3 Lebesgue Measure

1 Introduction

The length $l(I)$ of an interval I is defined, as usual, to be the difference of the endpoints of the interval. Length is an example of a *set function*, that is, a function which associates an extended real number to each set in some collection of sets. In the case of length the domain is the collection of all intervals. We should like to extend the notion of length to more complicated sets than intervals. For instance, we could define the "length" of an open set to be the sum of the lengths of the open intervals of which it is composed. Since the class of open sets is still too restricted for our purposes, we would like to construct a set function m which assigns to each set E in some collection $\mathfrak{M}$ of sets of real numbers a nonnegative extended real number mE called the measure of E. Ideally, we should like m to have the following properties:

i. mE is defined for each set E of real numbers; that is, $\mathfrak{M} = \mathcal{P}(R)$;
ii. for an interval I, $mI = l(I)$;
iii. if $\langle E_n \rangle$ is a sequence of disjoint sets (for which m is defined), $m(\bigcup E_n) = \sum mE_n$;
iv. m is translation invariant; that is, if E is a set for which m is defined and if $E + y$ is the set $\{x + y: x \in E\}$, obtained by replacing each point x in E by the point $x + y$, then

$$m(E + y) = mE.$$

Unfortunately, as we shall see in Section 4, it is impossible to construct a set function having all four of these properties, and it is not known whether there is a set function satisfying the first three properties.[1] Consequently, one of these properties must be weakened, and it is most useful to retain the last three properties and to weaken the first condition so that mE need not be defined for all sets E of real numbers.[2] We shall want mE to be defined for as many sets as possible and will find it convenient to require the family $\mathfrak{M}$ of sets for which m is defined to be a σ-algebra. Thus we shall say that m is a **countably additive measure** if it is a nonnegative extended real-valued function whose domain of definition is a σ-algebra $\mathfrak{M}$ of sets (of real numbers) and we have $m(\bigcup E_n) = \sum mE_n$ for each sequence $\langle E_n \rangle$ of disjoint sets in $\mathfrak{M}$. Our goal in the next two sections will be the construction of a countably additive measure which is translation invariant and has the property that $mI = l(I)$ for each interval I.

Problems

Let m be a countably additive measure defined for all sets in a σ-algebra $\mathfrak{M}$.

1. If A and B are two sets in $\mathfrak{M}$ with $A \subset B$, then $mA \leq mB$. This property is called *monotonicity*.

2. Let $\langle E_n \rangle$ be any sequence of sets in $\mathfrak{M}$. Then $m(\bigcup E_n) \leq \sum mE_n$. [Hint: Use Proposition 1.2.] This property of a measure is called *countable subadditivity*.

3. If there is a set A in $\mathfrak{M}$ such that $mA < \infty$, then $m\varnothing = 0$.

4. Let nE be ∞ for an infinite set E and be equal to the number of elements in E for a finite set. Show that n is a countably additive set function which is translation invariant and defined for all sets of real numbers. This measure is called the **counting measure.**

[1] If we assume the continuum hypothesis (that every noncountable set of real numbers can be put in one-to-one correspondence with the set of all real numbers), then such a measure is impossible.

[2] Weakening property (*i*) is not the only approach; it is also possible to replace property (*iii*) of countable additivity by the weaker property of finite additivity: for each finite sequence $\langle E_n \rangle$ of disjoint sets we have $m(\bigcup E_n) = \sum mE_n$ (see Problem 10.21). Another possible alternative to property (*iii*) is countable subadditivity, which is satisfied by the outer measure we construct in the next section (see Problem 2).

2 Outer Measure

For each set A of real numbers consider the countable collections $\{I_n\}$ of open intervals which cover A, that is, collections for which $A \subset \bigcup I_n$, and for each such collection consider the sum of the lengths of the intervals in the collection. Since the lengths are positive numbers, this sum is uniquely defined independently of the order of the terms. We define the **outer measure**[3] m^*A of A to be the infimum of all such sums. In an abbreviated notation

$$m^*A = \inf_{A \subset \cup I_n} \sum l(I_n).$$

It follows immediately from the definition of m^* that $m^*\varnothing = 0$ and that if $A \subset B$, then $m^*A \leq m^*B$. Also each set consisting of a single point has outer measure zero. We establish two propositions concerning outer measure:

1. Proposition: *The outer measure of an interval is its length.*

Proof: We begin with the case in which we have a closed finite interval, say $[a, b]$. Since the open interval $(a - \epsilon, b + \epsilon)$ contains $[a, b]$ for each positive ϵ, we have $m^*[a, b] \leq l(a - \epsilon, b + \epsilon) = b - a + 2\epsilon$. Since $m^*[a, b] \leq b - a + 2\epsilon$ for each positive ϵ, we must have $m^*[a, b] \leq b - a$. Thus we have only to show that $m^*[a, b] \geq b - a$. But this is equivalent to showing that if $\{I_n\}$ is any countable collection of open intervals covering $[a, b]$, then

$$\sum l(I_n) \geq b - a. \tag{1}$$

By the Heine-Borel theorem, any collection of open intervals covering $[a, b]$ contains a finite subcollection which also covers $[a, b]$, and since the sum of the lengths of the finite subcollection is no greater than the sum of the lengths of the original collection, it suffices to prove the inequality (1) for finite collections $\{I_n\}$ which cover $[a, b]$. Since a is contained in $\bigcup I_n$, there must be one of the I_n's which contains a. Let this be the interval (a_1, b_1). We have $a_1 < a < b_1$. If $b_1 \leq b$, then $b_1 \in [a, b]$, and since $b_1 \notin (a_1, b_1)$, there must be an

[3] In order to distinguish this outer measure from the more general outer measures to be considered in Chapter 12, we call this outer measure *Lebesgue* outer measure, after Henri Lebesgue. Since we consider no other outer measure in this chapter, we refer to m^* simply as outer measure.

interval (a_2, b_2) in the collection $\{I_n\}$ such that $b_1 \in (a_2, b_2)$; that is, $a_2 < b_1 < b_2$. Continuing in this fashion we obtain a sequence $(a_1, b_1), \ldots, (a_k, b_k)$ from the collection $\{I_n\}$ such that $a_i < b_{i-1} < b_i$.

Since $\{I_n\}$ is a finite collection, our process must terminate with some interval (a_k, b_k). But it terminates only if $b \in (a_k, b_k)$, that is, if $a_k < b < b_k$. Thus

$$\begin{aligned}\sum l(I_n) &\geq \sum l(a_i, b_i)\\ &= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \ldots + (b_1 - a_1)\\ &= b_k - (a_k - b_{k-1}) - (a_{k-1} - b_{k-2})\\ &\quad - \ldots - (a_2 - b_1) - a_1 > b_k - a_1,\end{aligned}$$

since $a_i < b_{i-1}$. But $b_k > b$ and $a_1 < a$, and so we have $b_k - a_1 > b - a$, whence $\sum l(I_n) > (b - a)$. This shows that $m^*[a, b] = b - a$.

If I is any finite interval, then, given $\epsilon > 0$, there is a closed interval $J \subset I$ such that $l(J) > l(I) - \epsilon$. Hence

$$l(I) - \epsilon < l(J) = m^*J \leq m^*I \leq m^*\bar{I} = l(\bar{I}) = l(I).$$

Thus for each $\epsilon > 0$,

$$l(I) - \epsilon < m^*I \leq l(I),$$

and so $m^*I = l(I)$.

If I is an infinite interval, then given any real number Δ, there is a closed interval $J \subset I$ with $l(J) = \Delta$. Hence $m^*I \geq m^*J = l(J) = \Delta$. Since $m^*I \geq \Delta$ for each Δ, $m^*I = \infty = l(I)$. ∎

2. Proposition: *Let $\{A_n\}$ be a countable collection of sets of real numbers. Then*

$$m^*\left(\bigcup A_n\right) \leq \sum m^*A_n.$$

Proof: If one of the sets A_n has infinite outer measure, the inequality holds trivially. If m^*A_n is finite, then, given $\epsilon > 0$, there is a countable collection $\{I_{n,i}\}_i$ of open intervals such that $A_n \subset \bigcup_i I_{n,i}$ and $\sum_i l(I_{n,i}) < m^*A_n + 2^{-n}\epsilon$. Now the collection $\{I_{n,i}\}_{n,i} = \bigcup_n \{I_{n,i}\}_i$ is countable, being the union of a countable number of countable collections, and covers $\bigcup A_n$. Thus

$$m^*\left(\bigcup A_n\right) \leq \sum_{n,i} l(I_{n,i}) = \sum_n \sum_i l(I_{n,i}) < \sum_n (m^*A_n + \epsilon 2^{-n})$$
$$= \sum m^*A_n + \epsilon.$$

Since ϵ was an arbitrary positive number,

$$m^*\left(\bigcup A_n\right) \leq \sum m^*A_n. \blacksquare$$

3. Corollary: *If A is countable, $m^*A = 0$.*

4. Corollary: *The set* $[0, 1]$ *is not countable.*

5. Proposition: *Given any set A and any $\epsilon > 0$, there is an open set O such that $A \subset O$ and $m^*O \leq m^*A + \epsilon$. There is a $G \in \mathcal{G}_\delta$ such that $A \subset G$ and $m^*A = m^*G$.*

Problems

5. Let A be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a *finite* collection of open intervals covering A. Then $\sum l(I_n) \geq 1$.
6. Prove Proposition 5.
7. Prove that m^* is translation invariant.
8. Prove that if $m^*A = 0$, then $m^*(A \cup B) = m^*B$.

3 Measurable Sets and Lebesgue Measure

While outer measure has the advantage that it is defined for all sets, it is not countably additive. It becomes countably additive, however, if we suitably reduce the family of sets on which it is defined. Perhaps the best way of doing this is to use the following definition due to Carathéodory:

Definition: *A set E is said to be* ***measurable***[4] *if for each set A we have $m^*A = m^*(A \cap E) + m^*(A \cap \tilde{E})$.*

Since we always have $m^*A \leq m^*(A \cap E) + m^*(A \cap \tilde{E})$, we see that E is measurable if (and only if) for each A we have $m^*A \geq$

[4] In the present case m^* is Lebesgue outer measure, and we say E is *Lebesgue* measurable. More general notions of measurable set are considered in Chapters 11 and 12.

$m^*(A \cap E) + m^*(A \cap \tilde{E})$. Since the definition of measurability is symmetric in E and $\tilde{E}$, we have $\tilde{E}$ measurable whenever E is. Clearly $\varnothing$ and the set $\mathbf{R}$ of all real numbers are measurable.

6. Lemma: *If $m^*E = 0$, then E is measurable.*

Proof: Let A be any set. Then $A \cap E \subset E$, and so $m^*(A \cap E) \leq m^*E = 0$. Also $A \supset A \cap \tilde{E}$, and so

$$m^*A \geq m^*(A \cap \tilde{E}) = m^*(A \cap \tilde{E}) + m^*(A \cap E),$$

and therefore E is measurable. ∎

7. Lemma: *If E_1 and E_2 are measurable, so is $E_1 \cup E_2$.*

Proof: Let A be any set. Since E_2 is measurable, we have

$$m^*(A \cap \tilde{E}_1) = m^*(A \cap \tilde{E}_1 \cap E_2) + m^*(A \cap \tilde{E}_1 \cap \tilde{E}_2),$$

and since $A \cap (E_1 \cup E_2) = [A \cap E_1] \cup [A \cap E_2 \cap \tilde{E}_1]$, we have

$$m^*(A \cap [E_1 \cup E_2]) \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap \tilde{E}_1).$$

Thus

$$\begin{aligned} m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap \tilde{E}_1 \cap \tilde{E}_2) &\leq m^*(A \cap E_1) \\ &\quad + m^*(A \cap E_2 \cap \tilde{E}_1) + m^*(A \cap \tilde{E}_1 \cap \tilde{E}_2) \\ &= m^*(A \cap E_1) + m^*(A \cap \tilde{E}_1) = m^*A, \end{aligned}$$

by the measurability of E_1. Since $\sim(E_1 \cup E_2) = \tilde{E}_1 \cap \tilde{E}_2$, this shows that $E_1 \cup E_2$ is measurable. ∎

8. Corollary: *The family $\mathfrak{M}$ of measurable sets is an algebra of sets.*

9. Lemma: *Let A be any set, and $E_1, \ldots, E_n$ a finite sequence of disjoint measurable sets. Then*

$$m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

Proof: We prove the lemma by induction on n. It is clearly true for $n = 1$, and we assume it is true if we have $n - 1$ sets E_i. Since the E_i are disjoint sets, we have

$$A \cap \left[\bigcup_{i=1}^{n} E_i\right] \cap E_n = A \cap E_n$$

and

$$A \cap \left[\bigcup_{i=1}^{n} E_i\right] \cap \tilde{E}_n = A \cap \left[\bigcup_{i=1}^{n-1} E_i\right].$$

Hence the measurability of E_n implies

$$m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_{i=1}^{n-1} E_i\right]\right)$$

$$= m^*(A \cap E_n) + \sum_{i=1}^{n-1} m^*(A \cap E_i)$$

by our assumption of the lemma for $n - 1$ sets. ▮

10. Theorem: *The collection $\mathfrak{M}$ of measurable sets is a σ-algebra; that is, the complement of a measurable set is measurable and the union (and intersection) of a countable collection of measurable sets is measurable. Moreover, every set with outer measure zero is measurable.*

Proof: We have already observed that $\mathfrak{M}$ is an algebra of sets, and so we have only to prove that if a set E is the union of a countable collection of measurable sets it is measurable. By Proposition 1.2 such an E must be the union of a sequence $\langle E_n \rangle$ of pairwise disjoint measurable sets. Let A be any set, and let $F_n = \bigcup_{i=1}^{n} E_i$. Then F_n is measurable, and $\tilde{F}_n \supset \tilde{E}$. Hence

$$m^*A = m^*(A \cap F_n) + m^*(A \cap \tilde{F}_n) \geq m^*(A \cap F_n) + m^*(A \cap \tilde{E}).$$

By Lemma 9

$$m^*(A \cap F_n) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

Thus

$$m^*A \geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap \tilde{E}).$$

Since the left side of this inequality is independent of n, we have

$$m^*A \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$$
$$\geq m^*(A \cap E) + m^*(A \cap \tilde{E})$$

by the countable subadditivity of m^*. ∎

11. Lemma: *The interval (a, ∞) is measurable.*

Proof: Let A be any set, $A_1 = A \cap (a, \infty)$, $A_2 = A \cap (-\infty, a]$. Then we must show $m^*A_1 + m^*A_2 \leq m^*A$. If $m^*A = \infty$, then there is nothing to prove. If $m^*A < \infty$, then, given $\epsilon > 0$, there is a countable collection $\{I_n\}$ of open intervals which cover A and for which

$$\sum l(I_n) \leq m^*A + \epsilon.$$

Let $I_n' = I_n \cap (a, \infty)$ and $I_n'' = I_n \cap (-\infty, a]$. Then I_n' and I_n'' are intervals (or empty) and

$$l(I_n) = l(I_n') + l(I_n'') = m^*I_n' + m^*I_n''.$$

Since $A_1 \subset \bigcup I_n'$, we have

$$m^*A_1 \leq m^*\left(\bigcup I_n'\right) \leq \sum m^*I_n',$$

and, since $A_2 \subset \bigcup I_n''$, we have

$$m^*A_2 \leq m^*\left(\bigcup I_n''\right) \leq \sum m^*I_n''.$$

Thus

$$m^*A_1 + m^*A_2 \leq \sum (m^*I_n' + m^*I_n'')$$
$$\leq \sum l(I_n) \leq m^*A + \epsilon.$$

But ϵ was an arbitrary positive number, and so we must have $m^*A_1 + m^*A_2 \leq m^*A$. ∎

12. Theorem: *Every Borel set is measurable. In particular each open set and each closed set is measurable.*

Proof: Since the collection $\mathfrak{M}$ of measurable sets is a σ-algebra, we have $(-\infty, a]$ measurable for each a since $(-\infty, a] = \sim(a, \infty)$. Since $(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - 1/n]$, we have $(-\infty, b)$ measurable. Hence each open interval $(a, b) = (-\infty, b) \cap (a, \infty)$ is measurable. But each open set is the union of a countable number of open intervals and so must be measurable. Thus $\mathfrak{M}$ is a σ-algebra containing the open sets and must therefore contain the family $\mathcal{B}$ of Borel sets, since $\mathcal{B}$ is the smallest σ-algebra containing the open sets. [*Note:* The theorem also follows immediately from the fact that $\mathfrak{M}$ is a σ-algebra containing each interval of the form (a, ∞) and the fact that $\mathcal{B}$ is the smallest σ-algebra containing all such intervals.] ▌

If E is a measurable set, we define the *Lebesgue measure* mE to be the outer measure of E. Thus m is the set function obtained by restricting the set function m^* to the family $\mathfrak{M}$ of measurable sets. Two important properties of Lebesgue measure are summarized by the following propositions:

13. Proposition: *Let* $\langle E_i \rangle$ *be a sequence of measurable sets. Then*

$$m(\bigcup E_i) \leq \sum mE_i.$$

If the sets E_n *are pairwise disjoint, then*

$$m(\bigcup E_i) = \sum mE_i.$$

Proof: The inequality is simply a restatement of the subadditivity of m^* given by Proposition 2. If $\langle E_i \rangle$ is a finite sequence of disjoint measurable sets, then Lemma 9 with $A = \mathbf{R}$ implies that

$$m(\bigcup E_i) = \sum mE_i,$$

and so m is finitely additive. Let $\langle E_i \rangle$ be an infinite sequence of pairwise disjoint measurable sets. Then

$$\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} E_i,$$

and so

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} mE_i.$$

Since the left side of this inequality is independent of n, we have

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} mE_i.$$

The reverse inequality follows from countable subadditivity, and we have

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} mE_i. \blacksquare$$

14. Proposition: *Let $\langle E_n \rangle$ be an infinite decreasing sequence of measurable sets, that is, a sequence with $E_{n+1} \subset E_n$ for each n. Let mE_1 be finite. Then*

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} mE_n.$$

Proof: Let $E = \bigcap_{i=1}^{\infty} E_i$, and let $F_i = E_i \sim E_{i+1}$. Then

$$E_1 \sim E = \bigcup_{i=1}^{\infty} F_i,$$

and the sets F_i are pairwise disjoint. Hence

$$m(E_1 \sim E) = \sum_{i=1}^{\infty} mF_i = \sum_{i=1}^{\infty} m(E_i \sim E_{i+1}).$$

But $mE_1 = mE + m(E_1 \sim E)$, and $mE_i = mE_{i+1} + m(E_i \sim E_{i+1})$, since $E \subset E_1$ and $E_{i+1} \subset E_i$. Since $mE_i \leq mE_1 < \infty$, we have $m(E_1 \sim E) = mE_1 - mE$ and $m(E_i \sim E_{i+1}) = mE_i - mE_{i+1}$. Thus

$$\begin{aligned} mE_1 - mE &= \sum_{i=1}^{\infty} (mE_i - mE_{i+1}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} (mE_i - mE_{i+1}) \\ &= \lim_{n\to\infty} (mE_1 - mE_n) \\ &= mE_1 - \lim_{n\to\infty} mE_n. \end{aligned}$$

Since $mE_1 < \infty$, we have

$$mE = \lim_{n\to\infty} mE_n. \blacksquare$$

The following proposition expresses a number of ways in which a measurable set is very nearly a nice set. The proof is left to the reader (Problem 13).

15. Proposition: *Let E be a given set. Then the following five statements are equivalent:*

i. *E is measurable;*
ii. *given $\epsilon > 0$, there is an open set $O \supset E$ with $m^*(O \sim E) < \epsilon$;*
iii. *given $\epsilon > 0$, there is a closed set $F \subset E$ with $m^*(E \sim F) < \epsilon$;*
iv. *there is a G in $\mathcal{G}_\delta$ with $E \subset G$, $m^*(G \sim E) = 0$;*
v. *there is an F in $\mathcal{F}_\sigma$ with $F \subset E$, $m^*(E \sim F) = 0$;*

If m^*E is finite, the above statements are equivalent to:

vi. *given $\epsilon > 0$, there is a finite union U of open intervals such that $m^*(U \triangle E) < \epsilon$.*

Problems

9. Show that if E is a measurable set, then each translate $E + y$ of E is also measurable.

10. Show that if E_1 and E_2 are measurable, then $m(E_1 \cup E_2) + m(E_1 \cap E_2) = mE_1 + mE_2$.

11. Show that the condition $mE_1 < \infty$ is necessary in Proposition 14 by giving a decreasing sequence $\langle E_n \rangle$ of measurable sets with $\varnothing = \bigcap E_n$ and $mE_n = \infty$ for each n.

12. Let $\langle E_i \rangle$ be a sequence of disjoint measurable sets and A any set. Then $m^*(A \cap \bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$.

13. Prove Proposition 15. [Hints:

a. Show that for $m^*E < \infty$, (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (vi) (cf. Proposition 5).

b. Use (a) to show that for arbitrary sets E, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

c. Use (b) to show that (i) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (i).]

14. a. Show that the Cantor ternary set (Problem 2.36) has measure zero.

b. Let F be a subset of $[0, 1]$ constructed in the same manner as the Cantor ternary set except that each of the intervals removed at the n^{th} step has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Then F is a closed set, $\tilde{F}$ dense in $[0, 1]$ and $mF = 1 - \alpha$. Such a set F is called a *generalized Cantor set.*

*4 A Nonmeasurable Set

We are going to show the existence of a nonmeasurable set. If x and y are real numbers in $[0, 1)$, we define the *sum modulo* 1 of x and y to be $x + y$, if $x + y < 1$, and to be $x + y - 1$ if $x + y \geq 1$. Let us denote the sum modulo 1 of x and y by $x \overset{\circ}{+} y$. Then $\overset{\circ}{+}$ is a commutative and associative operation taking pairs of numbers in $[0, 1)$ into numbers in $[0, 1)$. If we assign to each $x \in [0, 1)$ the angle $2\pi x$, then addition modulo 1 corresponds to the addition of angles. If E is a subset of $[0, 1)$, we define the translate modulo 1 of E to be the set $E \overset{\circ}{+} y = \{z\colon z = x \overset{\circ}{+} y \text{ for some } x \in E\}$. If we consider addition modulo 1 as addition of angles, translation modulo 1 by y corresponds to rotation through an angle of $2\pi y$. The following lemma shows that Lebesgue measure is invariant under translation modulo 1.

16. Lemma: *Let $E \subset [0, 1)$ be a measurable set. Then for each $y \in [0, 1)$ the set $E \overset{\circ}{+} y$ is measurable and $m(E \overset{\circ}{+} y) = mE$.*

Proof: Let $E_1 = E \cap [0, 1 - y)$ and $E_2 = E \cap [1 - y, 1)$. Then E_1 and E_2 are disjoint measurable sets whose union is E, and so

$$mE = mE_1 + mE_2.$$

Now $E_1 \overset{\circ}{+} y = E_1 + y$, and so $E_1 \overset{\circ}{+} y$ is measurable and we have $m(E_1 \overset{\circ}{+} y) = mE_1$, since m is translation invariant. Also $E_2 \overset{\circ}{+} y = E_2 + (y - 1)$, and so $E_2 \overset{\circ}{+} y$ is measurable and $m(E_2 \overset{\circ}{+} y) = mE_2$. But $E \overset{\circ}{+} y = (E_1 \overset{\circ}{+} y) \cup (E_2 \overset{\circ}{+} y)$ and the sets $(E_1 \overset{\circ}{+} y)$ and $(E_2 \overset{\circ}{+} y)$ are disjoint measurable sets. Hence $E \overset{\circ}{+} y$ is measurable and

$$\begin{aligned} m(E \overset{\circ}{+} y) &= m(E_1 \overset{\circ}{+} y) + m(E_2 \overset{\circ}{+} y) \\ &= mE_1 + mE_2 \\ &= mE. \blacksquare \end{aligned}$$

We are now in a position to define a nonmeasurable set. If $x - y$ is a rational number, we say that x and y are equivalent and write $x \sim y$. This is an equivalence relation and hence partitions $[0, 1)$ into equivalence classes, that is, classes such that any two elements of one class differ by a rational number, while any two elements of different classes differ by an irrational number. By the axiom of choice there is a set P which contains exactly one element from each equivalence class. Let $\langle r_i \rangle_{i=0}^{\infty}$ be an enumeration of the rational numbers in $[0, 1)$ with $r_0 = 0$, and define $P_i = P \overset{\circ}{+} r_i$. Then $P_0 = P$. Let $x \in P_i \cap P_j$. Then $x = p_i + r_i = p_j + r_j$ with p_i and p_j belonging to P. But $p_i - p_j = r_j - r_i$ is a rational number, whence $p_i \sim p_j$. Since P has only one element from each equivalence class, we must have $i = j$. This implies that if $i \neq j$, $P_i \cap P_j = \varnothing$, that is, that $\langle P_i \rangle$ is a pairwise disjoint sequence of sets. On the other hand, each real number x in $[0, 1)$ is in some equivalence class and so is equivalent to an element in P. But if x differs from an element in P by the rational number r_i, then $x \in P_i$. Thus $\bigcup P_i = [0, 1)$. Since each P_i is a translation modulo 1 of P, each P_i will be measurable if P is and will have the same measure. But if this were the case,

$$m[0, 1) = \sum_{i=1}^{\infty} mP_i = \sum_{i=1}^{\infty} mP,$$

and the right side is either zero or infinite, depending on whether mP is zero or positive. But this is impossible since $m[0, 1) = 1$, and consequently P cannot be measurable.

While the above proof that P is not measurable is a proof by contradiction, it should be noted that (until the last sentence) we have made no use of properties of Lebesgue measure other than translation invariance and countable additivity. Hence the foregoing argument gives a direct proof of the following theorem:

17. Theorem: *If m is a countably additive, translation invariant measure defined on a σ-algebra containing the set P, then $m[0, 1)$ is either zero or infinite.*

The nonmeasurability of P with respect to any translation invariant countably additive measure m for which $m[0, 1)$ is 1 follows by contraposition.

Problems

15. Show that if E is measurable and $E \subset P$, then $mE = 0$. [Hint: Let $E_i = E \overset{\circ}{+} r_i$. Then $\langle E_i \rangle$ is a disjoint sequence of measurable sets and $mE_i = mE$. Thus $\sum mE_i = m \bigcup E_i \leq m[0, 1]$.]

16. Show that, if A is any set with $m^*A > 0$, then there is a non-measurable set $E \subset A$. [Hint: If $A \subset (0, 1)$, let $E_i = A \cap P_i$. The measurability of E_i implies $mE_i = 0$, while $\sum m^*E_i \geq m^*A > 0$.]

17. **a.** Give an example where $\langle E_i \rangle$ is a disjoint sequence of sets and $m^*(\bigcup E_i) < \sum m^*E_i$.

b. Give an example of a sequence of sets $\langle E_i \rangle$ with $E_i \supset E_{i+1}$, $m^*E_i < \infty$, and $m^*(\bigcap E_i) < \lim m^*E_i$.

5 Measurable Functions

Since not all sets are measurable, it is of great importance to know that sets which arise naturally in certain constructions are measurable. If we start with a function f the most important sets which arise from it are those listed in the following proposition:

18. Proposition: *Let f be an extended real-valued function whose domain is measurable. Then the following statements are equivalent:*

i. *For each real number α the set $\{x: f(x) > \alpha\}$ is measurable.*
ii. *For each real number α the set $\{x: f(x) \geq \alpha\}$ is measurable.*
iii. *For each real number α the set $\{x: f(x) < \alpha\}$ is measurable.*
iv. *For each real number α the set $\{x: f(x) \leq \alpha\}$ is measurable.*

These statements imply

v. *For each extended real number α the set $\{x: f(x) = \alpha\}$ is measurable.*

Proof: Let the domain of f be D. We have (i) $\Rightarrow$ (iv), since $\{x: f(x) \leq \alpha\} = D \sim \{x: f(x) > \alpha\}$ and the difference of two measurable sets is measurable. Similarly, (iv) $\Rightarrow$ (i) and (ii) $\Leftrightarrow$ (iii). Now (i) $\Rightarrow$ (ii), since $\{x: f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x: f(x) > \alpha - 1/n\}$, and the intersection of a sequence of measurable sets is measurable. Similarly, (ii) $\Rightarrow$ (i), since $\{x: f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x: f(x) \geq \alpha + 1/n\}$, and the union of a sequence of measurable sets is measurable. This shows that the first four statements are equivalent. If α is a real number, $\{x: f(x) = \alpha\} = \{x: f(x) \geq \alpha\} \cap \{x: f(x) \leq \alpha\}$, and so (ii) and (iv) $\Rightarrow$ (v) for α real. Since

$$\{x: f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x: f(x) \geq n\},$$

(ii) $\Rightarrow$ (v) for $\alpha = \infty$. Similarly, (iv) $\Rightarrow$ (v) for $\alpha = -\infty$, and we have (ii) & (iv) $\Rightarrow$ (v). ▮

Definition: *An extended real-valued function f is said to be (Lebesgue) measurable if its domain is measurable and if it satisfies one of the first four statements of Proposition* 18.

Thus if we restrict ourselves to measurable functions, the most important sets connected with them are measurable. It should be noted that a continuous function (with a measurable domain) is measurable, and of course each step function is measurable. If f is a measurable function and E is a measurable subset of the domain of f, then the function obtained by restricting f to E is also measurable. The following proposition tells us that certain operations performed on measurable functions lead again to measurable functions:

19. Proposition: *Let c be a constant and f and g two measurable real-valued functions defined on the same domain. Then the functions* $f + c$, cf, $f + g$, $g - f$, *and* fg *are also measurable.*

Proof: We shall use condition (iii) of Proposition 18. Then

$$\{x: f(x) + c < \alpha\} = \{x: f(x) < \alpha - c\},$$

and so $f + c$ is measurable when f is. A similar argument shows cf to be measurable.

If $f(x) + g(x) < \alpha$, then $f(x) < \alpha - g(x)$ and by the corollary to the axiom of Archimedes there is a rational number r such that

$$f(x) < r < \alpha - g(x).$$

Hence

$$\{x: f(x) + g(x) < \alpha\} = \bigcup_r (\{x: f(x) < r\} \cap \{x: g(x) < \alpha - r\}).$$

Since the rationals are countable, this set is measurable and so $f + g$

is measurable. Since $-g = (-1)g$ is measurable when g is, we have $f - g$ measurable.

The function f^2 is measurable, since

$$\{x: f^2(x) > \alpha\} = \{x: f(x) > \sqrt{\alpha}\} \cup \{x: f(x) < -\sqrt{\alpha}\}$$

for $\alpha \geq 0$ and

$$\{x: f^2(x) > \alpha\} = D$$

if $\alpha < 0$, where D is the domain of f. Thus

$$fg = \tfrac{1}{2}[(f+g)^2 - f^2 - g^2]$$

is measurable. ∎

We will often want to use Proposition 19 for extended real-valued functions f and g. Unfortunately, $f + g$ is not defined at points where it is of the form $\infty - \infty$. However, fg is always measurable and $f + g$ is measurable if we always take the same value for $f + g$ at points where it is undefined. Also, $f + g$ is measurable no matter what values we take at the points where it is not defined, provided these points are a set of measure zero. See Problem 22.

20. Theorem: *Let $\langle f_n \rangle$ be a sequence of measurable functions (with the same domain of definition). Then the functions* $\sup\{f_1, \ldots, f_n\}$, $\inf\{f_1, \ldots, f_n\}$, $\sup_n f_n$, $\inf_n f_n$, $\overline{\lim} f_n$, *and* $\underline{\lim} f_n$ *are all measurable.*

Proof: If h is defined by $h(x) = \sup\{f_1(x), \ldots, f_n(x)\}$, then $\{x: h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x: f_i(x) > \alpha\}$. Hence the measurability of the f_i implies that of h. Similarly, if g is defined by $g(x) = \sup f_n(x)$, then $\{x: g(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x: f_n(x) > \alpha\}$, and so g is measurable. A similar argument establishes the corresponding statements for inf. Since $\overline{\lim} f_n = \inf_n \sup_{k \geq n} f_k$, we have $\overline{\lim} f_n$ measurable, and similarly for $\underline{\lim} f_n$. ∎

A property is said to hold **almost everywhere**[5] (abbreviated a.e.) if the set of points where it fails to hold is a set of measure zero. Thus in particular we say that $f = g$ a.e. if f and g have the same domain

[5] French: *presque partout* (p.p.).

and $m\{x: f(x) \neq g(x)\} = 0$. Similarly, we say that f_n converges to g almost everywhere if there is a set E of measure zero such that $f_n(x)$ converges to $g(x)$ for each x not in E. One consequence of equality a.e. is the following:

21. Proposition: *If f is a measurable function and $f = g$ a.e., then g is measurable.*

Proof: Let E be the set $\{x: f(x) \neq g(x)\}$. By hypothesis $mE = 0$. Now

$$\{x: g(x) > \alpha\} = \{x: f(x) > \alpha\} \cup \{x \in E: g(x) > \alpha\} \sim \{x \in E: g(x) \leq \alpha\}.$$

The first set on the right is measurable, since f is a measurable function. The last two sets on the right are measurable since they are subsets of E and $mE = 0$. Thus $\{x: g(x) > \alpha\}$ is measurable for each α, and so g is measurable. ∎

The following proposition tells us that a measurable function is "almost" a continuous function. The proof is left to the reader (cf. Problem 23).

22. Proposition: *Let f be a measurable function defined on an interval $[a, b]$, and assume that f takes the values $\pm\infty$ only on a set of measure zero. Then given $\epsilon > 0$, we can find a step function g and a continuous function h such that*

$$|f - g| < \epsilon \quad \textit{and} \quad |f - h| < \epsilon$$

except on a set of measure less than ϵ; i.e. $m\{x: |f(x) - g(x)| \geq \epsilon\} < \epsilon$ and $m\{x: |f(x) - h(x)| \geq \epsilon\} < \epsilon$. If in addition $m \leq f \leq M$, then we may choose the functions g and h so that $m \leq g \leq M$ and $m \leq h \leq M$.

If A is any set, we define the **characteristic function** χ_A of the set A to be the function given by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The function χ_A is measurable if and only if A is measurable. Thus the existence of a nonmeasurable set implies the existence of a nonmeasurable function.

A real-valued function φ is called **simple** if it is measurable and assumes only a finite number of values. If φ is simple and has the values $\alpha_1, \ldots, \alpha_n$ then $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, where $A_i = \{x: \varphi(x) = \alpha_i\}$. The sum, product, and difference of two simple functions are simple.

Problems

18. Show that (v) does not imply (iv) in Proposition 18 by constructing a function f such that $\{x: f(x) > 0\} = E$, a given nonmeasurable set, and such that f assumes each value at most once.

19. Let D be a dense set of real numbers, that is, a set of real numbers such that every interval contains an element of D. Let f be an extended real-valued function on **R** such that $\{x: f(x) > \alpha\}$ is measurable for each $\alpha \in D$. Then f is measurable.

20. Show that the sum and product of two simple functions are simple. Show that

$$\chi_{A \cap B} = \chi_A \cdot \chi_B$$
$$\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$$
$$\chi_{\tilde{A}} = 1 - \chi_A.$$

21. a. Let D and E be measurable sets and f a function with domain $D \cup E$. Show that f is measurable if and only if its restrictions to D and E are measurable.

b. Let f be a function with measurable domain D. Show that f is measurable iff the function g defined by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

22. a. Let f be an extended real-valued function with measurable domain D, and let $D_1 = \{x: f(x) = \infty\}$, $D_2 = \{x: f(x) = -\infty\}$. Then f is measurable if and only if D_1 and D_2 are measurable and the restriction of f to $D \sim (D_1 \cup D_2)$ is measurable.

b. Prove that the product of two measurable extended real-valued functions is measurable.

c. If f and g are measurable extended real-valued functions and α a fixed number, then $f + g$ is measurable if we define $f + g$ to be α whenever it is of the form $\infty - \infty$ or $-\infty + \infty$.

d. Let f and g be measurable extended real-valued functions which are finite almost everywhere. Then $f + g$ is measurable no matter how it is defined at points where it has the form $\infty - \infty$.

23. Prove Proposition 22 by establishing the following lemmas:

a. Given a measurable function f on $[a, b]$ which takes the values $\pm\infty$ only on a set of measure zero, and given $\epsilon > 0$, there is an M such that $|f| \leq M$ except on a set of measure less than $\epsilon/3$.

b. Let f be a measurable function on $[a, b]$. Given $\epsilon > 0$ and M, there is a simple function φ such that $|f(x) - \varphi(x)| < \epsilon$ except where $|f(x)| \geq M$. If $m \leq f \leq M$, then we may take φ so that $m \leq \varphi \leq M$.

c. Given a simple function φ on $[a, b]$, there is a step function g on $[a, b]$ such that $g(x) = \varphi(x)$ except on a set of measure less than $\epsilon/3$. [Hint: Use Proposition 15.] If $m \leq \varphi \leq M$, then we can take g so that $m \leq g \leq M$.

d. Given a step function g on $[a, b]$, there is a continuous function h such that $g(x) = h(x)$ except on a set of measure less than $\epsilon/3$. If $m \leq g \leq M$, then we may take h so that $m \leq h \leq M$.

24. Let f be measurable and B a Borel set. Then $f^{-1}[B]$ is a measurable set. [Hint: The class of sets for which $f^{-1}[E]$ is measurable is a σ-algebra.]

25. Show that if f is a meaurable real-valued function and g a continuous function defined on $(-\infty, \infty)$, then $g \circ f$ is measurable.

26. *Borel measurability.* A function f is said to be **Borel measurable** if for each α the set $\{x: f(x) > \alpha\}$ is a Borel set. Verify that Propositions 18 and 19 and Theorem 20 remain valid if we replace "measurable set" by "Borel set" and "(Lebesgue) measurable" by "Borel measurable." Every Borel measurable function is Lebesgue measurable. If f is Borel measurable, and B is a Borel set, then $f^{-1}[B]$ is a Borel set. If f and g are Borel measurable, so is $f \circ g$. If f is Borel measurable and g is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.

27. How much of the preceding problem can be carried out if we replace the class $\mathcal{B}$ of Borel sets by an arbitrary σ-algebra $\mathcal{A}$ of sets?

28. Let f_1 be the Cantor ternary function (cf. Problem 2.46), and define f by $f(x) = f_1(x) + x$.

a. Show that f is a homeomorphism of $[0, 1]$ onto $[0, 2]$.

b. Show that f maps the Cantor set onto a set F of measure 1.

c. Let $g = f^{-1}$. Show that there is a measurable set A such that $g^{-1}[A]$ is not measurable.

d. Give an example of a continuous function g and a measurable function h such that $h \circ g$ is not measurable. Compare with Problems 25 and 26.

e. Show that there is a measurable set which is not a Borel set.

6 Littlewood's Three Principles

Speaking of the theory of functions of a real variable, J. E. Littlewood says,[6] "The extent of knowledge required is nothing like so great as is sometimes supposed. There are three principles, roughly expressible in the following terms: Every (measurable) set is nearly a finite union of intervals; every [measurable] function is nearly continuous; every convergent sequence of [measurable] functions is nearly uniformly convergent. Most of the results of [the theory] are fairly intuitive applications of these ideas, and the student armed with them should be equal to most occasions when real variable theory is called for. If one of the principles would be the obvious means to settle the problem if it were 'quite' true, it is natural to ask if the 'nearly' is near enough, and for a problem that is actually solvable it generally is."

We have already met two of Littlewood's principles: Various forms of the first principle are given by Proposition 15. One version of the second principle is given by Proposition 22, another version by Problem 31, and a third is given by Problems 4.15 and 6.14. The following proposition gives one version of the third principle. A slightly stronger form is given by Egoroff's theorem (Problem 30), but you will generally find the weak form adequate.

23. Proposition: *Let E be a measurable set of finite measure, and $\langle f_n \rangle$ a sequence of measurable functions defined on E. Let f be a measurable real-valued function such that for each x in E we have $f_n(x) \to f(x)$. Then, given $\epsilon > 0$ and $\delta > 0$, there is a measurable set $A \subset E$ with $mA < \delta$ and an integer N such that for all $x \not\in A$ and all $n \geq N$,*

$$|f_n(x) - f(x)| < \epsilon.$$

Proof: Let

$$G_n = \{x \varepsilon E: |f_n(x) - f(x)| \geq \epsilon\},$$

and set

$$E_N = \bigcup_{n=N}^{\infty} G_n = \{x \varepsilon E: |f_n(x) - f(x)| \geq \epsilon \text{ for some } n \geq N\}.$$

We have $E_{N+1} \subset E_N$, and for each $x \varepsilon E$ there must be some E_N to which x does not belong, since $f_n(x) \to f(x)$. Thus $\bigcap E_N = \varnothing$, and so, by Proposition 14, $\lim mE_N = 0$. Hence given $\delta > 0$, $\exists N$ so that

[6] *Lectures on the Theory of Functions*, Oxford, 1944, p. 26.

$mE_N < \delta$; that is,

$$m\{x \in E: |f_n(x) - f(x)| \geq \epsilon \text{ for some } n \geq N\} < \delta.$$

If we write A for this E_N, then $mA < \delta$ and

$$\tilde{A} = \{x \in E: |f_n(x) - f(x)| < \epsilon \text{ for all } n \geq N\}. \blacksquare$$

If, as in the hypothesis of the proposition, we have $f_n(x) \to f(x)$ for each x, we say that the sequence $\langle f_n \rangle$ converges **pointwise** to f on E. If there is a subset B of E with $mB = 0$ such that $f_n \to f$ pointwise on $E \sim B$, we say that $f_n \to f$ a.e. on E. We have the following trivial modification of the last proposition:

24. Proposition: *Let E be a measurable set of finite measure, and $\langle f_n \rangle$ a sequence of measurable functions which converge to a real-valued function f a.e. on E. Then, given $\epsilon > 0$ and $\delta > 0$, there is a set $A \subset E$ with $mA < \delta$, and an N such that for all $x \notin A$ and all $n \geq N$,*

$$|f_n(x) - f(x)| < \epsilon.$$

Problems

29. Give an example to show that we must require $mE < \infty$ in Proposition 23.

30. Prove *Egoroff's Theorem:* If $\langle f_n \rangle$ is a sequence of measurable functions which converge to a real-valued function f a.e. on a measurable set E of finite measure, then, given $\eta > 0$, there is a subset $A \subset E$ with $mA < \eta$ such that f_n converges to f *uniformly* on $E \sim A$. [Hint: Apply Proposition 24 repeatedly with $\epsilon_n = 1/n$ and $\delta_n = 2^{-n}\eta$.]

31. Prove *Lusin's Theorem:* Let f be a measurable real-valued function on an interval $[a, b]$. Then given $\delta > 0$, there is a continuous function φ on $[a, b]$ such that $m\{x: f(x) \neq \varphi(x)\} < \delta$. Can you do the same on the interval $(-\infty, \infty)$? [Hint: Use Egoroff's theorem, Propositions 15 and 22, and Problem 2.39.]

32. Show that Proposition 23 need not be true if the integer variable n is replaced by a real variable t; that is, construct a family $\langle f_t \rangle$ of measurable real-valued functions on $[0, 1]$ such that for each x we have $\lim_{t \to 0} f_t(x) = 0$, but for some $\delta > 0$ we have $m^*\{x: f_t(x) > \frac{1}{2}\} > \delta$. Hint: Let P_ι be the sets in Section 4. For $2^{-\iota-1} \leq t < 2^{-\iota}$ define f_t by

$$f_t(x) = \begin{cases} 1 & \text{if } x \in P_\iota \text{ and } x = 2^{\iota+1}t - 1 \\ 0 & \text{otherwise.} \end{cases}$$