Homework #7-solution

1-

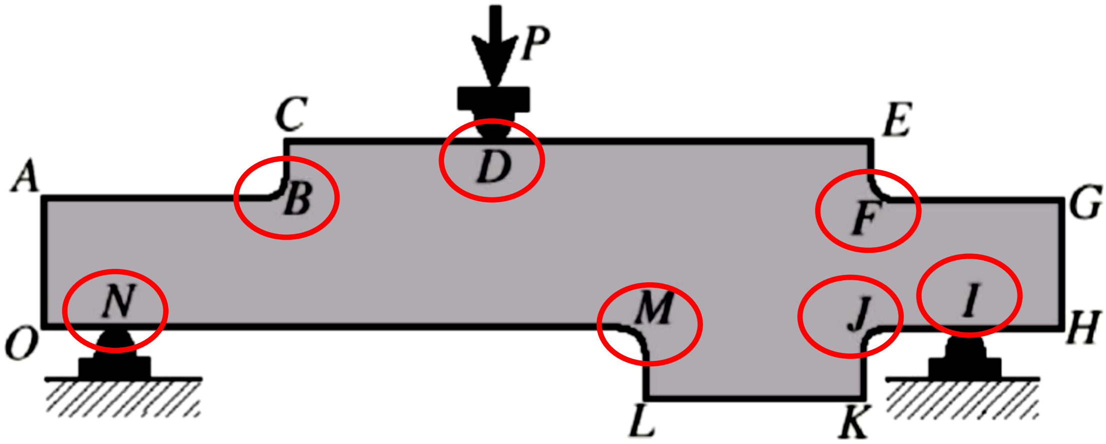
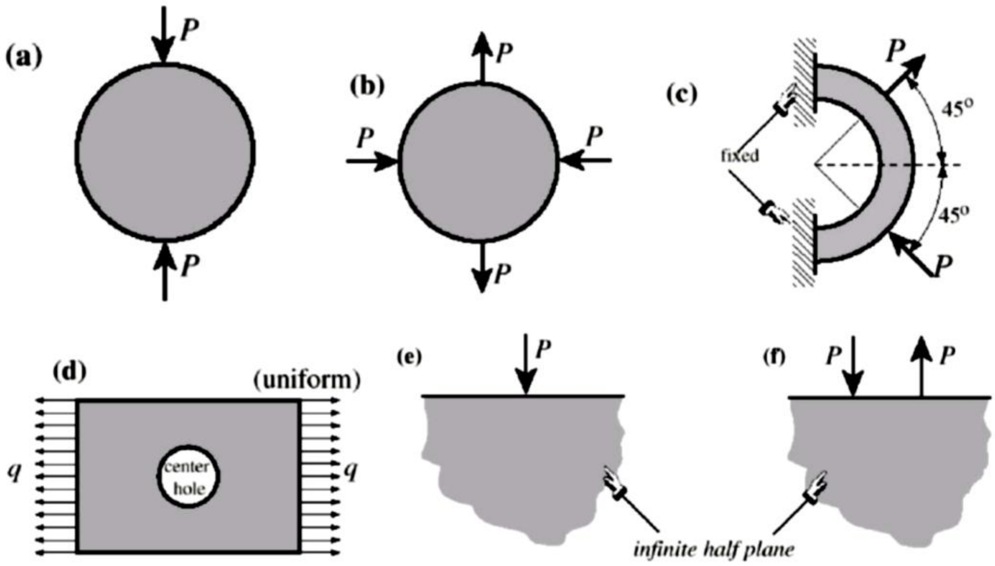
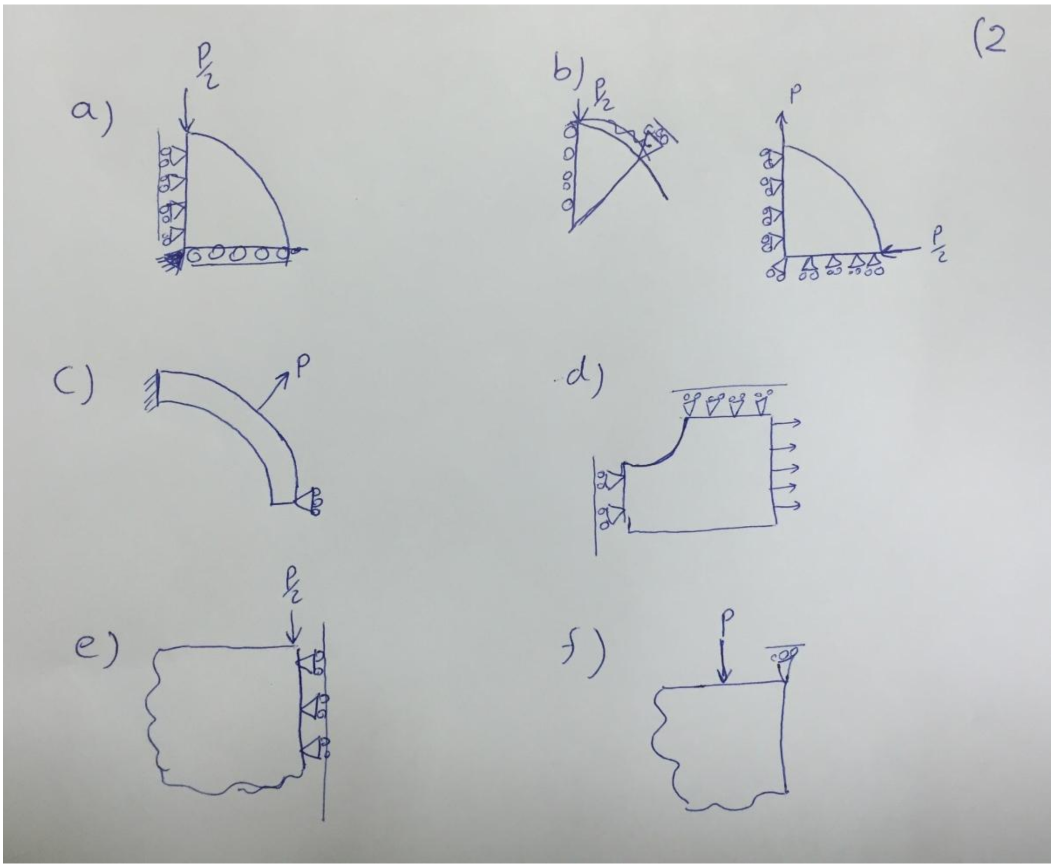


Fig. 1 The plate structure

2-





3-

$$u = [(1-x)y \quad x(1-y)]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$u_{,x} = [(1-x) \quad (1-y)]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$u_{,y} = [(1-x) \quad -x]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$v = v_{,x} = v_{,y} = 0$$

$$\varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} u_{,x} \\ v,y \\ u_{,y} + v_{,x} \end{bmatrix} = \begin{bmatrix} -y & 1-y \\ 0 & 0 \\ (1-x) & -x \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$U = \frac{1}{2}\int_0^1 \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}^T \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dx\, dy - \int_0^1 (f_x u + f_y v) dx\, dy$$

$$U = \frac{1}{2} \times \frac{E}{1-\nu 2}\int_0^1 \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}^T \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} dx\, dy - \int_0^1 (f_x [u_1 \quad u_2]\begin{bmatrix} -y & 0 & 1-x \\ 1-y & 0 & -x \end{bmatrix}) dx\, dy U$$

$$U = \frac{1}{2} \times \frac{E}{1-\nu 2} \int_0^1 [u_1 \quad u_2] \begin{bmatrix} -y & 0 & 1-x \\ 1-y & 0 & -x \end{bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \begin{bmatrix} -y & 1-y \\ 0 & 0 \\ (1-x) & -x \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} hdxdy - \int_0^1 (f_x [u_1 \quad u_2] \begin{bmatrix} -y & 0 & 1-x \\ 1-y & 0 & -x \end{bmatrix}) hdx\, dy$$

$$\frac{\partial U}{\partial u_i} = 0$$

$$=> [K] = \frac{hE}{1-\nu 2} \int_0^1 \begin{bmatrix} -y & 0 & 1-x \\ 1-y & 0 & -x \end{bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \begin{bmatrix} -y & 1-y \\ 0 & 0 \\ (1-x) & -x \end{bmatrix}$$

$$[K] = \frac{hE}{1-\nu 2} \int_0^1 \begin{bmatrix} y^2 + \frac{1-\nu}{2}(1-x)^2 & -y(1-y) - \frac{1-\nu}{2} x(1-x) \\ -y(1-y) - \frac{1-\nu}{2} x(1-x) & y^2 + \frac{1-\nu}{2}(x)^2 \end{bmatrix}$$

$$[K] = \frac{hE}{1-\nu 2} \begin{bmatrix} \frac{1}{3}(1 + \frac{1-\nu}{2}) & \frac{-1}{6}(1 + \frac{1-\nu}{2}) \\ \frac{-1}{6}(1 + \frac{1-\nu}{2}) & \frac{1}{3}(1 + \frac{1-\nu}{2}) \end{bmatrix}$$

راه دوم:

می دانیم که ماتریس سفتی برای حالت تنش صفحه ای از رابطه زیر محاسبه می شود:

$$[K] = \int_0^1 [B]^T [D]\, [B] h\, dxdy$$

که:

$$B = \begin{bmatrix} -y & 1-y \\ 0 & 0 \\ (1-x) & -x \end{bmatrix} \quad ; \quad D = \frac{E}{1-\nu 2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

که با جایگذاری این دو ماتریس در فرمون ماتریس سفتی، و انتگرال گیری نسبت به x و y به رابطه بالا خواهیم رسید.

4-

a) *The Timoshenko (shear-deformable) beam theory:*

$$\left.\begin{aligned} -\frac{d}{dx}\left[GKA\left(\frac{dw}{dx}+\Psi\right)\right]=f \\ -\frac{d}{dx}\left(EI\frac{d\Psi}{dx}\right)+GKA\left(\frac{dw}{dx}+\Psi\right)=0 \end{aligned}\right\} \quad \text{for } 0<x<L$$

$$w(0)=w(L)=0, \quad \left(EI\frac{d\Psi}{dx}\right)\bigg|_{x=0}=\left(EI\frac{d\Psi}{dx}\right)\bigg|_{x=L}=0$$

where G, K, A, E, I, and f are functions of x.

$$\int v_1\left(-\frac{d}{dx}\left[GKA\left(\frac{dw}{dx}+\psi\right)\right]\right)dx-\int v_1\, f dx=0$$

$$\int \frac{dv_1}{dx}\left(\left[GKA\left(\frac{dw}{dx}+\psi\right)\right]\right)dx-\left(v_1\left[GKA\left(\frac{dw}{dx}+\psi\right)\right]\right)\bigg|_0^L-\int v_1\, f dx=0$$

So : $v_1(0)=v_1(L)=0$

And:

$$\int \frac{dv_1}{dx}\left(\left[GKA\left(\frac{dw}{dx}+\psi\right)\right]\right)dx-\int v_1\, f dx=0 \quad (1)$$

$$\int v_2\left(-\frac{d}{dx}\left[EI\left(\frac{d\psi}{dx}\right)\right]+v_2\left[GKA\left(\frac{dw}{dx}+\psi\right)\right]\right)dx=0$$

$$\int \frac{dv_2}{dx}\left(\left[EI\left(\frac{d\psi}{dx}\right)\right]+v_2\left[GKA\left(\frac{dw}{dx}+\psi\right)\right]\right)dx-\left(v_2\left[EI\left(\frac{d\psi}{dx}\right)\right]\right)\bigg|_0^L=0$$

Which : $\left[EI\left(\frac{d\psi}{dx}\right)\right](0)=\left[EI\left(\frac{d\psi}{dx}\right)\right](L)=0$

$$\int \frac{dv_2}{dx}\left(\left[EI\left(\frac{d\psi}{dx}\right)\right]+v_2\left[GKA\left(\frac{dw}{dx}+\psi\right)\right]\right)dx=0 \quad (2)$$

$$\begin{cases} v_1=\delta w \\ v_2=\delta\psi \end{cases}$$

$$(1)+(2) \quad => \int GKA\left(\left[\left(\frac{dw}{dx}+\psi\right)\right]\right)\ \delta\left(\left[\left(\frac{dw}{dx}+\psi\right)\right]\right)\ dx + \int EI\left(\frac{d\psi}{dx}\right)\ \delta\left(\left[\left(\frac{d\psi}{dx}\right)\right]\right)\ dx - \int \delta w\, f dx = 0$$

$$\delta\left(\int\left(GKA\left(\left[\left(\frac{dw}{dx}+\psi\right)\right]\right)+EI\left(\frac{d\psi}{dx}\right)^2 - wf\right)\ dx\right) = 0$$

$$I = \int\left(GKA\left(\left[\left(\frac{dw}{dx}+\psi\right)\right]\right)+EI\left(\frac{d\psi}{dx}\right)^2 - wf\right)\ dx$$

The Euler-Bernoulli-von Kármán nonlinear beam theory

$$-\frac{d}{dx}\left\{EA\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\} = f \quad \text{for} \quad 0 < x < L$$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - \frac{d}{dx}\left\{EA\frac{dw}{dx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\} = q$$

$$u = w = 0 \quad \text{at} \quad x = 0,\ L; \quad \left(\frac{dw}{dx}\right)\Big|_{x=0} = 0; \quad \left(EI\frac{d^2w}{dx^2}\right)\Big|_{x=L} = M_0$$

where EA, EI, f, and q are functions of x, and M_0 is a constant. Here u denotes the axial displacement and w the transverse deflection of the beam.

Solution: The first step of the formulation is to multiply each equation with a weight function, say v_1 for the first equation and v_2 for the second equation, and integrate over the interval $(0, L)$. In the second step, carry out the integration-by-parts once in the first equation, twice in the first term of the second equation, and once in the second part of the second equation. Then use the fact that $v_1(0) = v_1(L) = 0$ (because u is specified there), $v_2(0) = v_2(L) = 0$ (because w is specified), and $(dv_2/dx)(0) = 0$

(because dw/dx is specified at $x = 0$). In addition, we have $EI(d^2w/dx^2) = M_0$ at $x = L$. The final weak forms are given by

$$0 = \int_0^L \left\{ EA\frac{dv_1}{dx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] - v_1 f \right\} dx \tag{1a}$$

$$0 = \int_0^L \left\{ EI\frac{d^2v_2}{dx^2}\frac{d^2w}{dx^2} + EA\frac{dv_2}{dx}\frac{dw}{dx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] - v_2 q \right\} dx$$
$$- \left(\frac{dv_2}{dx}\right)\bigg|_L M_0 \tag{1b}$$

Note that for this case the weak form is not linear in u or w. However, a functional can be constructed for this using the potential operator theory (see: J. T. Oden and J. N. Reddy, *Variational Methods in Theoretical Mechanics*, 2nd ed., Springer-Verlag, Berlin, 1983 and Reddy [3]). The functional is given by

$$\Pi(u, w) = \int_0^L \left\{ \frac{EA}{2}\left[\left(\frac{du}{dx}\right)^2 + \frac{du}{dx}\left(\frac{dw}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^4\right] + \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 \right.$$
$$\left. + uf + wq \right\} dx - \frac{dw}{dx}\bigg|_L M_0$$

5-

$$-\frac{d}{dx}\left[(1+x)\frac{du}{dx}\right] = 0 \quad \text{for} \quad 0 < x < 1$$

$$u(0) = 0, \quad u(1) = 1$$

Use algebraic polynomials for the approximation functions. Specialize your result for $N = 2$ and compute the Ritz coefficients.

Solution: The weak form for this problem is given by

$$0 = \int_0^1 (1+x)\frac{dv}{dx}\frac{du}{dx}dx$$

$$B_{ij} = B(\phi_i, \phi_j) = \int_0^1 (1+x)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx \tag{1a}$$

$$F_i = -B(\phi_i, \phi_0) = -\int_0^1 (1+x)\frac{d\phi_i}{dx}\frac{d\phi_0}{dx}dx \tag{1b}$$

The approximation functions ϕ_0 and ϕ_i should be chosen such that

$$\phi_0(0) = 0,\ \phi_0(1) = 1\ ;\quad \phi_i(0) = \phi_i(1) = 0,\ (i = 1, 2, ..., n) \tag{2}$$

The following algebraic polynomials satisfy the above requirements:

$$\phi_0 = x\ ,\quad \phi_i = x^i(1-x) \tag{3}$$

Substitution of Eq.(3) into Eqs.(1a,b) and evaluating the integrals, we obtain

$$B_{ij} = \frac{ij}{i+j-1} - \frac{ij+i+j}{i+j} + \frac{1-ij}{i+j+1} + \frac{(i+1)(j+1)}{i+j+2} \tag{4a}$$

$$F_i = \frac{1}{(1+i)(2+i)} \tag{4b}$$

For the two-parameter $(N = 2)$ case, we have

$$B_{11} = \frac{1}{2}\ ,\quad B_{12} = B_{21} = \frac{17}{60}\ ,\quad B_{22} = \frac{7}{30}\ ,\quad F_1 = \frac{1}{6}\ ,\quad F_2 = \frac{1}{12}$$

and the parameters c_1 and c_2 are given by

$$c_1 = \frac{55}{131}\ ,\quad c_2 = -\frac{20}{131}$$

The two-parameter Ritz solution becomes

$$\begin{aligned} u(x) &= \phi_0 + c_1\phi_1 + c_2\phi_2 \\ &= x + \frac{55}{131}(x - x^2) - \frac{20}{131}(x^2 - x^3) \\ &= \frac{1}{131}(186x - 75x^2 + 20x^3) \end{aligned}$$

The exact solution is given by

$$u_{exact} = \frac{\log\ (1+x)}{\log\ 2}$$

6-

$$-k\nabla^2 T = g_0$$

$$T = 0 \quad \text{on sides} \quad x = 1 \quad \text{and} \quad y = 1 \tag{1}$$

$$\frac{\partial T}{\partial n} = 0 \quad \text{(insulated) on sides} \quad x = 0 \quad \text{and} \quad y = 0 \tag{2}$$

using a one-parameter Ritz approximation of the form

$$T_1(x, y) = c_1(1 - x^2)(1 - y^2) \tag{3}$$

Solution: The weak form of the equation is given by

$$0 = \int_0^1 \int_0^1 \left[k \left(\frac{\partial v}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial T}{\partial y} \right) - v g_0 \right] dxdy \tag{4}$$

The coefficients B_{11} and F_1 are given by

$$\begin{aligned} B_{11} &= \int_0^1 \int_0^1 k \left(\frac{\partial \phi_1}{\partial x}\frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_1}{\partial y}\frac{\partial \phi_1}{\partial y} \right) \, dxdy \\ &= \int_0^1 \int_0^1 k \left[4x^2(1-y^2)^2 + 4y^2(1-x^2)^2 \right] dxdy = \frac{64}{45}k \end{aligned} \tag{5a}$$

$$\begin{aligned} F_1 &= \int_0^1 \int_0^1 g_0 \phi_1 \, dxdy \\ &= \int_0^1 \int_0^1 g_0 (1-x^2)(1-y^2) \, dxdy = \frac{4}{9} g_0 \end{aligned} \tag{5b}$$

and the parameter c_1 is given by

$$c_1 = \frac{F_1}{B_{11}} = \frac{5 g_0}{16 k} \tag{6}$$

$$-2u\frac{d^2u}{dx^2} + \left(\frac{du}{dx}\right)^2 = 4 \quad \text{for} \quad 0 < x < 1$$

subject to the boundary conditions $u(0) = 1$ and $u(1) = 0$, and compare it with the exact solution $u_0 = 1 - x^2$. Use (a) the Galerkin method, (b) the least-squares method, and (c) the Petrov–Galerkin method with weight function $w = 1$.

7-

Solution: We must choose ϕ_0 such that it satisfies *all* specified boundary conditions:

$$\phi_0(0) = 1\ ,\ \ \phi_0(1) = 0 \tag{1}$$

and ϕ_i must be selected such that it satisfies the homogeneous form of *all* specified boundary conditions:

$$\phi_i(0) = 0\ ,\ \ \phi_i(1) = 0 \tag{2}$$

Obviously, the following choice would meet the requirements,

$$\phi_0 = 1 - x\ ,\ \ \phi_1 = x(1-x) \tag{3}$$

The residual is given by

$$\begin{aligned} R &= -2c_1(c_1\phi_1 + \phi_0)\frac{d^2\phi_1}{dx^2} + (c_1\frac{d\phi_1}{dx} + \frac{d\phi_0}{dx})^2 - 4 \\ &= -2\left[(1-x) + c_1(x - x^2)\right](-2c_1) + [-1 + c_1(1-2x)]^2 - 4 \\ &= -3 + 2c_1 + (c_1)^2 \end{aligned} \tag{4}$$

(a) The weighted-residual statement for the Galerkin method is given by

$$0 = \int_0^1 (x - x^2)R\ dx = \frac{1}{6}\left[-3 + 2c_1 + (c_1)^2\right]$$

which gives two solutions, $(c_1)_1 = 1$ and $(c_1)_2 = -3$. We choose $c_1 = 1$ on the basis of the criterion that $\int_0^1 R\ dx$ is a minimum. For $c_1 = 1$, the Galerkin solution coincides with the exact solution, $u(x) = 1 - x^2$.

(b) The least-squares statement is given by

$$0 = \int_0^1 \frac{dR}{dc_1} R\ dx = \int_0^1 2(1 + c_1)\left[-3 + 2c_1 + (c_1)^2\right]\ dx$$

which gives three solutions, $(c_1)_1 = 1$, $(c_1)_2 = -3$, and $(c_1)_3 = -1$. Once again, we choose $c_1 = 1$.