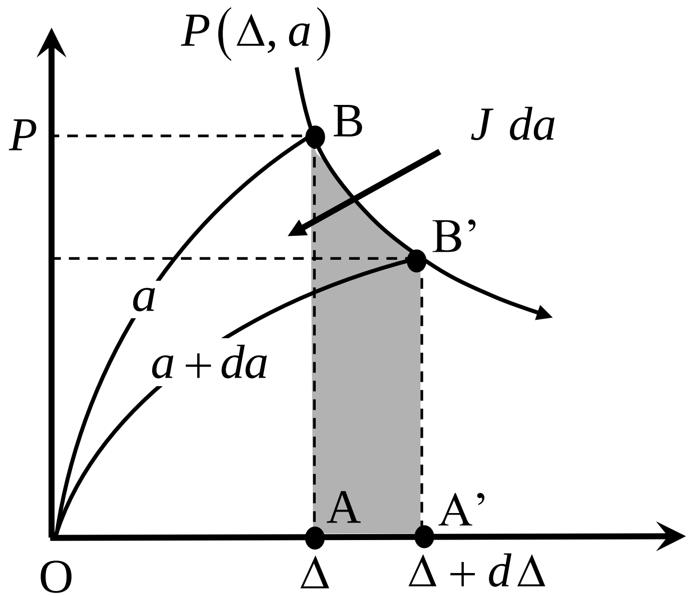
U^*: Complementary energy

- Definition of J using the potential energy Π :

$$J = -\frac{d\Pi}{dA}$$

$A = a\,B$: for a cracked plate with through crack

- Geometrical interpretation:



OB and OB' :

loading/unloading for the given body with crack lengths *a* and *a+da*

$P(\Delta, a)$:

Possible relationship between the load *P* and the displacement Δ while the crack is moving.

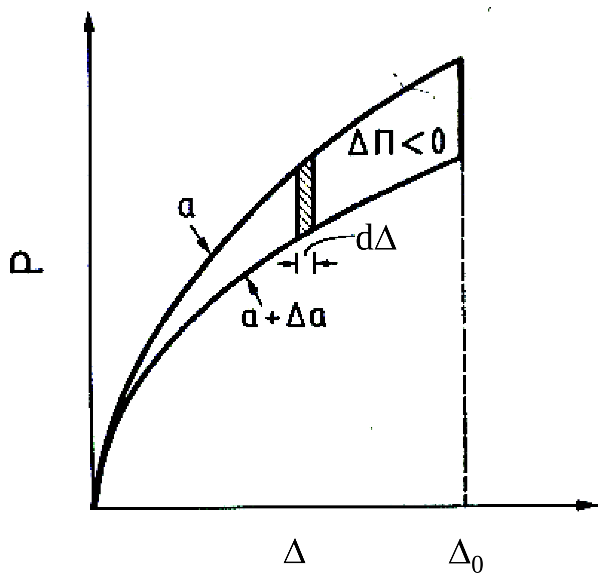
We have $J\ dA = Pd\Delta - dU$

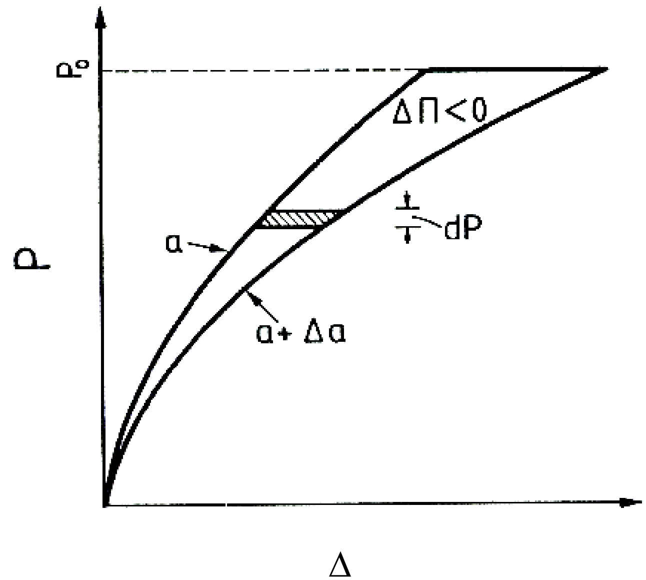
d*U* is the difference between the areas under OB' and OB : OA'B' – OAB

PdΔ appears as the area AA'B'B

Thus, $J\ dA = J\ B\,da$ = AA'B'B + OAB – OA'B' = OBB'

- In particular ,





At constant displacement:

$$J = -\frac{1}{B}\left(\frac{\partial U}{\partial a}\right)_{\Delta} \quad = -\frac{1}{B}\int_{0}^{\Delta_0}\left(\frac{\partial P}{\partial a}\right)_{\Delta} d\Delta$$

At constant force (dual form):

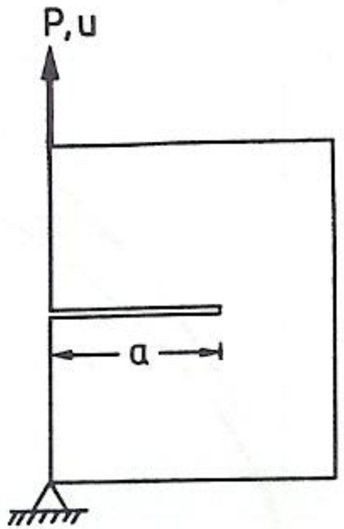
$$J = \frac{1}{B}\left(\frac{\partial U^{*}}{\partial a}\right)_{P} \quad = \frac{1}{B}\int_{0}^{P_0}\left(\frac{\partial \Delta}{\partial a}\right)_{P} dP$$

Useful expressions for the experimental determination of J

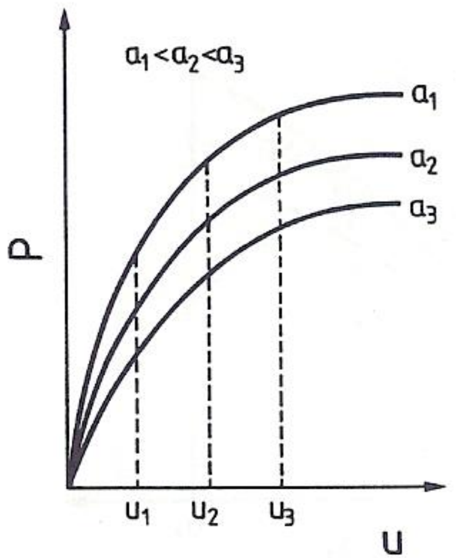
- Experimental determination of the J-integral :
  - Multiple-specimen method (Begley and Landes (1972)) :

Procedure

(1) Consider cracked specimens with different crack lengths a_i



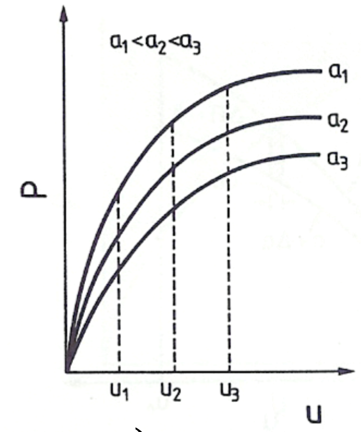
(1)



(2)

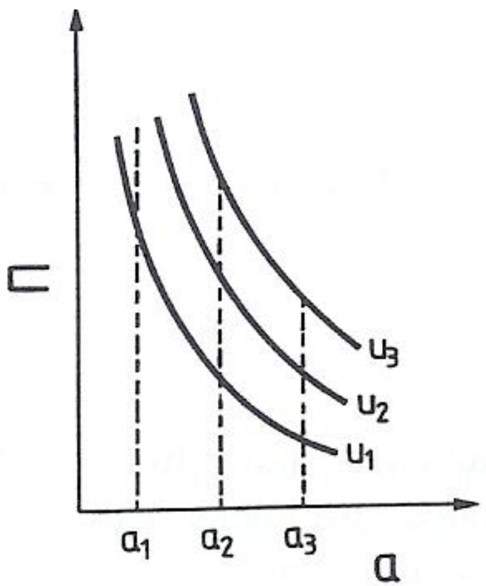
(3) Calculation of the potential energy Π for given values of displacement u

= area under the load-displacement curve

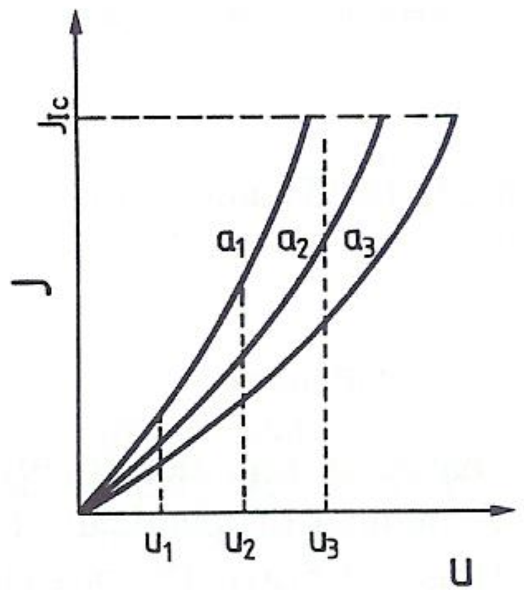


(4) Negative slopes of the P – a curves determined and plotted versus displacement for different crack lengths :

Critical value J_{Ic} of J at the onset of crack extension (material constant)



(3)



(4)

The J contour integral as yield criterion

- J as a path-independent line integral

$$J = \int_\Gamma \left(w\,dy - T_i \frac{\partial u_i}{\partial x} ds \right) \quad \text{with} \quad w(\varepsilon_{mn}) = \int_0^{\varepsilon_{mn}} \sigma_{ij}\, d\varepsilon_{ij} \quad \text{strain energy density}$$

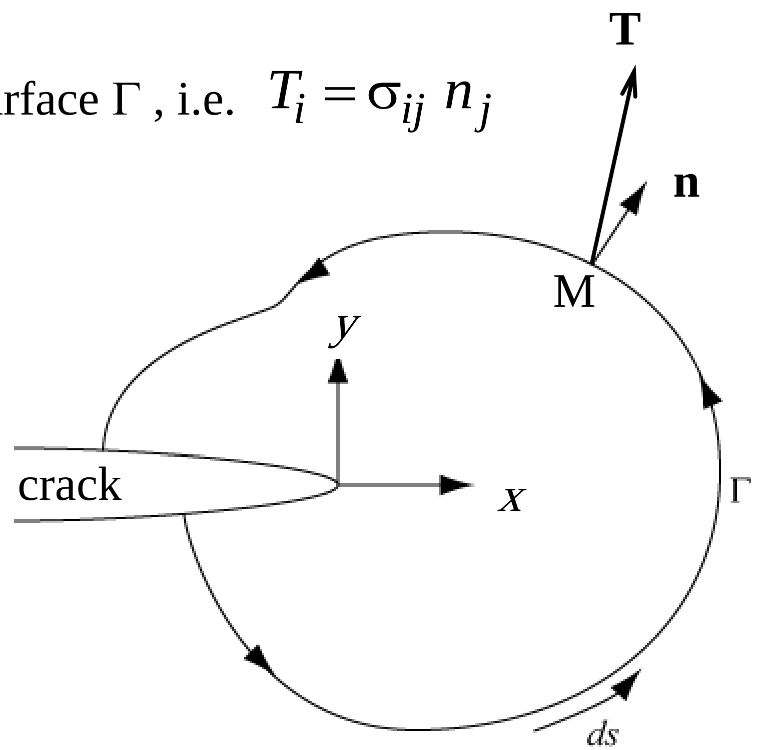
$$= \int_\Gamma \left(w\,dy - \mathbf{T} \cdot \frac{\partial \mathbf{u}}{\partial x} ds \right)$$

T : traction vector at a point M on the bounding surface Γ , i.e. $T_i = \sigma_{ij}\, n_j$

u : displacement vector at the same point M.

n : unit *outward* normal.

The contour Γ is followed in the *counter-clockwise* direction.



The J contour integral as yield criterion

Equivalence of the two definitions

- 2D solid of unit thickness of area S ,

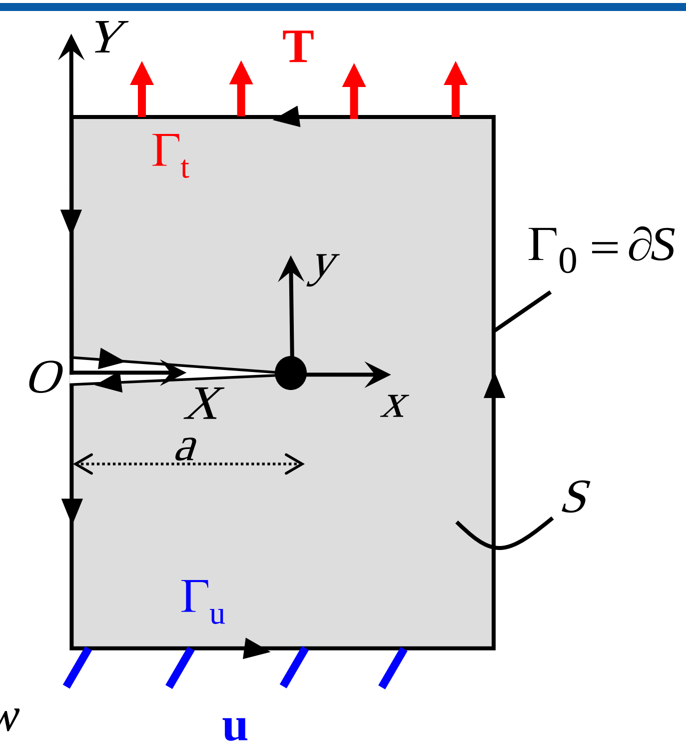
  with a linear crack of length a along OX (fixed)

- Crack faces are traction-free.

- Total contour of the solid Γ_0 *including* the crack tip:

  Imposed tractions on the part of the contour Γ_t

  Displacements applied on Γ_u



Proof : Recall for the potential energy (per unit thickness),

$$\Pi(a) = \iint_S w\, dS - \int_{\Gamma_t} T_i u_i \, ds \qquad T_i = \sigma_{ij} n_j \qquad \sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}$$

The tractions and displacements imposed on Γ_t and Γ_u are independent of a

$$\frac{dT_i}{da} = 0, \quad on \quad \Gamma_t$$

$$\frac{du_i}{da} = 0 \quad on \quad \Gamma_u$$

$$\Rightarrow \quad \frac{d\Pi}{da} = \iint_S \frac{dw}{da} dS - \int_{\Gamma_0} T_i \frac{du_i}{da} ds$$

The J contour integral as yield criterion

Considering the moving coordinate system x, y (attached to the crack tip), $x = X - a$

$\frac{d}{da}$: total derivative/crack length

$$\frac{d}{da} = \left(\frac{\partial}{\partial a}\right)_x + \left(\frac{\partial x}{\partial a}\right)_X \left(\frac{\partial}{\partial x}\right)_a \qquad = \frac{\partial}{\partial a} - \frac{\partial}{\partial x}$$

Thus,

$$\frac{d\Pi}{da} = \iint_S \left(\frac{\partial w}{\partial a} - \frac{\partial w}{\partial x}\right) dS - \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial a} - \frac{\partial u_i}{\partial x}\right) ds$$

However,

$$\frac{\partial w}{\partial a} = \frac{\partial w}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial a} = \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial a} = \sigma_{ij} \frac{\partial}{\partial a}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] \qquad \text{since} \quad \sigma_{ij} = \sigma_{ji}$$

$$= \sigma_{ij} \frac{\partial}{\partial a} \frac{\partial u_i}{\partial x_j} = \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial a}\right)$$

The J contour integral as yield criterion

Thus,

$$\iint_S \frac{\partial w}{\partial a} dS = \iint_S \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial a}\right) dS$$

We have,

$$\iint_S \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial a}\right) dS = \int_{\Gamma_0} \sigma_{ij} \frac{\partial u_i}{\partial a} n_j \, ds = \int_{\Gamma_0} T_i \frac{\partial u_i}{\partial a} \, ds$$

The derivative of J reduces to,

$$\frac{d\Pi}{da} = \iint_S \left(\frac{\partial w}{\partial a} - \frac{\partial w}{\partial x}\right) dS - \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial a} - \frac{\partial u_i}{\partial x}\right) ds$$

$$= -\iint_S \left(\frac{\partial w}{\partial x}\right) dS + \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial a}\right) ds - \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial a} - \frac{\partial u_i}{\partial x}\right) ds$$

$$= -\left(\iint_S \left(\frac{\partial w}{\partial x}\right) dS - \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial x}\right) ds\right)$$

The J contour integral as yield criterion

Using the Green Theorem, i.e. $\oint_{\Gamma} P(x,y)dx + Q(x,y)dy = \iint_{A} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$

$$-\frac{d\Pi}{da} = \iint_{S} \left(\frac{\partial w}{\partial x} \right) dS - \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial x} \right) ds$$

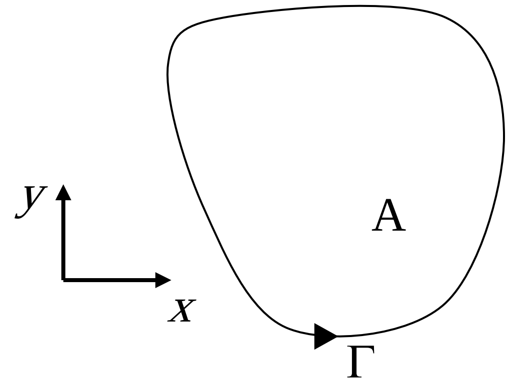
$$= \int_{\Gamma_0} \left(w\, dy - T_i \left(\frac{\partial u_i}{\partial x} \right) ds \right)$$

➡ J derives from a potential

Properties of the J-integral

1) J is *zero* for any closed contour containing *no crack tip*.

Closed contour around A



Consider $J\big|_{\Gamma} = \oint_{\Gamma} \left(w\,dy - T_i \dfrac{\partial u_i}{\partial x} ds \right)$

Using the Green Theorem, i.e. $\oint_{\Gamma} P(x,y)\,dx + Q(x,y)\,dy = \iint_{A} \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right) dx\,dy$

We have $J\big|_{\Gamma} = \int_{A} \dfrac{\partial w}{\partial x} dx dy - \oint_{\Gamma} T_i \dfrac{\partial u_i}{\partial x} ds \quad = \int_{A} \dfrac{\partial w}{\partial x} dx dy - \oint_{\Gamma} \sigma_{ij} \dfrac{\partial u_i}{\partial x} n_j\, ds$

From the divergence theorem,

$$\oint_{\Gamma} \sigma_{ij} \frac{\partial u_i}{\partial x} n_j\, ds \quad = \int_{A} \frac{\partial}{\partial x_j} \left(\sigma_{ij} \frac{\partial u_i}{\partial x} \right) dx dy$$

The integral becomes,

$$J\big|_{\Gamma} = \int_A \left[\frac{\partial w}{\partial x} - \frac{\partial}{\partial x_j}\left(\sigma_{ij} \frac{\partial u_i}{\partial x} \right) \right] dxdy$$

However,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial x} = \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial x} = \sigma_{ij} \frac{\partial}{\partial x}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] \qquad \text{since} \quad \sigma_{ij} = \sigma_{ji}$$

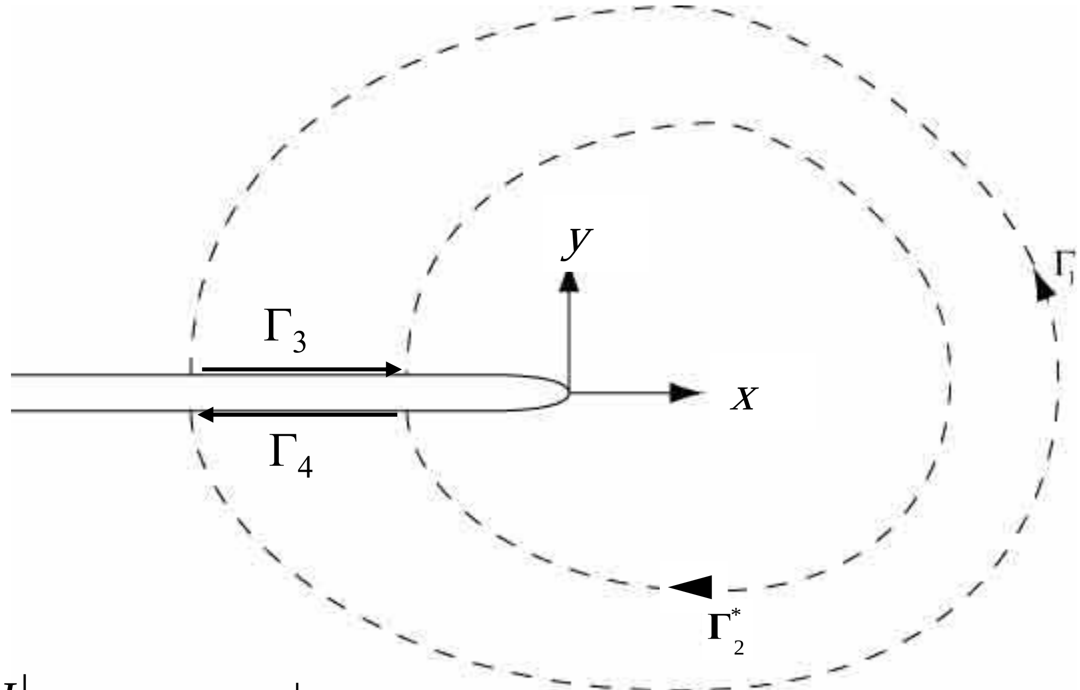
$$= \sigma_{ij} \frac{\partial}{\partial x} \frac{\partial u_i}{\partial x_j} = \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x} \right)$$

Invoking the equilibrium equation, $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$

$$\frac{\partial}{\partial x_j}\left(\sigma_{ij} \frac{\partial u_i}{\partial x} \right) = \underbrace{\frac{\partial \sigma_{ij}}{\partial x_j} \frac{\partial u_i}{\partial x}}_{0} + \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x} \right) = \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x} \right)$$

Replacing in the integral, $J\big|_{\Gamma} = 0$

2) J is path-independent



Consider the *closed* contour:

$$\Gamma = \Gamma_1 + \Gamma_3 + \Gamma_2^* + \Gamma_4$$

We have $J\big|_{\Gamma} = J\big|_{\Gamma_1} + J\big|_{\Gamma_2^*} + J\big|_{\Gamma_3} + J\big|_{\Gamma_4}$ and $J\big|_{\Gamma} = 0$

The crack faces are traction free :

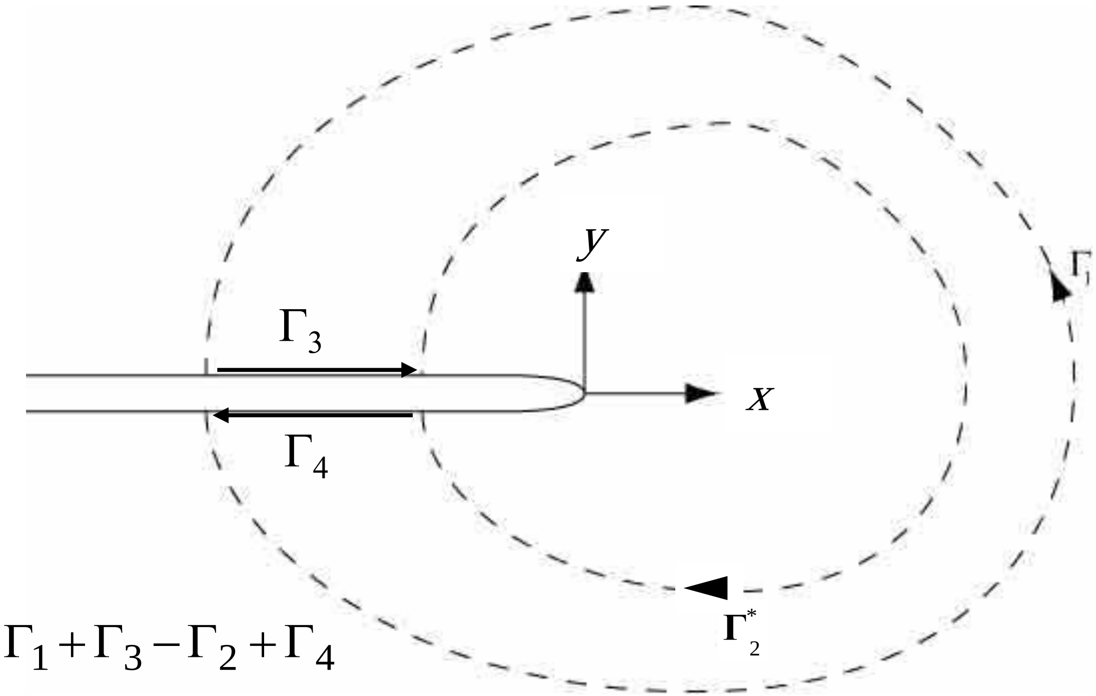
$T_i = \sigma_{ij}\, n_j = 0$ on Γ_3 and Γ_4

$dy = 0$ along these contours

$\Rightarrow$ $J\big|_{\Gamma_3} = J\big|_{\Gamma_4} = 0$

The J contour integral as yield criterion

2) J is path-independent



Note that,

$$J\big|_{\Gamma_2^*} = -J\big|_{\Gamma_2}$$

and $\quad J\big|_{\Gamma} = J\big|_{\Gamma_1} - J\big|_{\Gamma_2} = 0$

$\Rightarrow \quad J\big|_{\Gamma_1} = J\big|_{\Gamma_2}$

$$\Gamma = \Gamma_1 + \Gamma_3 - \Gamma_2 + \Gamma_4$$

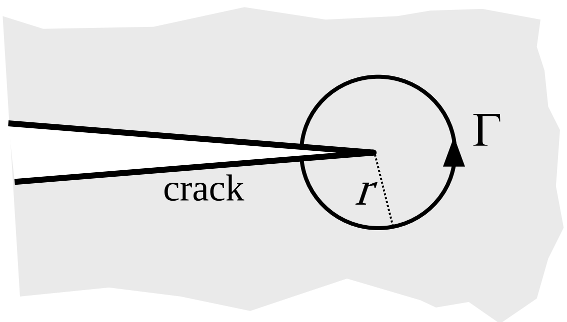
Γ_2 followed in the *counter-clockwise* direction.

Any arbitrary (counterclockwise) path around a crack gives the same value of J

$\Rightarrow$ J is *path*-independent

The J contour integral as yield criterion

J can be evaluated when the path is a circle of radius r around the crack tip



Γ is followed from $\theta = -\pi$ to $\theta = \pi$

We have, $ds = r d\theta$

$dy = r \cos\theta \, d\theta$

J integral becomes,

$$J = \int_{-\pi}^{\pi} \left[w(r,\theta)\cos\theta - T_i(r,\theta) \frac{\partial u_i(r,\theta)}{\partial x} \right] r \, d\theta$$

When $r \rightarrow 0$ only the singular terms remain

For LEFM , we can obtain : $J = G = \dfrac{K_I^2}{E'}$ (if mode I loading)

HRR theory

Hutchinson **R**ice and **R**osengren: J characterizes the crack-tip field in a non-linear elastic material.

- For uniaxial deformation:

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha\left(\frac{\sigma}{\sigma_0}\right)^n$$ Ramberg-Osgood equation

σ_0 = yield strength

$\varepsilon_0 = \sigma_0 / E$

α : dimensionless constant

n : strain-hardening exponent

} material properties

Power law relationship assumed between plastic strain and stress.

For a linear elastic material $n = 1$.

The J contour integral as yield criterion

- Asymptotic field derived by **H**utchinson **R**ice and **R**osengren:

$$\varepsilon_{ij} = A_2 \left(\frac{J}{r}\right)^{n/(n+1)} \qquad \sigma_{ij} = A_1 \left(\frac{J}{r}\right)^{1/(n+1)} \qquad u_i = A_3\, J^{\,n/(n+1)}\, r^{1/(n+1)}$$

A_i are regular functions that depend on θ and the previous parameters.

The $1/\sqrt{r}$ singularity is recovered when n = 1.

Path independence of J ⇨ The product $\sigma_{ij}\,\varepsilon_{ij}$ varies as $1/r$:

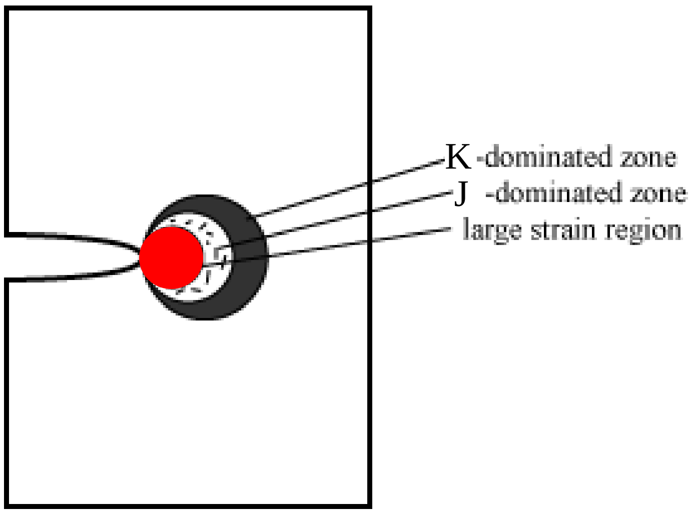
From $$J = r\int_{-\pi}^{\pi}\left[w(r,\theta)\cos\theta - T_i(r,\theta)\,\frac{\partial u_i(r,\theta)}{\partial x}\right]d\theta$$

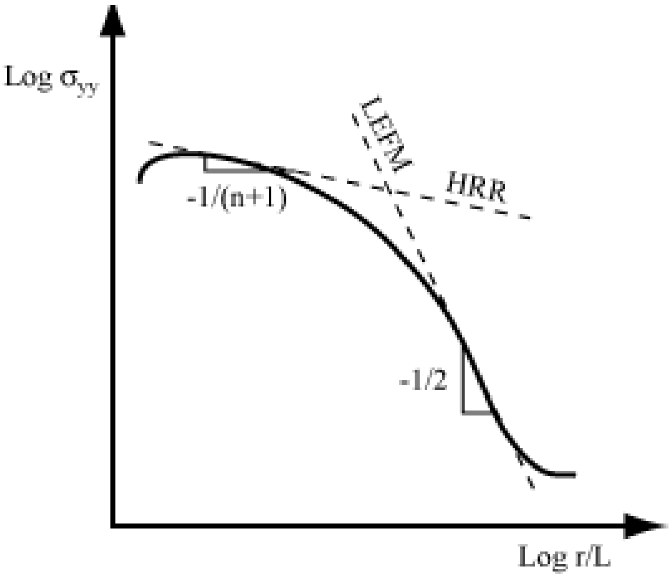
$$\sigma_{ij}\,\varepsilon_{ij} \rightarrow \frac{f(\theta)}{r} \quad as \quad r \rightarrow 0$$

J defines the amplitude of the HRR field as K does in the linear case.

The J contour integral as yield criterion

Two singular zones can be identified:





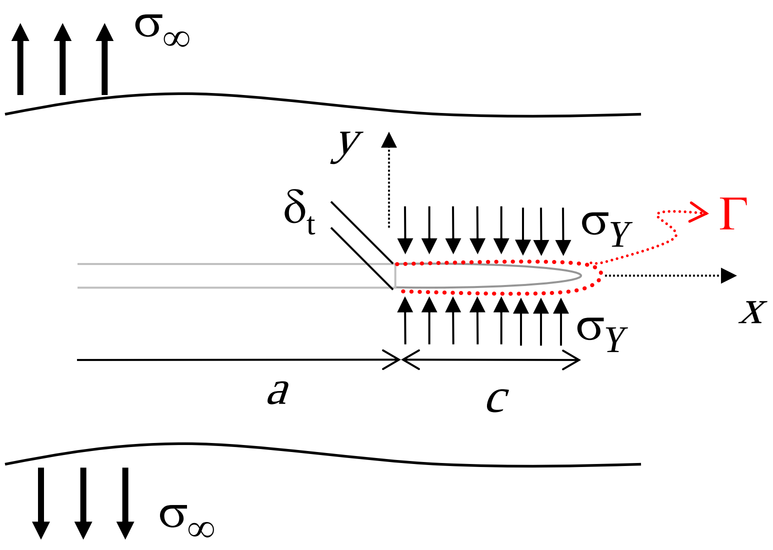
Small region where crack blunting occurs.

Large deformation

HRR based upon small displacements non applicable.

- Relationship between J and CTOD

Consider again the strip-yield problem,



The first term in the J integral vanishes because *dy= 0* (slender zone)

$$J = -\int_{\Gamma} \sigma_{ij} n_j \frac{\partial u_i}{\partial x} ds$$

$$\text{but} \quad \sigma_{ij} n_j \frac{\partial u_i}{\partial x} ds = \sigma_{yy} n_y \frac{\partial u_y}{\partial x} ds = -\sigma_Y \frac{\partial u_y}{\partial x} dx$$

$$J = \int_{\Gamma} \sigma_Y \frac{\partial u_y}{\partial x} dx = \int_{-\delta_t}^{\delta_t} \sigma_Y du_y = \sigma_Y \delta_t$$

The J contour integral as yield criterion

General unique relationship between *J* and CTOD:

$$J = m\,\sigma_Y\,\delta_t$$

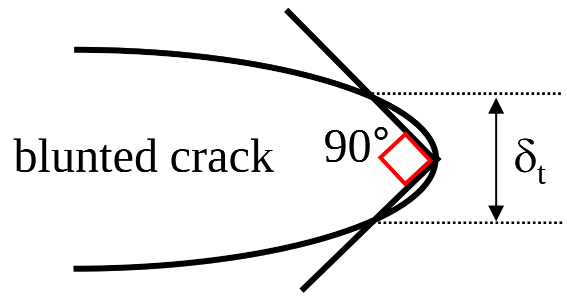
m : dimensionless parameter depending on the stress state and materials properties

- The strip-yield model predicts that m=1 (non-hardening material, plane stress condition)

- This relation is more generally derived for *hardening* materials (n >1) using the HRR displacements near the crack tip, i.e.

$$u_i = A_3\,J^{n/(n+1)}\,r^{1/(n+1)}$$

Shih proposed this definition for δ_t :



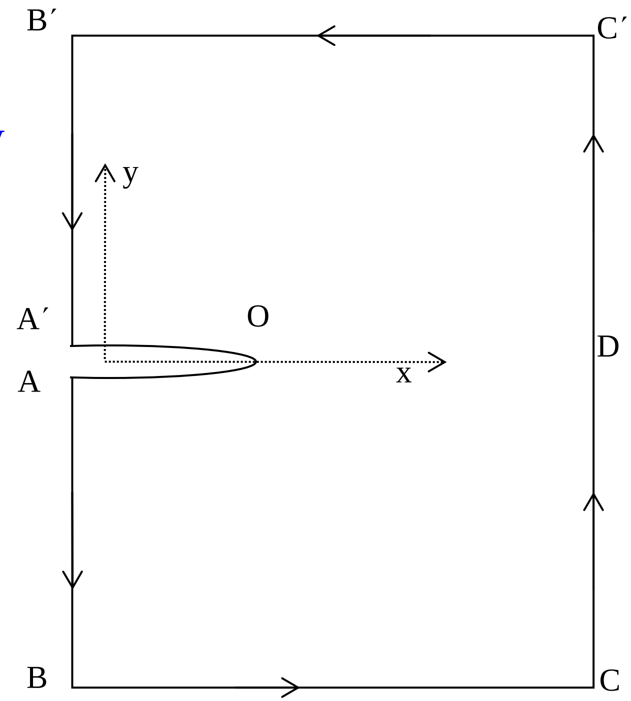
➡ m becomes a (complicated) function of n

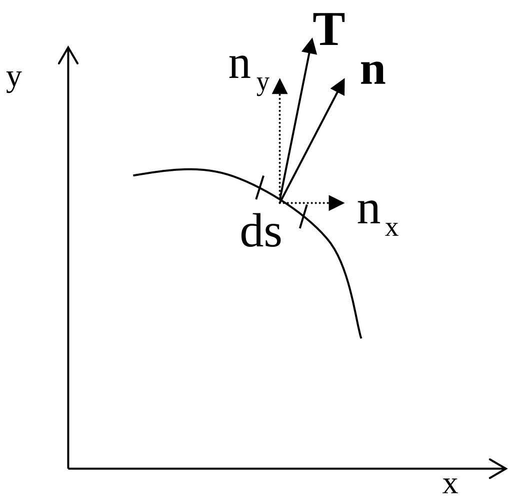
➡ The proposed definition of δ_t agrees with the one of the Irwin model

Moreover, $G = \frac{\pi}{4}\sigma_Y\delta_t$ $\qquad m = \frac{\pi}{4}$ in this case

Applications the J-integral

J-integral evaluated explicitly along specific contours





Loads and geometry symmetric / Ox

$$J = \int_{\Gamma} \left(w\, dy - \sigma_{ij} n_j \frac{\partial u_i}{\partial x} ds \right) \quad ?$$

$$w = \int \sigma_{ij} d\varepsilon_{ij} = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{1}{2} \left(\sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + 2\sigma_{xy} \varepsilon_{xy} \right)$$

for a plane stress, linear elastic problem

From stress-strain relation,

$$w=\frac{1}{2E}\left(\sigma_{xx}^2+\sigma_{yy}^2-2\nu\sigma_{xx}\sigma_{yy}\right)+\frac{1+\nu}{E}\sigma_{xy}^2$$

Expanded form for $\sigma_{ij}n_j\frac{\partial u_i}{\partial x}ds$

$$=\sigma_{xx}n_x\frac{\partial u_x}{\partial x}ds+\sigma_{xy}n_y\frac{\partial u_x}{\partial x}ds+\sigma_{yx}n_x\frac{\partial u_y}{\partial x}ds+\sigma_{yy}n_y\frac{\partial u_y}{\partial x}ds$$ (2D problem)

Simplification :

Along AB or B´A´ $n_x = -1,\ n_y = 0$ and $ds=-dy \neq 0$

$$=\sigma_{xx}\frac{\partial u_x}{\partial x}dy+\sigma_{yx}n_x\frac{\partial u_y}{\partial x}dy$$

Along CD or DC´ $n_x = 1,\ n_y = 0$ and $ds=dy \neq 0$

$$=\sigma_{xx}\frac{\partial u_x}{\partial x}dy+\sigma_{yx}\frac{\partial u_y}{\partial x}dx$$

Along BC or C´B´

BC : $n_x = 0,\ n_y = -1$ and $ds=dx \neq 0$

$$=-\sigma_{xy}\frac{\partial u_x}{\partial x}dx-\sigma_{yy}\frac{\partial u_y}{\partial x}dx$$

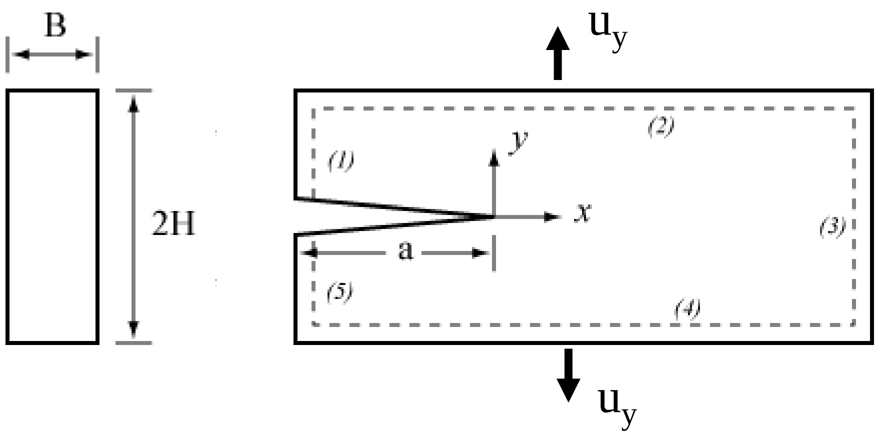
C'B' : $n_x = 0,\ n_y = 1$ and $ds=-dx \neq 0$

Along OA and A´O J is zero since $dy = 0$ and $T_i = 0$

Finally,

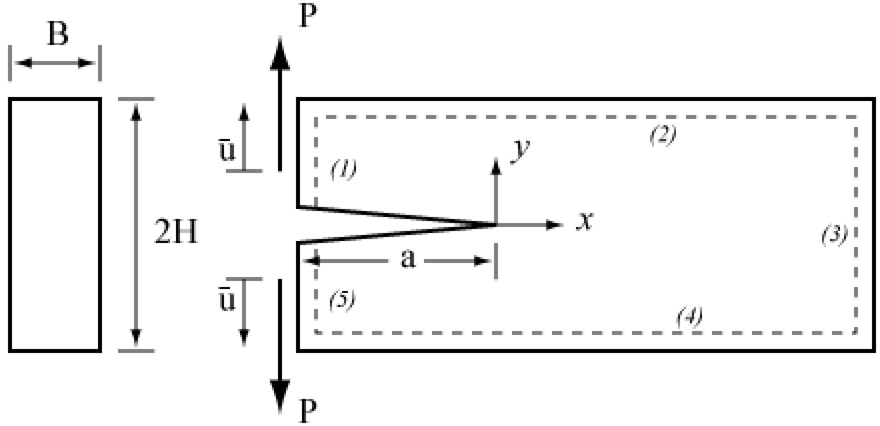
$$J=2\int_A^B\left[w-\sigma_{xx}\frac{\partial u_x}{\partial x}-\sigma_{xy}\frac{\partial u_y}{\partial x}\right]dy+2\int_B^C\left[\sigma_{xy}\frac{\partial u_x}{\partial x}+\sigma_{yy}\frac{\partial u_y}{\partial x}\right]dx+2\int_C^D\left[w-\sigma_{xx}\frac{\partial u_x}{\partial x}-\sigma_{xy}\frac{\partial u_y}{\partial x}\right]dy$$

Example 1



$$J=2hw=\frac{(1-\nu)Eu_y^2}{(1+\nu)(1-2\nu)h}$$

Example 2



$$J=\frac{12P^2a^2}{EB^2h^3}$$