Stability and Phase Plane Analysis

Objectives

Objectives of the section:

- ❑ Introducing the Phase Plane Analysis
- ❑ Introducing the Concept of stability
- ❑ Stability Analysis of Linear Time Invariant Systems
- ❑ Lyapunov Indirect Method in Stability Analysis of Nonlinear Sys.
- ❑ Lyapunov Direct Method in Stability Analysis of Nonlinear Sys.
- ❑ Invariant Sets and Stability Analysis of Invariant Sets

Introducing

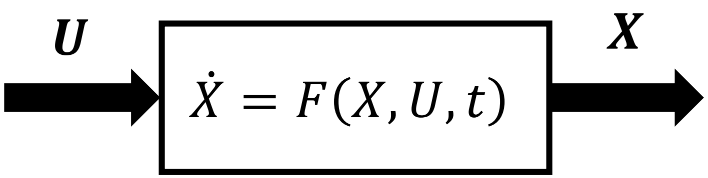
the Phase Plane Analysis

Phase Plane Analysis

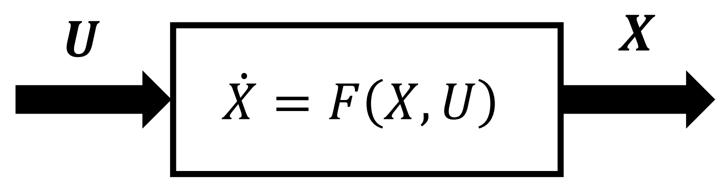
Phase Space form of a Dynamical System:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t) \\ \dot{x}_2 = f_2(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t) \\ \vdots \\ \dot{x}_n = f_n(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t) \end{cases} \longrightarrow \quad \dot{X} = F(X, U, t)$$

$$X \in \mathbb{R}^n \quad U \in \mathbb{R}^m$$



Time-Varying System



Time-Invariant System

Phase Space form of a Linear Time Invariant (LTI) System:

$$\dot{\boldsymbol{X}} = \boldsymbol{AX} + \boldsymbol{BU}$$

$$X \in \mathbb{R}^n \quad U \in \mathbb{R}^m$$

Special Properties of Nonlinear Systems:

- ❑ Multiple isolated equilibria
- ❑ Limit Cycle
- ❑ Finite escape time
- ❑ Harmonic, sub-harmonic and almost periodic Oscillation
- ❑ Chaos
- ❑ Multiple modes of behavior

Phase Plane Analysis

Phase Plane Analysis is a graphical method for studying <u>second-order systems</u> respect to initial conditions by:

- providing motion trajectories corresponding to various initial conditions.
- examining the qualitative features of the trajectories
- obtaining information regarding the stability of the equilibrium points

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

Advantages of Phase Plane Analysis:

- It is graphical analysis and the solution trajectories can be represented by curves in a plane
- Provides easy visualization of the system qualitative
- Without solving the nonlinear equations analytically, one can study the behavior of the nonlinear system from various initial conditions.
- It is not restricted to small or smooth nonlinearities and applies equally well to strong and hard nonlinearities.
- There are lots of practical systems which can be approximated by second-order systems, and apply phase plane analysis.

Disadvantage of Phase Plane Method:

- It is restricted to at most second-order
- graphical study of higher-order is computationally and geometrically complex.

Example: First Order NLTI System

$$\dot{x} = \sin(x)$$

Analytical Solution

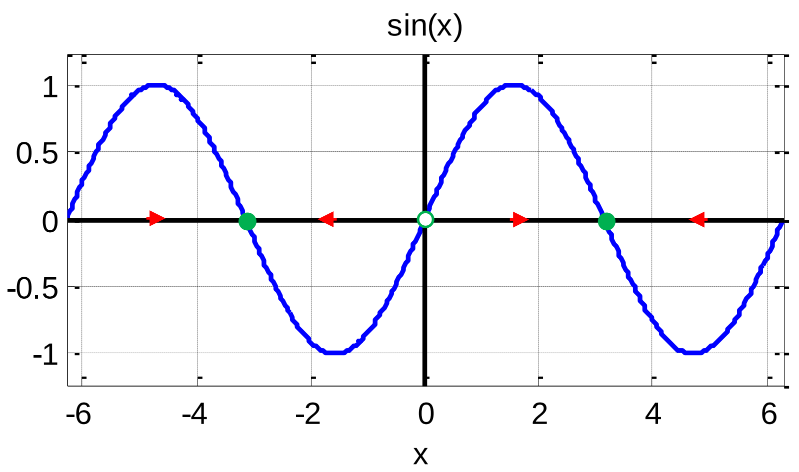
$$\frac{dx}{dt} = \sin(x)$$

$$\frac{dx}{\sin(x)} = dt$$

$$\int_{x_0}^{x} \frac{dx}{\sin(x)} = \int_0^t dt$$

$$t = \ln\left|\frac{\cos(x_0) + \cot(x_0)}{\cos(x) + \cot(x)}\right|$$

Graphical Solution



Concept of Phase Plane Analysis:

- Phase plane method is applied to Autonomous Second Order System

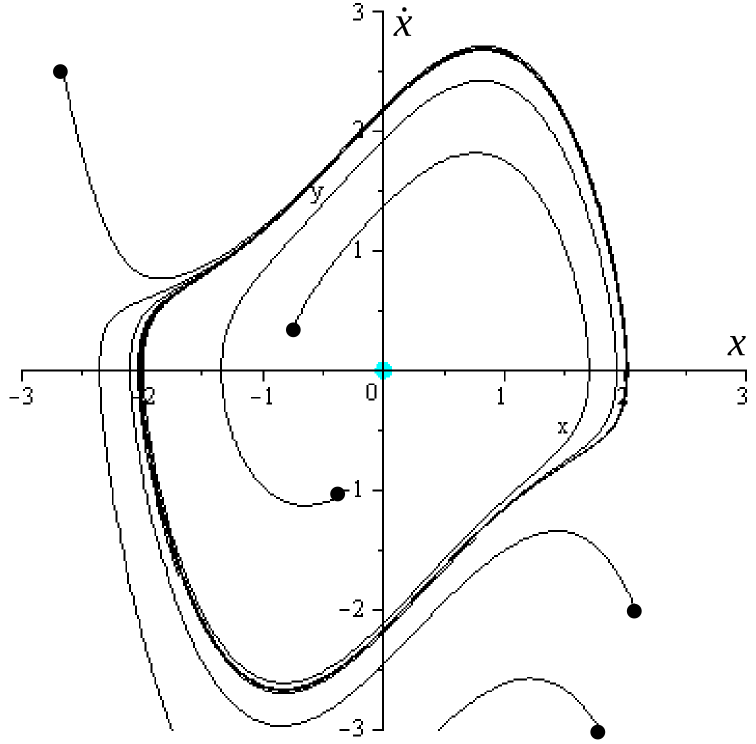
$$\dot{x}_1 = f_1(x_1, x_2) \qquad \dot{x}_2 = f_2(x_1, x_2)$$

- System response $X(t) = (x_1(t), x_2(t))$ to initial condition $X_0 = \big(x_1(0), x_2(0)\big)$ is a mapping from $\mathbb{R}$(Time) to $\mathbb{R}^2(x_1, x_2)$
- The solution can be plotted in the $x_1 - x_2$ plane called ***State Plane*** or ***Phase Plane***
- The locus in the $x_1 - x_2$ plane is a curved named ***Trajectory*** that pass through point X_0
- The family of the phase plane trajectories corresponding to various initial conditions is called ***Phase portrait*** of the system.
- For a single DOF mechanical system, the phase plane is in fact $(x, \dot{x})$ plane

Example: Van der Pol Oscillator Phase Portrait

$$\ddot{x} - (1 - x^2)\dot{x} + x = 0$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (1 - x_1^2)x_2 - x_1 \end{cases}$$



Plotting Phase Plane Diagram:

- ❑ Analytical Method
- ❑ Numerical Solution Method
- ❑ Isocline Method
- ❑ Vector Field Diagram Method
- ❑ Delta Method
- ❑ Lienard's Method
- ❑ Pell's Method

Analytical Method

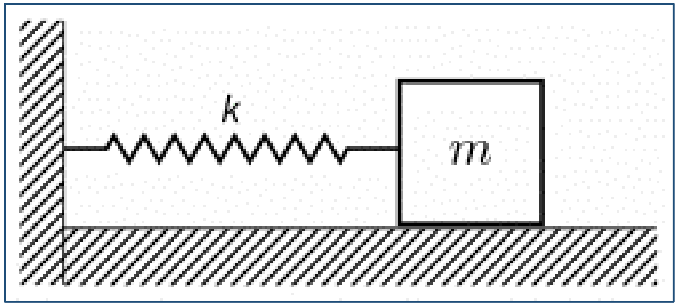
- ❑ Dynamic equations of the system is solved, then time parameter is omitted to obtain relation between two states for various initial conditions

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

$$\xrightarrow{\text{Solve}} \quad \begin{matrix} x_1(t, X_0) = g_1(t, X_0) \\ x_2(t, X_0) = g_2(t, X_0) \end{matrix} \quad \longrightarrow \quad F(x_1, x_2) = 0$$

✓ For linear or partially linear systems

Example: Mass Spring System

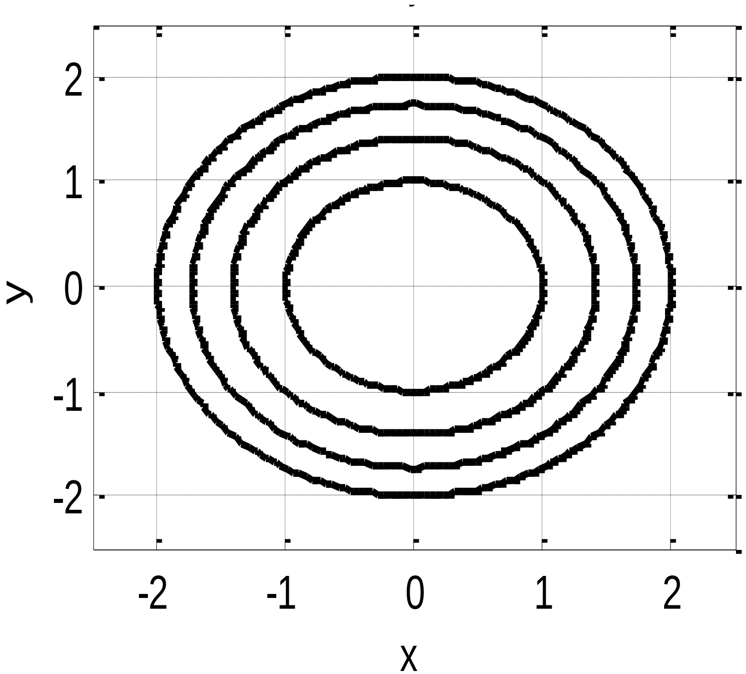


$$m\ddot{x} + kx = 0$$

For $m = k = 1$: $\ddot{x} + x = 0$

$$\begin{cases} x(t) = x_0 \cos(t) + \dot{x}_0 \sin(t) \\ \dot{x}(t) = -x_0 \sin(t) + \dot{x}_0 \cos(t) \end{cases}$$

$$\boxed{x^2 + \dot{x}^2 = x_0^2 + \dot{x}_0^2}$$



Analytical Method

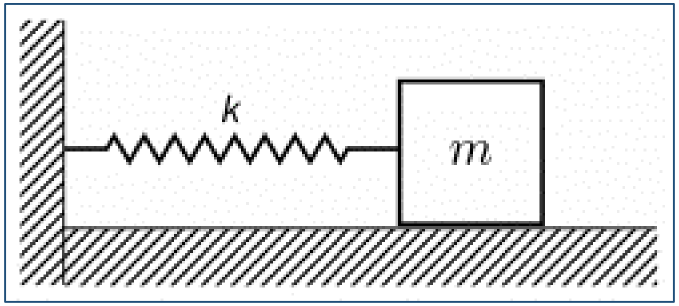
- Time differential is omitted from dynamic equations of the system, then partial differential equation is solved

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \xrightarrow{\text{Solve}} F(x_1, x_2) = 0$$

✓ For linear or partially linear systems

Example: Mass Spring System



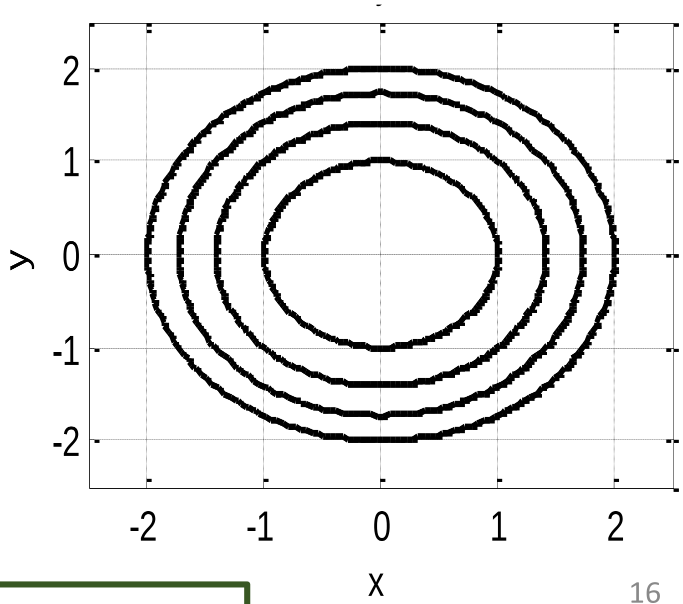
$$m\ddot{x} + kx = 0$$

For $m = k = 1$: $\quad \ddot{x} + x = 0 \longrightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases}$

$$\frac{dx_2}{dx_1} = \frac{-x_1}{x_2}$$

$$\int_{x_{20}}^{x_2} x_2 dx_2 = \int_{x_{10}}^{x_1} -x_1 dx_1$$

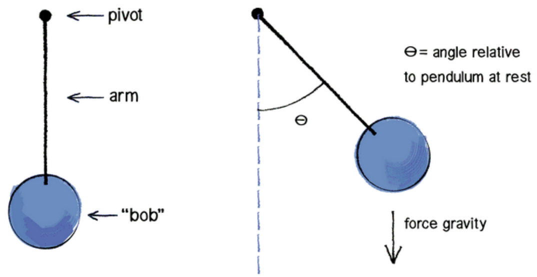
$$\boxed{x^2 + \dot{x}^2 = x_0^2 + \dot{x}_0^2}$$



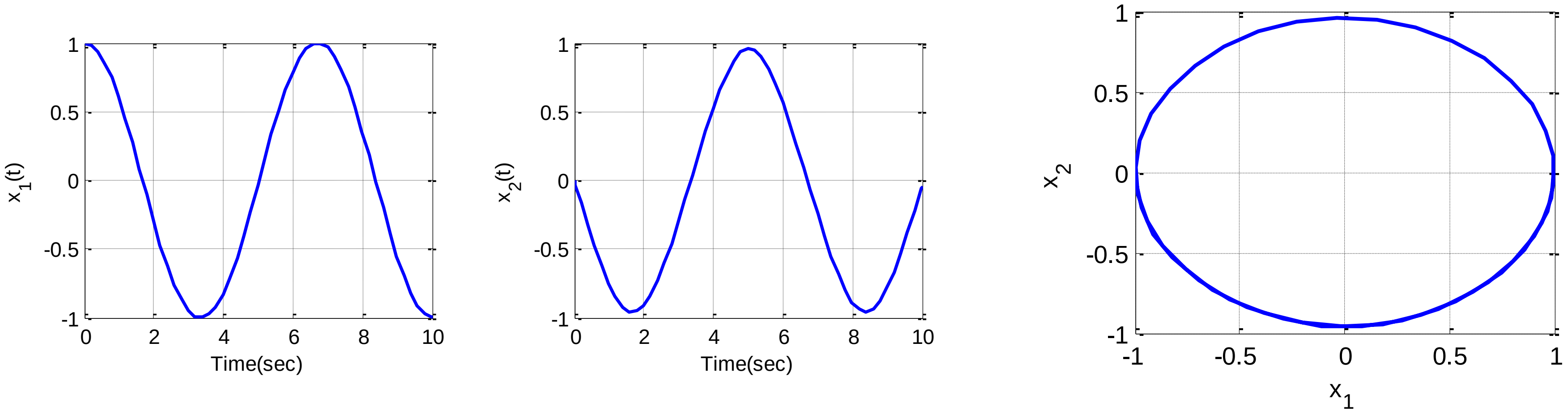
Numerical Solution Method

Dynamic equations of the system is solved numerically (e.g. ode45) for various initial conditions and time response is obtained, then two states are plotted in each time.

Example: Pendulum



$$\ddot{\theta} + \sin(\theta) = 0$$



Isocline Method

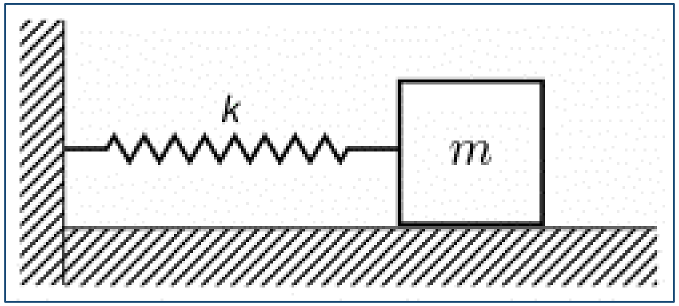
Isocline: The set of all points which have same trajectory slope

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = \alpha \longrightarrow f_2(x_1, x_2) = \alpha f_1(x_1, x_2)$$

First various isoclines are plotted, then trajectories are drawn.

Example: Mass Spring System

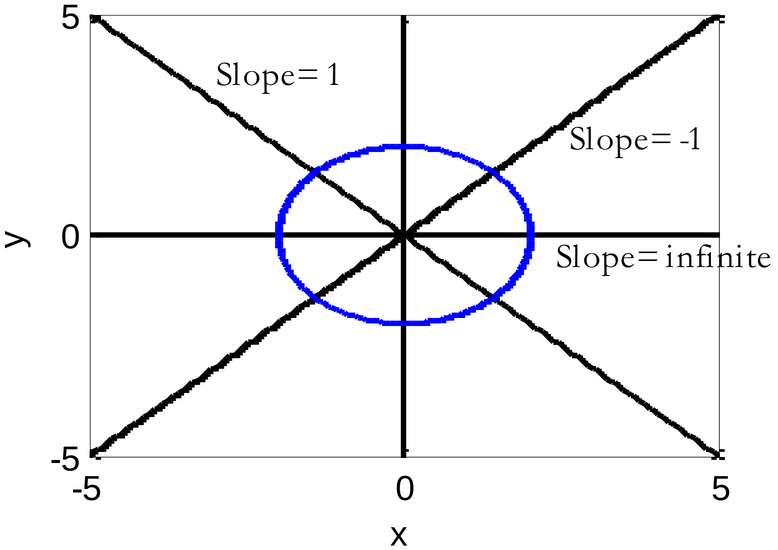


$$m\ddot{x} + kx = 0$$

For $m = k = 1$: $\ddot{x} + x = 0 \longrightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases}$

$$\frac{dx_2}{dx_1} = \frac{-x_1}{x_2} = \alpha$$

$$x_1 + \alpha x_2 = 0$$



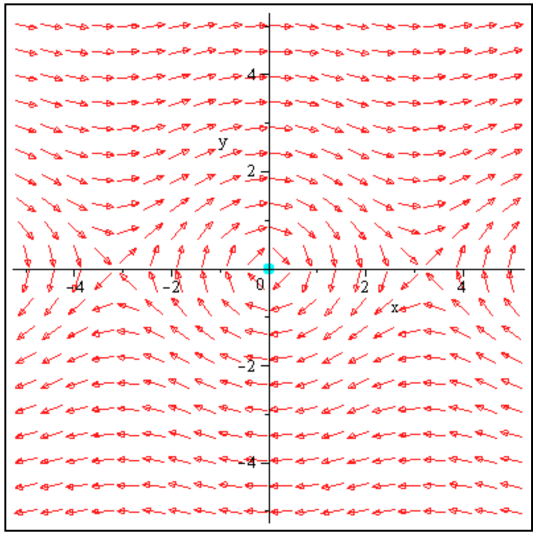
Vector Field Diagram Method

Vector Field: A set of vectors that is tangent to the trajectory

- At each point (x_1, x_2) vector $\begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix}$ is tangent to the trajectories
- Hence vector field can be constructed in the phase plane and direction of the trajectories can be easily realized with that

$$\ddot{\theta} + \sin(\theta) = 0$$

$$\left.\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \end{aligned}\right\} \Rightarrow \mathbf{f} = \begin{bmatrix} x_2 \\ -\sin(x_1) \end{bmatrix}$$



Singular Points in the Phase Plane Diagram:

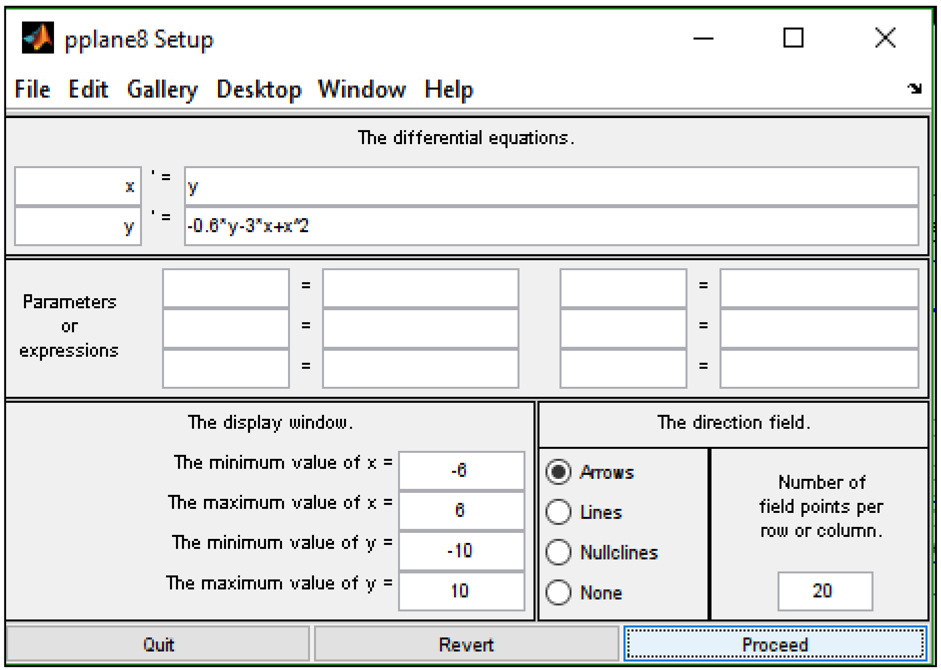
Equilibrium points are in fact singular points in the phase plane diagram

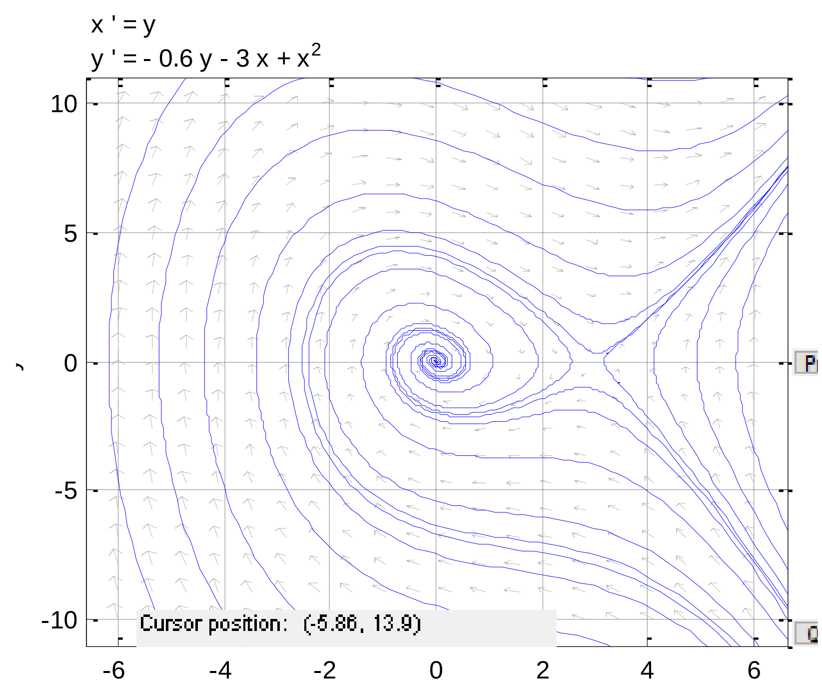
$$\left.\begin{aligned} f_1(x_1, x_2) &= 0 \\ f_2(x_1, x_2) &= 0 \end{aligned}\right\} \xrightarrow[\text{equilibrium points}]{\text{Slope of the trajectories at}} \frac{dx_2}{dx_1} = \frac{0}{0}$$

- Singular point is an important concept which reveals great info about properties of system such as stability.
- Singular points are only points which several trajectories pass/approach them (i.e. trajectories intersect).

Example: Using Matlab

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

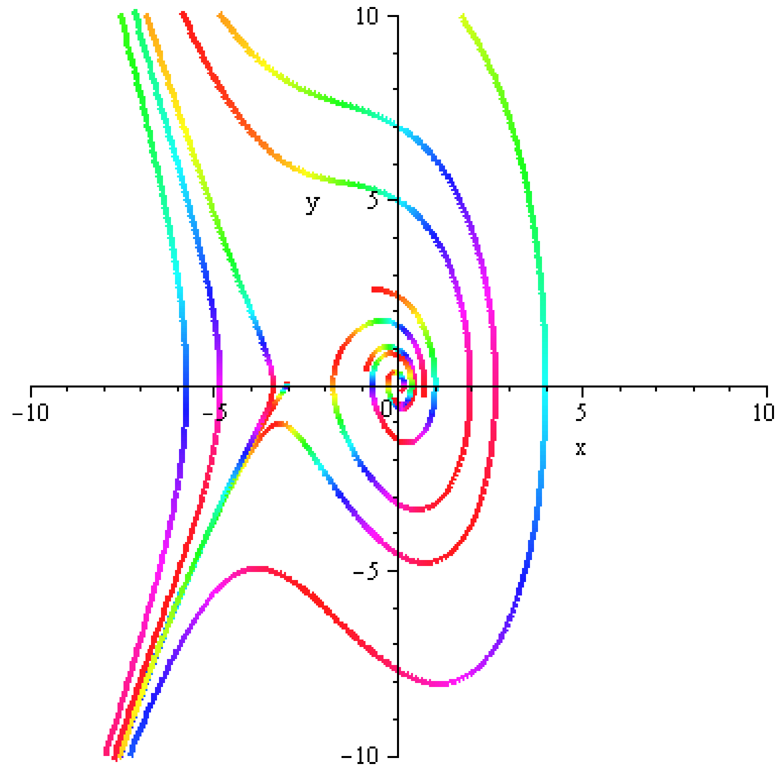




Example: Using Maple Code

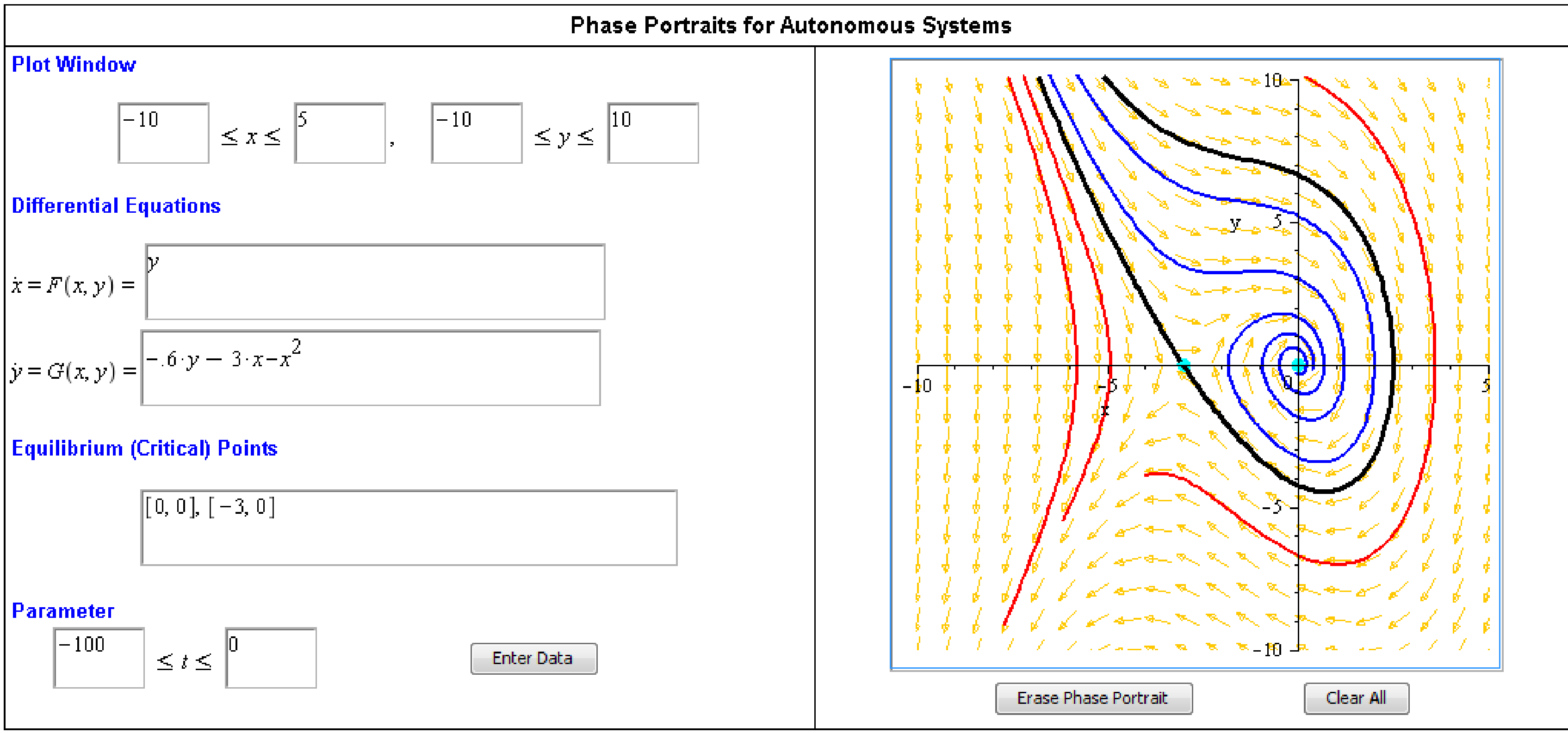
$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

```
with(DEtools):
xx:=x(t):   yy:=y(t):
dx:=diff(xx,t):    dy:=diff(yy,t):
e0:=diff(dx,t)+.6*dx+3*xx+xx^2:
e1:=dx-yy=0:
e2:=dy+0.6*yy+3*xx+xx^2=0:
eqn:=[e1,e2]:  depvar:=[x,y]:
rang:=t=-1..5:  stpsz:=stepsize=0.005:
IC1:=[x(0)=0,y(0)=1]:
IC2:=[x(0)=0,y(0)=5]:
IC3:=[x(0)=0,y(0)=7]:
IC4:=[x(0)=0,y(0)=7]:
IC5:=[x(0)=-3.01,y(0)=0]:
IC6:=[x(0)=-4,y(0)=2]:
IC7:=[x(0)=1,y(0)=0]:
IC8:=[x(0)=4,y(0)=0]:
IC9:=[x(0)=-6,y(0)=3]:
IC10:=[x(0)=-6,y(0)=6]:
ICs:=[IC||(1..10)]:
lincl:=linecolour=sin((1/2)*t*Pi):
mtd:=method=classical[foreuler]:
phaseportrait(eqn,depvar,rang,ICs,stpsz,lincl,mtd);
```



Example: Using Maple Tools

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$



Phase Plane Analysis for Single DOF Mechanical System

In the case of single DOF mechanical system

$$\ddot{x} + g(x, \dot{x}) = 0 \xrightarrow[x_2 = \dot{x}]{x_1 = x} \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g(x_1, x_2) \end{cases}$$

- The phase plane is in fact $(x - \dot{x})$ plane and every point shows the position and velocity of the system.
- Trajectories are always clockwise. This is not true in the general phase plane $(x_1 - x_2)$

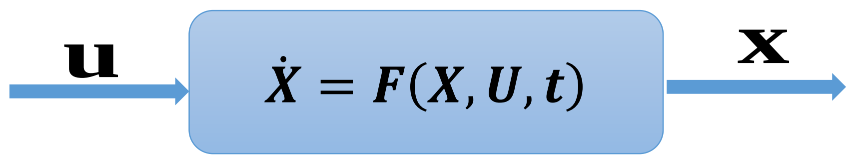
Introducing
the Concept of Stability

- Stability analysis of a dynamic system is normally introduced in the state space form of the equations.

$$\dot{X} = F(X, U, t)$$

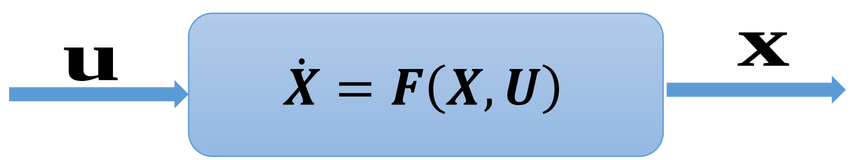
$$X \in \mathbb{R}^n \quad U \in \mathbb{R}^m$$

Time-varying Dynamic System



- Most of the concepts in this chapter are introduced for autonomous systems

Autonomous Dynamic System



Stability analysis of a dynamic system is divided in three categories:

1. Stability analysis of the equilibrium points of the systems. We study the behavior (dynamics) of the free (unforced, $u = 0$) system when it is perturbed from its equilibrium point.

2. Input-output stability analysis. We study the system (forced system $u \neq 0$) output behavior in response to bounded inputs.

3. Stability analysis of periodic orbits. This analysis is for those systems which perform a periodic or cyclic motion like walking of a biped or orbital motion of a space object.

✓ <u>Our main concern is the first type analysis.</u> Some preliminary issues of the third type analysis will be also discussed.

Reminder:

X_e is said an equilibrium point of the system if once the system reaches this position it stays there for ever, i.e. $f(X_e) = 0$

Definition (Lyapunov Stability):

The equilibrium point X_e is said to be stable (in the sense of Lyapunov stability) or motion of the system about its equilibrium point is said to be stable if the system states (X) is perturbed away from X_e then it stays close to X_e. Mathematically X_e is stable if

$$\varepsilon > 0,\ \exists\ \delta = \delta(\varepsilon) > 0 \qquad \left\|\mathbf{x}(0) - \mathbf{x}_e\right\| < \delta \Rightarrow \left\|\mathbf{x}(t) - \mathbf{x}_e\right\| < \varepsilon \quad \forall t \geq 0$$

- Without loss of generality we can present our analysis about equilibrium point $X_e = 0$, since the system equation can be transferred to a new form with zero as the equilibrium point of the system.

$$\begin{matrix} \dot{Y} = F(Y) \\ Y_e \neq 0 \end{matrix} \xrightarrow{X = Y - Y_e} \begin{matrix} \dot{X} = F(X) \\ X_e = 0 \end{matrix}$$

A more precise definition:

The equilibrium point X_e is said to be stable (in the sense of Lyapunov stability) or motion of the system about its equilibrium point is said to be stable if for any $R > 0$, there exists $0 < r < R$ such that

$$\|\mathbf{x}(0) - \mathbf{x}_e\| < r \Rightarrow \|\mathbf{x}(t) - \mathbf{x}_e\| < R \quad \forall t \geq 0$$

Definition (Lyapunov Stability):

The equilibrium point $X_e = 0$ is said to be

- **Stable** if

$$\forall R > 0 \quad \exists 0 < r < R \quad s.t.$$

$$\|X(0)\| < r \quad \Rightarrow \quad \|X(t)\| < R \quad \forall t > 0$$

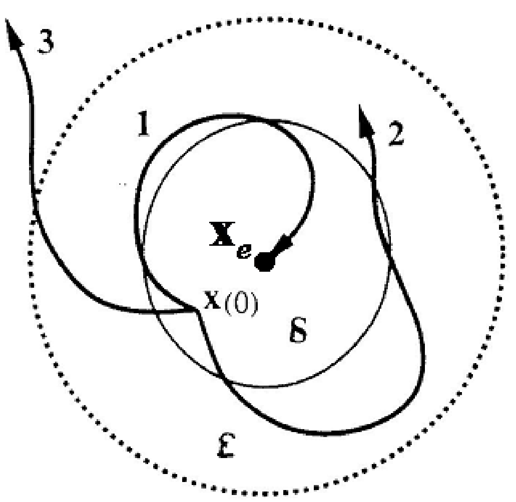
- **Unstable** if it is not stable.

- **Asymptotically stable** if it is stable and

$$\forall r > 0 \;\; s.t.$$

$$\|X(0)\| < r \quad \Rightarrow \quad \lim_{t\to\infty} X(t) = 0$$

- **Marginally stable** if it is stable and not asymptotically stable



curve 1 - asymptotically stable

curve 2 - marginally stable

curve 3 - unstable

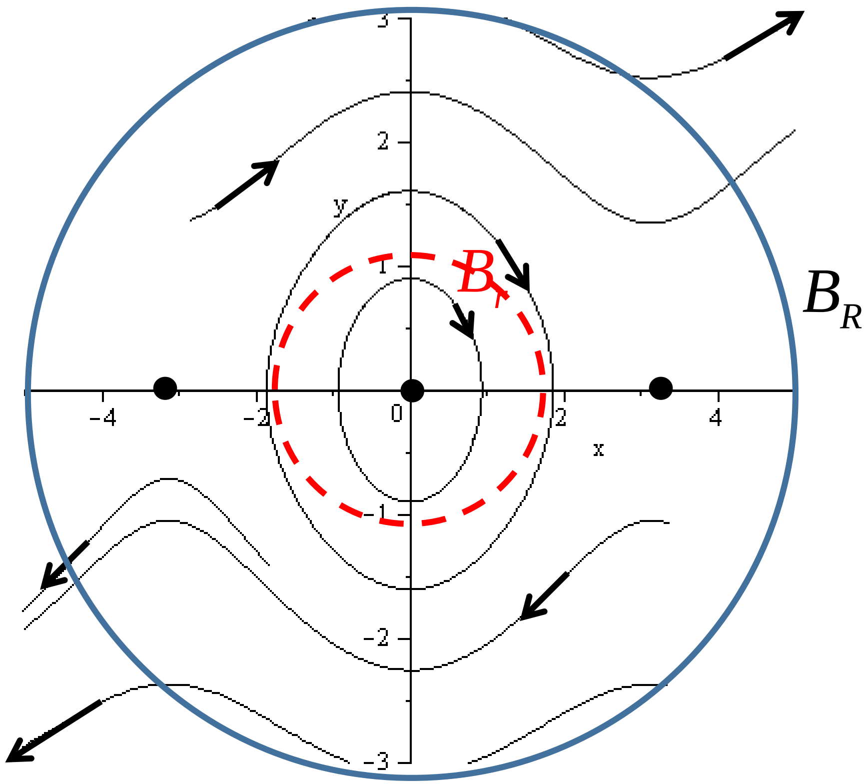
- **Exponentially stable** if it is asymptotically stable with an exponential rate

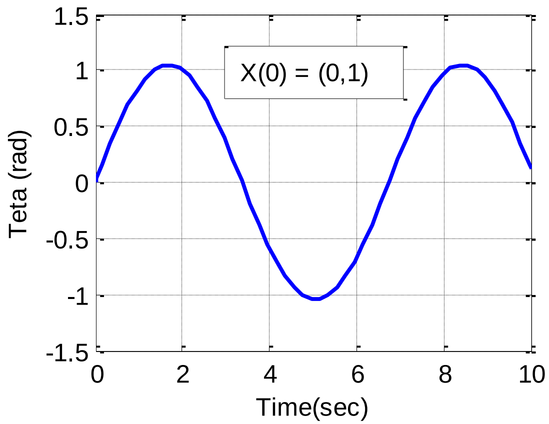
$$\|X(0)\| < r \quad \Rightarrow \quad \|X(t)\| < \alpha e^{-\beta t}\|X(0)\| \quad \alpha, \beta > 0$$

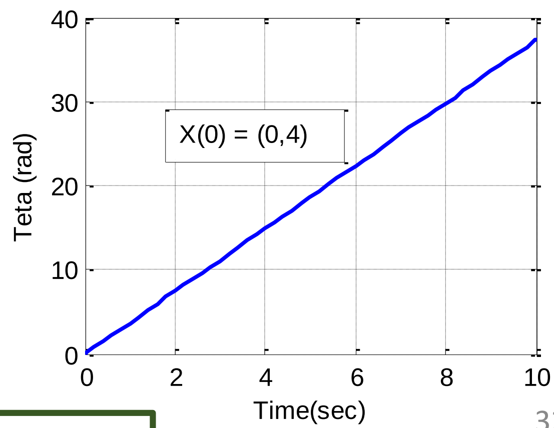
Example: Undamped Pendulum

$$\ddot{\theta} + \frac{g}{l}\sin(\theta) = 0 \qquad \theta_{e1} = 0 \ , \qquad \theta_{e2} = \pi$$

✓ θ_{e1} is a marginally stable point and θ_{e2} is an unstable point





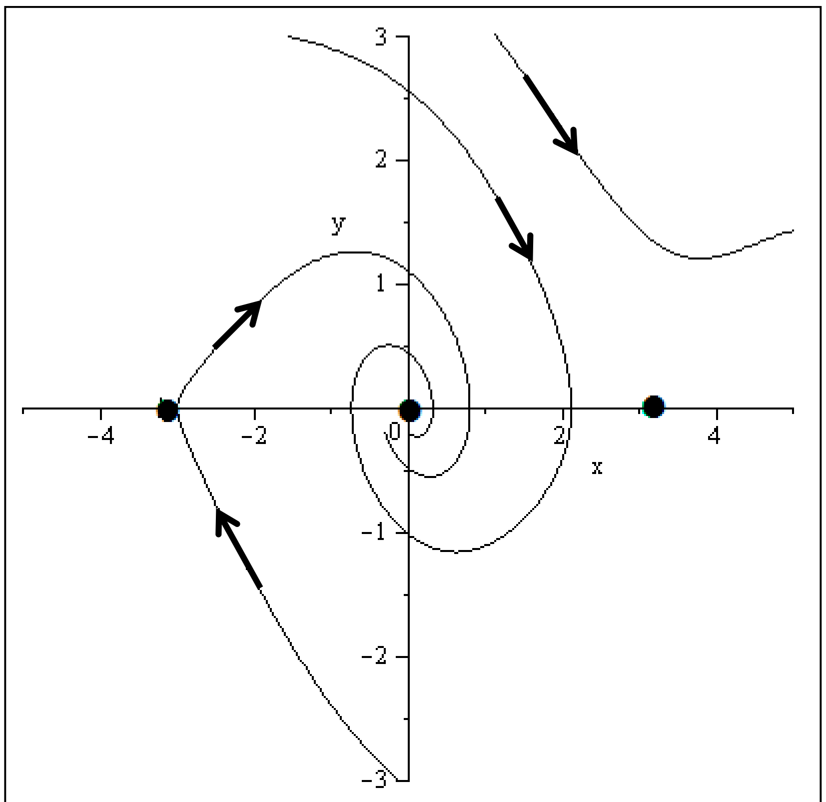


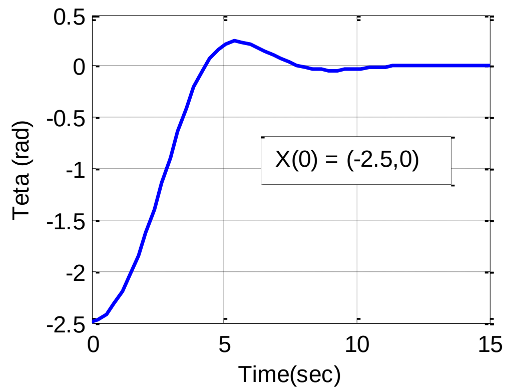
Stability, Definitions and Examples

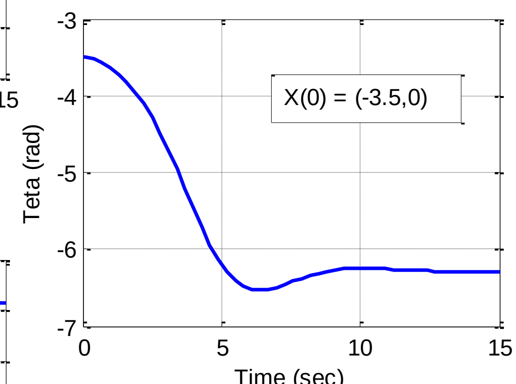
Example: damped Pendulum

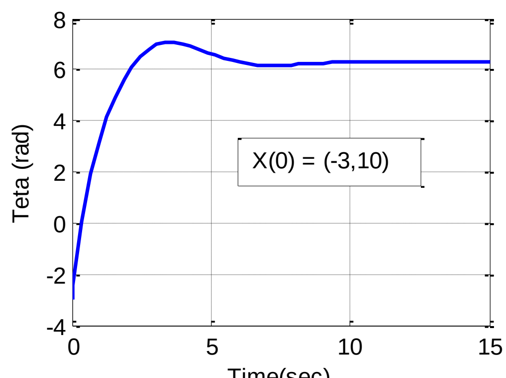
$$\ddot{\theta} + C\dot{\theta} + \frac{g}{l}\sin(\theta) = 0 \qquad\qquad \theta_{e1} = 0 \;, \qquad \theta_{e2} = \pi$$

- ✓ θ_{e1} is an exponentially stable point and θ_{e2} is an unstable point





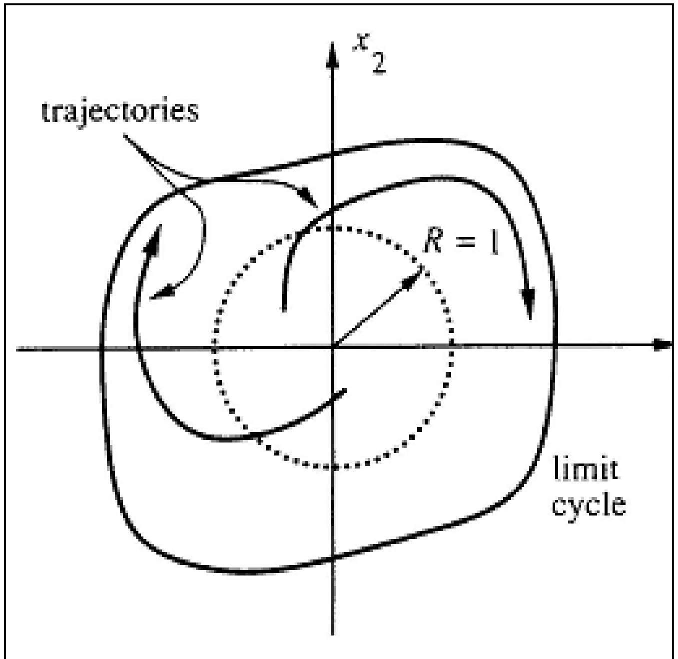




Example: Van Der Pol Oscillator

$$\ddot{x} - (1-x)^2 \dot{x} + x = 0 \overset{\text{state form}}{\Longrightarrow} \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + (1-x_1)^2 x_2 \end{cases} \Rightarrow x_{1_e} = x_{2_e} = 0$$

✓ $x_e = 0$ is an unstable point



Unstable origin of the Van der Pol

Definition:

if the equil. point X_e is asymptotically stable, then the set of all points that trajectories initiated at these point eventually converge to the origin is called ***domain of attraction***.

Definition:

if the equil. point X_e is asymptotically/exponentially stable, then the equil. point is called ***globally stable*** if the whole space is domain of attraction. Otherwise it is called ***locally stable***.

Example 1:

The origin in the first order system of $\dot{x} = -x$ is globally exponentially stable.

$$\dot{x} = -x \Rightarrow x(t) = x_0 e^{-t} \Rightarrow \lim_{t\to\infty} x(t) = 0 \quad \forall\, x_0 \neq 0$$

Example 2:

The origin in the first order system $\dot{x} = -x^3$ is globally asymptotically but not exponentially stable.

$$\dot{x} = -x^3 \Rightarrow x(t) = \frac{x_0}{\sqrt{1+2tx_0^2}} \Rightarrow \lim_{t\to\infty} x(t) = 0 \quad \forall\, x_0$$

Example 3:

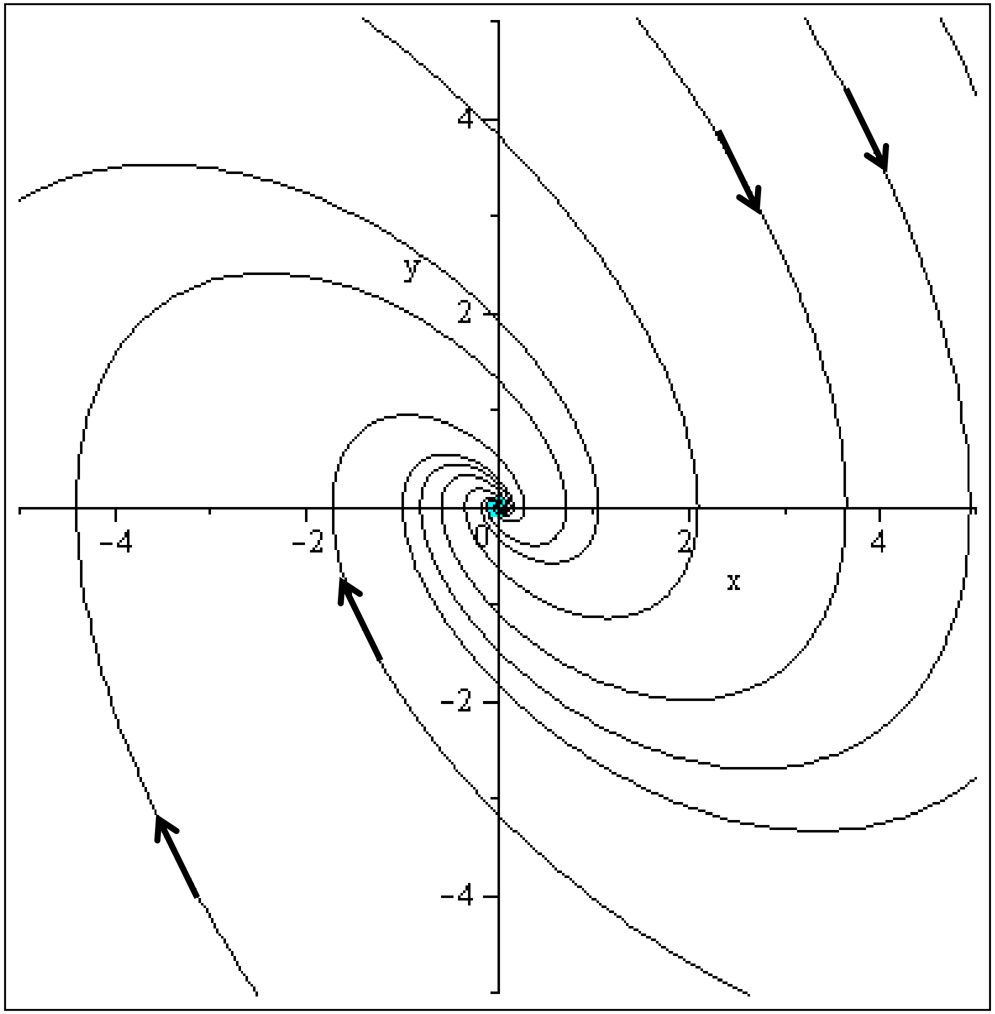
The origin in the first order system $\dot{x} = -x^2$ is semi-asymptotically but not exponentially stable.

$$\dot{x} = -x^2 \Rightarrow x(t) = \frac{x_0}{1+tx_0} \Rightarrow \begin{cases} \lim\limits_{t\to\infty} x(t) = 0 & \text{if} \quad x_0 > 0 \\ \lim\limits_{t\to -1/x_0} x(t) \to \infty & \text{if} \quad x_0 < 0 \end{cases}$$

Domain of attraction is $x_0 > 0$.

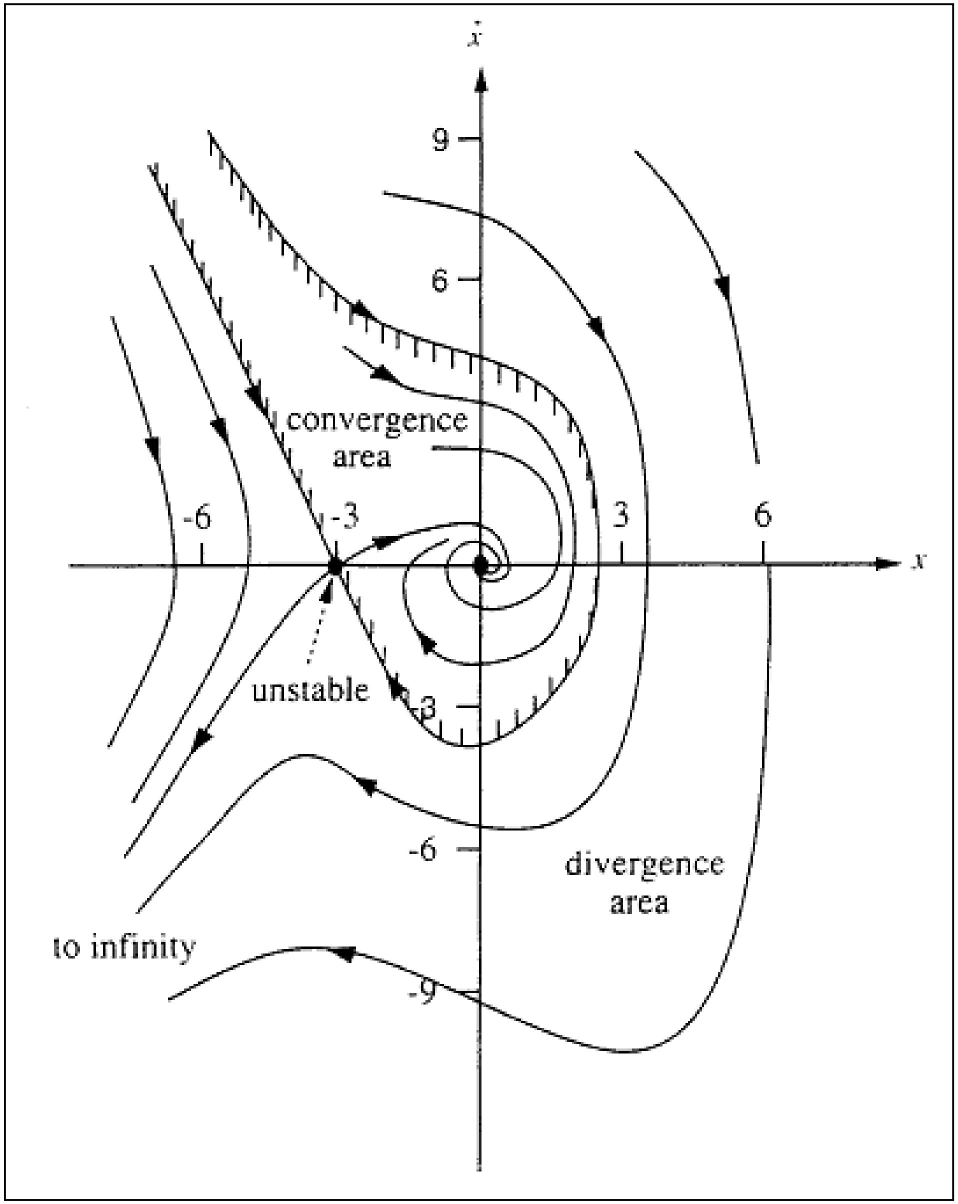
Example 4:

$$\ddot{x} + \dot{x} + x = 0$$



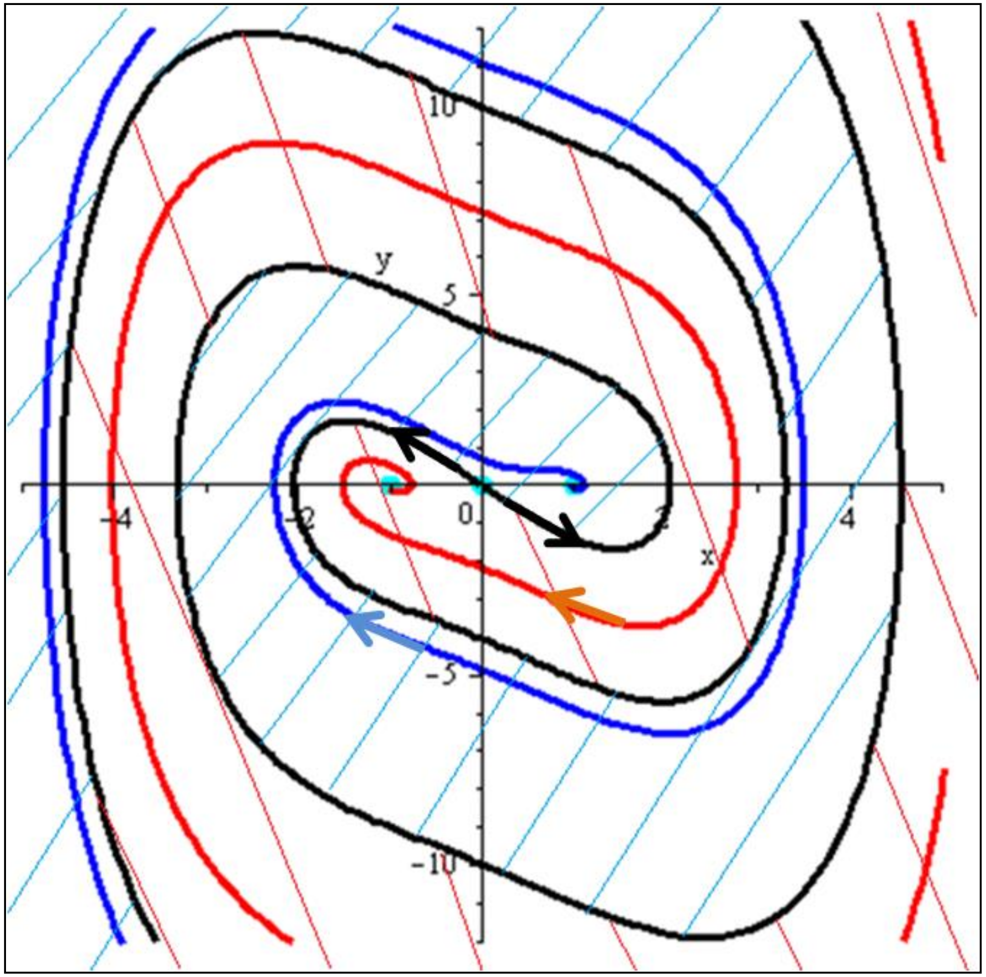
Example 5:

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$



Example 6:

$$\ddot{x} + \dot{x} + x^3 - x = 0$$



Stability Analysis of

Linear Time Invariant Systems

Phase Plane Analysis of LTI Systems

- It is the best tool for study of the linear system graphically
- This analysis gives a very good insight of linear systems behavior
- The analysis can be extended for higher order linear system
- Local behavior of the nonlinear systems can be understood from this analysis
- The analysis is performed based on the system eigenvalues and eigenvectors.

Consider a second order linear system:

$$\dot{x} = Ax \qquad\qquad A \in \mathbb{R}^{2\times 2}\ , x \in \mathbb{R}^2$$

- If the **A** matrix is nonsingular, origin is the only equilibrium point of the system

$$\mathbf{A} \text{ is non-singular} \Rightarrow \mathbf{x}_e = \mathbf{0}$$

- If the **A** matrix is singular then the system has infinite number of equilibrium points. In fact all of the points belonging to the null space of **A** are the equilibrium point of the system.

$$\mathbf{A} \text{ is singular} \Rightarrow \mathbf{x}_e = \left\{ \mathbf{x}_* \mid \mathbf{x}_* \in Null(\mathbf{A}) \right\}$$

Phase Plane Analysis of LTI Systems

Consider a second order linear system:

$$\dot{x} = Ax$$

$$A \in \mathbb{R}^{2\times 2} \ , x \in \mathbb{R}^2$$

The analytical solution can be obtained based on eigenvalues (λ_1, λ_2):

- If λ_1, λ_2 are **real** and **distinct** $\qquad x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$
- If λ_1, λ_2 are **real** and **similar** $\qquad x(t) = (A + Bt)e^{\lambda t}$
- If λ_1, λ_2 are **complex conjugate** $\qquad x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

$$= e^{\alpha t}(A\sin(\beta t) + B\cos(\beta t))$$

Jordan Form (almost diagonal form)

This representation has the system eigenvalues on the leading diagonal, and either 0 or 1 on the super diagonal.

$$\dot{x} = Ax \quad \longleftrightarrow \quad \dot{y} = Jy$$

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_n \end{bmatrix} \qquad J_i = \begin{bmatrix} \lambda_i & 1 & \\ & \ddots & 1 \\ & & \lambda_i \end{bmatrix}$$

Obtaining Jordan form:

$$y = P^{-1}x$$

$$J = P^{-1}AP$$

$$P = [v_1 \quad \cdots \quad v_n]$$

A is non-singular:

The A matrix has 2 eigenvalues (either two real, or two complex conjugates) and can have either two eigenvectors or one eigenvectors. Four categories can be realized

1. Two distinct real eigenvalues and two real eigenvectors
2. Two complex conjugate eigenvalues and two complex eigenvectors
3. Two similar (real) eigenvalues and two eigenvectors
4. Two similar (real) eigenvalues and one eigenvectors

1. Two distinct real eigenvalues and two real eigenvectors

$$\dot{y} = Jy \qquad J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\begin{cases} \dot{y}_1 = \lambda_1 y_1 & y_1 = y_{10} e^{\lambda_1 t} \\ \dot{y}_2 = \lambda_2 y_2 & y_2 = y_{20} e^{\lambda_2 t} \end{cases}$$

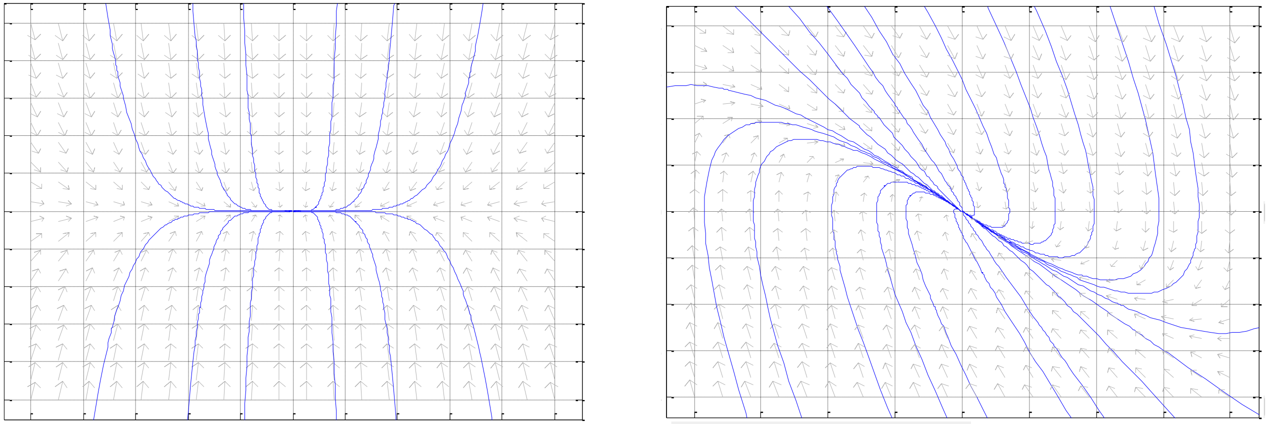
$$\left.\begin{array}{l} \ln\left(\dfrac{y_1}{y_{10}}\right) = \lambda_1 t \\ \ln\left(\dfrac{y_2}{y_{20}}\right) = \lambda_2 t \end{array}\right] \longrightarrow \frac{\lambda_2}{\lambda_1} \ln\left(\frac{y_1}{y_{10}}\right) = \ln\left(\frac{y_2}{y_{20}}\right) \qquad \left(\frac{y_1}{y_{10}}\right)^{\frac{\lambda_2}{\lambda_1}} = \frac{y_2}{y_{20}}$$

$$\longrightarrow y_2 = \frac{y_{20}}{{y_{10}}^{\lambda_2/\lambda_1}} {y_1}^{\lambda_2/\lambda_1} \qquad \boxed{y_2 = K {y_1}^{\lambda_2/\lambda_1}}$$

1. A) $\boldsymbol{\lambda_2 < \lambda_1 < 0}$

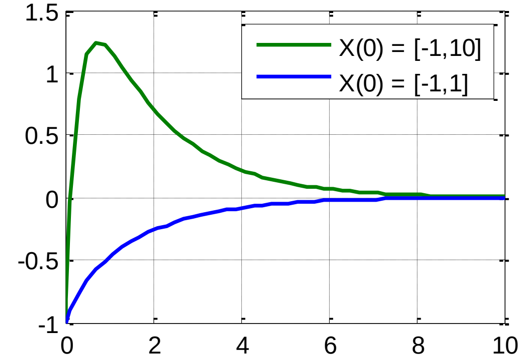
$$\boxed{y_2 = K {y_1}^{\lambda_2/\lambda_1}}$$

- System has two eigenvectors v_1, v_2 the phase plane portrait is as the following



- Trajectories are:
  - ✓ tangent to the slow eigenvector (v_1) for near the origin
  - ✓ parallel to the fast eigenvector (v_2) for far from the origin
- The equilibrium point $X_e = 0$ is called **stable node**

Example 7: $\ddot{x} + 4\dot{x} + 2x = 0$



$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix} X$$

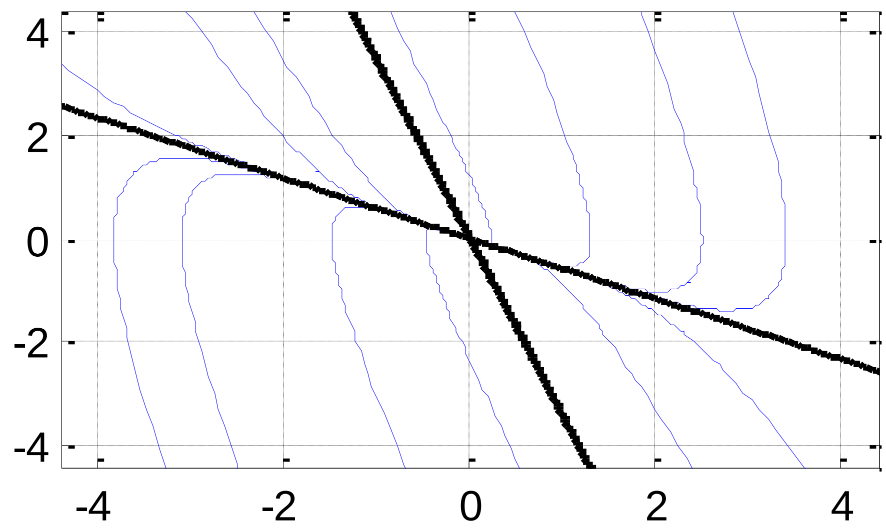
$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix}\right) = \lambda^2 + 4\lambda + 2 = 0 \qquad \lambda_1 = -0.59 \ , \lambda_2 = -3.41$$

$$\begin{bmatrix} -0.59 & -1 \\ 2 & -0.59+4 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

$$\boxed{v_1 = \begin{pmatrix} 0.86 \\ -0.51 \end{pmatrix}}$$

$$\begin{bmatrix} -3.41 & -1 \\ 2 & -3.41+4 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

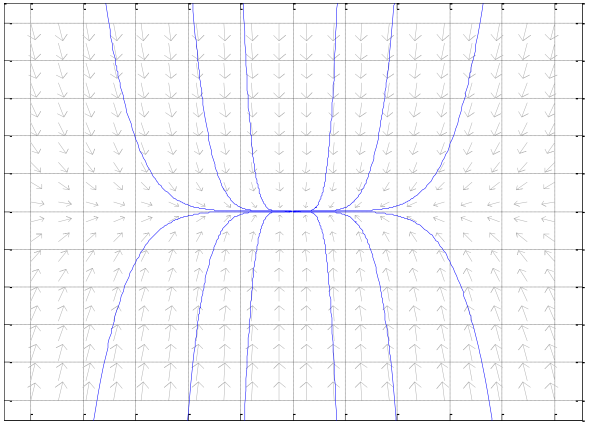
$$\boxed{v_2 = \begin{pmatrix} -0.28 \\ 0.96 \end{pmatrix}}$$

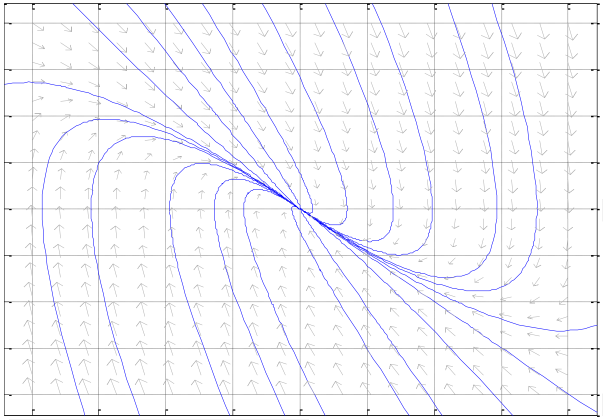


1. B) $\lambda_2 > \lambda_1 > 0$

$$\boxed{y_2 = K {y_1}^{\lambda_2/\lambda_1}}$$

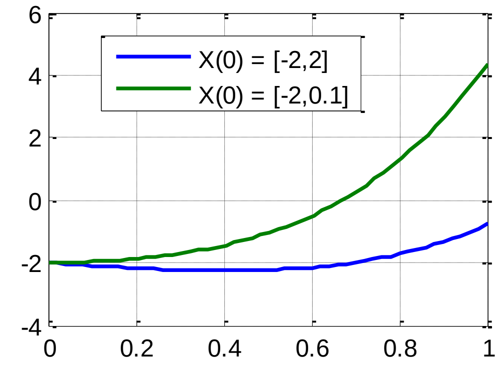
- System has two eigenvectors v_1 and v_2 the phase plane portrait is opposite as the previous one





- Trajectories are:
  - ✓ tangent to the slow eigenvector v_1 for near origin
  - ✓ parallel to the fast eigenvector v_2 for far from origin
- The equilibrium point $X_e = 0$ is called **unstable node**

Example 8: $\ddot{x} - 3\dot{x} + 2x = 0$



$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} X$$

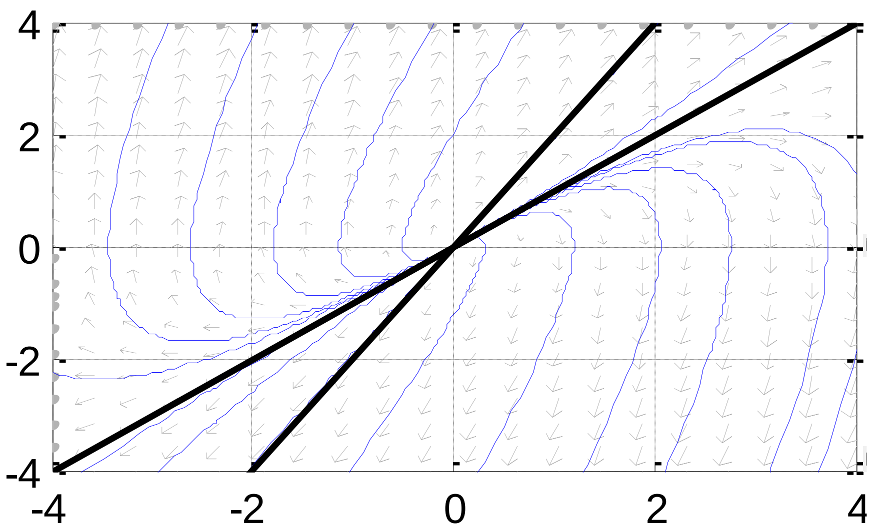
$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\right) = \lambda^2 - 3\lambda + 2 = 0 \qquad \lambda_1 = 1 \;, \lambda_2 = 2$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 1-3 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

$$\boxed{v_1 = \begin{pmatrix} -0.71 \\ -0.71 \end{pmatrix}}$$

$$\begin{bmatrix} 2 & -1 \\ 2 & 2-3 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

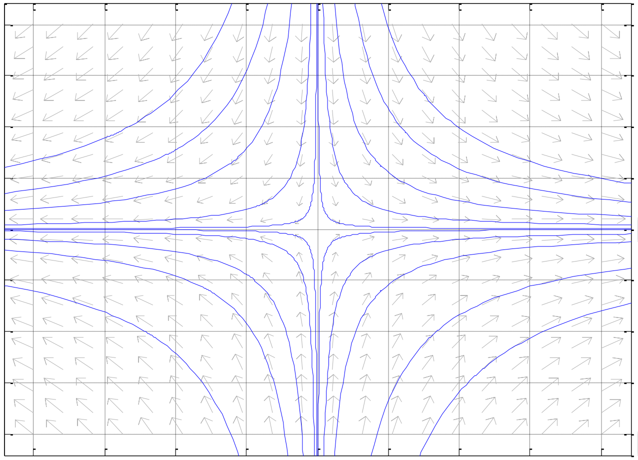
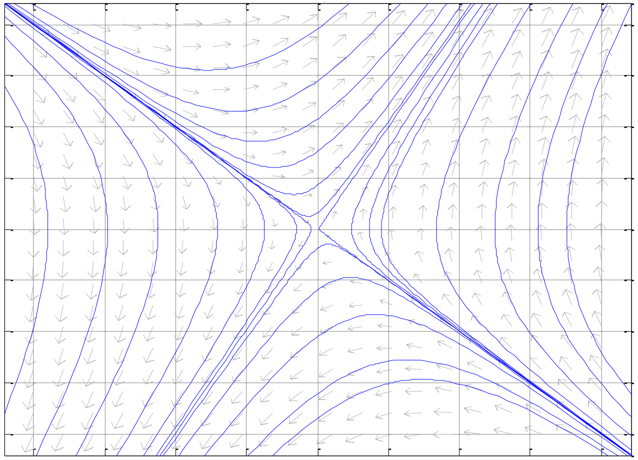
$$\boxed{v_2 = \begin{pmatrix} -0.45 \\ -0.89 \end{pmatrix}}$$



1. C) $\lambda_2 < 0 < \lambda_1$

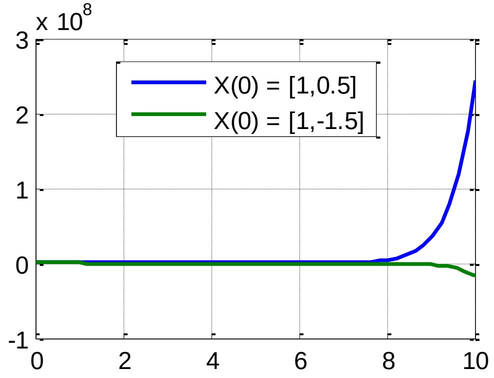
$$\boxed{y_2 = K {y_1}^{\lambda_2/\lambda_1}}$$

- System has two eigenvectors v_1 and v_2, the phase plane portrait is as the following

- Only trajectories along v_2 are stable trajectories
- All other trajectories at start are tangent to v_2 and at the end are tangent to v_1
- This equilibrium point is unstable and is called **saddle point**

Example 9: $\ddot{x} - \dot{x} - 2x = 0$



$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} X$$

$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = 0$$

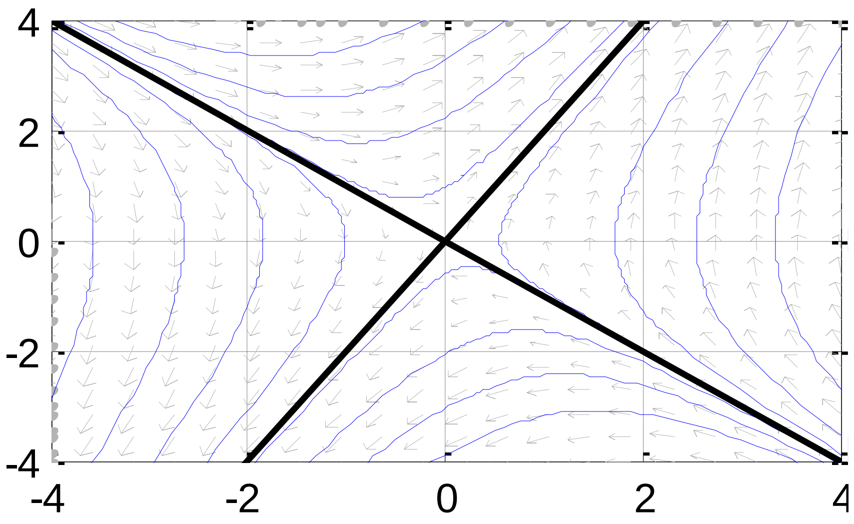
$$\lambda_1 = -1 \quad , \lambda_2 = 2$$

$$\begin{bmatrix} -1 & -1 \\ -2 & -1-1 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

$$\boxed{v_1 = \begin{pmatrix} -0.71 \\ 0.71 \end{pmatrix}}$$

$$\begin{bmatrix} 2 & -1 \\ -2 & 2-1 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

$$\boxed{v_2 = \begin{pmatrix} -0.45 \\ -0.89 \end{pmatrix}}$$



2. Two complex conjugate eigenvalues and two complex eigenvectors

$$\dot{y} = Jy \qquad J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$r \equiv \sqrt{y_1^2 + y_2^2}$$

$$\theta \equiv \tan^{-1}(\frac{y_2}{y_1})$$

$$r\dot{r} = y_1\dot{y}_1 + y_2\dot{y}_2 = y_1(\alpha y_1 - \beta y_2) + y_2(\beta y_1 + \alpha y_2) = \alpha r^2$$

$$\dot{\theta}(1 + \tan\theta^2) = \frac{y_1\dot{y}_2 - y_2\dot{y}_1}{y_1^2} = \frac{y_1(\beta y_1 + \alpha y_2) - y_2(\alpha y_1 - \beta y_2)}{y_1^2} = \beta(1 + \tan\theta^2)$$

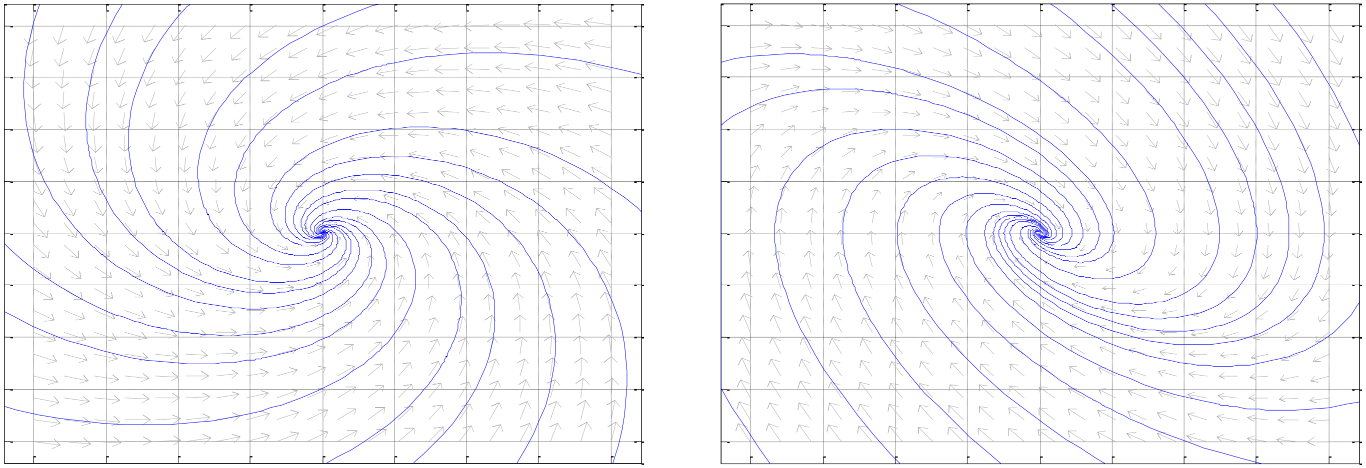
$$\begin{cases} \dot{r} = \alpha r \\ \dot{\theta} = \beta \end{cases} \longrightarrow \qquad r(t) = r_0 e^{\alpha t} \qquad \theta(t) = \theta_0 + \beta t$$

$$r(t) = r_0 e^{\alpha t}$$

$$\theta(t) = \theta_0 + \beta t$$

2. A) $\boldsymbol{\lambda_2,\ \lambda_1 = \alpha \pm \beta i \quad , \quad \alpha < 0\ ,\beta \neq 0}$

- System has no real eigenvectors the phase plane portrait is as the following

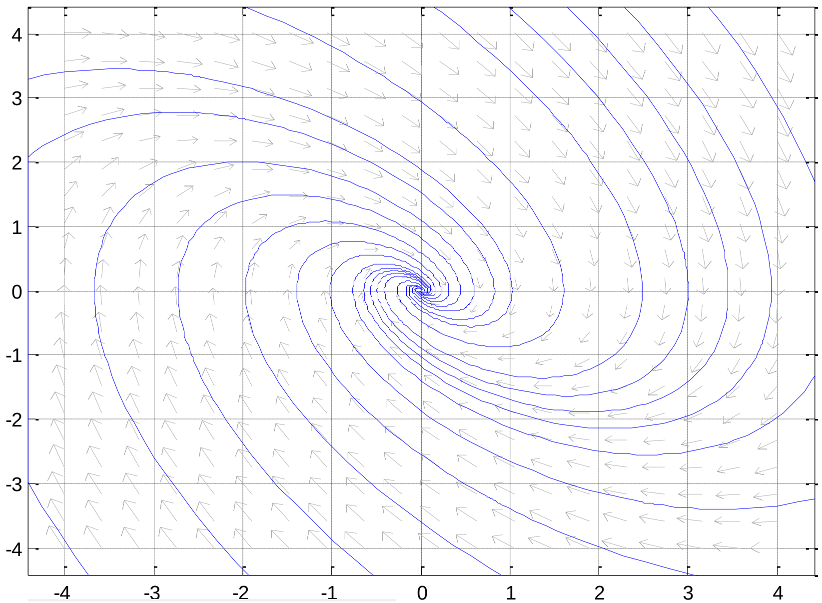


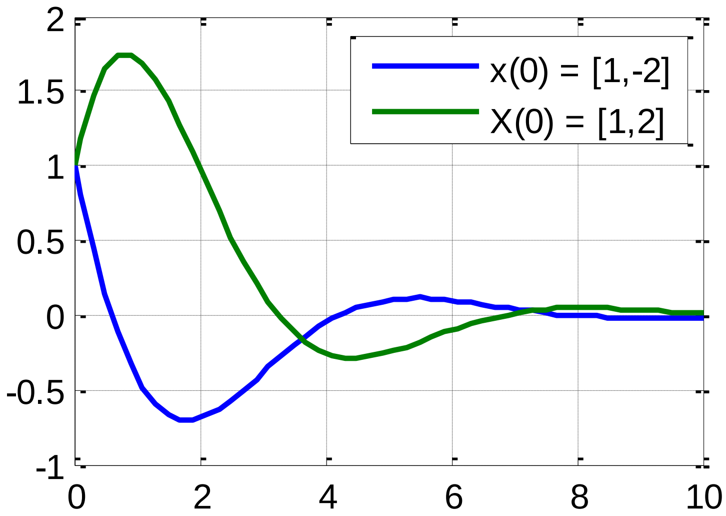
- The trajectories are spiral around the origin and toward the origin.
- This equilibrium point is called **stable focus.**

Example 10: $\ddot{x} + \dot{x} + x = 0$

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} X$$

$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\right) = \lambda^2 + \lambda + 1 = 0 \qquad \lambda_1, \lambda_2 = -0.5 \pm 0.866i$$



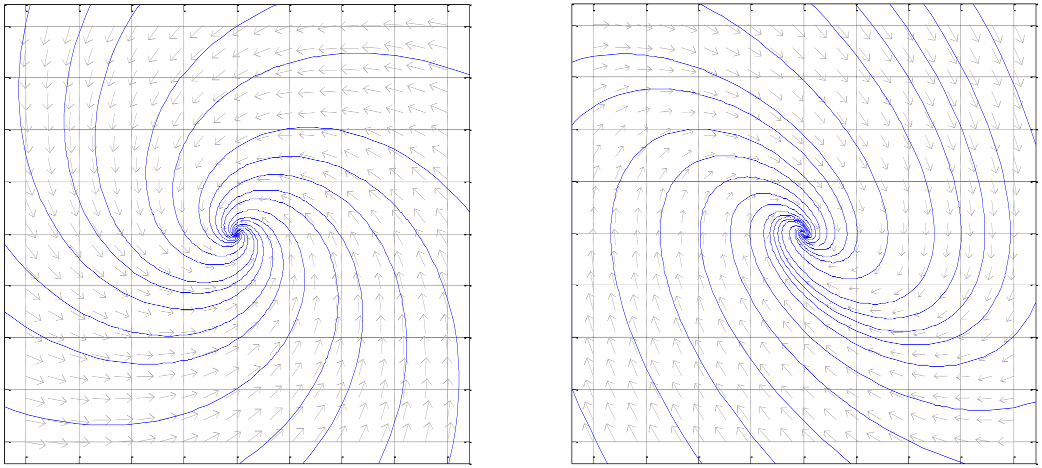


2. B) $\lambda_2,\ \lambda_1 = \alpha \pm \beta i \quad , \quad \alpha > 0\,, \beta \neq 0$

$$r(t) = r_0 e^{\alpha t}$$

$$\theta(t) = \theta_0 + \beta t$$

- System has no real eigenvectors the phase plane portrait is as the following



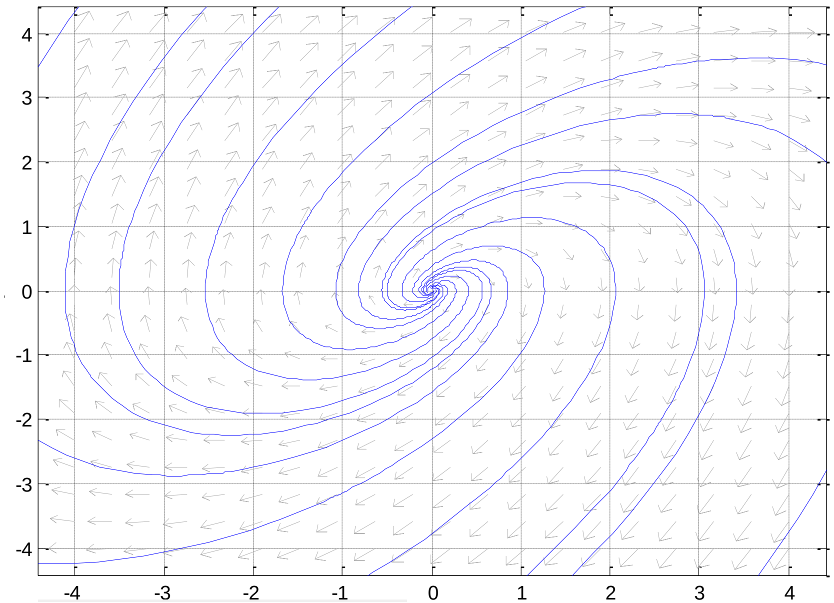
- The trajectories are spiral around the origin and diverge from the origin.

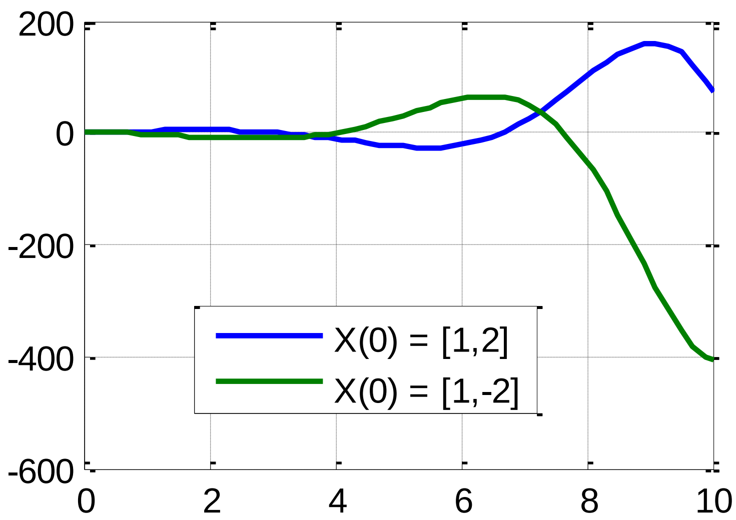
- This equilibrium point is called **unstable focus.**

Example 11: $$\ddot{x} - \dot{x} + x = 0$$

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} X$$

$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\right) = \lambda^2 - \lambda + 1 = 0 \qquad \lambda_1, \lambda_2 = 0.5 \pm 0.866i$$



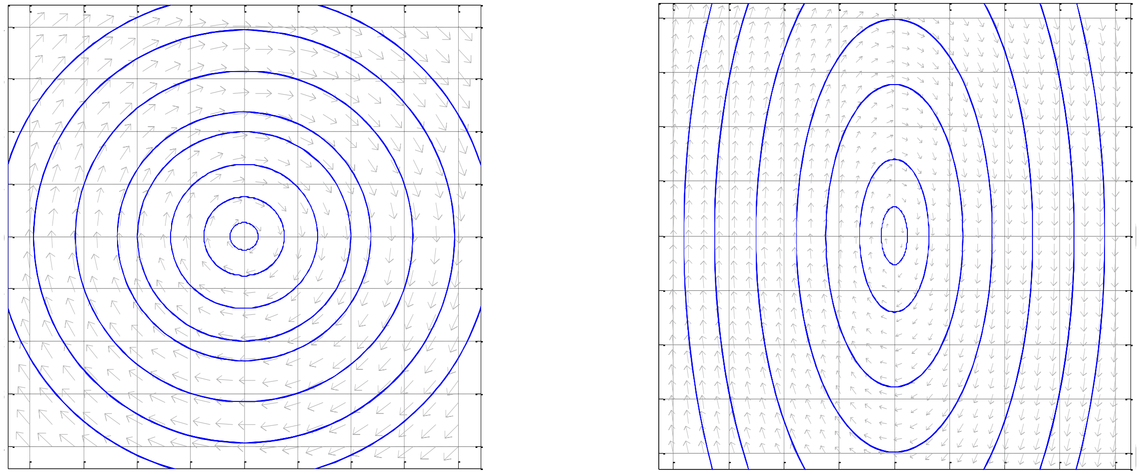


2. C) $\boldsymbol{\lambda_2, \ \lambda_1 = \pm \beta i}$, $\boldsymbol{\alpha = 0 \, , \beta \neq 0}$

$$r(t) = r_0 e^{\alpha t}$$

$$\theta(t) = \theta_0 + \beta t$$

- System has two imaginary eigenvalues and no real eigenvectors the phase plane portrait is as the following



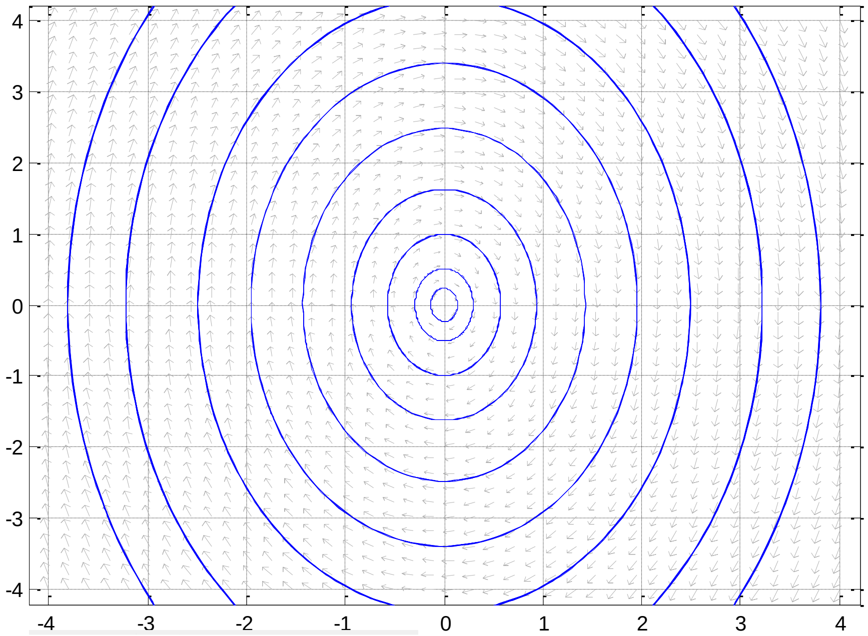
- The trajectories are closed trajectories around the origin.
- This equilibrium point is marginally stable and is called **center.**

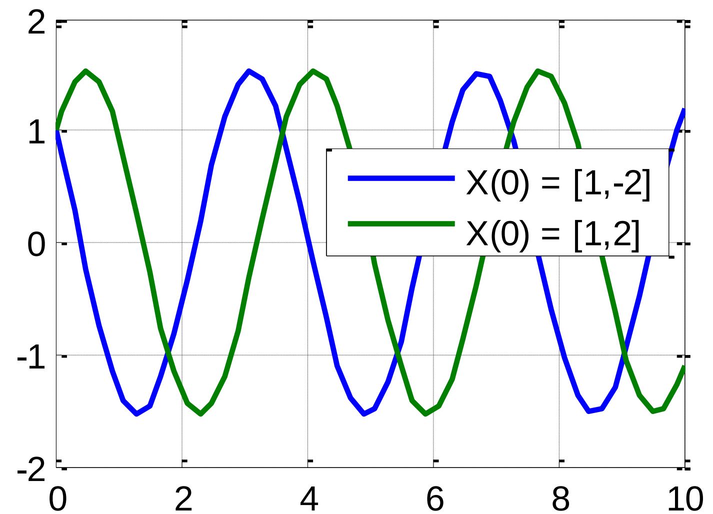
Example 12: $\ddot{x} + 3x = 0$

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix} X$$

$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}\right) = \lambda^2 + 3 = 0$$

$$\lambda_1, \lambda_2 = \pm 1.732i$$





3. Two similar (real) eigenvalues and two eigenvectors

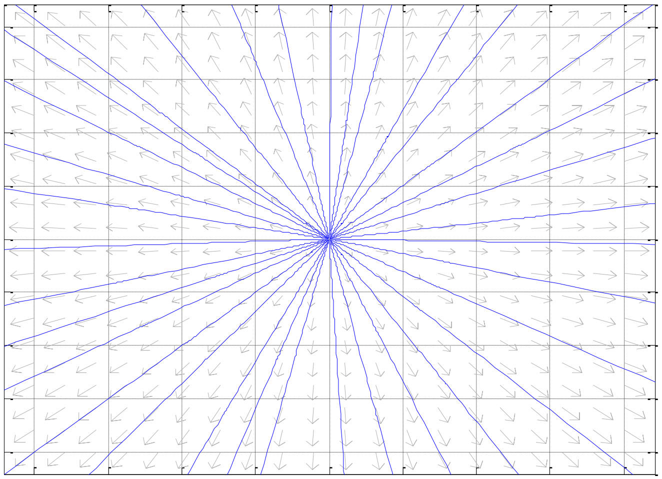
$$\dot{y} = Jy \qquad J = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\begin{cases} \dot{y}_1 = \lambda y_1 \\ \dot{y}_2 = \lambda y_2 \end{cases} \qquad \begin{aligned} y_1 &= y_{10} e^{\lambda t} \\ y_2 &= y_{20} e^{\lambda t} \end{aligned}$$

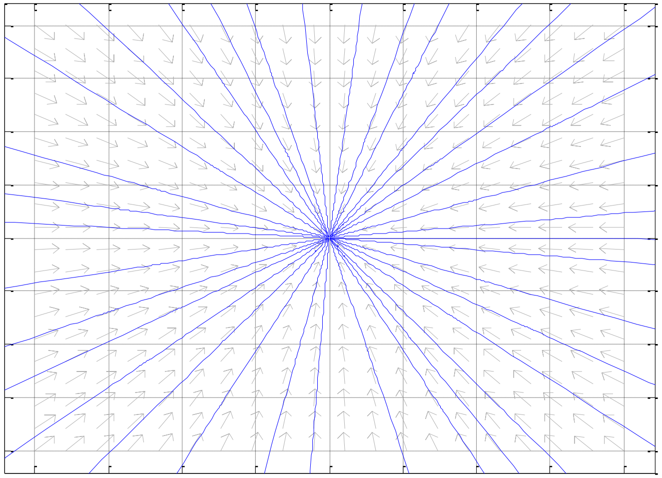
$$\frac{y_1}{y_2} = \frac{y_{10}}{y_{20}} \longrightarrow \boxed{y_2 = K y_1}$$

3) $\boldsymbol{\lambda_2 = \lambda_1 = \lambda \neq 0}$

- System has two similar eigenvalues and two different eigenvectors. The phase plane portrait is as the following, depending to the sign of λ



$\lambda > 0$



$\lambda < 0$

- The trajectories are all along the initial conditions and they are $\lambda < 0$ toward $\lambda > 0$ or outward the origin

4. Two similar (real) eigenvalues and One eigenvectors

$$\dot{y} = Jy \qquad J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

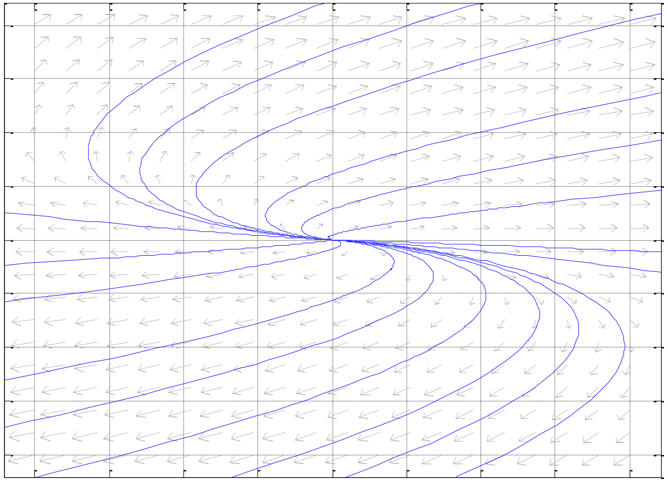
$$\begin{cases} \dot{y}_1 = \lambda y_1 + y_2 \\ \dot{y}_2 = \lambda y_2 \end{cases} \qquad \begin{aligned} y_1 &= y_{10} e^{\lambda t} + y_{20} t e^{\lambda t} \\ y_2 &= y_{20} e^{\lambda t} \end{aligned}$$

$$y_1 = y_{10} \frac{y_2}{y_{20}} + y_2 \frac{1}{\lambda} \ln(\frac{y_2}{y_{20}})$$

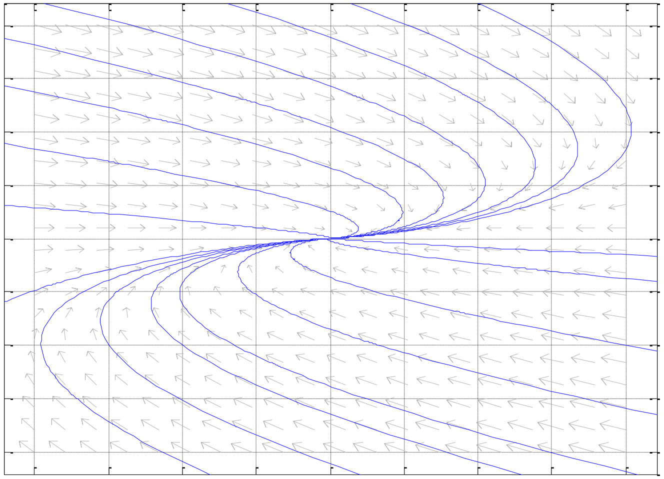
$$\boxed{y_1 = y_2 (\frac{y_{10}}{y_{20}} + \frac{1}{\lambda} \ln\left(\frac{y_2}{y_{20}}\right))}$$

4) $\lambda_2 = \lambda_1 = \lambda \neq 0$

- System has two similar eigenvalues and only one eigenvector. The phase plane portrait is as the following, depending to the sign of λ



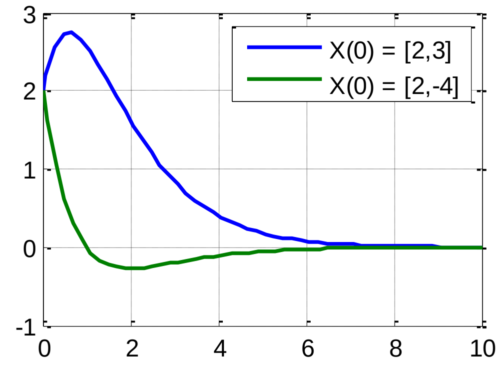
$\lambda > 0$



$\lambda < 0$

- The trajectories converge to zero or diverge to infinity along the system eigenvector.

Example 13: $\ddot{x} + 2\dot{x} + x = 0$

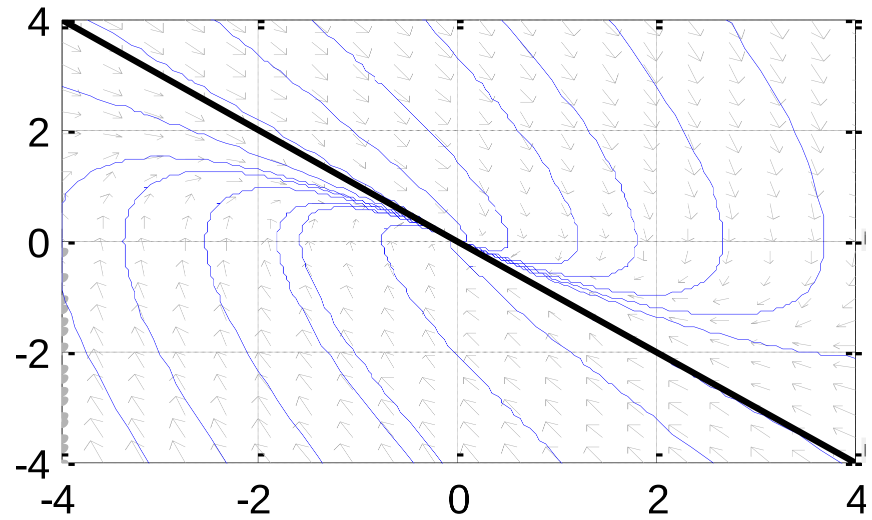


$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} X$$

$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\right) = \lambda^2 + 2\lambda + 1 = 0 \qquad \lambda_1, \lambda_2 = -1$$

$$\begin{bmatrix} -1 & -1 \\ 1 & -1+2 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

$$\boxed{v_1 = \begin{pmatrix} -0.71 \\ 0.71 \end{pmatrix}}$$



A is singular ($\boldsymbol{det(A) = 0}$):

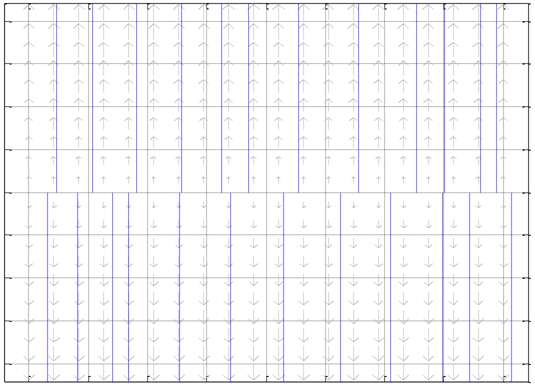
- System has at least one eigenvalue equal to zero and therefore infinite number of equilibrium points. Three different categories can be specified

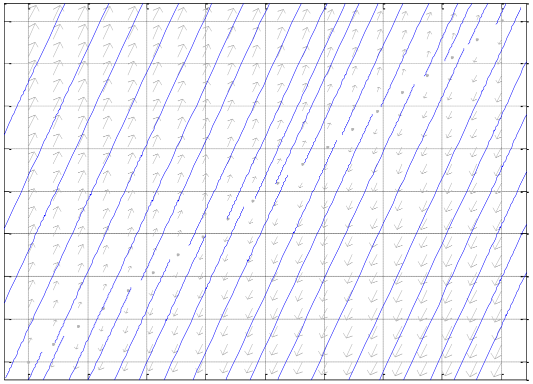
  - $\lambda_1 = 0 \quad , \quad \lambda_2 \neq 0$
  - $\lambda_1, \lambda_2 = 0 \quad , \quad Rank(A) = 1$
  - $\lambda_1, \lambda_2 = 0 \quad , \quad Rank(A) = 0$

Phase Plane Analysis of LTI Systems

1) $\boldsymbol{\lambda_1 = 0 \quad , \quad \lambda_2 \neq 0}$ $\qquad \dot{y} = Jy \qquad J = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$

$$\begin{cases} \dot{y}_1 = 0 \\ \dot{y}_2 = \lambda y_2 \end{cases} \qquad \begin{array}{l} y_1 = y_{10} \\ y_2 = y_{20} e^{\lambda t} \end{array}$$

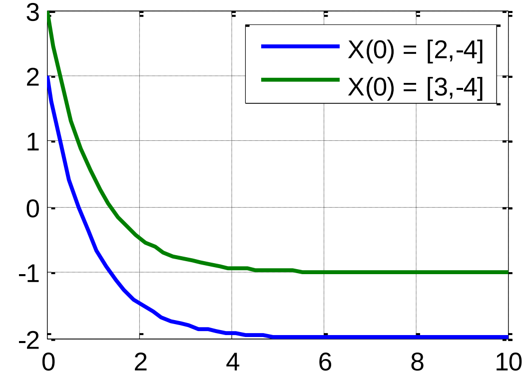




- System has infinite number of non-isolated equilibrium points along a line
- System has two eigenvectors. Eigenvector corresponding to zero eigenvalue is in fact loci of the equilibrium points
- Depending on the sign of the second eigenvalue, all the trajectories move inward or outward to v_1 along v_2

Phase Plane Analysis of LTI Systems

Example 14: $\ddot{x} + \dot{x} = 0$



$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} X$$

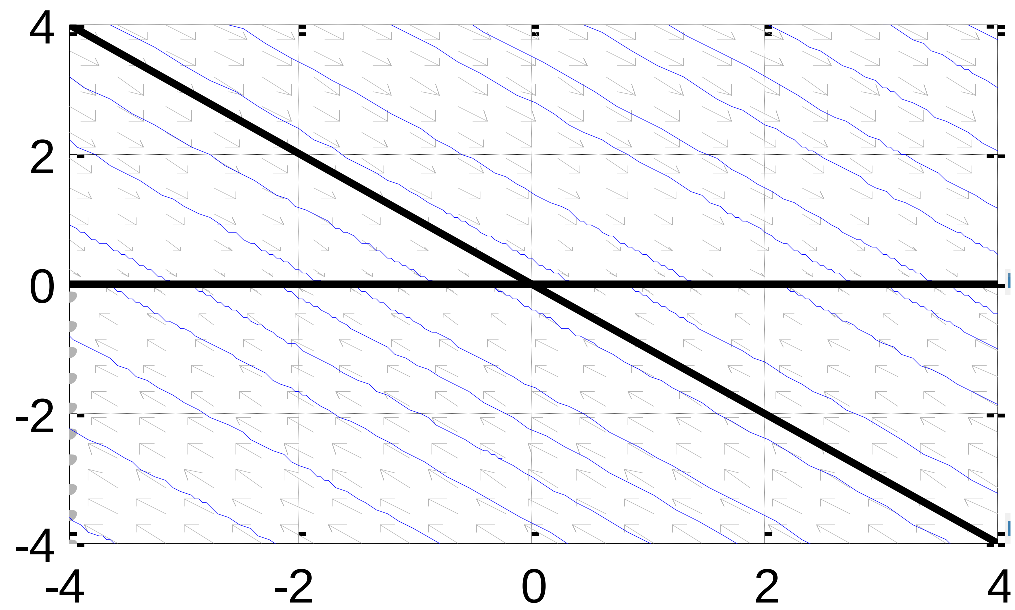
$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\right) = \lambda^2 + \lambda = 0 \qquad \lambda_1 = 0 \quad , \lambda_2 = -1$$

$$\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

$$\boxed{v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

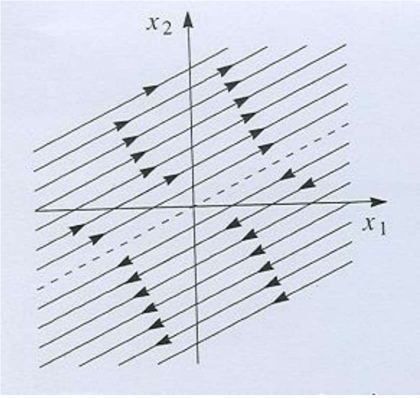
$$\begin{bmatrix} -1 & -1 \\ 0 & -1+1 \end{bmatrix} \begin{pmatrix} v_2^1 \\ v_2^2 \end{pmatrix} = 0$$

$$\boxed{v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$



2) $\boldsymbol{\lambda_1, \lambda_2 = 0}$, $Rank(A) = 1$ $\qquad \dot{y} = Jy \qquad J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = 0 \end{cases} \qquad \begin{aligned} y_1 &= y_{20}t + y_{10} \\ y_2 &= y_{20} \end{aligned}$$



- System has infinite number of non-isolated equilibrium points along a line
- System has only one eigenvector, and it is loci of the equilibrium points
- All the trajectories move toward infinity along the system eigenvector (unstable system).

2) $\boldsymbol{\lambda_1, \lambda_1 = 0}$, $Rank(A) = 0$

$$\dot{y} = Jy \qquad J = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} \dot{y}_1 = 0 \\ \dot{y}_2 = 0 \end{cases} \qquad \begin{aligned} y_1 &= y_{10} \\ y_2 &= y_{20} \end{aligned}$$

- System is a static system. All the points are equilibrium points

Summary

Six different type of isolated equilibrium points can be identified

- ❑Stable/unstable node
- ❑Saddle point
- ❑Stable/ unstable focus
- ❑Center

Stability Analysis of Higher Order Systems:

- Analysis and results for the second order LTI system can be extended to higher order LTI system
- Graphical tool is not useful for higher order LTI system except for third order systems.
- This means stability analysis of mechanical system with more than one DOF can not be materialized graphically
- Stability analysis is performed through the eigenvalue analysis of the **A** matrix.

Phase Plane Analysis of LTI Systems

❑Consider a linear time invariant (LTI) system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

❑Origin is the only equilibrium point of the system if **A** is non-singular

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad \overset{\det(\mathbf{A})\neq 0}{\Longrightarrow} \quad \mathbf{x}_e = \mathbf{0}$$

❑Otherwise the system has infinite number of equilibrium points, all the points on null-space of A are in fact equil. points of the system.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad \overset{\det(\mathbf{A})= 0}{\Longrightarrow} \quad \mathbf{x}_e = \left\{\mathbf{x}_* \mid \mathbf{x}_* \in \text{Nullspace}(\mathbf{A})\right\}$$

Details for Case of Non-Singular A

- Origin is the only equilibrium point of the system
- This equilibrium point (system) is
  - Exponentially stable if all eigenvalues of **A** are either real negative or complex with negative real part.
  - Marginally stable if eigenvalues of **A** have non-positive real part and $rank(A - \lambda I) = n - r$ for all repeated imaginary eigenvalues, λ with multiplicity of r
  - Unstable, otherwise.

A is Non-Singular

- Classification of the equilibrium point of higher order system into node, focus, and saddle point is not as easy as second order system. However some points can be emphasized:
  - ✓ The equilibrium point is **stable/unstable node** if all eigenvalues are real and have the negative/positive sign.
  - ✓ The equilibrium point is **center** if a pair of eigenvalues are pure imaginary complex conjugate and all other eigenvalues have negative real
  - ✓ In the case of different sign in the real part of the eigenvalues trajectories have the **saddle** type behavior near the equilibrium point

✓Trajectories are along the eigenvector with minimum absolute real part near the equilibrium point and along the eigenvector with maximum absolute real part.

✓Trajectories have **spiral** behavior if there exist some complex (obviously conjugates) eigenvalues.

✓Spiral behavior is toward/outward depending on the sign (negative and positive) of real part of the complex conjugate eigenvalues.

❑ These concepts can be visualized and better understood in three dimensional case

Lyapunov Indirect Method in Stability Analysis of Nonlinear Systems

Lyapunov Indirect Method

- There are two conventional approaches in the stability analysis of nonlinear systems:

  - ✓ Lyapunov direct method

  - ✓ Lyapunov indirect method or linearization approach

- The direct method analyzes stability of the system (equilibrium point) using the nonlinear equations of the system

- The indirect method analyzes the system stability using the linearized equations about the equilibrium point.

Motivation:

A nonlinear system near its equilibrium point behaves like a linear:

- Nonlinear system: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
- Equilibrium point: $\mathbf{f}(\mathbf{x}) = 0 \Rightarrow \mathbf{x}_e$
- Motion about equilibrium point: $\mathbf{x} = \mathbf{x}_e + \hat{\mathbf{x}}$
- Linearized motion:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \Rightarrow \dot{\hat{\mathbf{x}}} = \mathbf{f}(\mathbf{x}_e + \hat{\mathbf{x}}) = \cancel{\mathbf{f}(\mathbf{x}_e)} + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_e} \hat{\mathbf{x}} + \text{H.O.T} \Rightarrow \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}}$$

- It means near x_e :

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \cong \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} \overset{\text{if } \mathbf{x}_e = \mathbf{0}}{\Longrightarrow} \mathbf{x} \approx \hat{\mathbf{x}}$$

Lyapunov Indirect Method

- This means stability of the equilibrium point may be studied through the stability analysis of the linearized system.

- This is the base of the Lyapunov Indirect Method

- **<u>Example:</u>** in the nonlinear second order system

$$\dot{x}_1 = x_2^2 + x_1 \cos x_2$$
$$\dot{x}_2 = x_2 + (1 + x_1)x_1 + x_1 \sin x_2$$

origin is the equilibrium point and the <u>linearized system</u> is given by

$$\begin{Bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{Bmatrix} = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}_e = 0} \begin{Bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{Bmatrix} \Rightarrow \begin{cases} \dot{\hat{x}}_1 = \hat{x}_1 \\ \dot{\hat{x}}_2 = \hat{x}_1 + \hat{x}_2 \end{cases}$$

Theorem (Lyapunov Linearization Method):

- If the linearized system is strictly stable (i.e. all eigenvalues of $\mathbf{A}$ are strictly in the left half complex plane) then the equilibrium point in the original nonlinear system is asymptotically stable.

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} \ \text{ is strictly stable} \Rightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \ \text{ is asymptotically stable}$$

- If the linearized system is unstable (i.e. in the case of right half plane eigenvalue(s) or repeated eigenvalues on the imaginary axis with geometrical deficiency ($r > n - rank(\lambda I - A)$), then the equilibrium point in the original nonlinear system is unstable.

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} \ \text{ is unstable} \Rightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \ \text{ is unstable}$$

Theorem (Lyapunov Linearization Method):

- If the linearized system is marginally stable (i.e. all eigenvalues of **A** are in the left half complex plane and eigenvalues on the imaginary axis have no geometrical deficiency) then one cannot conclude anything from the linear approximation. The equilibrium point in the original nonlinear system may be stable, asymptotically stable, or unstable.

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} \quad \text{is marginally stable} \Rightarrow \begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) & \text{is asymptotically stable} \\ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) & \text{is marginally stable} \\ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) & \text{is unstable} \end{cases}$$

- The Lyapunov linearized approximation method only talks about the local stability of the nonlinear system, if anything can be concluded.

Example 15:

- The nonlinear system $\dot{x} = ax + bx^5$ is
  - ✓ Asymptotically stable if $a < 0$
  - ✓ Unstable if $a > 0$
  - ✓ No conclusion from linear approximation can be drawn if
- The origin in the nonlinear second order system

$$\dot{x}_1 = x_2^2 + x_1 \cos x_2$$
$$\dot{x}_2 = x_2 + (1 + x_1)x_1 + x_1 \sin x_2$$

is unstable because the linearized system $\begin{Bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{Bmatrix}$ is unstable

Lyapunov Direct Method in Stability Analysis of Nonlinear Systems

Physical Motivation: Consider a single DOF damped nonlinear system

$$m\ddot{x} + c\dot{x} + kx^3 = 0 \quad m, c, k > 0$$

- ❑ Kinetic energy of the system: $T = 1/2\, m\dot{x}^2$
- ❑ Potential energy of the system: $U = \int_0^x k\xi^3 d\xi = 1/4\, kx^4$

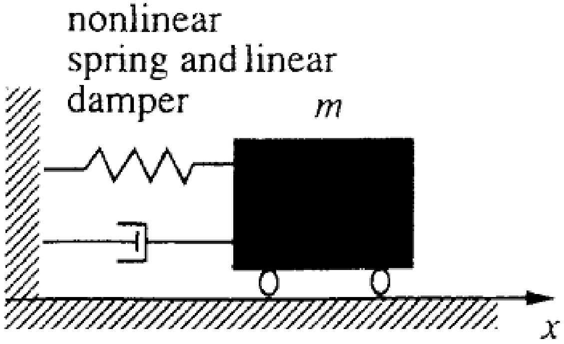
➢ Let us define:

$$V(x, \dot{x}) = E = T + U = 1/2\, m\dot{x}^2 + 1/4\, kx^4$$

- ❑ It is clear that:

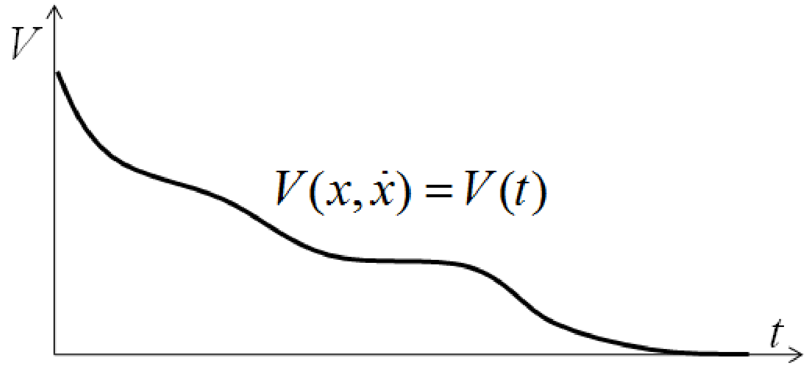
1. $V(0,0) = 0$

2. $V(x, \dot{x}) > 0 \quad \forall (x, \dot{x}) \neq (0,0)$



3. Furthermore: $\dot{V}(x,\dot{x}) = (m\ddot{x} + kx^3)\dot{x} = -c\dot{x}^2 \leq 0$

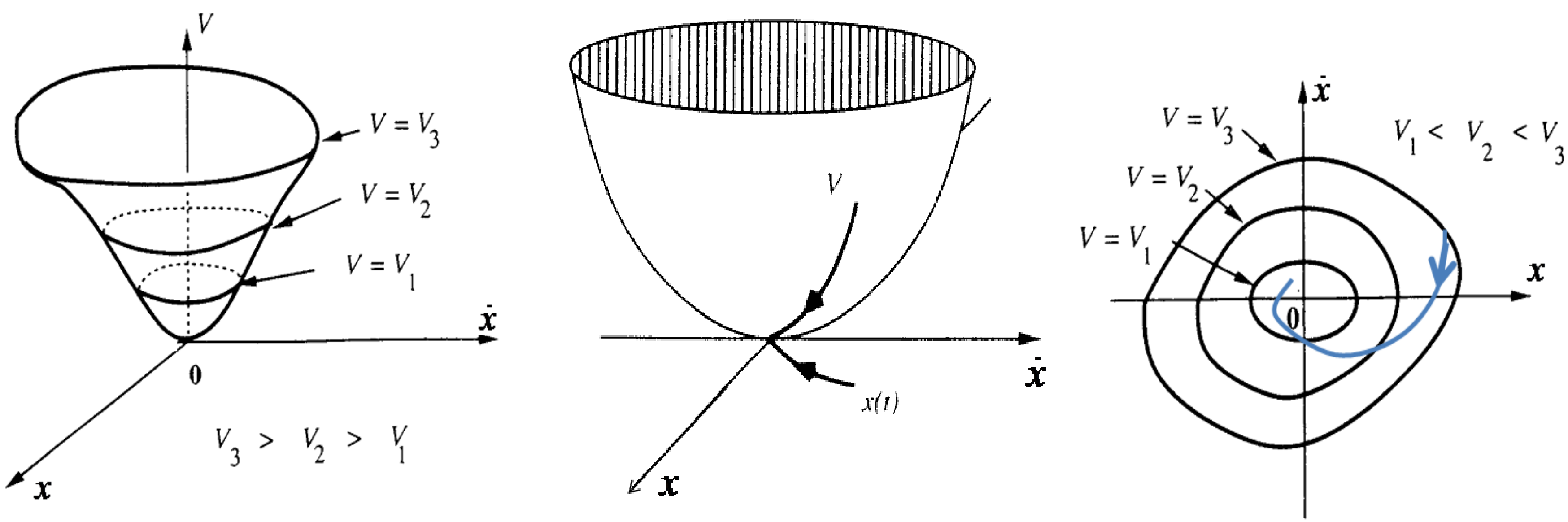
- This means if we consider the mechanical energy of the system:
  - It is always positive except at the equilibrium point which is zero
  - It is decreasing while the system is in motion.
  - Rate of the decrease equals to the power of the damping force
- Therefore V converges to zero



Lyapunov Direct Method

- It declares that once the system is perturbed from its equilibrium point at origin, the motion of the system is such that the total energy is decreasing and the system moves toward the rest position

- This is the meaning of the stability of the equilibrium point.



Lyapunov Direct Method

<u>Theorem:</u> Let $\mathbf{x}=\mathbf{0}$ be an equilibrium point for $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$ and let

$$V(\mathbf{x}): D \rightarrow R,\ D \subset R^n$$

be a continuously differentiable function on a neighborhood D of $\mathbf{x}=\mathbf{0}$ such that:

1. $V(0)=0$
2. $V(\mathbf{x})>0 \quad \forall \mathbf{x} \in D-\{\mathbf{0}\}$
3. $\dot{V}(\mathbf{x}) \leq 0 \quad \forall \mathbf{x} \in D$

then $\mathbf{x}=\mathbf{0}$ is <u>locally</u> (in D) <u>stable</u>.

Moreover it is <u>asymptotically stable</u> if $\dot{V}(\mathbf{x})<0 \quad \forall \mathbf{x} \in D-\{\mathbf{0}\}$

Definition: $V(\mathbf{x})$ with the above characteristics is called *Lyapunov function*

The surface $V(\mathbf{x}) = c$ for some $c > 0$ is called *Lyapunov surface* or *level surface*

❑Note that for a dynamic system with state-space equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

$\dot{V}(\mathbf{x})$ is calculated as the following

$$\dot{V}(\mathbf{x}) = \left(\frac{\partial V}{\partial \mathbf{x}}\right)\dot{\mathbf{x}} = \left(\frac{\partial V}{\partial \mathbf{x}}\right)\mathbf{f}(\mathbf{x})$$

Lyapunov Direct Method

<u>Example 1 (local stability):</u> $\ddot{\theta} + \dot{\theta} + \sin(\theta) = 0$

- Lyapunov candidate function: $V(\mathbf{x}) = (1 - \cos\theta) + \dfrac{\dot{\theta}^2}{2}$

- For this candidate function

$$\Rightarrow \begin{cases} 1. \;\; V(0,0) = V(\theta = 0, \dot{\theta} = 0) = 0 \\ 2. \;\; V > 0 \quad \forall \;\; (\theta, \dot{\theta}) \neq (0,0) \;\; \& \; -\pi < \theta < \pi \\ 3. \;\; \dot{V} = (\sin\theta + \ddot{\theta})\dot{\theta} = -\dot{\theta}^2 \leq 0 \;\; \forall \;\; (\theta, \dot{\theta}) \neq (0,0) \end{cases}$$

- Therefore $V(\mathbf{x})$ is a Lyapunov function and the equilibrium point is **<u>locally stable</u>**

Example 2 (asymptotic stability):

- System equations

$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2$$

$$\dot{x}_2 = x_1(x_1^2 + x_2^2 - 2) + 4x_1^2x_2$$

- Lyapunov candidate function $V(\mathbf{x}) = x_1^2 + x_2^2$

- Then

$$\Rightarrow \begin{cases} 1.\ \ V(0,0) = 0 \\ 2.\ \ V > 0 \quad \forall\ (x_1, x_2) \neq (0,0) \\ 3.\ \ \dot{V} = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2) < 0 \quad \forall\ (x_1, x_2) \neq (0,0)\ \& \ (x_1, x_2) \in \left|x_1^2 + x_2^2\right| < 2 \end{cases}$$

- Therefore $V(\mathbf{x})$ is a Lyapunov function and the equilibrium point is **locally asymptotic stable**

<u>Positive Definite Function:</u>

- function $V(\mathbf{x}) : D \to R,\ D \subset R^n$ satisfying :

  1. $V(0) = 0$
  2. $V(\mathbf{x}) > 0 \quad \forall \mathbf{x} \in D - \{\mathbf{0}\}$

  is called **positive definite** in *D*.

- If it satisfies a weaker condition $V(\mathbf{x}) \geq 0 \quad \forall \mathbf{x} \in D - \{\mathbf{0}\}$ it is **positive semi-definite**

- Function $V(\mathbf{x})$ is **negative (semi) definite** if $-V(\mathbf{x})$ is positive (semi) definite

Lyapunov Direct Method

- As an example we are already familiar with positive (semi) definite quadratic function

$$V(\mathbf{x}) = \mathbf{x}^T \mathbf{P} \mathbf{x}$$

where the symmetric matrix $\mathbf{P} = \mathbf{P}^T$ is positive (semi) definite matrix, i.e. for all eigenvalues of $\mathbf{P}$ we have

$$\forall i \in (1, 2, \ldots, n) \Rightarrow \lambda_i \geq 0$$

Stability Theorem: The origin is stable if there is a continuously differentiable , P.D. function $V(\boldsymbol{X})$ such that $\dot{V}(\boldsymbol{X})$ is N.S.D., and it is asymptotically stable if $\dot{V}(\boldsymbol{X})$ is N.D.

Lyapunov Direct Method

Global Asymptotic Stability: Being $D = R^n$ is not enough to have global stability. The Lyapunov function should be radially unbounded as well, i.e.

$$\|\mathbf{x}\| \to \infty \Rightarrow V(\mathbf{x}) \to \infty$$

Theorem: Assume there exists a scalar function $V(\boldsymbol{X})$ with continuous first order derivative in $\mathbf{x}$ such that

1. $V(\boldsymbol{X})$ is positive definite
2. $\dot{V}(\boldsymbol{X})$ is negative definite
3. $V(\boldsymbol{X})$ is radially bounded

then the origin is globally asymptotically stable

Example 3:

A class of first order system defined by:

$$\dot{x} = -c(x) \quad \text{with} \qquad xc(x) > 0 \quad \forall x \neq 0$$

- Since $c(x)$ is continuous, it means $c(0) = 0$
- Intuitively this condition implies that $-c(x)$ pushes the system back to the origin (rest position)
- We can see that $V(x) = x^2$ is a Lyapunov function and satisfies all the globally asymptotically stability conditions.

$$V(0) = 0, \quad V > 0 \ (\forall x \neq 0), \quad |x| \to \infty \Rightarrow V(x) \to \infty$$

$$\dot{V} = 2x\dot{x} = -2xc(x) < 0 \quad (\forall x \neq 0)$$

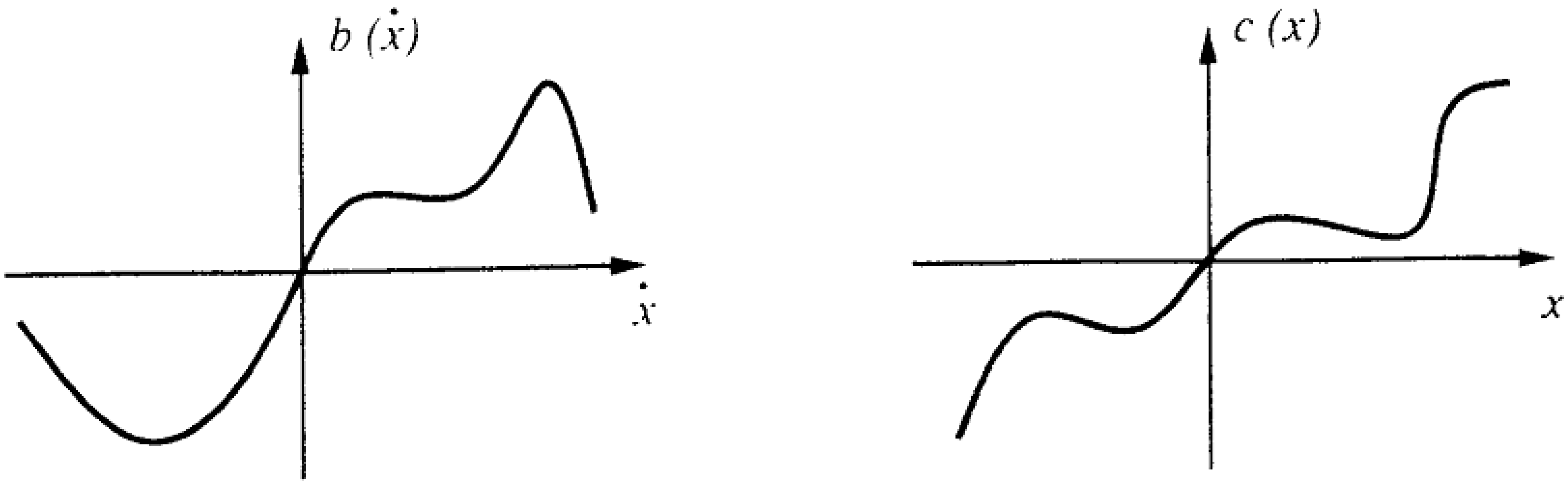
- Therefore the origin is globally asymptotically stable

- $\dot{x} = \sin^2 x - x$ and $\dot{x} = -x^3$ are two other examples of this type of first order systems

Example 4:

2. A class of second order system defined by

$$\ddot{x} + b(\dot{x}) + c(x) = 0 \quad \text{with} \quad \begin{cases} \dot{x}b(\dot{x}) > 0 & \forall \dot{x} \neq 0 \\ xc(x) > 0 & \forall x \neq 0 \end{cases}$$



Example 4:

- The candidate function

$$V(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + \int_0^x c(\sigma)d\sigma$$

is a Lyapunov function and satisfies the following conditions

$$V(0,0) = 0,$$

$$V > 0 \quad \forall (x,\dot{x}) \neq (0,0),$$

$$\dot{V} = \left(\ddot{x} + c(x)\right)\dot{x} = -\dot{x}b(\dot{x}) \leq 0 \quad \forall (x,\dot{x}) \neq (0,0)$$

Example 4:

- This means the origin is at least stable (nothing can be said about the asymptotically stability so far).

- Clearly V is decreasing till $\dot{V}=0$. At this state

$$\dot{V}=0 \Rightarrow \dot{x}=0$$

- There can be two cases for this zero velocity state:

  a) $x=0$ then the system is at its equilibrium point and stays there

  b) $x \neq 0$ then since $c(x) \neq 0$ we conclude $\ddot{x} \neq 0$

Example 4:

- This means the system moves away from this $(x,0)$ state and again $V(x,\dot{x})$ decreases till it comes to the rest position $(0,0)$.

- Therefore the origin complies with the asymptotic stability conditions as well.

- Since $\int_0^x c(\sigma)d\sigma$ is unbounded as $|x| \to \infty$ then $V(x,\dot{x})$ also satisfies the radially unbounded condition, i.e.,

$$\left\| [x,\dot{x}]^T \right\| \to \infty \Rightarrow V(x,\dot{x}) \to \infty$$

Hence it can be stated that the origin is globally asymptotically stable (**g.a.s.**)equilibrium point.

Example 4:

- Following examples are from this category

$$\ddot{x}+\dot{x}+x=0$$

$$\ddot{x}+\dot{x}^3+x^5-x^4\sin^2 x=0$$

Facts about Lyapunov Direct Method and Lyapunov Functions:

- Lyapunov direct method is established by introducing a positive definite candidate function, $V(\boldsymbol{X})$, and evaluation of negative difinite-ness of $\dot{V}(\boldsymbol{X})$.
- This method is totally depended on introduction of the Lyapunov function.
- Lyapunov direct method is a necessary condition. Failing in the candidate function does not mean instability
- A Lyapunov function is strictly defined based on the give system equations

Facts about Lyapunov Direct Method and Lyapunov Functions:

- If $V(x)$ is a Lyapunov function (positive definite function with negative (semi) definite time derivative), then

$$V_1(x) = \rho V^{\alpha}(x) \qquad \text{with} \quad \rho > 0, \alpha > 0$$

is also a Lyapunov function.

- Addition, and multiplication of two Lyapunov functions construct a new Lyapunov function

$$V(\mathbf{x}) = V_1(\mathbf{x}) + V_2(\mathbf{x}) > 0 \Rightarrow \dot{V}(\mathbf{x}) = \dot{V}_1(\mathbf{x}) + \dot{V}_2(\mathbf{x}) \le 0$$

$$V(\mathbf{x}) = V_1(\mathbf{x}) V_2(\mathbf{x}) > 0 \Rightarrow \dot{V}(\mathbf{x}) = \dot{V}_1(\mathbf{x}) V_2(\mathbf{x}) + V_1(\mathbf{x}) \dot{V}_2(\mathbf{x}) \le 0$$

Facts about Lyapunov Direct Method and Lyapunov Functions:

- Conclusions like local stability or marginal stability are the least conclusions and a better selection of candidate function may lead to stronger conclusions.

- In general a Lyapunov function may be pure mathematical function, however a physical understanding of the system can help in selection of good candidate for Lyapunov Function

Example 5:

for a damped pendulum, $\ddot{\theta}+\dot{\theta}+\sin(\theta)=0$

- Candidate function, $V(\mathbf{x})=(1-\cos\theta)+\dot{\theta}^2/2$ which is the mechanical energy of the system leads to $\dot{V}=-\dot{\theta}^2$ which is negative semi definite. Therefore only local stability is concluded.

- A new candidate function,

$$V(\mathbf{x})=2(1-\cos\theta)+\frac{1}{2}\left[\dot{\theta}^2+\left(\dot{\theta}+\theta\right)^2\right]$$

leads to $\dot{V}=-(\dot{\theta}^2+\theta\sin\theta)$ which is a negative definite function and concludes locally asymptotically stability.

- This new function has less physical meaning to the system

The **main question** in the analysis of dynamic system using Lyapunov direct method is how to obtain the Lyapunov function

<u>Linear system analysis based on Lyapunov Direct Method:</u>

- We already know how to analyze stability of a linear system, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ through its eigenvalues.
- We know that for any positive definite matrix, $\mathbf{P}$, function $V = \mathbf{x}^T\mathbf{P}\mathbf{x}$ is a positive definite.
- Starting from a positive definite matrix, $\mathbf{P}$ we may come up with a negative semi definite derivative for

$$V = \mathbf{x}^T\mathbf{P}\mathbf{x} \Rightarrow \dot{V} = \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} \Rightarrow \dot{V} = \mathbf{x}^T\mathbf{A}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\mathbf{A}\mathbf{x}$$

$$\Rightarrow \dot{V} = \mathbf{x}^T\underbrace{\left(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}\right)}_{-\mathbf{Q}}\mathbf{x} = -\mathbf{x}^T\mathbf{Q}\mathbf{x}$$

<u>Theorem:</u> A necessary and sufficient condition for a LTI system to be strictly stable (asymptotically stable) is that, for any given P.D. matrix **Q**, the unique matrix P, solution of the Lyapunov equation, $\mathbf{A}^T\mathbf{P}+\mathbf{P}\mathbf{A}=-\mathbf{Q}$ to be symmetric positive definite.

- Since **Q** can be any positive definite matrix, a simple choice for **Q** is the identity matrix.

- Short conclusion:

$$\text{select}\,\mathbf{Q}>0 \rightarrow \text{solve}\,\mathbf{A}^T\mathbf{P}+\mathbf{P}\mathbf{A}=-\mathbf{Q} \rightarrow \begin{cases} \mathbf{P}>0 \Rightarrow \text{strictly stable} \\ \mathbf{P}\geq 0 \Rightarrow \text{no conclusion} \\ \mathbf{P}\not\geq 0 \Rightarrow \text{unstable system} \end{cases}$$

- This theorem is very much applicable in the control design for linear systems

Example 6:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \qquad \mathbf{A} = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}$$

$$\mathbf{Q} = \mathbf{I}$$

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$$

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix} + \begin{bmatrix} 0 & -8 \\ 4 & -12 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solution $\mathbf{P} = \dfrac{1}{16}\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} > 0$, therefore the system is asym. stable.

Analysis of some nonlinear systems:

❑ **Theorem (Krasovskii):** consider the autonomous system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with origin as the equilibrium point. Let $\mathbf{A} = [\partial \mathbf{f}(\mathbf{x}) / \partial \mathbf{x}]$ denotes the Jacobian matrix of the system at this equilibrium point. If the matrix $\mathbf{F} = \mathbf{A} + \mathbf{A}^T$ is negative definite in a neighborhood Ω then the equilibrium point is asymptotically stable. A Lyapunov function for the system is $V = \mathbf{f}^T(\mathbf{x})\mathbf{f}(\mathbf{x})$. Moreover if Ω is the whole space and $\lim V \to \infty$ as $\|\mathbf{x}\| \to \infty$ Then the equilibrium point is globally asymptotically stable.

Example 7:

- System: $$\dot{x}_1 = -6x_1 + 2x_2$$ $$\dot{x}_2 = 2x_1 - 6x_2 - 2x_2^3$$

- Jacobian: $\mathbf{A} = \dfrac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} -6 & 2 \\ 2 & -6-6x_2^3 \end{bmatrix}$

- **F** matrix: $\mathbf{F} = \mathbf{A}^T + \mathbf{A} = \begin{bmatrix} -12 & 4 \\ 4 & -12-12x_2^3 \end{bmatrix} < 0$

- Lyapunov function: $V(x) = \mathbf{f}^T\mathbf{f} = (-6+2x_2)^2 + (2x_1 - 6x_2 - 2x_2^3)^2$

- Asymptotically stability: $\lim V \to \infty$ as $\|\mathbf{x}\| \to \infty$

- **<u>Theorem (Generalized Krasovskii Thm):</u>** consider the autonomous system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with origin as the equilibrium point and the Jacobian matrix, $\mathbf{A} = \left[\partial \mathbf{f}(\mathbf{x}) / \partial \mathbf{x}\right]_{\mathbf{x}=\mathbf{0}}$. Then a sufficient condition for the origin to be asymptotically stable is that there exist two positive definite matrices, $\mathbf{P}$ and $\mathbf{Q}$, such that $\forall \mathbf{x} \neq \mathbf{0}$, the matrix $\mathbf{F} = \mathbf{A}^T \mathbf{P} + \mathbf{PA} + \mathbf{Q}$ is negative semi definite in some neighborhood Ω of the origin. The function $V = \mathbf{f}^T \mathbf{P} \mathbf{f}$ is then a Lyapunov function for the system. Moreover if Ω is the whole space and $\lim V \to \infty$ as $\|\mathbf{x}\| \to \infty$ Then the equilibrium point is globally asymptotically stable.

Physically motivated Lyapunov function (Positioning a robotic system:

- Robotic system: $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}$

- Controller (PD controller with gravity compensator):

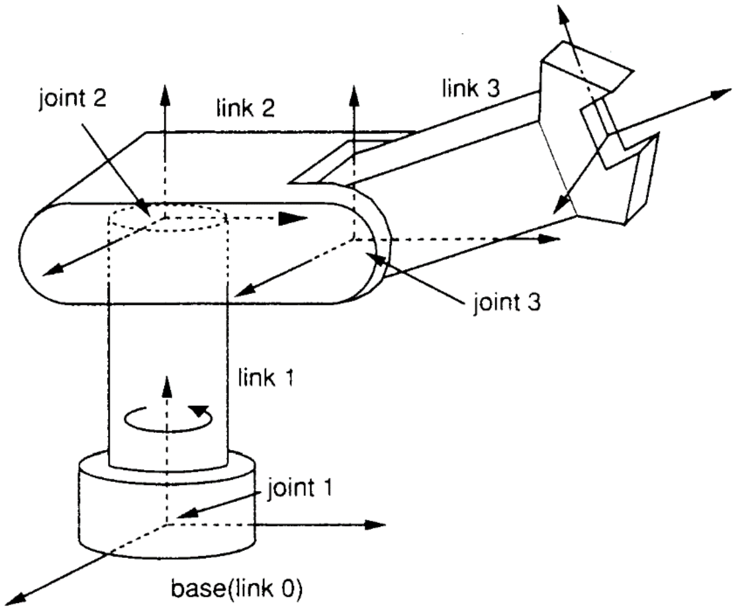
$$\boldsymbol{\tau} = -\mathbf{K}_D\dot{\mathbf{q}} - \mathbf{K}_p\mathbf{q} + \mathbf{g}(\mathbf{q})$$

- Closed loop system dynamics:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}_D\dot{\mathbf{q}} + \mathbf{K}_p\mathbf{q} = \mathbf{0}$$

- Lyapunov candidate function:

$$V(\dot{\mathbf{q}}, \mathbf{q}) = \frac{1}{2}\left(\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} + \mathbf{q}^T\mathbf{K}_p\mathbf{q}\right)$$



- Time derivative of V

$$\dot{V} = \frac{d}{dt}(KE) + \dot{\mathbf{q}}^T \mathbf{K}_p \mathbf{q} = \underbrace{\dot{\mathbf{q}}^T \left(\boldsymbol{\tau} - \mathbf{g}(\mathbf{q})\right)}_{\text{input power}} + \dot{\mathbf{q}}^T \mathbf{K}_p \mathbf{q} = -\dot{\mathbf{q}}^T \mathbf{K}_D \dot{\mathbf{q}} \le 0$$

- Therefore the system is locally stable.

- Both from physics and mathematics of the system it can be realized that the system cannot be stuck at non-zero position, i.e., $\mathbf{q} \neq \mathbf{0}$. This means motion of the system is continued to V=0. Therefore, the system is locally asymptotically stable.

- Since the Lyapunov function satisfies the radially unbounded condition, the system is in fact globally asymptotically condition.

Controller Design Using Lyapunov Direct Method :

- Dynamic system: $\ddot{x} - \dot{x}^3 + x^2 = u$
- Proposed controller: $u = u_1(\dot{x}) + u_2(x)$
- Closed loop system dynamics: $\ddot{x} - \left(\dot{x}^3 + u_1(\dot{x})\right) + \left(x^2 - u_2(x)\right) = 0$
- $u_1(\dot{x})$ and $u_2(x)$ are selected such that:

$$x\left(x^2 - u_2(x)\right) > 0 \quad \forall x \neq 0$$

$$\dot{x}\left(\dot{x}^3 + u_1(\dot{x})\right) < 0 \quad \forall \dot{x} \neq 0$$

- Then the closed loop system will be globally asymptotically stable
- For example: $u_2(x) = -\dfrac{1}{2}(x + x^3) \qquad u_1(\dot{x}) = -2\dot{x}^3$

Invariant Sets

and

Stability Analysis of Invariant Sets

Invariant Set

Definition: a subset $\mathcal{M}$ of the state space is called an *Invariant Set* for the dynamic system of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ if

$$\mathbf{x}(0) \in \mathcal{M} \Rightarrow \mathbf{x}(t) \in \mathcal{M} \quad \forall t > 0$$

Examples: Equilibrium points, Whole state space, Attraction region, Limit cycle, A set of trajectories.

- Invariant set is an extension to the equilibrium point
- Analysis for the equilibrium points, like stability analysis, can be extended to the invariant sets.

Periodic or Closed Orbital Motions:

- A system oscillates when it has a nontrivial (non-constant) periodic solution

$$\mathbf{x}(t+T)=\mathbf{x}(t)$$

- The image of a periodic solution in the phase portrait is a closed trajectory, called periodic orbit or closed orbit.

- There are two types of closed trajectory in dynamical systems

  1. Closed trajectories around center equilibrium point (harmonic oscillation).
  2. Closed trajectories called *limit cycles*

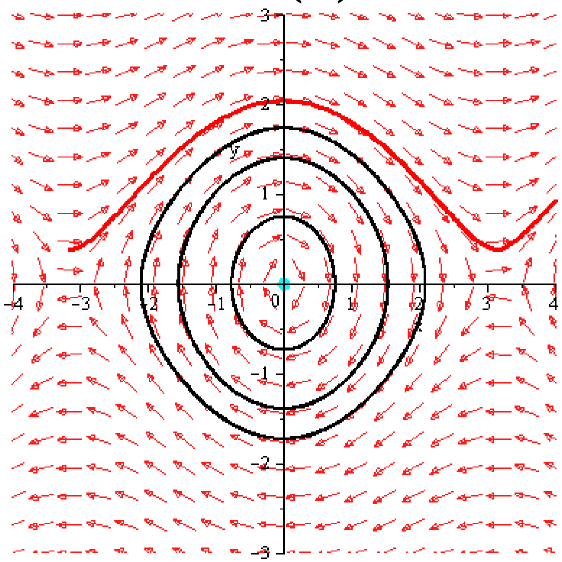
Invariant Set

Harmonic oscillations in a dynamic system

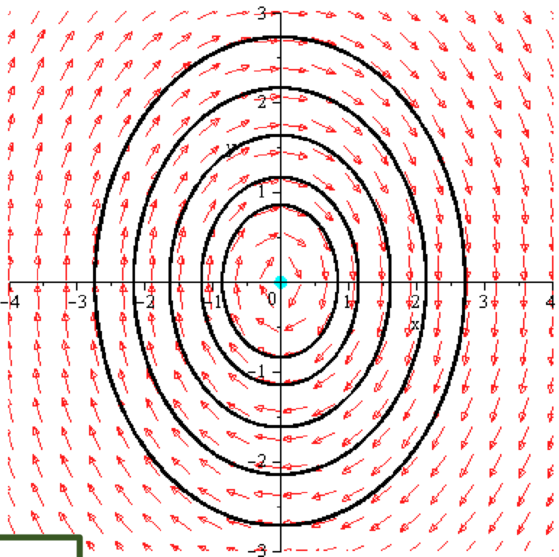
- are not unique or isolated.
- make a continuum of closed orbit.
- have amplitudes depended on the initial conditions.
- are not robust and any perturbation, i.e. the system moves in a new closed trajectory

Example: non-damped pendulum, spring-mass systems

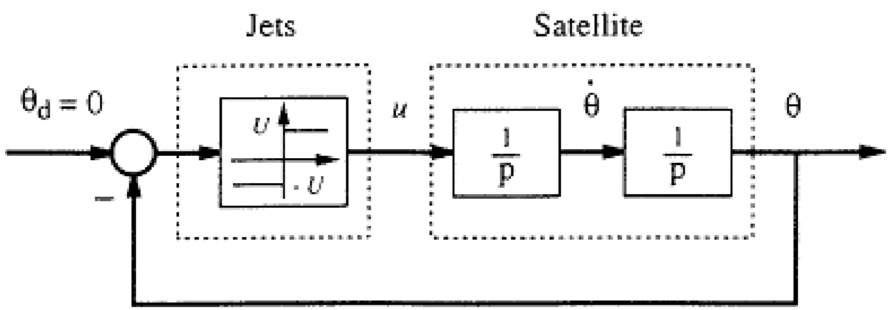
$$\ddot{x} + \sin(x) = 0$$



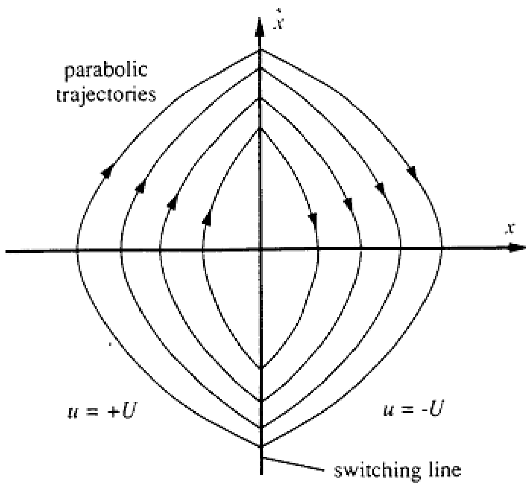
$$\ddot{x} + x = 0$$

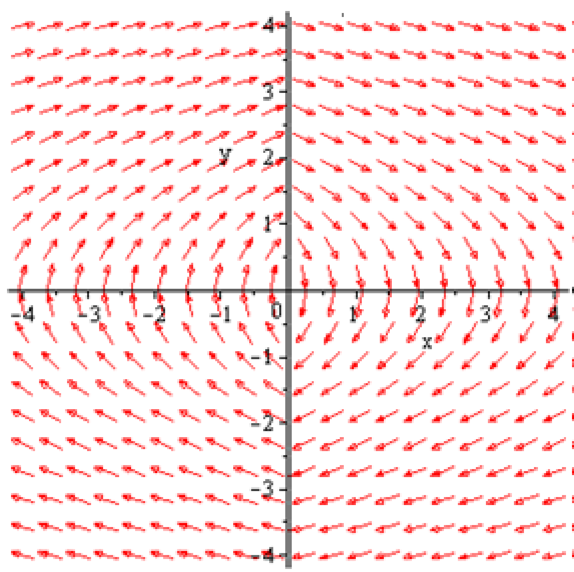


Another example

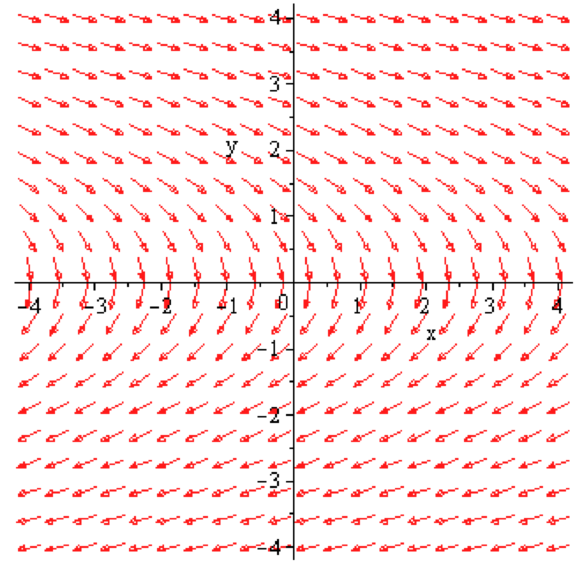


$$\ddot{\theta} = \begin{cases} -U_0 & \theta > 0 \\ U_0 & \theta < 0 \end{cases}$$

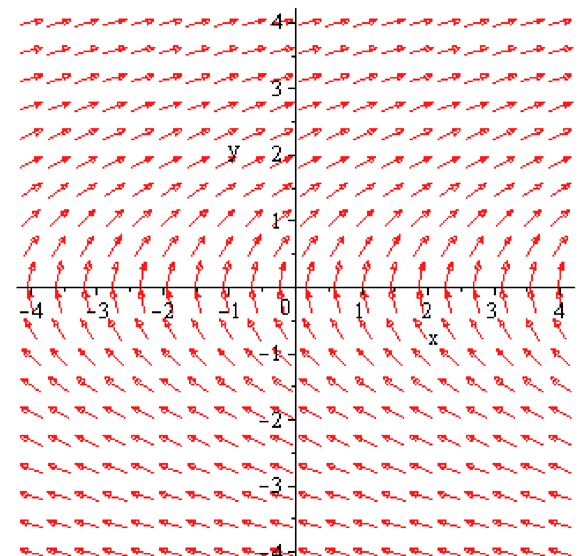




$\ddot{\theta} = -1$

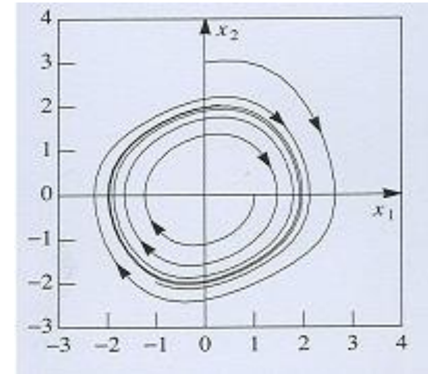


$\ddot{\theta} = 1$

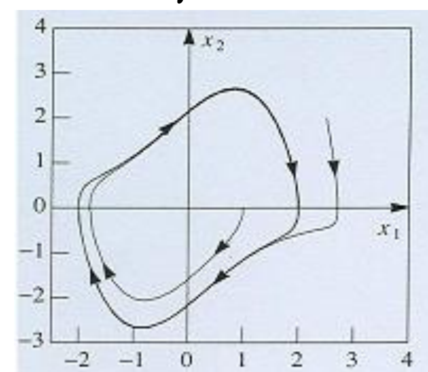


Invariant Set

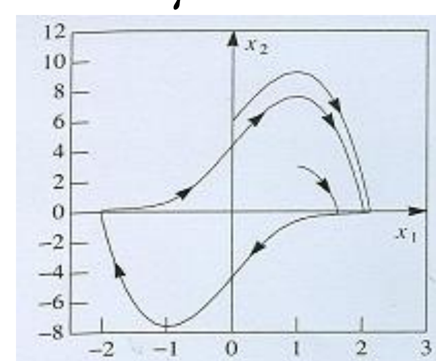
$\mu = 0.2$



$\mu = 1$



$\mu = 5$



Limit cycles:

- can be realized only in nonlinear systems
- are isolated closed orbits and do not make a continuum set of closed orbits.
- have amplitude independent of the initial conditions.
- are difficult both in realization and analysis.

Example: Van Der Pole system

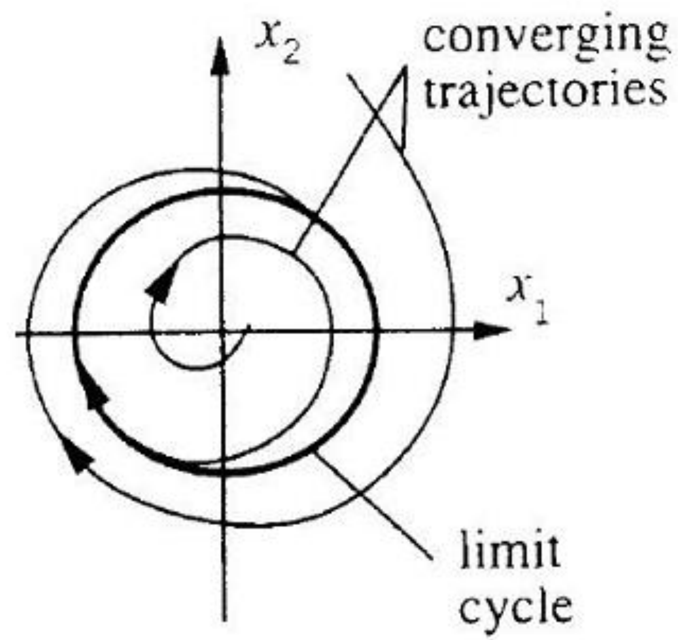
$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

➢ There are three type of limit cycles:

1) Stable Limit Cycle

Example:

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1)\\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2 - 1)\end{aligned}$$

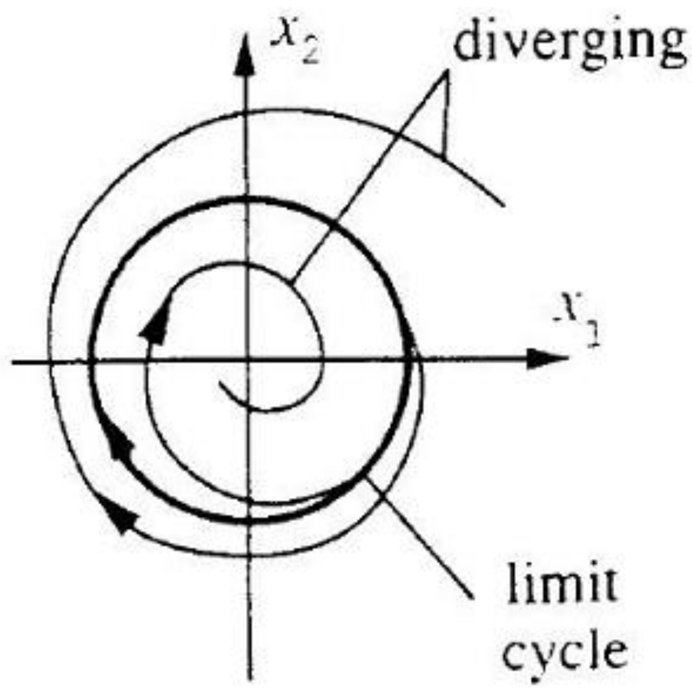


Polar Coordinates

$$\begin{aligned}\dot{r} &= -r(r^2 - 1)\\ \dot{\theta} &= -1\end{aligned}$$

- There are three type of limit cycles:

1) Unstable Limit Cycle



Example:

$$\dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2 - 1)$$
$$\dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2 - 1$$

Polar Coordinates

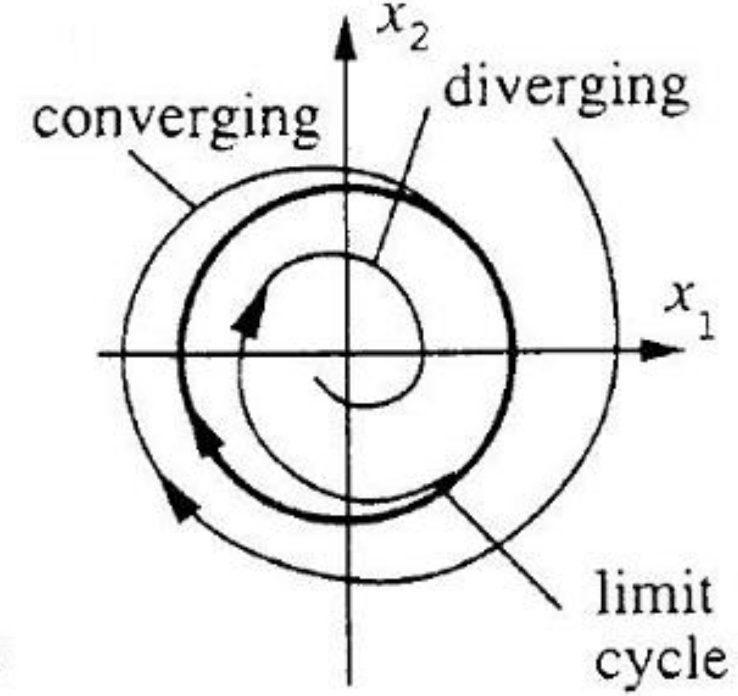
$$\dot{r} = r(r^2 - 1)$$
$$\dot{\theta} = -1$$

- There are three type of limit cycles:

1) Semi-stable Limit Cycle

Example:

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2 - 1)^2\end{aligned}$$



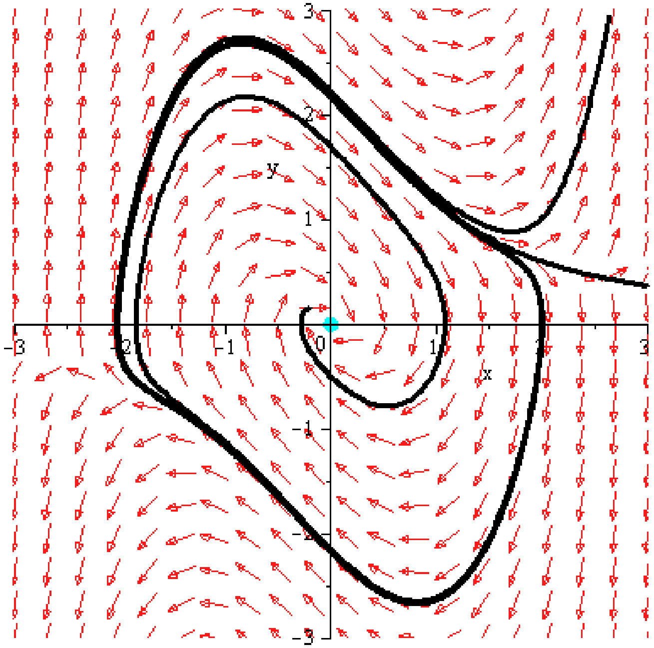
Polar Coordinates

$$\begin{aligned}\dot{r} &= -r(r^2 - 1)^2 \\ \dot{\theta} &= -1\end{aligned}$$

Example 7: Van Der Pole system

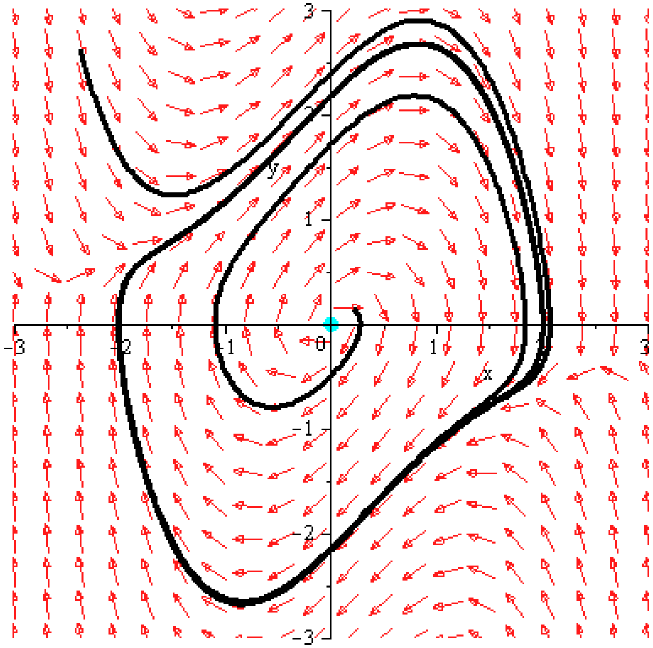
$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

$\mu > 0$ Unstable Limit Cycle



Stable Equilibrium Point

$\mu < 0$ Stable Limit Cycle



Unstable Equilibrium Point

Example 9: The system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_1 &= x_1 + x_2 - 2x_1 x_2\end{aligned}$$

has two equilibrium points at (0, 0) and (1, 1). The Jacobian:

$$\left[\frac{\partial f}{\partial x}\right]\bigg|_{(0,0)} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}; \quad \left[\frac{\partial f}{\partial x}\right]\bigg|_{(1,1)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

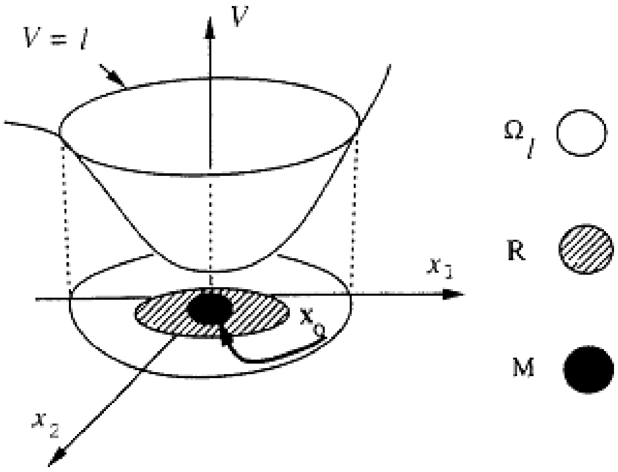
- (0; 0) is a saddle point and (1; 1) is a stable focus.
- Only a single focus can be encircled by a stable focus.
- Periodic orbit in other region such as that encircling both Eq. points are ruled out.

Invariant Set

Theorem (Local Invariant Set Theorem): Consider an autonomous system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, with f continuous, and let $V(\mathbf{x})$ be a scalar function with continuous first partial derivatives. Assume that

- for some $\ell > 0$, the region Ω_ℓ defined by $V(\mathbf{x}) < \ell$ is bounded
- $\dot{V}(\mathbf{x}) \le 0$ for all $\mathbf{x}$ in Ω_ℓ

Let **R** be the set of all points within Ω_ℓ where $\dot{V}(\mathbf{x}) = 0$ and **M** be the largest invariant set in **R**. then every trajectory $\mathbf{x}(t)$ originating in Ω_ℓ tends to **M** as $t \to \infty$



Example 10: mass-spring-damper

- System: $m\ddot{x}+b\dot{x}\left|\dot{x}\right|+k_0x+k_1x^3=0$
- Lyapunov function: $V=\dfrac{1}{4}(2m\dot{x}^2+2k_0x^2+k_1x^4)$
- Time derivative of V: $\dot{V}=(m\ddot{x}+k_0x+k_1x^3)\dot{x}=-b\dot{x}^2\left|\dot{x}\right|$
- At least, locally stable by Lyapunov direct method
- Set of $\mathbf{R}$: $\dot{V}=-b\dot{x}^2\left|\dot{x}\right|=0\Rightarrow\mathbf{R}=x-\text{axis}$
- Set of $\mathbf{M}$, the biggest invariant set in $\mathbf{R}$ (all the trajectories and equilibrium points, limit cycles and … in $\mathbf{R}$):
- Conclusion: asymptotically stability $\lim_{t\to\infty}x(t)=0$

Example 11: attractive limit cycle

System:
$$\dot{x}_1 = x_2 - x_1^7(x_1^4 + 2x_2^2 - 10)$$
$$\dot{x}_2 = -x_1^3 - 3x_2^5(x_1^4 + 2x_2^2 - 10)$$

Consider a set defined by: $L: x_1^4 + 2x_2^2 - 10 = 0$

This set is invariant set:

$$\frac{d}{dt}(x_1^4 + 2x_2^2 - 10) = -(4x_1^{10} + 12x_2^6)(x_1^4 + 2x_2^2 - 10) = 0$$

Motion on the set: a closed orbit motion which is in fact a limit cycle

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 \end{aligned}\right\} \Rightarrow \dot{x}_1 + x_1^3 = 0$$

Invariant Set

Lyapunov function: $V = \left(x_1^4 + 2x_2^2 - 10\right)^2$

$$\Rightarrow \dot{V} = -8(x_1^{10} + 3x_2^6)\left(x_1^4 + 2x_2^2 - 10\right)^2$$

It is negative definite for all points except for origin and the L set.

Therefore:

$$\dot{V} = 0 \Rightarrow \mathbf{R} = \left\{(x_1, x_2) \mid (0,0) \text{ and } x_1^4 + 2x_2^2 - 10 = 0 \right\}$$

The **M** set: **M=R**

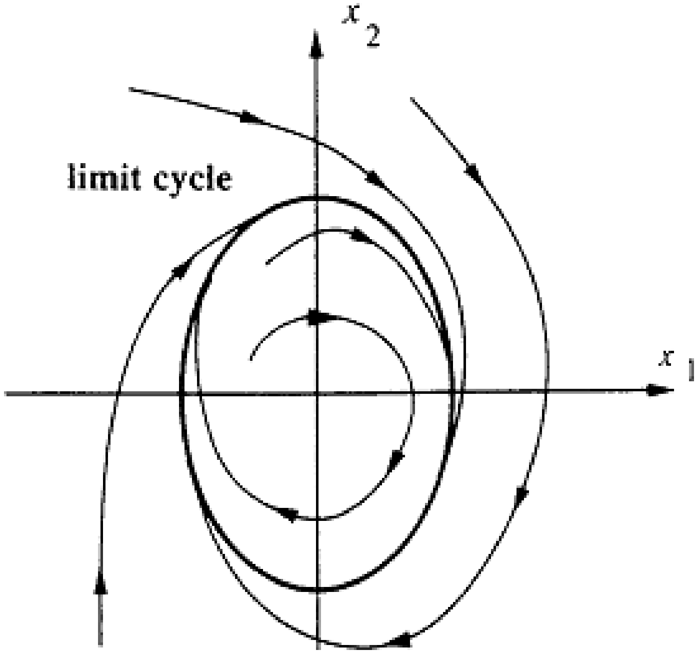
This means starting trajectory in every bounded region Ω_ℓ defined by $V(\mathbf{x}) < \ell$, for any $\ell > 0$ will converge to either the limit cycle and origin (equilibrium point), obviously if this invariants exist there.

Invariant Set

Now question is which way trajectories go

1. Clearly trajectories from out side the limit cycle converge to the limit cycle

2. Selecting $\ell > 100$ excludes origin from Ω_ℓ therefore trajectories inside the limit cycle except at the origin converge to the limit cycle as well.



this concludes the limit cycle is a stable limit cycle and the origin is an unstable equilibrium point

Invariant Set

Some Notes:

- Function V is not necessarily positive definite, from the properties assumed for this function, it is lower bounded.

- Lyapunov direct theorem is a special case of this theorem.

Theorem (Global Invariant Set Theorem.): Consider an autonomous system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, with f continuous, and let $V(\mathbf{x})$ be a scalar function with continuous first partial derivatives. Assume that

$$\lim V \to \infty \text{ as } \|\mathbf{x}\| \to \infty$$

$$\dot{V}(\mathbf{x}) < 0 \text{ for all } \mathbf{x} \text{ in } \Omega_\ell$$

Let $\mathbf{R}$ be the set of all points where $\dot{V}(\mathbf{x}) = 0$ and $\mathbf{M}$ be the largest invariant set in $\mathbf{R}$. Then all the trajectories asymptotically converge to $\mathbf{M}$ as $t \to \infty$