Proof. Fix x in $\mathscr{X}$. By the hypothesis and Corollary 14.3, $\sup\{\ \|A(x)\|$: A $\in \mathscr{A}\} <$ co. By (14.1) $\sup\{\|A\|$: A $\in \mathscr{A}\} < \infty$. ■

A special form of the PUB that is quite useful is the following.

14.6. The Banach-Steinhaus Theorem. *Zf X and $\mathscr{Y}$ are Banach spaces and {A,} is a sequence in $\mathscr{B}(\mathscr{X},\mathscr{Y})$ with the property that for every x in X there is a y in $\mathscr{Y}$ such that $\|A_n x - y\| \to 0$, then there is an A in $\mathscr{B}(\mathscr{X},\mathscr{Y})$ such that $\|A_n x - Ax\| \to 0$ for every x in X and* $\sup\|A_n\| < \infty$.

Proof. If x $\in$ X, let Ax = lim $A_n x$. By hypothesis $A: \mathscr{X} \to \mathscr{Y}$ is defined and it is easy to see that it is linear. To show that A is bounded, note that the PUB implies that there is a constant $M > 0$ such that $\|A_n\| \le M$ for all n. If x $\in$ X and $\|x\| \le 1$, then for any $n \ge 1$, $\|Ax\| \le \|Ax - A_n x\| + \|A_n x\| \le \|Ax - A_n x\| + M$. Letting $n \to \infty$ shows that $\|Ax\| \le M$ whenever $\|x\| \le 1$. ■

The Banach-Steinhaus Theorem is a result about sequences, not nets. Note that if Z is the identity operator on X and for each $n \ge 1$, A,, = $n^{-1}I$ and for $n \le 0$, A,, = nI, then $\{A_n : n \in \mathbb{Z}\}$ is a countable net that converges in norm to 0. but the net is not bounded.

14.7. Proposition. *Let X be locally compact and let $\{f_n\}$ be a sequence in C,(X). Then $\int f_n\, d\mu \to \int f\, d\mu$ for every μ in M(X) if and only if* $\sup_n\|f_n\| < \infty$ *and $f_n(x) \to f(x)$ for every x in X.*

Proof. Suppose $\int f_n\, d\mu \to \int f\, d\mu$ for every μ in M(X). Since M(X) = C,(X)*, (14.3) implies that $\sup_n\|f_n\| < \infty$. By letting $\mu = \delta_x$, the unit point mass at x, we see that $\int f_n\, d\delta_x$ = f,(x) $\to f(x)$. The converse follows by the Lebesgue Dominated Convergence Theorem. ■

Exercises

1. Here is another proof of the PUB using the **Baire** Category Theorem. With the notation of **(14.1)**, let $B_n \equiv \{x \in \mathscr{X} : \|Ax\| \le n$ for all A in $\mathscr{A}\}$. By hypothesis, $\bigcup_{n=1}^{\infty} B_n = \mathscr{X}$. Now apply the Baire Category Theorem.
2. If $1 < p < \infty$ and $\{x_n\} \subseteq l^p$, then $\sum_{j=1}^{\infty} x_n(j)y(j) \to 0$ for every y in l^q, $1/p +$ l/q = 1, if and only if $\sup_n\|x_n\|_p <$ cc and $x_n(j) \to 0$ for every j $\ge$ 1.
3. If {x,} $\subseteq l^1$, then $\sum_{j=1}^{\infty} x_n(j)y(j) \to 0$ for every y in c_0 if and only if $\sup_n\|x_n\|_1 < \infty$ and x,,(j) $\to$ 0 for every j $\ge$ 1.
4. If (X, Ω, μ) is a measure space, $1 < p < \infty$, and $\{f_n\} \subseteq L^p(X, \Omega, \mu)$, then $\int fg\, d\mu \to 0$ for every g in $L^q(\mu), 1/p +$ l/q = 1, if and only if $\sup\{\|f_n\|_p$: $n \ge 1\} < \infty$ **and for every set** E **in** Ω **with** $\mu(E) < \infty$, $\int_E f_n\, d\mu \to 0$ as $n \to \infty$.
5. If (X, Ω, μ) is a u-finite measure space and $\{f_n\}$ is a sequence in $L^1(X, \Omega, \mu)$, then $\int f_n g\, d\mu \to 0$ for every g in $L^\infty(\mu)$ if and only if $\sup\{\|f_n\|_1 : n \ge 1\} < \infty$ and $\int_E f_n\, d\mu \to 0$ for every E in Ω.

6. Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{E}$ be an orthonormal basis for $\mathscr{H}$. Show that a sequence $\{h_n\}$ in $\mathscr{H}$ satisfies $\langle h_n, h\rangle \to 0$ for every h in $\mathscr{H}$ if and only if $\sup\{\|h_n\|: n \geq 1\} <$ cc and $\langle h_n, e\rangle \to 0$ for every e in $\mathscr{E}$.

7. If X is locally compact and $\{\mu_n\}$ is a sequence in M(X), then $L(\mu_n) \to 0$ for every L in M(X)* if and only if $\sup\{\|\mu_n\|: n \geq 1\} <$ cc and $\mu_n(E) \to 0$ for every Bore l set E.

8. In (14.6), show that $\|A\| \leq \limsup \|A_n\|$.

9. If (S, d) is a metric space and X is a normed space, say that a function $f: S \to$ X is a *Lipschitz function* if there is a constant $M > 0$ such that $\|f(x) - f(t)\| \leq Md(s, t)$ for all s, t in S. Show that if $f: S \to$ X is a function such that for all L in $\mathscr{X}^*$, $L \circ f: S \to \mathbb{F}$ is Lipschitz, then $f: S \to$ X is a Lipschitz function.

10. Let X be a Banach space and suppose $\{x_n\}$ is a sequence in X that is linearly independent and such that for each x in X there are scalars {a, } such that lim,, $_{\to \infty} \|x - \Sigma_{k=1}^n \alpha_k x_k\| = 0$. Such a sequence is called a *basis*. (a) Prove that X is separable. (b) Let $\mathscr{Y}$ = {{a,, }∈ IF : $\Sigma_{n=1}^{\infty} \alpha_n x_n$ converges in X} and for y = {a,,} in $\mathscr{Y}$ define $\|y\| = \sup_n \|\Sigma_{k=1}^n \alpha_k x_k\|$. Show that $\mathscr{Y}$ is a Banach space. (c) Show that there is a bounded bijection $T: \mathscr{X} \to \mathscr{Y}$. (d) If $n \geq 1$ and f_n: $\mathscr{X} \to \mathbb{F}$ is defined by $f_n(\Sigma_{k=1}^{\infty} \alpha_k x_k) = \alpha_n$, show that $f_n \in \mathscr{X}^*$. (e) Show that $x_n \notin$ the closed linear span of $\{x_k : k \neq n\}$.