Stability and Phase Plane Analysis

Objectives

Objectives of the section:

- Introducing the Phase Plane Analysis
- Introducing the Concept of stability
- Stability Analysis of Linear Time Invariant Systems
- Lyapunov Indirect Method in Stability Analysis of Nonlinear Systems

Introducing

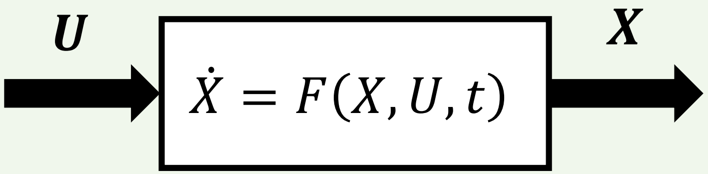
the Phase Plane Analysis

Phase Plane Analysis

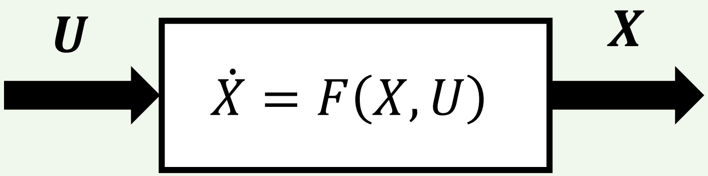
Phase Space form of a Dynamical System:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t) \\ \dot{x}_2 = f_2(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t) \\ \vdots \\ \dot{x}_n = f_n(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m, t) \end{cases} \longrightarrow \quad \dot{X} = F(X, U, t)$$

$$X \in \mathbb{R}^n \quad U \in \mathbb{R}^m$$



Time-Varying System



Time-Invariant System

Phase Plane Analysis

Phase Space form of a Linear Time Invariant (LTI) System:

$$\dot{\boldsymbol{X}} = \boldsymbol{AX} + \boldsymbol{BU}$$

$$X \in \mathbb{R}^n \quad U \in \mathbb{R}^m$$

Special Properties of Nonlinear Systems:

- ❑ Multiple isolated equilibria
- ❑ Limit Cycle
- ❑ Finite escape time
- ❑ Harmonic, sub-harmonic and almost periodic Oscillation
- ❑ Chaos
- ❑ Multiple modes of behavior

Phase Plane Analysis

Phase Plane Analysis is a graphical method for studying second-order systems respect to initial conditions by:

- providing motion trajectories corresponding to various initial conditions.
- examining the qualitative features of the trajectories
- obtaining information regarding the stability of the equilibrium points

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

Advantages of Phase Plane Analysis:

- It is graphical analysis and the solution trajectories can be represented by curves in a plane
- Provides easy visualization of the system qualitative
- Without solving the nonlinear equations analytically, one can study the behavior of the nonlinear system from various initial conditions.
- It is not restricted to small or smooth nonlinearities and applies equally well to strong and hard nonlinearities.
- There are lots of practical systems which can be approximated by second-order systems, and apply phase plane analysis.

Disadvantage of Phase Plane Method:

- It is restricted to at most second-order
- graphical study of higher-order is computationally and geometrically complex.

Example: First Order LTI System

$$\dot{x} = \sin(x)$$

Analytical Solution

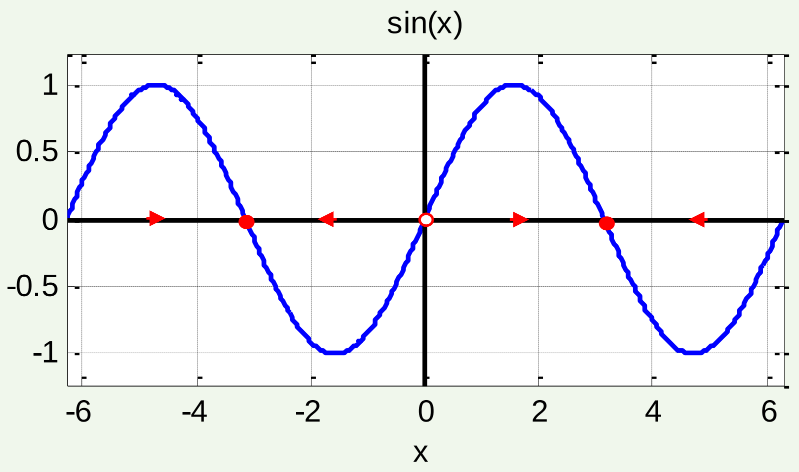
$$\frac{dx}{dt} = \sin(x)$$

$$\frac{dx}{\sin(x)} = dt$$

$$\int_{x_0}^{x} \frac{dx}{\sin(x)} = \int_{0}^{t} dt$$

$$t = \ln\left|\frac{\cos(x_0) + \cot(x_0)}{\cos(x) + \cot(x)}\right|$$

Graphical Solution



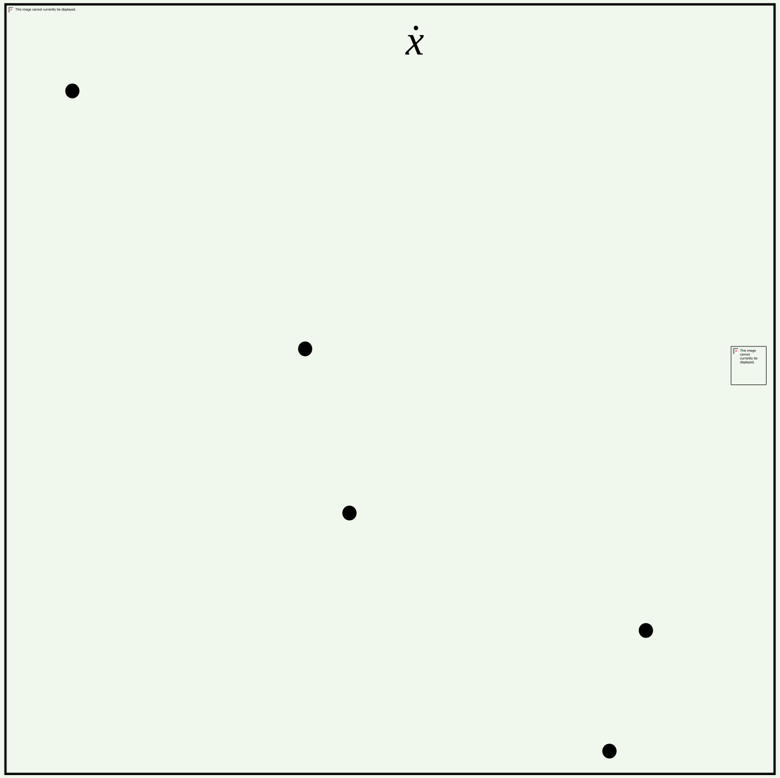
Concept of Phase Plane Analysis:

- Phase plane method is applied to Autonomous Second Order System

$$\dot{x}_1 = f_1(x_1, x_2) \qquad \dot{x}_2 = f_2(x_1, x_2)$$

- System response $X(t) = (x_1(t), x_2(t))$ to initial condition $X_0 = \left(x_1(0), x_2(0)\right)$ is a mapping from $\mathbb{R}$(Time) to $\mathbb{R}^2(x_1, x_2)$
- The solution can be plotted in the $x_1 - x_2$ plane called ***State Plane*** or ***Phase Plane***
- The locus in the $x_1 - x_2$ plane is a curved named ***Trajectory*** that pass through point X_0
- The family of the phase plane trajectories corresponding to various initial conditions is called ***Phase portrait*** of the system.
- For a single DOF mechanical system, the phase plane is in fact is $(x, \dot{x})$ plane

Example: Van der Pol Oscillator Phase Portrait



Plotting Phase Plane Diagram:

- ❑ Analytical Method
- ❑ Numerical Solution Method
- ❑ Isocline Method
- ❑ Vector Field Diagram Method
- ❑ Delta Method
- ❑ Lienard’s Method
- ❑ Pell’s Method

Analytical Method

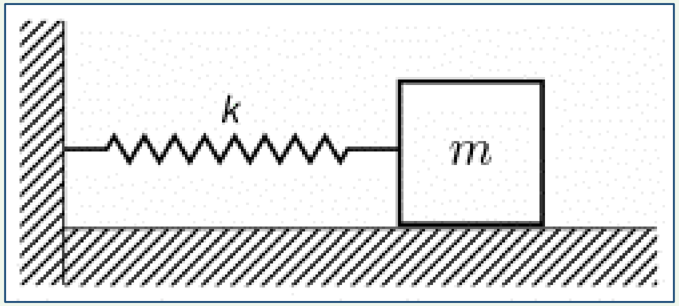
- Dynamic equations of the system is solved, then time parameter is omitted to obtain relation between two states for various initial conditions

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

$$\xrightarrow{\text{Solve}} \quad \begin{array}{l} x_1(t, X_0) = g_1(t, X_0) \\ x_2(t, X_0) = g_2(t, X_0) \end{array} \quad \longrightarrow \quad F(x_1, x_2) = 0$$

✓ For linear or partially linear systems

Example: Mass Spring System

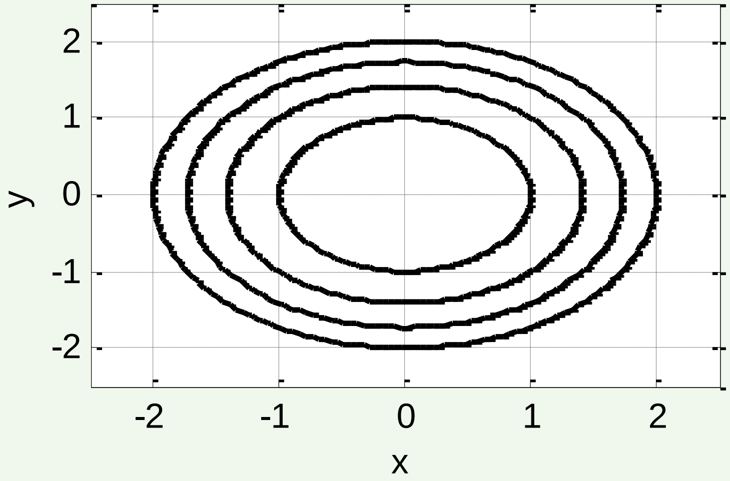


$$m\ddot{x} + kx = 0$$

For $m = k = 1:$ $\qquad \ddot{x} + x = 0$

$$\begin{cases} x(t) = x_0 \cos(t) + \dot{x}_0 \sin(t) \\ \dot{x}(t) = -x_0 \sin(t) + \dot{x}_0 \cos(t) \end{cases}$$

$$\boxed{x^2 + \dot{x}^2 = x_0^2 + \dot{x}_0^2}$$



Analytical Method

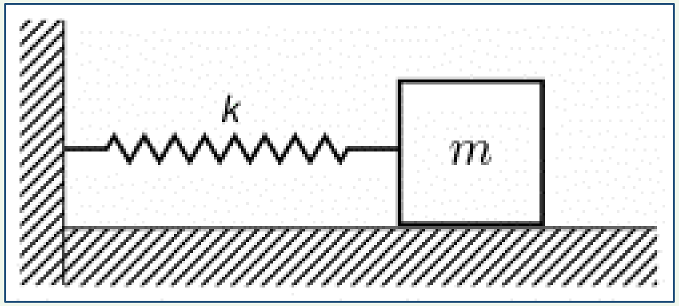
- Time differential is omitted from dynamic equations of the system, then partial differential equation is solved

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \xrightarrow{\text{Solve}} F(x_1, x_2) = 0$$

✓ For linear or partially linear systems

Example: Mass Spring System



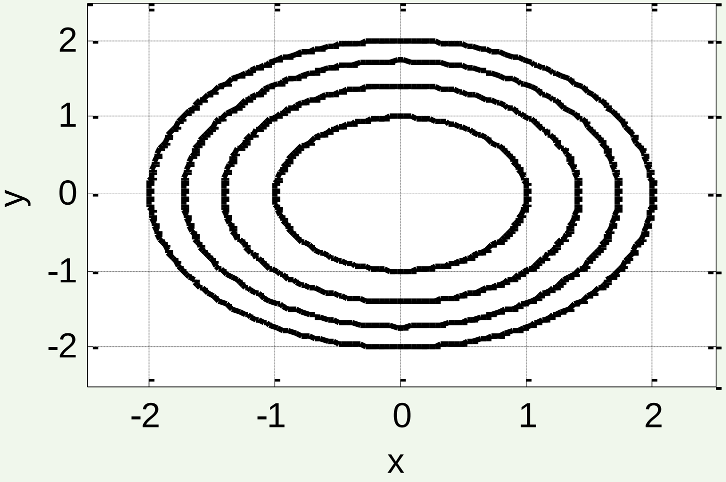
$$m\ddot{x} + kx = 0$$

For $m = k = 1$: $\quad \ddot{x} + x = 0 \longrightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases}$

$$\frac{dx_2}{dx_1} = \frac{-x_1}{x_2}$$

$$\int_{x_{20}}^{x_2} x_2 dx_2 = \int_{x_{10}}^{x_1} -x_1 dx_1$$

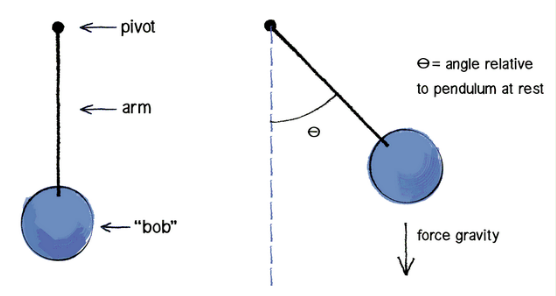
$$\boxed{x^2 + \dot{x}^2 = x_0^2 + \dot{x}_0^2}$$



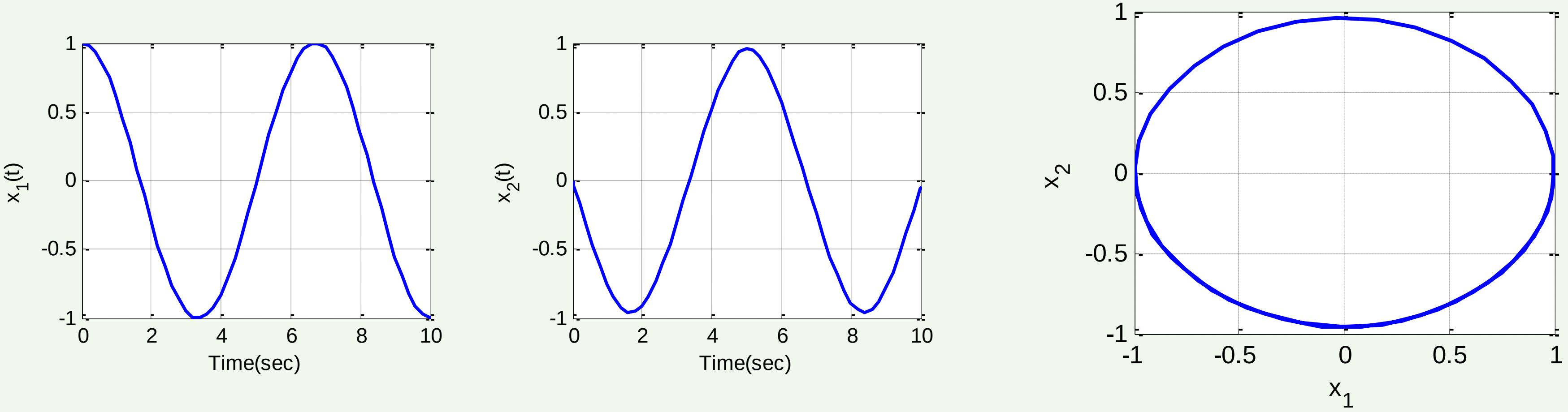
Numerical Solution Method

Dynamic equations of the system is solved numerically (e.g. Ode45) for various initial conditions and time response is obtained, then two states are plotted in each time.

Example: Pendulum



$$\ddot{\theta} + \sin(\theta) = 0$$



Isocline Method

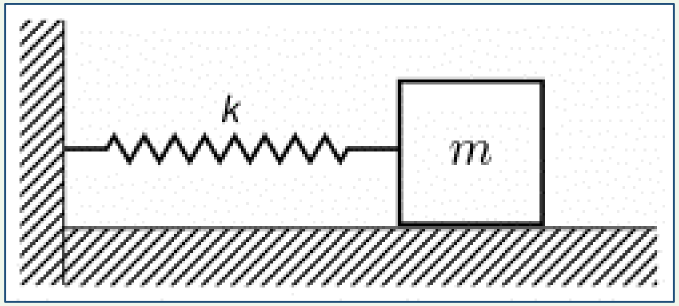
Isocline: The set of all points which have same trajectory slope

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = \alpha \longrightarrow f_2(x_1, x_2) = \alpha f_1(x_1, x_2)$$

First various isoclines are plotted, then trajectories are drawn.

Example: Mass Spring System

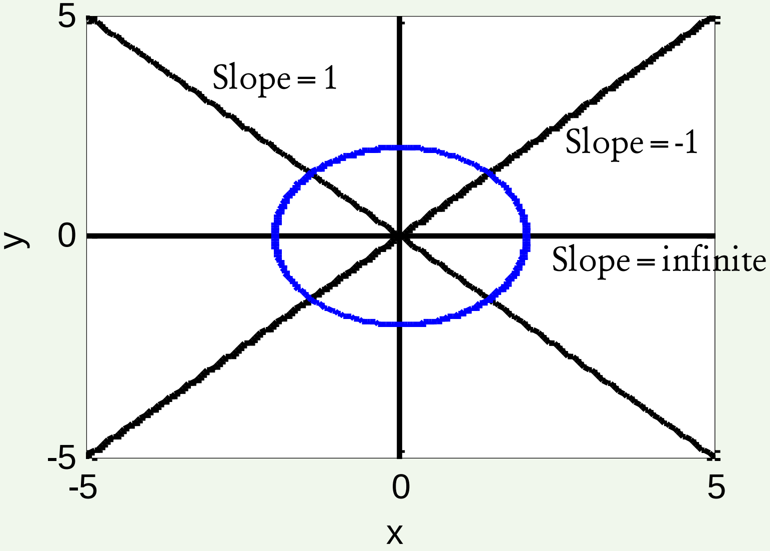


$$m\ddot{x} + kx = 0$$

For $m = k = 1$: $\quad \ddot{x} + x = 0 \longrightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases}$

$$\frac{dx_2}{dx_1} = \frac{-x_1}{x_2} = \alpha$$

$$x_1 + \alpha x_2 = 0$$



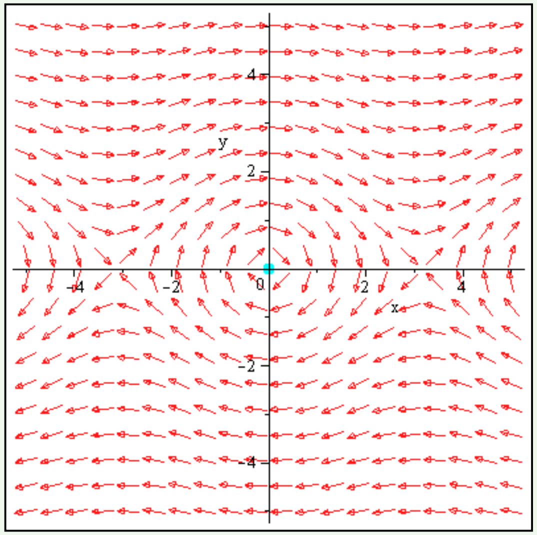
Vector Field Diagram Method

Vector Field: A set of vectors that is tangent to the trajectory

- At each point (x_1, x_2) vector $\begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix}$ is tangent to the trajectories
- Hence vector field can be constructed in the phase plane and direction of the trajectories can be easily realized with that

$$\ddot{\theta} + \sin(\theta) = 0$$

$$\left.\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \end{aligned}\right\} \Rightarrow \mathbf{f} = \begin{bmatrix} x_2 \\ -\sin(x_1) \end{bmatrix}$$



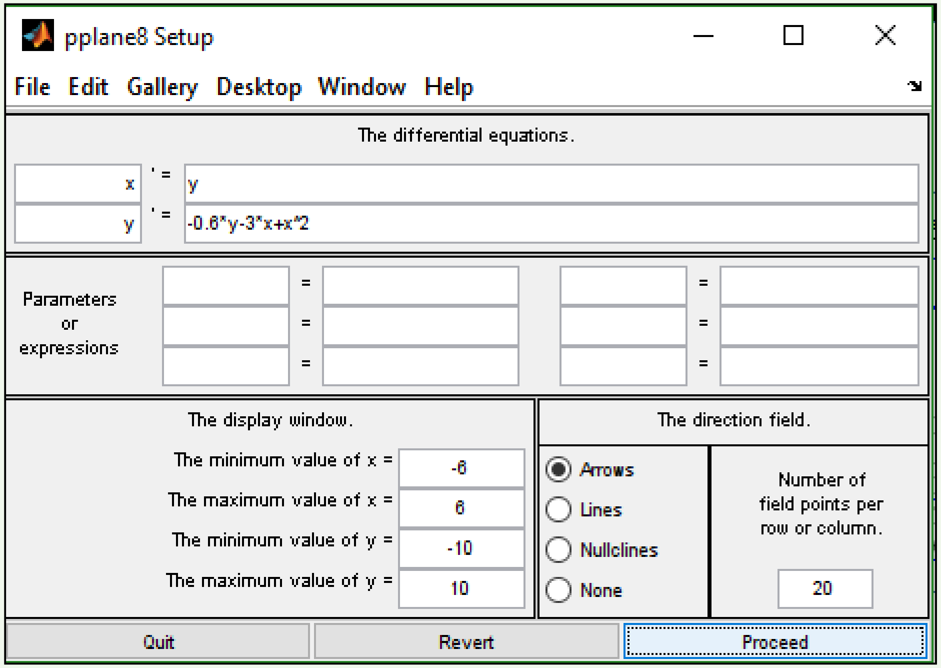
Singular Points in the Phase Plane Diagram:

Equilibrium points are in fact singular points in the phase plane diagram

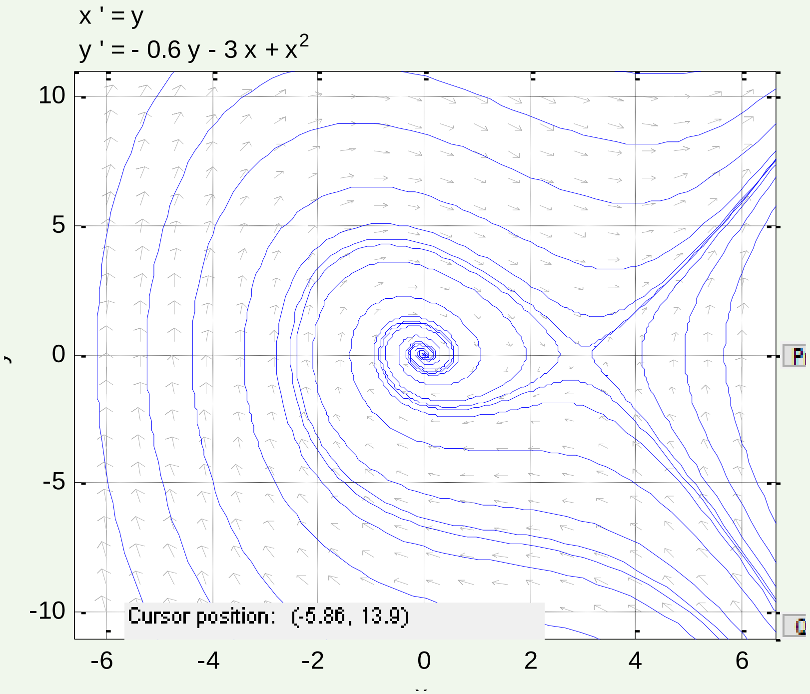
$$\left.\begin{matrix} f_1(x_1, x_2) = 0 \\ \\ f_1(x_1, x_2) = 0 \end{matrix}\right\} \xrightarrow[\text{equilibrium points}]{\text{Slope of the trajectories at}} \frac{dx_2}{dx_1} = \frac{0}{0}$$

- ❑ Singular point is an important concept which reveals great info about properties of system such as stability.
- ❑ Singular points are only points which several trajectories pass/approach them (i.e. trajectories intersect).

Example: Using Matlab



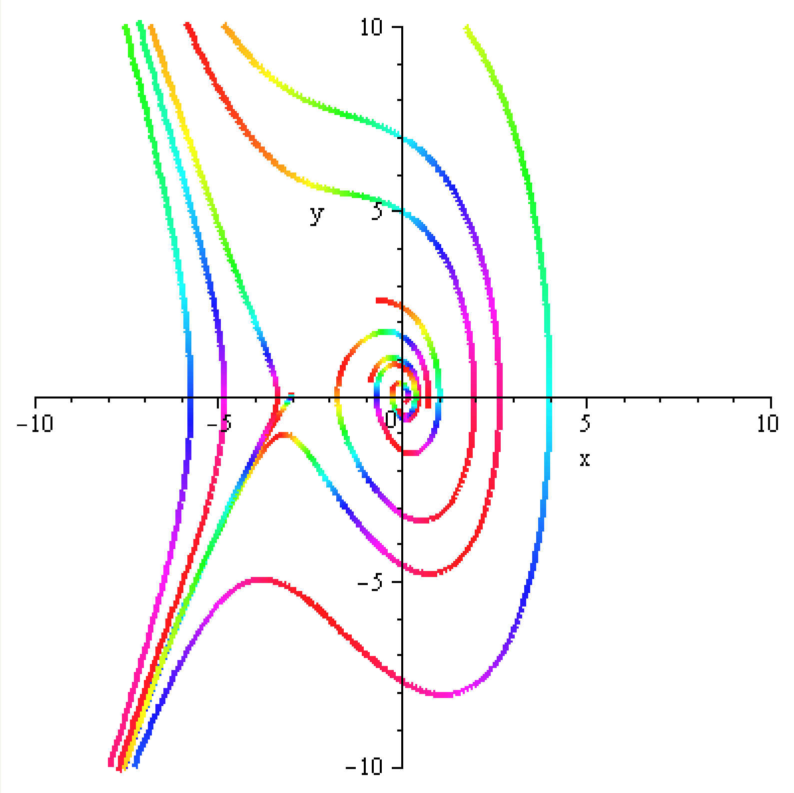
$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$



Example: Using Maple Code

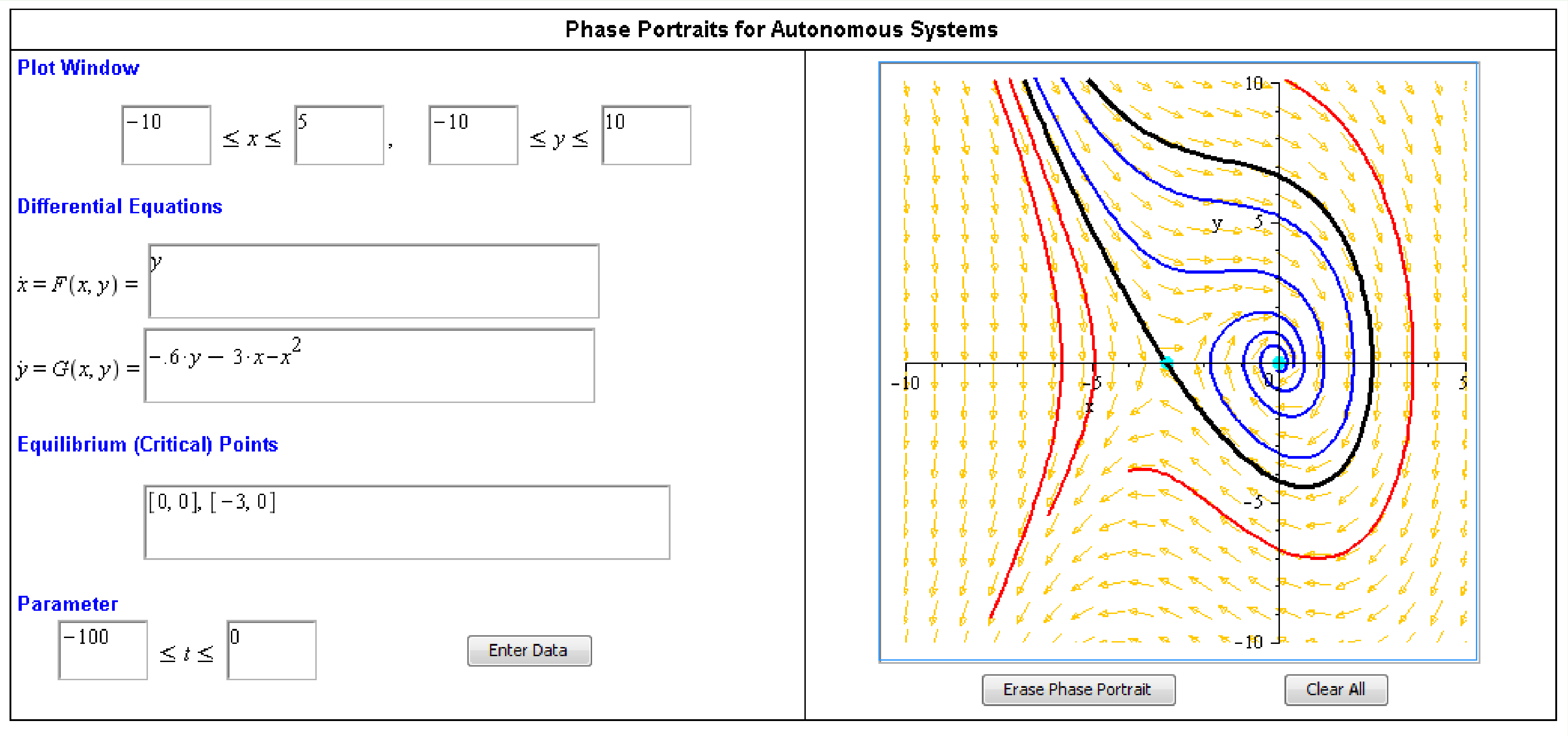
$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

```
with(DEtools):
xx:=x(t):   yy:=y(t):
dx:=diff(xx,t):    dy:=diff(yy,t):
e0:=diff(dx,t)+.6*dx+3*xx+xx^2:
e1:=dx-yy=0:
e2:=dy+0.6*yy+3*xx+xx^2=0:
eqn:=[e1,e2]:  depvar:=[x,y]:
rang:=t=-1..5:  stpsz:=stepsize=0.005:
IC1:=[x(0)=0,y(0)=1]:
IC2:=[x(0)=0,y(0)=5]:
IC3:=[x(0)=0,y(0)=7]:
IC4:=[x(0)=0,y(0)=7]:
IC5:=[x(0)=-3.01,y(0)=0]:
IC6:=[x(0)=-4,y(0)=2]:
IC7:=[x(0)=1,y(0)=0]:
IC8:=[x(0)=4,y(0)=0]:
IC9:=[x(0)=-6,y(0)=3]:
IC10:=[x(0)=-6,y(0)=6]:
ICs:=[IC||(1..10)]:
lincl:=linecolour=sin((1/2)*t*Pi):
mtd:=method=classical[foreuler]:
phaseportrait(eqn,depvar,rang,ICs,stpsz,lincl,mtd);
```



Example: Using Maple Tools

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$



Phase Plane Analysis for Single DOF Mechanical System

In the case of single DOF mechanical system

$$\ddot{x} + g(x, \dot{x}) = 0 \quad \xrightarrow[x_2 = \dot{x}]{x_1 = x} \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g(x_1, x_2) \end{cases}$$

- The phase plane is in fact $(x - \dot{x})$ plane and every point shows the position and velocity of the system.

- Trajectories are always clockwise. This is not true in the general phase plane $(x_1 - x_2)$

Introducing

the Concept of Stability

Stability, Definitions and Examples

- Stability analysis of a dynamic system is normally introduced in the state space form of the equations.

$$\dot{X} = F(X, U, t)$$

$$X \in \mathbb{R}^n \quad U \in \mathbb{R}^m$$

Dynamic System

$$\mathbf{u} \longrightarrow \boxed{\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)} \longrightarrow \mathbf{x}$$

- Most of the concepts in this chapter are introduced for autonomous systems

Autonomous Dynamic System

$$\mathbf{u} \longrightarrow \boxed{\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})} \longrightarrow \mathbf{x}$$

Stability analysis of a dynamic system is divided in three categories:

1. Stability analysis of the equilibrium points of the systems. We study the behavior (dynamics) of the free (unforced, $u = 0$) system when it is perturbed from its equilibrium point.

2. Input-output stability analysis. We study the system (forced system $u \neq 0$) output behavior in response to bounded inputs.

3. Stability analysis of periodic orbits. This analysis is for those systems which perform a periodic or cyclic motion like walking of a biped or orbital motion of a space object.

✓ Our main concern is the first type analysis. Some preliminary issues of the second type analysis will be also discussed.

Reminder:

X_e is said an equilibrium point of the system if once the system reaches this position it stays there for ever, i.e. $f(X_e) = 0$

Definition (Lyapunov Stability):

The equilibrium point X_e is said to be stable (in the sense of Lyapunov stability) or motion of the system about its equilibrium point is said to be stable if the system states (X) is perturbed away from X_e then it stays close to X_e. Mathematically X_e is stable if

$$\varepsilon > 0,\ \exists\ \delta = \delta(\varepsilon) > 0 \qquad \|\mathbf{x}(0) - \mathbf{x}_e\| < \delta \Rightarrow \|\mathbf{x}(t) - \mathbf{x}_e\| < \varepsilon \quad \forall t \geq 0$$

- Without loss of generality we can present our analysis about equilibrium point $X_e = 0$, since the system equation can be transferred to a new form with zero as the equilibrium point of the system.

$$\begin{matrix} \dot{Y} = F(Y) \\ Y_e \neq 0 \end{matrix} \quad \xrightarrow{X = Y - Y_e} \quad \begin{matrix} \dot{X} = F(X) \\ X_e \neq 0 \end{matrix}$$

A more precise definition:

The equilibrium point X_e is said to be stable (in the sense of Lyapunov stability) or motion of the system about its equilibrium point is said to be stable if for any $R > 0$, there exists $0 < r < R$ such that

$$\left\|\mathbf{x}(0) - \mathbf{x}_e\right\| < r \Rightarrow \left\|\mathbf{x}(t) - \mathbf{x}_e\right\| < R \quad \forall t \geq 0$$

Definition (Lyapunov Stability):

The equilibrium point $X_e = 0$ is said to be

- **Stable** if

$$\forall R > 0 \quad \exists 0 < r < R \quad s.t.$$

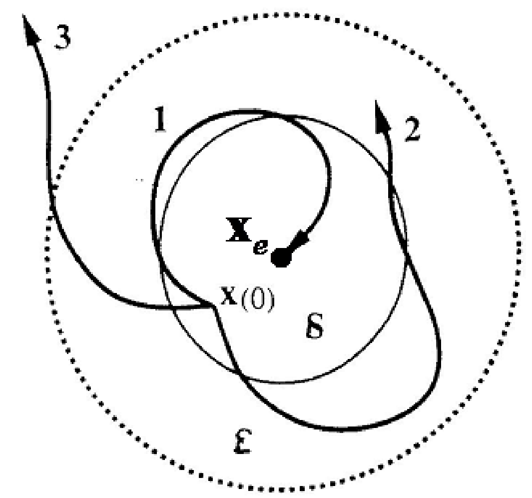
$$\|X(0)\| < r \quad \Rightarrow \quad \|X(t)\| < R \quad \forall t > 0$$

- **Unstable** if it is not stable.
- **Asymptotically stable** if it is stable and

$$\forall r > 0 \quad s.t.$$

$$\|X(0)\| < r \quad \Rightarrow \quad \lim_{t\to\infty} X(t) = 0$$

- **Marginally stable** if it is stable and not asymptotically stable



curve 1 - asymptotically stable

curve 2 - marginally stable

curve 3 - unstable

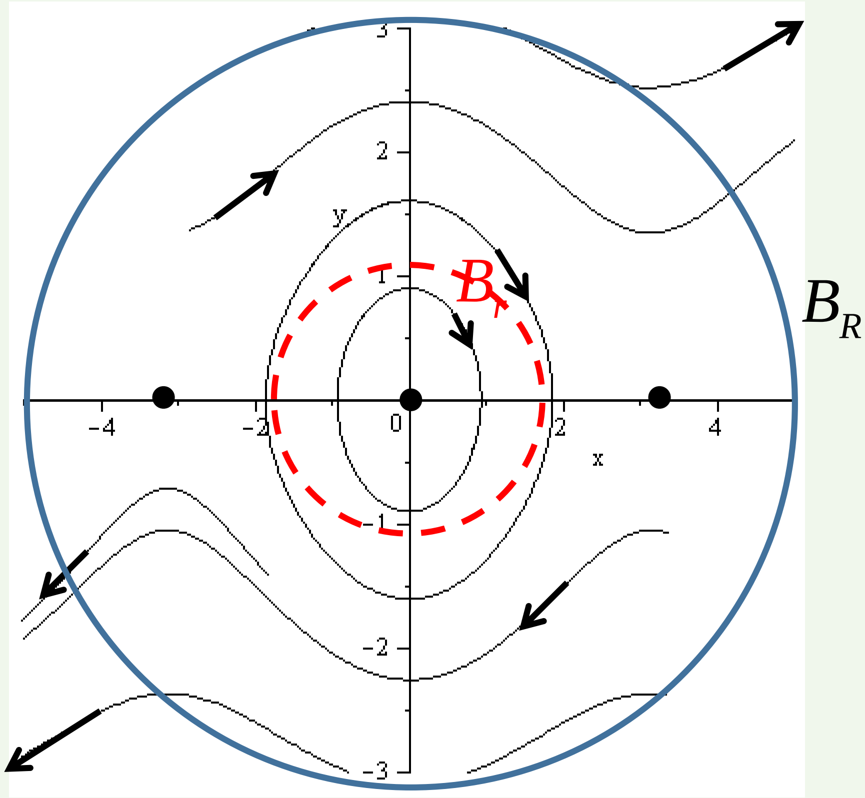
- **Exponentially stable** if it is asymptotically stable with an exponential rate

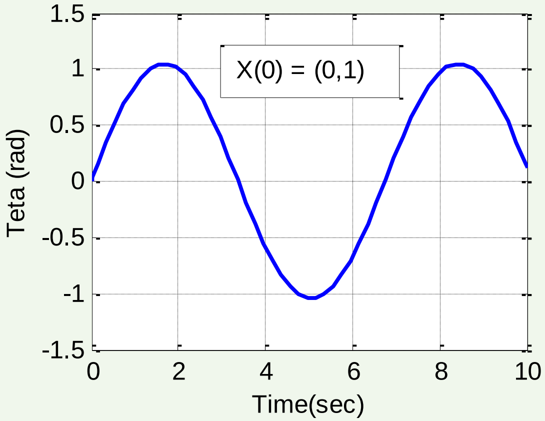
$$\|X(0)\| < r \quad \Rightarrow \quad \|X(t)\| < \alpha e^{-\beta t}\|X(0)\| \quad \alpha, \beta > 0$$

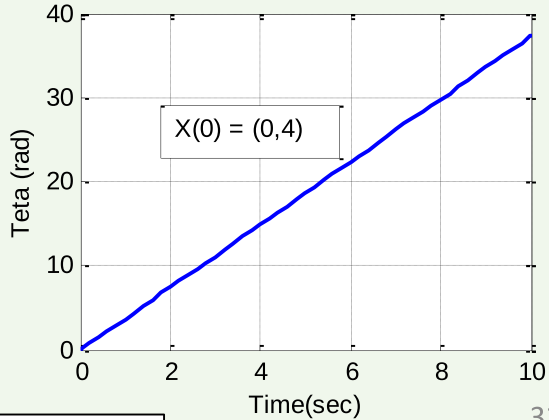
Example: Undamped Pendulum

$$\ddot{\theta} + \frac{g}{l}\sin(\theta) = 0 \qquad \theta_{e1} = 0 \ , \qquad \theta_{e2} = \pi$$

- ✓ θ_{e1} is a marginally stable point and θ_{e2} is an unstable point



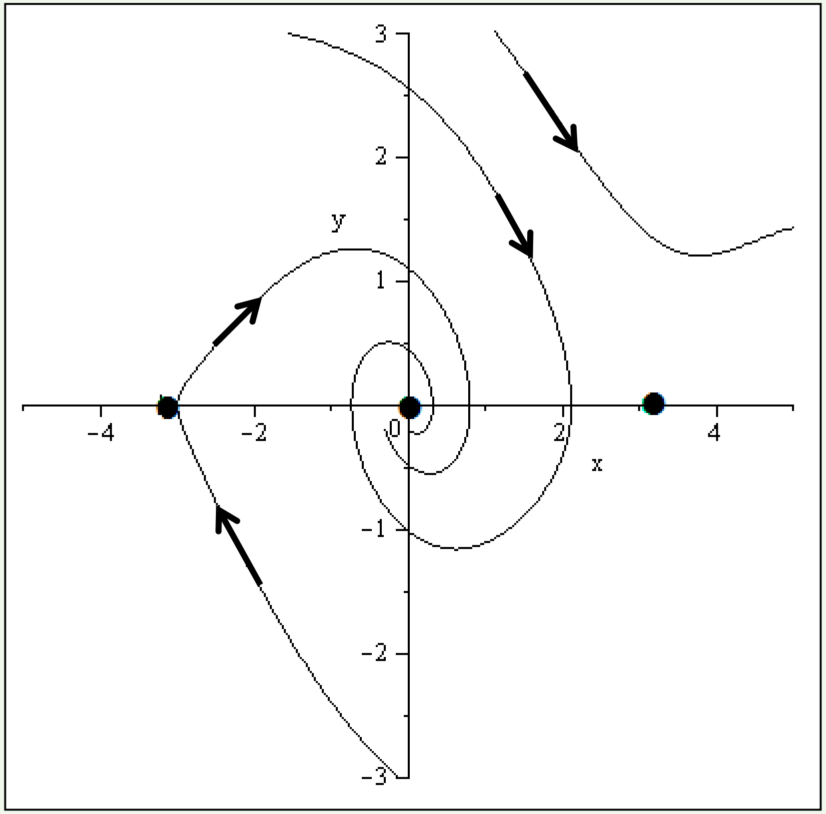


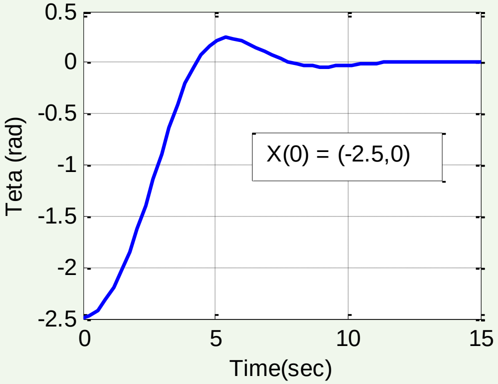


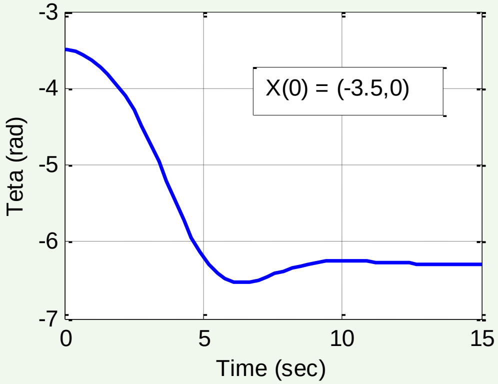
Example: damped Pendulum

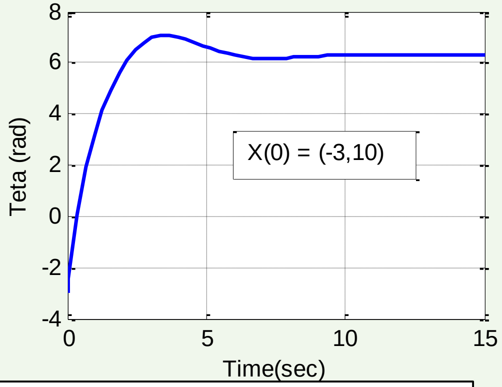
$$\ddot{\theta} + C\dot{\theta} + \frac{g}{l}\sin(\theta) = 0 \qquad\qquad \theta_{e1} = 0 \;, \qquad \theta_{e2} = \pi$$

✓ θ_{e1} is an exponentially stable point and θ_{e2} is an unstable point





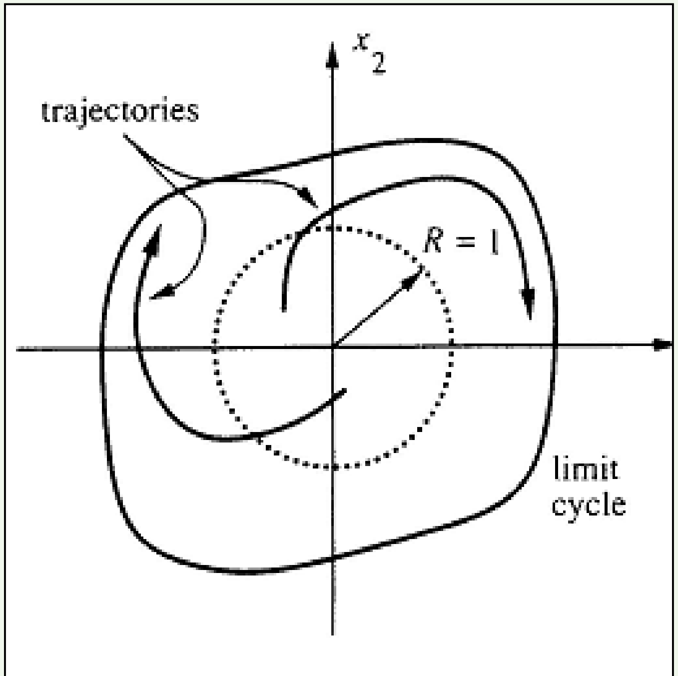




Example: Van Der Pol Oscillator

$$\ddot{x}-(1-x)^2\dot{x}+x=0 \overset{\text{state form}}{\Longrightarrow} \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1+(1-x_1)^2 x_2 \end{cases} \Rightarrow x_{1_e} = x_{2_e} = 0$$

✓ $x_e = 0$ is an unstable point



Unstable origin of the Van der Pol

Definition:

if the equil. point X_e is asymptotically stable, then the set of all points that trajectories initiated at these point eventually converge to the origin is called ***domain of attraction.***

Definition:

if the equil. point X_e is asymptotically/exponentially stable, then the equil. point is called ***globally stable*** if the whole space is domain of attraction. Otherwise it is called ***locally stable***.

Example 1:

The origin in the first order system of $\dot{x} = -x$ is globally exponentially stable.

$$\dot{x} = -x \Rightarrow x(t) = x_0 e^{-t} \Rightarrow \lim_{t\to\infty} x(t) = 0 \quad \forall\, x_0 \neq 0$$

Example 2:

The origin in the first order system $\dot{x} = -x^3$ is globally asymptotically but not exponentially stable.

$$\dot{x} = -x^3 \Rightarrow x(t) = \frac{x_0}{\sqrt{1+2tx_0^2}} \Rightarrow \lim_{t\to\infty} x(t) = 0 \quad \forall\, x_0$$

Example 3:

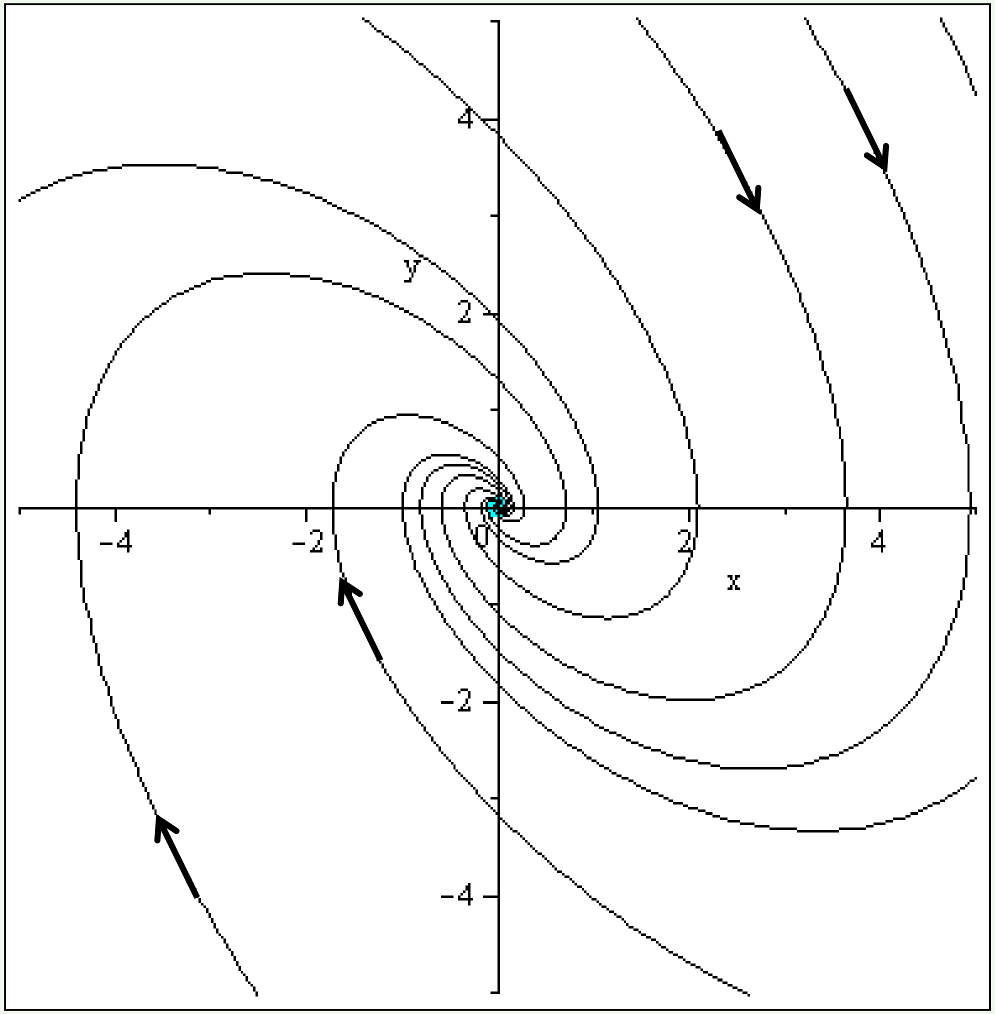
The origin in the first order system $\dot{x} = -x^2$ is semi-asymptotically but not exponentially stable.

$$\dot{x} = -x^2 \Rightarrow x(t) = \frac{x_0}{1+tx_0} \Rightarrow \begin{cases} \lim\limits_{t\to\infty} x(t) = 0 & \text{if} \quad x_0 > 0 \\ \lim\limits_{t\to -1/x_0} x(t) \to \infty & \text{if} \quad x_0 < 0 \end{cases}$$

Domain of attraction is $x_0 > 0$.

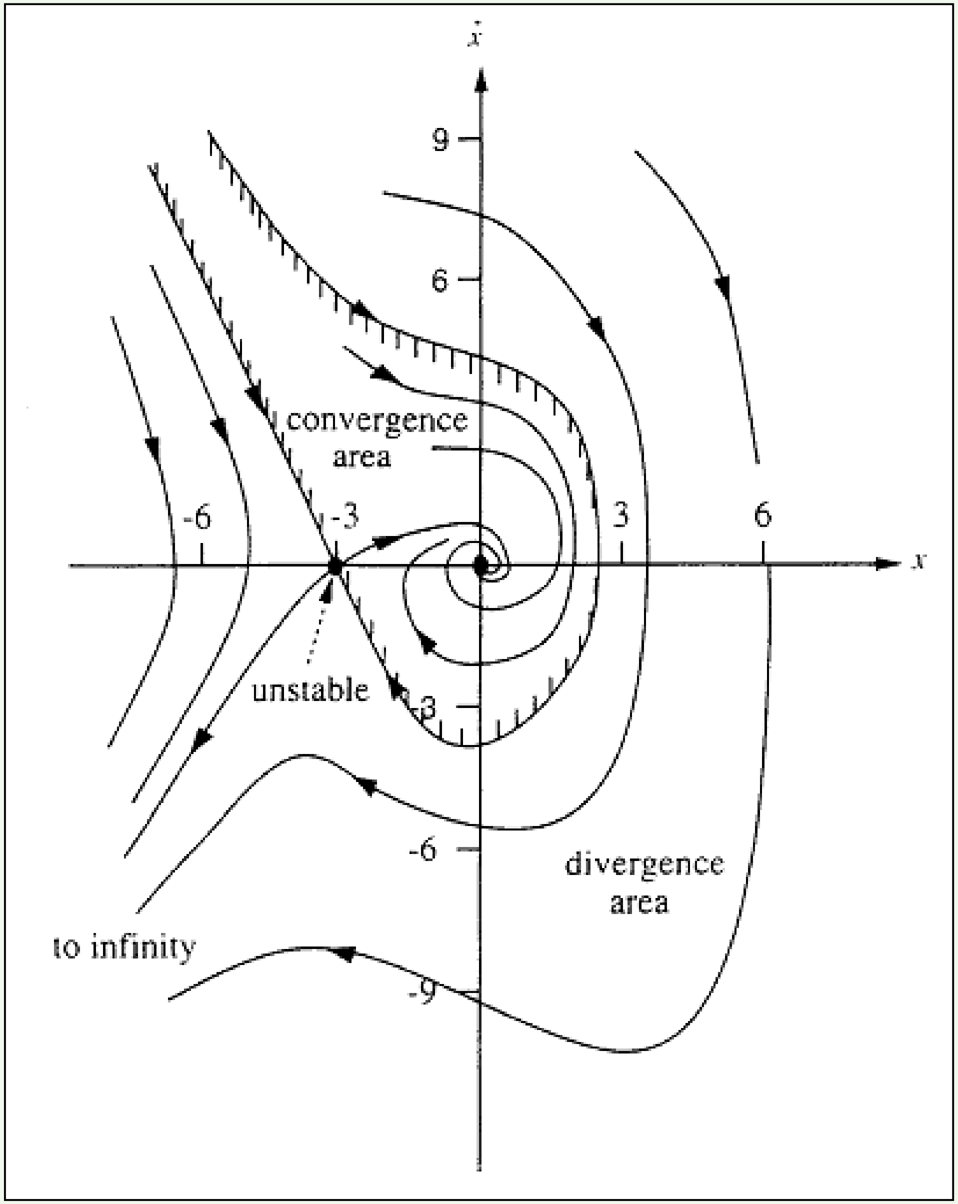
Example 4:

$$\ddot{x} + \dot{x} + x = 0$$



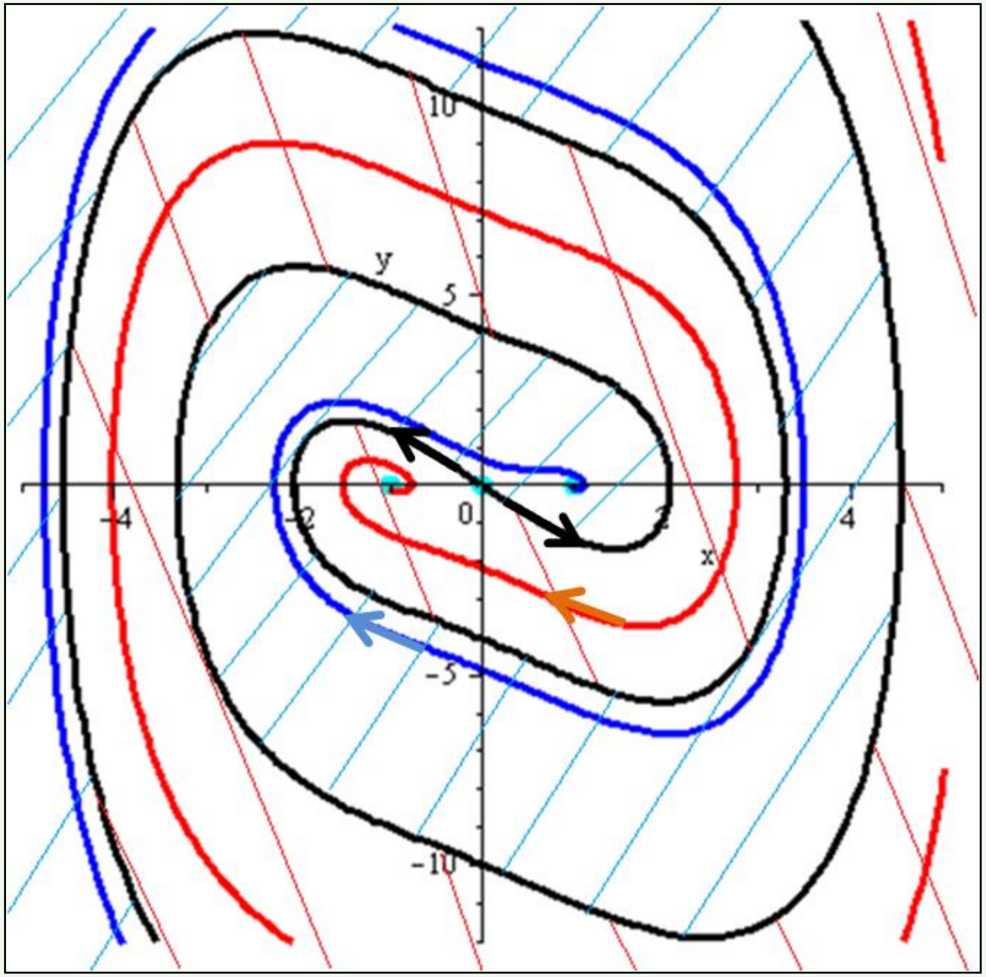
Example 5:

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$



Example 6:

$$\ddot{x} + \dot{x} + x^3 - x = 0$$



Stability Analysis of

Linear Time Invariant Systems

Phase Plane Analysis of LTI Systems

- It is the best tool for study of the linear system graphically
- This analysis gives a very good insight of linear systems behavior
- The analysis can be extended for higher order linear system
- Local behavior of the nonlinear systems can be understood from this analysis
- The analysis is performed based on the system eigenvalues and eigenvectors.

Phase Plane Analysis of LTI Systems

Consider a second order linear system:

$$\dot{x} = Ax \qquad A \in \mathbb{R}^{2\times 2},\ x \in \mathbb{R}^2$$

- If the **A** matrix is nonsingular, origin is the only equilibrium point of the system

$$\mathbf{A} \text{ is non-singular} \Rightarrow \mathbf{x}_e = \mathbf{0}$$

- If the **A** matrix is singular then the system has infinite number of equilibrium points. In fact all of the points belonging to the null space of **A** are the equilibrium point of the system.

$$\mathbf{A} \text{ is singular} \Rightarrow \mathbf{x}_e = \left\{ \mathbf{x}_* \mid \mathbf{x}_* \in Null(\mathbf{A}) \right\}$$

Phase Plane Analysis of LTI Systems

Consider a second order linear system:

$$\dot{x} = Ax$$

$$A \in \mathbb{R}^{2\times 2} \; , x \in \mathbb{R}^2$$

The analytical solution can be obtained based on eigenvalues (λ_1, λ_2):

- If λ_1, λ_2 are **real** and **distinct** $\qquad x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$
- If λ_1, λ_2 are **real** and **similar** $\qquad x(t) = (A + Bt)e^{\lambda t}$
- If λ_1, λ_2 are **complex conjugate** $\qquad x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

$$= e^{\alpha t}(A\sin(\beta t) + B\cos(\beta t))$$

Jordan Form (almost diagonal form)

This representation has the system eigenvalues on the leading diagonal, and either 0 or 1 on the super diagonal.

$$\dot{x} = Ax \quad \Longleftrightarrow \quad \dot{y} = Jy$$

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_n \end{bmatrix} \qquad J_i = \begin{bmatrix} \lambda_i & 1 & \\ & \ddots & 1 \\ & & \lambda_i \end{bmatrix}$$

Obtaining Jordan form:

$$y = P^{-1}x$$

$$J = P^{-1}AP$$

$$P = [v_1 \quad \cdots \quad v_n]$$

A is non-singular:

The A matrix has 2 eigenvalues (either two real, or two complex conjugates) and can have either two eigenvectors or one eigenvectors. Four categories can be realized

1. Two distinct real eigenvalues and two real eigenvectors
2. Two complex conjugate eigenvalues and two complex eigenvectors
3. Two similar (real) eigenvalues and two eigenvectors
4. Two similar (real) eigenvalues and one eigenvectors

1. Two distinct real eigenvalues and two real eigenvectors

$$\dot{y} = Jy \qquad J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\begin{cases} \dot{y}_1 = \lambda_1 y_1 \\ \dot{y}_2 = \lambda_2 y_2 \end{cases} \qquad \begin{aligned} y_1 &= y_{10} e^{\lambda_1 t} \\ y_2 &= y_{20} e^{\lambda_2 t} \end{aligned}$$

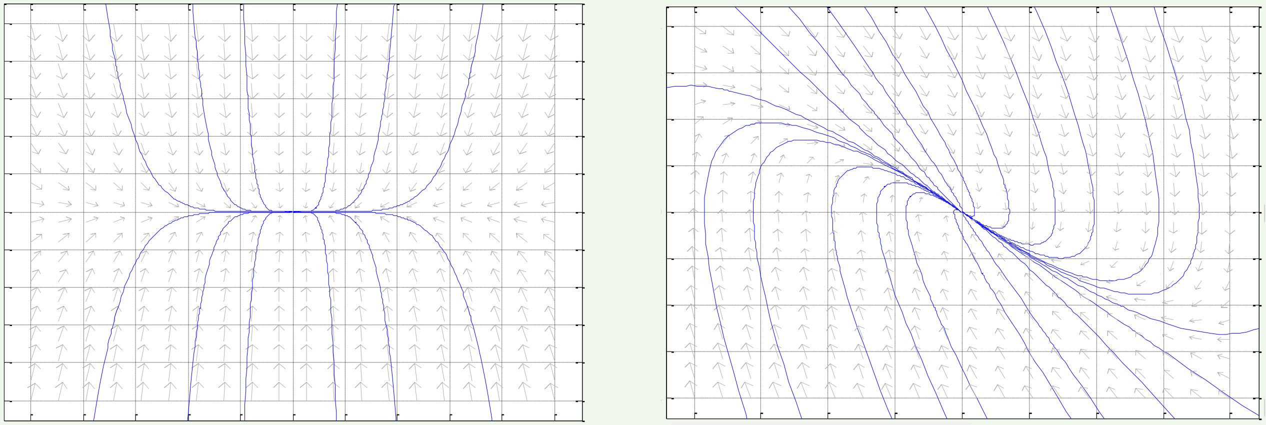
$$\left.\begin{aligned} \ln\left(\frac{y_1}{y_{10}}\right) &= \lambda_1 t \\ \ln\left(\frac{y_2}{y_{20}}\right) &= \lambda_2 t \end{aligned}\right] \longrightarrow \frac{\lambda_2}{\lambda_1} \ln\left(\frac{y_1}{y_{10}}\right) = \ln\left(\frac{y_2}{y_{20}}\right) \qquad \left(\frac{y_1}{y_{10}}\right)^{\frac{\lambda_2}{\lambda_1}} = \frac{y_2}{y_{20}}$$

$$\longrightarrow y_2 = \frac{y_{20}}{{y_{10}}^{\lambda_2/\lambda_1}} {y_1}^{\lambda_2/\lambda_1} \qquad \boxed{y_2 = K {y_1}^{\lambda_2/\lambda_1}}$$

1. A) $\lambda_2 < \lambda_1 < 0$

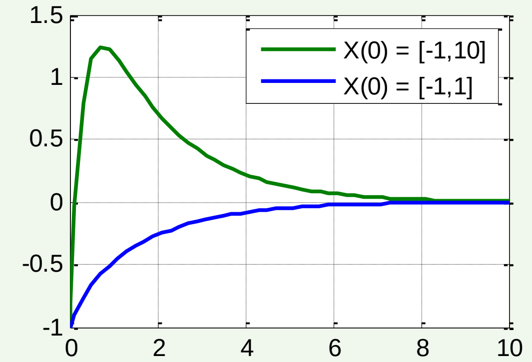
$$y_2 = K {y_1}^{\lambda_2/\lambda_1}$$

- System has two eigenvectors v_1, v_2 the phase plane portrait is as the following



- Trajectories are:
  - ✓ tangent to the slow eigenvector (v_1) for near the origin
  - ✓ parallel to the fast eigenvector (v_2) for far from the origin
- The equilibrium point $X_e = 0$ is called **stable node**

Phase Plane Analysis of LTI Systems



Example 7: $\ddot{x} + 4\dot{x} + 2x = 0$

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix} X$$

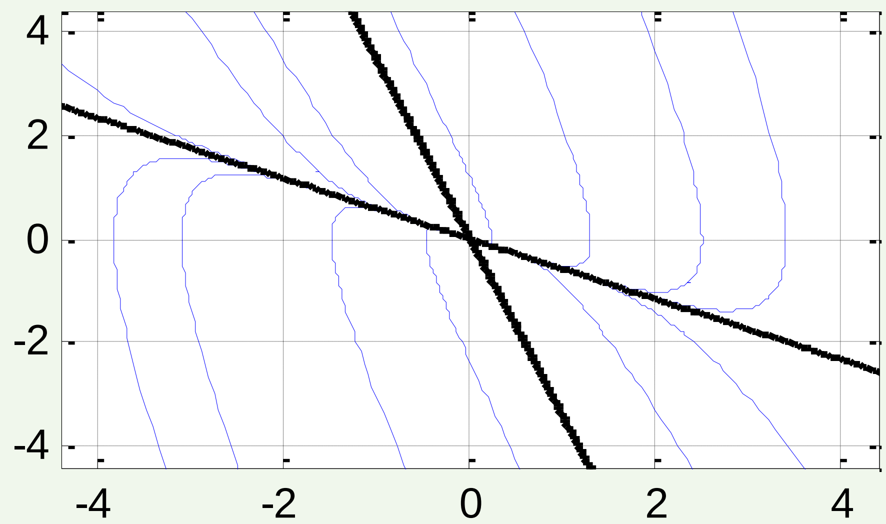
$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix}\right) = \lambda^2 + 4\lambda + 2 = 0 \qquad \lambda_1 = -0.59 \ \ ,\lambda_2 = -3.41$$

$$\begin{bmatrix} -0.59 & -1 \\ 2 & -0.59+4 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

$$\boxed{v_1 = \begin{pmatrix} 0.86 \\ -0.51 \end{pmatrix}}$$

$$\begin{bmatrix} -3.41 & -1 \\ 2 & -3.41+4 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

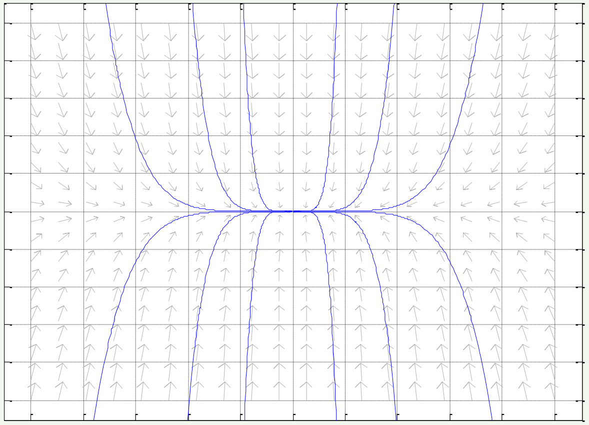
$$\boxed{v_2 = \begin{pmatrix} -0.28 \\ 0.96 \end{pmatrix}}$$

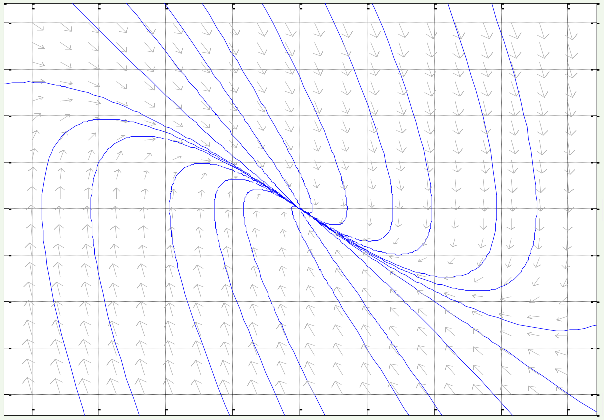


1. B) $\boldsymbol{\lambda_2 > \lambda_1 > 0}$

$$\boxed{y_2 = K {y_1}^{\lambda_2/\lambda_1}}$$

- System has two eigenvectors v_1 and v_2 the phase plane portrait is opposite as the previous one

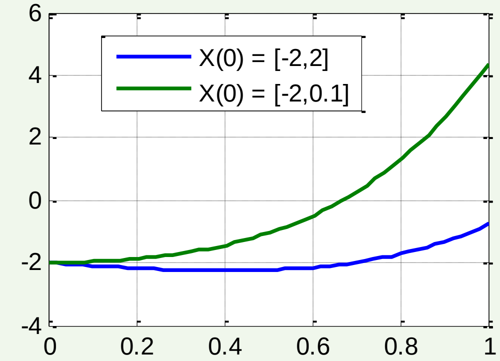




- Trajectories are:
  - ✓ tangent to the slow eigenvector v_1 for near origin
  - ✓ parallel to the fast eigenvector v_2 for far from origin
- The equilibrium point $X_e = 0$ is called **unstable node**

Phase Plane Analysis of LTI Systems

Example 8: $\ddot{x} - 3\dot{x} + 2x = 0$



$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} X$$

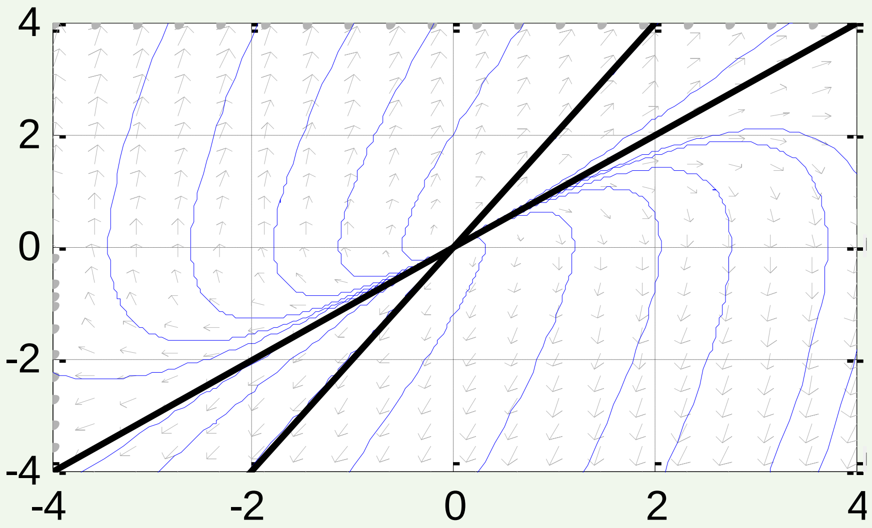
$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\right) = \lambda^2 - 3\lambda + 2 = 0 \qquad \lambda_1 = 1 \ \ , \lambda_2 = 2$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 1-3 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

$$\boxed{v_1 = \begin{pmatrix} -0.71 \\ -0.71 \end{pmatrix}}$$

$$\begin{bmatrix} 2 & -1 \\ 2 & 2-3 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

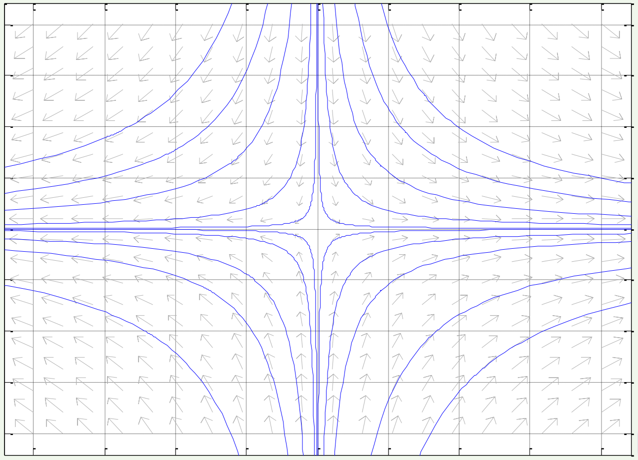
$$\boxed{v_2 = \begin{pmatrix} -0.45 \\ -0.89 \end{pmatrix}}$$

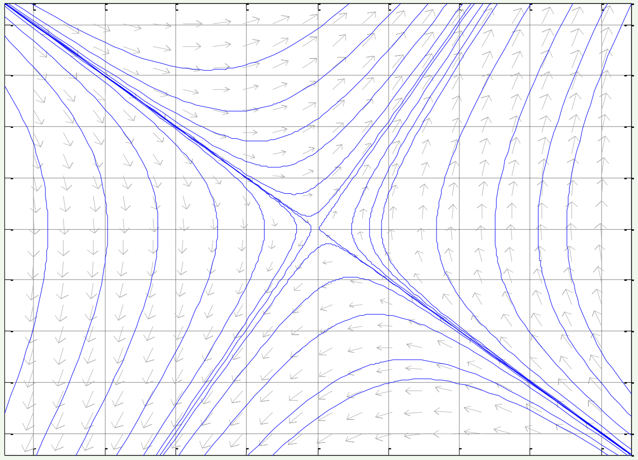


1. C) $\lambda_2 < 0 < \lambda_1$

$$\boxed{y_2 = K {y_1}^{\lambda_2/\lambda_1}}$$

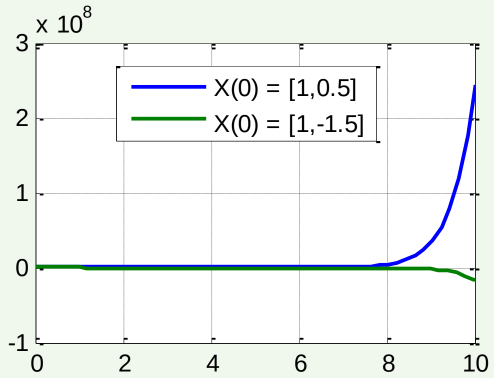
- System has two eigenvectors v_1 and v_2, the phase plane portrait is as the following





- Only trajectories along v_2 are stable trajectories
- All other trajectories at start are tangent to v_2 and at the end are tangent to v_1
- This equilibrium point is unstable and is called **saddle point**

Example 9: $\ddot{x} - \dot{x} - 2x = 0$



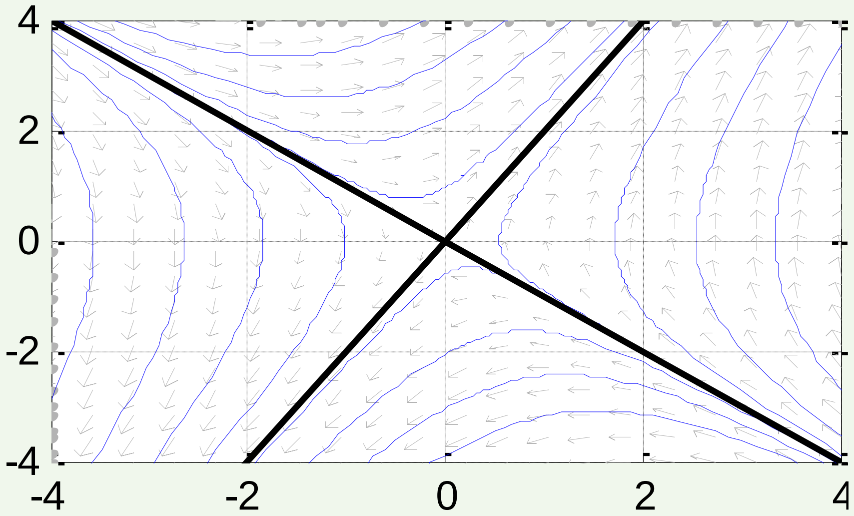
$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} X$$

$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = 0$$

$$\lambda_1 = -1 \quad , \lambda_2 = 2$$

$$\begin{bmatrix} -1 & -1 \\ -2 & -1-1 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

$$\boxed{v_1 = \begin{pmatrix} -0.71 \\ 0.71 \end{pmatrix}}$$



$$\begin{bmatrix} 2 & -1 \\ -2 & 2-1 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

$$\boxed{v_2 = \begin{pmatrix} -0.45 \\ -0.89 \end{pmatrix}}$$

2. Two complex conjugate eigenvalues and two complex eigenvectors

$$\dot{y} = Jy \qquad J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$r \equiv \sqrt{y_1^2 + y_2^2}$$

$$\theta \equiv \tan^{-1}\left(\frac{y_2}{y_1}\right)$$

$$r\dot{r} = y_1\dot{y}_1 + y_2\dot{y}_2 = y_1(\alpha y_1 - \beta y_2) + y_2(\beta y_1 + \alpha y_2) = \alpha r^2$$

$$\dot{\theta}(1 + \tan\theta^2) = \frac{y_1\dot{y}_2 - y_2\dot{y}_1}{y_1^2} = \frac{y_1(\beta y_1 + \alpha y_2) - y_2(\alpha y_1 - \beta y_2)}{y_1^2} = \beta(1 + \tan\theta^2)$$

$$\begin{cases} \dot{r} = \alpha r \\ \dot{\theta} = \beta \end{cases} \longrightarrow \qquad r(t) = r_0 e^{\alpha t} \qquad \theta(t) = \theta_0 + \beta t$$

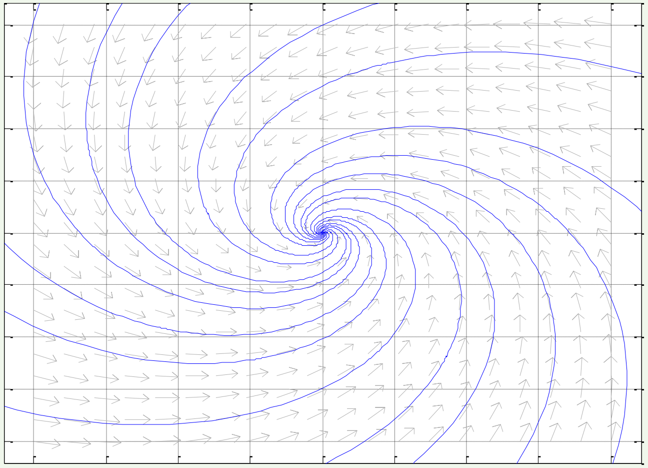
Phase Plane Analysis of LTI Systems

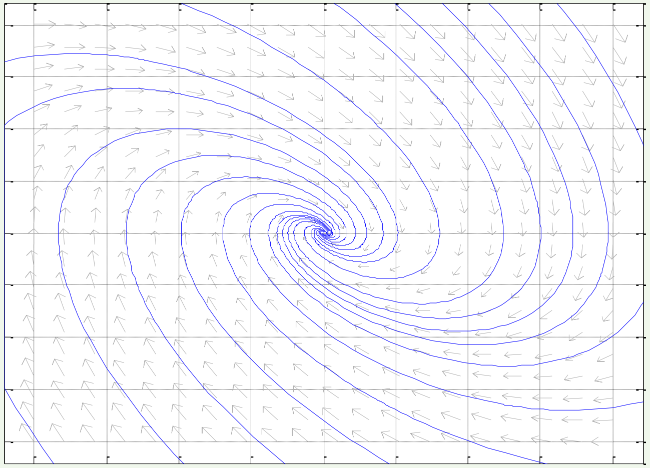
$$2.\ \mathrm{A})\ \boldsymbol{\lambda_2,\ \lambda_1 = \alpha \pm \beta i} \quad , \quad \boldsymbol{\alpha < 0\ ,\beta \neq 0}$$

$$r(t) = r_0 e^{\alpha t}$$

$$\theta(t) = \theta_0 + \beta t$$

- System has no real eigenvectors the phase plane portrait is as the following



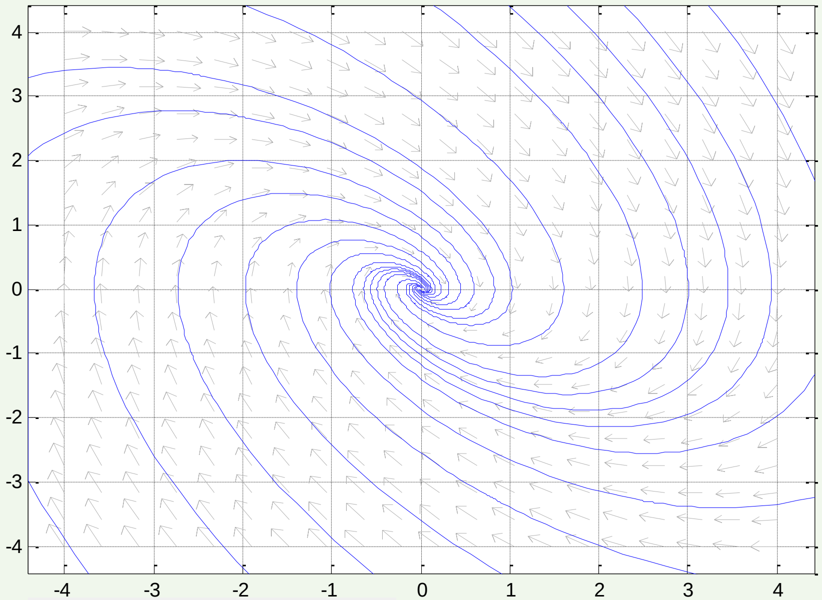


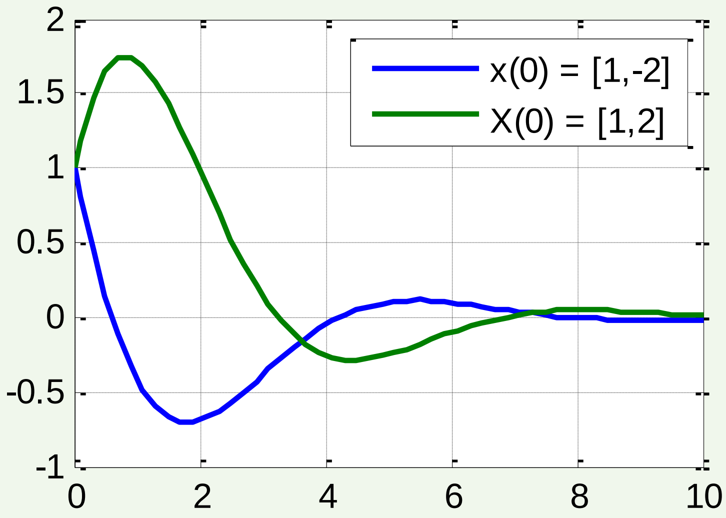
- The trajectories are spiral around the origin and toward the origin.
- This equilibrium point is called **stable focus.**

Example 10: $\ddot{x} + \dot{x} + x = 0$

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} X$$

$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\right) = \lambda^2 + \lambda + 1 = 0 \qquad \lambda_1, \lambda_2 = -0.5 \pm 0.866i$$





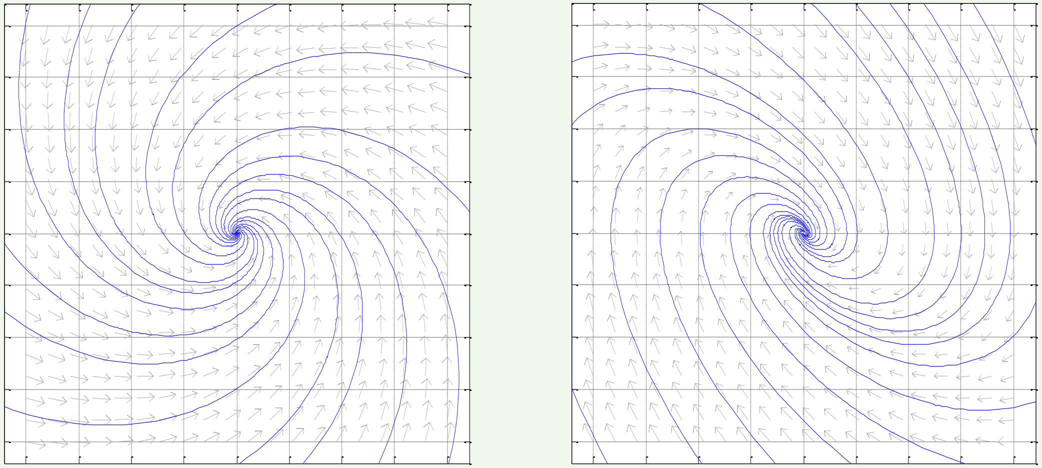
Phase Plane Analysis of LTI Systems

$$2.\ \text{B}\ )\ \boldsymbol{\lambda_2,\ \lambda_1 = \alpha \pm \beta i} \quad , \quad \boldsymbol{\alpha > 0\ , \beta \neq 0}$$

$$r(t) = r_0 e^{\alpha t}$$

$$\theta(t) = \theta_0 + \beta t$$

- System has no real eigenvectors the phase plane portrait is as the following

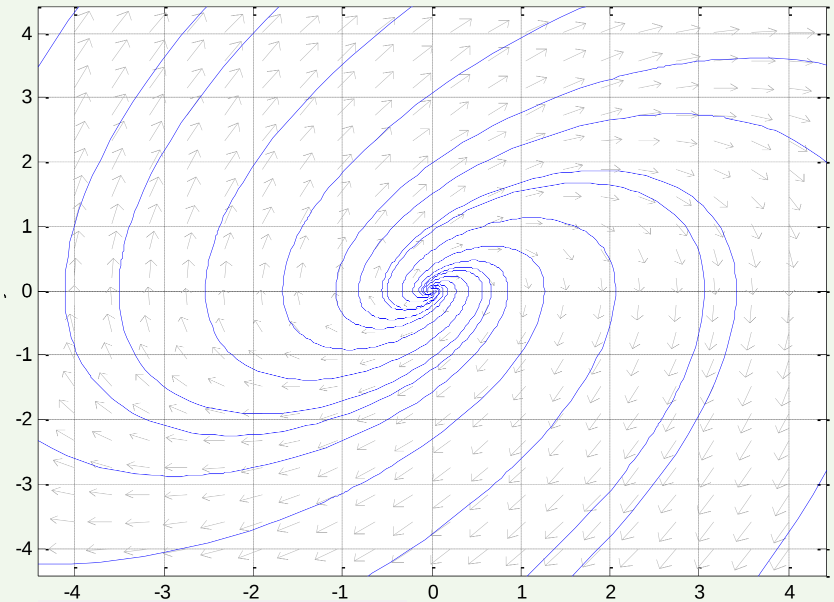


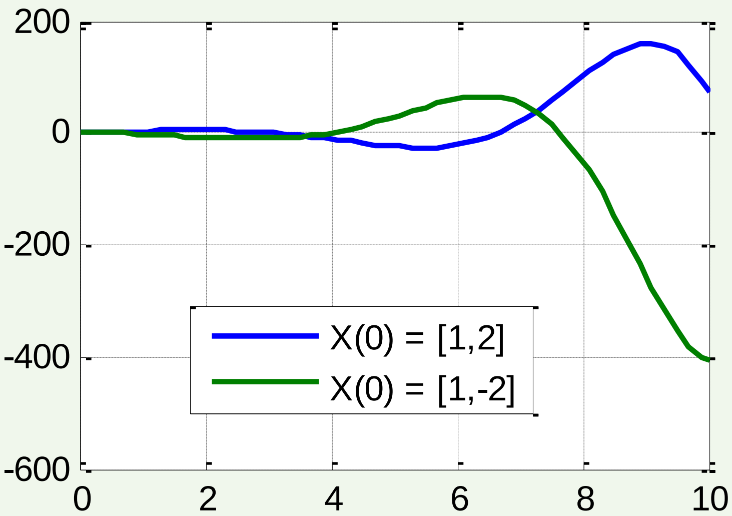
- The trajectories are spiral around the origin and diverge from the origin.
- This equilibrium point is called **unstable focus.**

Example 11: $\ddot{x} - \dot{x} + x = 0$

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} X$$

$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\right) = \lambda^2 - \lambda + 1 = 0 \qquad \lambda_1, \lambda_2 = 0.5 \pm 0.866i$$





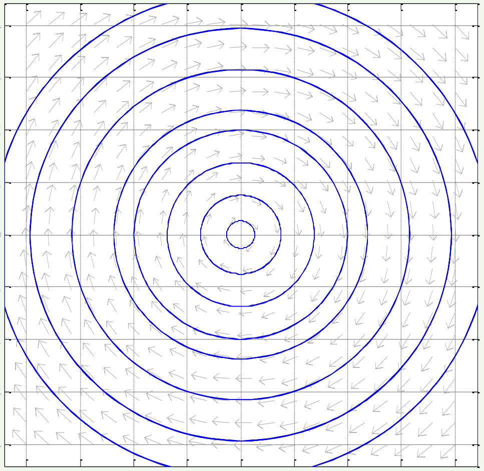
Phase Plane Analysis of LTI Systems

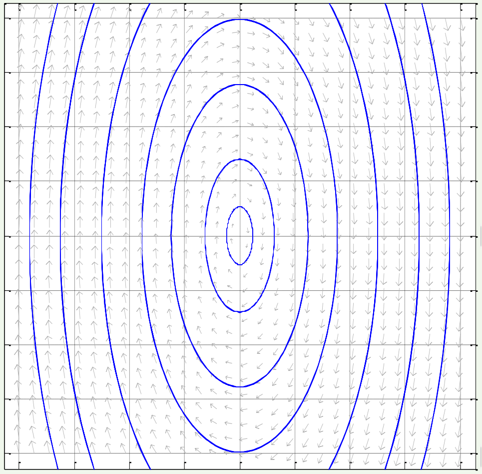
$$2.\ \mathrm{C}\ )\ \boldsymbol{\lambda_2,\ \lambda_1 = \pm\beta i \quad , \quad \alpha = 0\ ,\beta \neq 0}$$

$$r(t) = r_0 e^{\alpha t}$$

$$\theta(t) = \theta_0 + \beta t$$

- System has two imaginary eigenvalues and no real eigenvectors the phase plane portrait is as the following



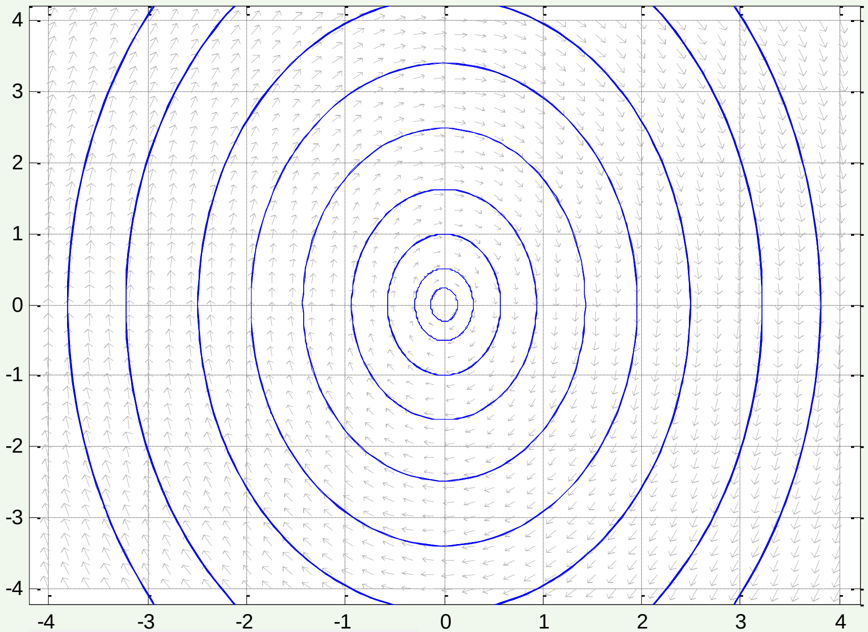


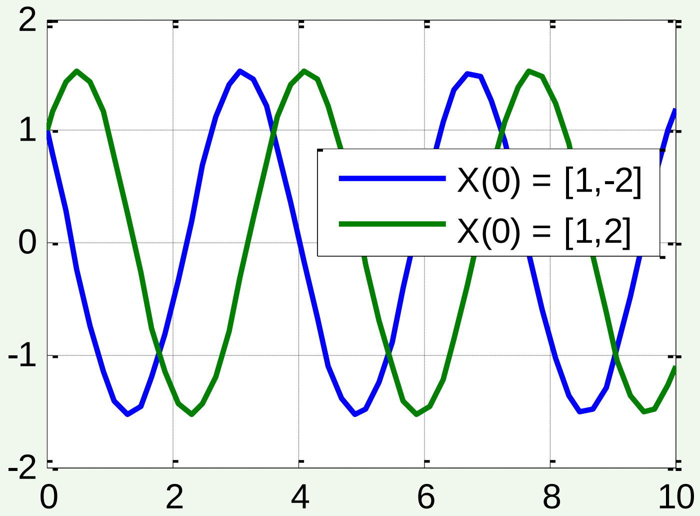
- The trajectories are closed trajectories around the origin.
- This equilibrium point is marginally stable and is called **center.**

Example 12: $\ddot{x} + 3x = 0$

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix} X$$

$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}\right) = \lambda^2 + 3 = 0 \qquad \lambda_1, \lambda_2 = \pm 1.732i$$





3. Two similar (real) eigenvalues and two eigenvectors

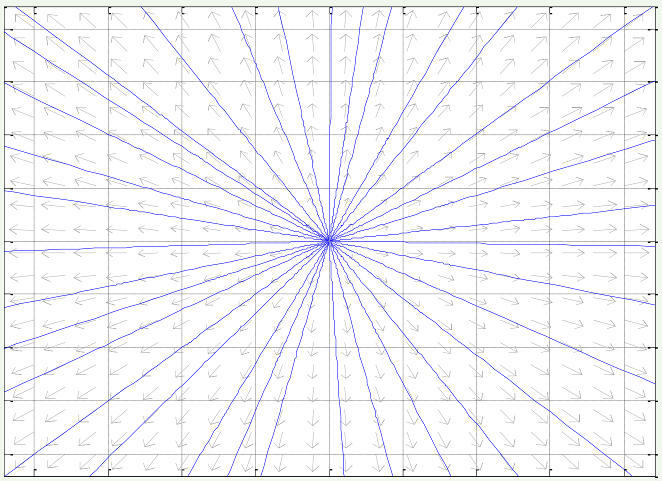
$$\dot{y} = Jy \qquad J = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\begin{cases} \dot{y}_1 = \lambda y_1 \\ \dot{y}_2 = \lambda y_2 \end{cases} \qquad \begin{aligned} y_1 &= y_{10} e^{\lambda t} \\ y_2 &= y_{20} e^{\lambda t} \end{aligned}$$

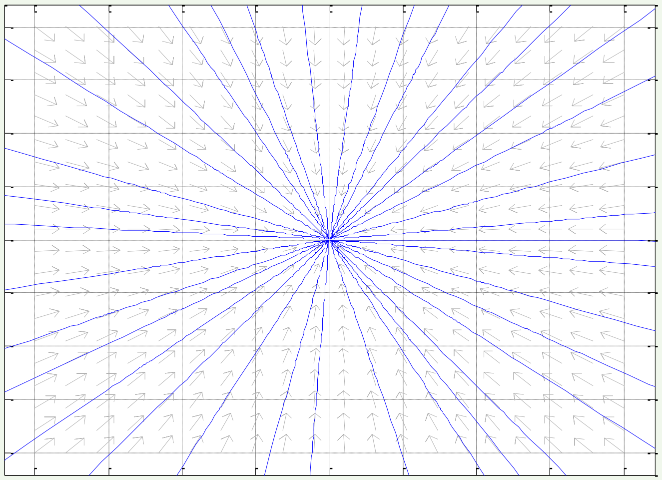
$$\frac{y_1}{y_2} = \frac{y_{10}}{y_{20}} \longrightarrow \boxed{y_2 = K y_1}$$

3) $\lambda_2 = \lambda_1 = \lambda \neq 0$

- System has two similar eigenvalues and two different eigenvectors. The phase plane portrait is as the following, depending to the sign of λ



$\lambda > 0$



$\lambda < 0$

- The trajectories are all along the initial conditions and they are $\lambda < 0$ toward $\lambda > 0$ or outward the origin

4. Two similar (real) eigenvalues and One eigenvectors

$$\dot{y} = Jy \qquad J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

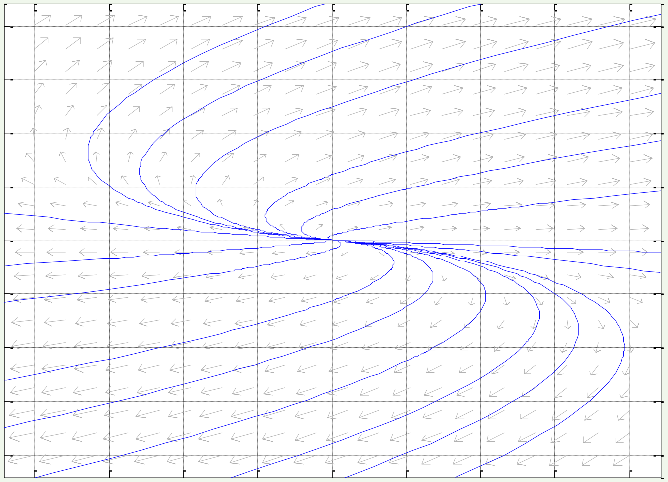
$$\begin{cases} \dot{y}_1 = \lambda y_1 + y_2 \\ \dot{y}_2 = \lambda y_2 \end{cases} \qquad \begin{aligned} y_1 &= y_{10} e^{\lambda t} + y_{20} t e^{\lambda t} \\ y_2 &= y_{20} e^{\lambda t} \end{aligned}$$

$$y_1 = y_{10} \frac{y_2}{y_{20}} + y_2 \frac{1}{\lambda} \ln\left(\frac{y_2}{y_{20}}\right)$$

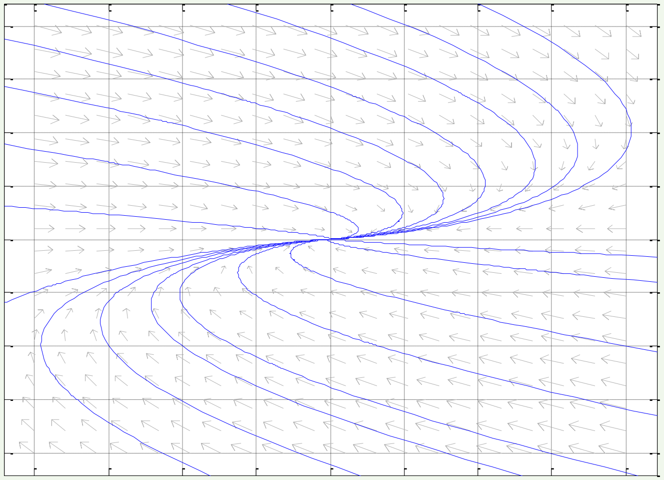
$$\boxed{y_1 = y_2\left(\frac{y_{10}}{y_{20}} + \frac{1}{\lambda} \ln\left(\frac{y_2}{y_{20}}\right)\right)}$$

4) $\lambda_2 = \lambda_1 = \lambda \neq 0$

- System has two similar eigenvalues and only one eigenvector. The phase plane portrait is as the following, depending to the sign of λ



$\lambda > 0$

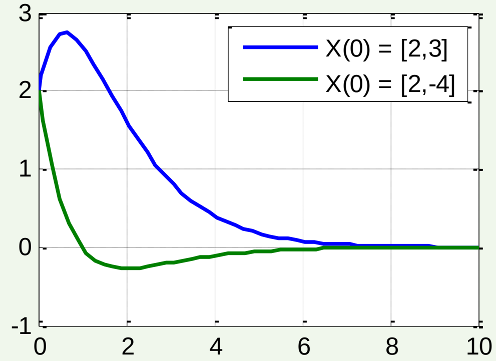


$\lambda < 0$

- The trajectories converge to zero or diverge to infinity along the system eigenvector.

Phase Plane Analysis of LTI Systems

Example 13: $$\ddot{x} + 2\dot{x} + x = 0$$

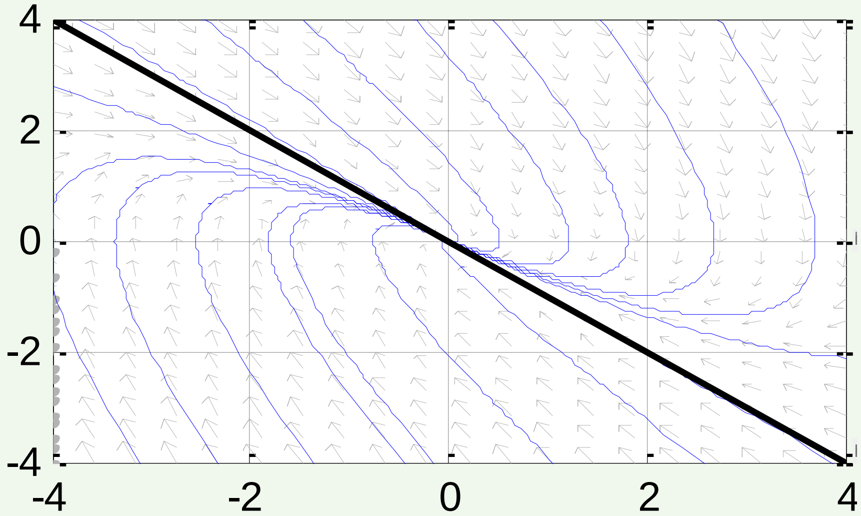


$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} X$$

$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\right) = \lambda^2 + 2\lambda + 1 = 0 \qquad \lambda_1, \lambda_2 = -1$$

$$\begin{bmatrix} -1 & -1 \\ 1 & -1+2 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

$$\boxed{v_1 = \begin{pmatrix} -0.71 \\ 0.71 \end{pmatrix}}$$



A is singular ($\boldsymbol{det(A) = 0}$):

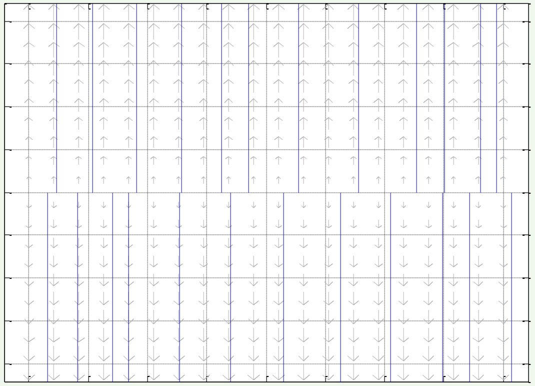
- System has at least one eigenvalue equal to zero and therefore infinite number of equilibrium points. Three different categories can be specified

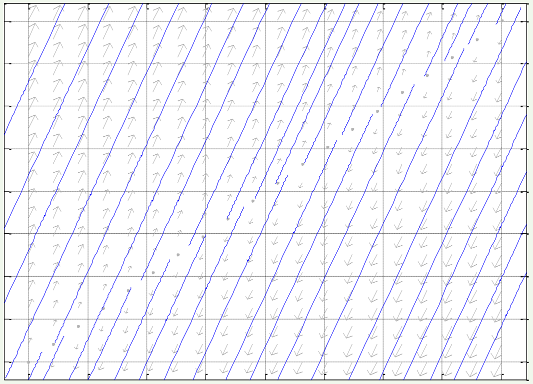
  - $\lambda_1 = 0 \;\;,\;\; \lambda_2 \neq 0$
  - $\lambda_1, \lambda_1 = 0 \;\;,\;\; Rank(A) = 1$
  - $\lambda_1, \lambda_1 = 0 \;\;,\;\; Rank(A) = 0$

Phase Plane Analysis of LTI Systems

1) $\lambda_1 = 0 \quad , \quad \lambda_2 \neq 0$ $\qquad \dot{y} = Jy \qquad J = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$

$$\begin{cases} \dot{y}_1 = 0 \\ \dot{y}_2 = \lambda y_2 \end{cases} \qquad \begin{aligned} y_1 &= y_{10} \\ y_2 &= y_{20} e^{\lambda t} \end{aligned}$$

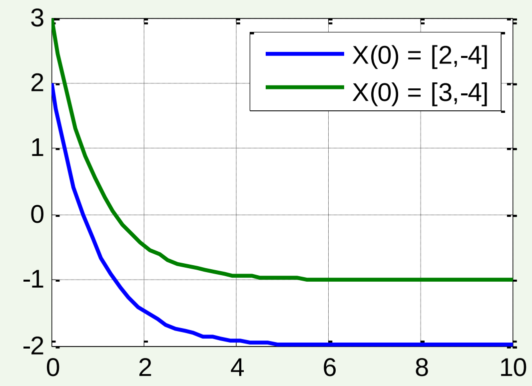




- System has infinite number of non-isolated equilibrium points along a line
- System has two eigenvectors. Eigenvector corresponding to zero eigenvalue is in fact loci of the equilibrium points
- Depending on the sign of the second eigenvalue, all the trajectories move inward or outward to v_1 along v_2

Phase Plane Analysis of LTI Systems

Example 14: $\ddot{x} + \dot{x} = 0$



$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} X$$

$$\det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\right) = \lambda^2 + \lambda = 0$$

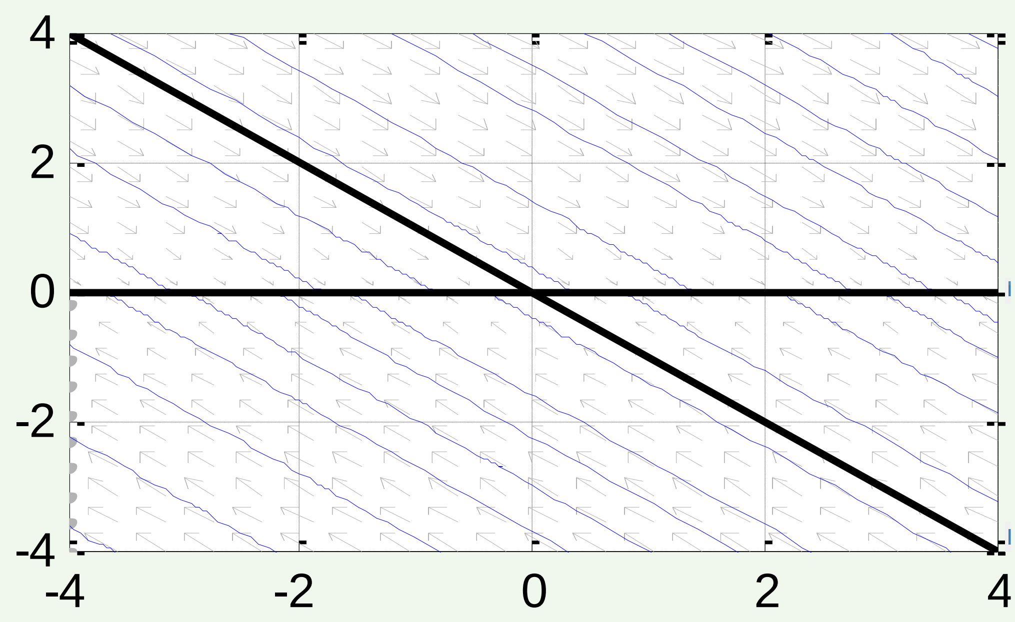
$$\lambda_1 = 0 \quad , \lambda_2 = -1$$

$$\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

$$\boxed{v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$\begin{bmatrix} -1 & -1 \\ 0 & -1+1 \end{bmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = 0$$

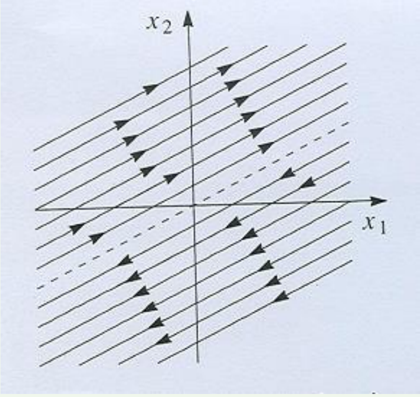
$$\boxed{v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$



Phase Plane Analysis of LTI Systems

2) $\lambda_1, \lambda_1 = 0 \quad , \quad Rank(A) = 1 \qquad\qquad \dot{y} = Jy \qquad J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = 0 \end{cases} \qquad \begin{aligned} y_1 &= y_{20}t + y_{10} \\ y_2 &= y_{20} \end{aligned}$$



- System has infinite number of non-isolated equilibrium points along a line
- System has only one eigenvector, and it is loci of the equilibrium points
- All the trajectories move toward infinity along the system eigenvector (unstable system).

Phase Plane Analysis of LTI Systems

2) $\lambda_1, \lambda_1 = 0 \quad , \quad Rank(A) = 0$

$$\dot{y} = Jy \qquad J = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} \dot{y}_1 = 0 \\ \dot{y}_2 = 0 \end{cases} \qquad \begin{aligned} y_1 &= y_{10} \\ y_2 &= y_{20} \end{aligned}$$

- ❑ System is a static system. All the points are equilibrium points

Summary

Six different type of isolated equilibrium points can be identified

- ❑Stable/unstable node
- ❑Saddle point
- ❑Stable/ unstable focus
- ❑Center

Stability Analysis of Higher Order Systems:

- Analysis and results for the second order LTI system can be extended to higher order LTI system

- Graphical tool is not useful for higher order LTI system except for third order systems.

- This means stability analysis of mechanical system with more than one DOF can not be materialized graphically

- Stability analysis is performed through the eigenvalue analysis of the **A** matrix.

Phase Plane Analysis of LTI Systems

- Consider a linear time invariant (LTI) system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- Origin is the only equilibrium point of the system if **A** is non-singular

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \overset{\det(\mathbf{A})\neq 0}{\Longrightarrow} \mathbf{x}_e = \mathbf{0}$$

- Otherwise the system has infinite number of equilibrium points, all the points on null-space of A are in fact equil. points of the system.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \overset{\det(\mathbf{A})=0}{\Longrightarrow} \mathbf{x}_e = \left\{\mathbf{x}_* \mid \mathbf{x}_* \in \mathrm{Nullspace}(\mathbf{A})\right\}$$

Details for Case of Non-Singular A

- Origin is the only equilibrium point of the system
- This equilibrium point (system) is
  - Exponentially stable if all eigenvalues of **A** are either real negative or complex with negative real part.
  - Marginally stable if eigenvalues of **A** have non-positive real part and $rank(A-\lambda I)=n-r$ for all repeated imaginary eigenvalues, λ with multiplicity of r
  - Unstable, otherwise.

A is Non-Singular

- Classification of the equilibrium point of higher order system into node, focus, and saddle point is not as easy as second order system. However some points can be emphasized:
  - ✓ The equilibrium point is **stable/unstable node** if all eigenvalues are real and have the negative/positive sign.
  - ✓ The equilibrium point is **center** if a pair of eigenvalues are pure imaginary complex conjugate and all other eigenvalues have negative real
  - ✓ In the case of different sign in the real part of the eigenvalues trajectories have the **saddle** type behavior near the equilibrium point

Phase Plane Analysis of LTI Systems

- ✓Trajectories are along the eigenvector with minimum absolute real part near the equilibrium point and along the eigenvector with maximum absolute real part.
- ✓Trajectories have **spiral** behavior if there exist some complex (obviously conjugates) eigenvalues.
- ✓Spiral behavior is toward/outward depending on the sign (negative and positive) of real part of the complex conjugate eigenvalues.

❑These concepts can be visualized and better understood in three dimensional case

Lyapunov Indirect Method in

Stability Analysis of

Nonlinear Systems

Phase Plane Analysis of LTI Systems

- There are two conventional approaches in the stability analysis of nonlinear systems:

  - ✓ Lyapunov direct method

  - ✓ Lyapunov indirect method or linearization approach

- The direct method analyzes stability of the system (equilibrium point) using the nonlinear equations of the system

- The indirect method analyzes the system stability using the linearized equations about the equilibrium point.

Phase Plane Analysis of LTI Systems

Motivation:

A nonlinear system near its equilibrium point behaves like a linear:

- Nonlinear system: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
- Equilibrium point: $\mathbf{f}(\mathbf{x}) = 0 \Rightarrow \mathbf{x}_e$
- Motion about equilibrium point: $\mathbf{x} = \mathbf{x}_e + \hat{\mathbf{x}}$
- Linearized motion:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \Rightarrow \dot{\hat{\mathbf{x}}} = \mathbf{f}(\mathbf{x}_e + \hat{\mathbf{x}}) = \cancel{\mathbf{f}(\mathbf{x}_e)} + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_e} \hat{\mathbf{x}} + \mathrm{H.O.T} \Rightarrow \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}}$$

- It means near x_e :

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad \square \quad \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} \xRightarrow{\text{if } \mathbf{x}_e = \mathbf{0}} \mathbf{x} \approx \hat{\mathbf{x}}$$

- This means stability of the equilibrium point may be studied through the stability analysis of the linearized system.

- This is the base of the Lyapunov Indirect Method

- **Example:** in the nonlinear second order system

$$\dot{x}_1 = x_2^2 + x_1 \cos x_2$$
$$\dot{x}_2 = x_2 + (1 + x_1) x_1 + x_1 \sin x_2$$

origin is the equilibrium point and the linearized system is given by

$$\begin{Bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{Bmatrix} = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]_{\mathbf{x}_e = 0} \begin{Bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{Bmatrix} \Rightarrow \begin{cases} \dot{\hat{x}}_1 = \hat{x}_1 \\ \dot{\hat{x}}_2 = \hat{x}_1 + \hat{x}_2 \end{cases}$$

Theorem (Lyapunov Linearization Method):

- If the linearized system is strictly stable (i.e. all eigenvalues of **A** are strictly in the left half complex plane) then the equilibrium point in the original nonlinear system is asymptotically stable.

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} \ \text{ is strictly stable} \Rightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \ \text{ is asymptotically stable}$$

- If the linearized system is unstable (i.e. in the case of right half plane eigenvalue(s) or repeated eigenvalues on the imaginary axis with geometrical deficiency $(r > n - rank(\lambda I - A))$, then the equilibrium point in the original nonlinear system is unstable.

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} \ \text{ is unstable} \Rightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \ \text{ is unstable}$$

Theorem (Lyapunov Linearization Method):

- If the linearized system is marginally stable (i.e. all eigenvalues of **A** are in the left half complex plane and eigenvalues on the imaginary axis have no geometrical deficiency) then one cannot conclude anything from the linear approximation. The equilibrium point in the original nonlinear system may be stable, asymptotically stable, or unstable.

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} \quad \text{is marginally stable} \Rightarrow \begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) & \text{is asymptotically stable} \\ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) & \text{is marginally stable} \\ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) & \text{is unstable} \end{cases}$$

- The Lyapunov linearized approximation method only talks about the local stability of the nonlinear system, if anything can be concluded.

Example 15:

- The nonlinear system $\dot{x} = ax + bx^5$ is
  - ✓ Asymptotically stable if $a < 0$
  - ✓ Unstable if $a > 0$
  - ✓ No conclusion from linear approximation can be drawn if
- The origin in the nonlinear second order system

$$\dot{x}_1 = x_2^2 + x_1 \cos x_2$$
$$\dot{x}_2 = x_2 + (1 + x_1)x_1 + x_1 \sin x_2$$

- is unstable because the linearized system $\begin{Bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{Bmatrix}$ is unstable