Option Pricing with Lévy-Stable Processes

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Abstract

In this paper we show how to calculate European-style option prices when the log-stock and stock returns processes follow a symmetric Lévy-Stable process. We extend our results to price European-style options when the log-stock process follows a skewed Lévy-Stable process.

Keywords: Lévy-Stable processes, stable Paretian hypothesis, stochastic volatility, α-stable processes, option pricing.

1 Introduction

Up until the early 1990's most of the underlying stochastic processes used in the financial literature were based on a combination of Brownian motion and Poisson processes. One of the most fundamental assumptions throughout has been that financial asset returns are the cumulative outcome of many small events that happen at a 'microscopic level' very often in time, so that their impact may be regarded as continuous. If these microscopic events are considered statistically independent with finite variance it is straightforward to characterise their cumulative behaviour by invoking the Central Limit Theorem (CLT). Hence, Gaussian-based distributions are a plausible class of models for financial processes.

But are there any other limiting distributions that characterise the behaviour of the sum of many 'microscopic' events? The answer is yes. The sum of many iid events always has, after appropriate

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scaling and shifting, a limiting distribution termed a Lévy-Stable law; this is the generalised version of the Central Limit Theorem, GCLT, [23]. The Gaussian distribution is one example. Based on this fundamental result, it is plausible to generalise the assumption of Gaussian price increments by modelling the 'formation' of prices in the market by the sum of many stochastic events with a Lévy-Stable limiting distribution.

Another important property of Lévy-Stable distributions is that of stability under addition. In other words, when two independent copies of a Lévy-Stable random variable are added then, up to scaling and shift, the resulting random variable is again Lévy-Stable with the same shape. This property is very desirable in models used in finance and particularly in portfolio analysis and risk management, see for example Fama [8], Ziemba [27] and the most recent work by Tokat and Schwartz [25], Ortobelli et al [21] and Mittnik et al [19]. Only for Lévy-Stable distributed returns do we have the property that linear combination of different return series, for example portfolios, again have a Lévy-Stable distribution [9].

Based on empirical data and with a profound belief in the importance of invariances and the possibility of identifying stationarity and scaling as invariance principles in economics, Mandelbrot [15] was the first to propose in the early 1960s the use of Lévy-Stable processes to model stock returns. For example, scaling is present in finance when 'short-term' pieces of a chart are compared to 'long-term' pieces and they look very 'similar'. One has to acknowledge that there is a limit as to how short or long are the terms upon which charts may look like down-sized versions of each other. If the short term is shorter than the physical time between trades in the market then the similarity argument would not hold. Similarly, for long time scales the forces acting behind changes, say in stock prices, are due more to macroeconomic fundamentals rather than speculation.

A clear example of the usefulness of the 'stable' property in financial markets is the shape of the implied volatility for different maturity prices. Carr and Wu [6] document that options on US equity indexes have an invariant smirk across different maturities. This invariance can only be obtained if innovations are Lévy-Stable distributed but not Gaussian. Standard option pricing models imply that the volatility smirk flattens as maturity increases; this is mainly a result of the CLT when the conditional moments for stock returns are finite.

Based on the GCLT we then have, in general terms, two ways of modelling stock prices or stock returns. If it is believed that *stock returns* are at least approximately governed by a Lévy-Stable distribution the accumulation of the random events is additive. On the other hand, if it is believed that the *logarithm of stock prices* are approximately governed by a Lévy-Stable distribution then the accumulation is multiplicative. In the literature most models have assumed that log-prices, instead of returns, follow a Lévy-Stable process. McCulloch [16] assumes that assets are log Lévy-Stable and prices options using a utility maximisation argument and in [17]. Recently Carr and Wu [6] priced European options when the log-stock price follows a maximally skewed Lévy-Stable process. Cartea and Howison [7] also assume that log prices follow a Lévy-Stable process and provide a solution to the pricing problem as a distinguished limit of the Lévy-Stable process.

Finally, based on Mandelbrot and Taylor [15], Platen, Hurst and Rachev [12] provide a model to price European options when returns follow a (symmetric) Lévy-Stable process. In their models the Brownian motion that drives the stochastic shocks to the stock process is subordinated to an intrinsic time process that represents 'operational time' on which the market operates. Option pricing can be done within the Black-Scholes framework and one can show that the subordinated Brownian motion is a symmetric Lévy-Stable motion.

In this paper we show that returns or log-prices can be modelled using a symmetric Lévy-Stable process and log-prices can be modelled using an asymmetric Lévy-Stable process.

Before proceeding with the results we give a brief description of the steps we take. First, we start by showing that if the log-stock process follows a Lévy-Stable process it is possible to price options within the Black-Scholes risk-neutral framework. The proof of existence relies mainly on the existence of the Laplace transform for totally skewed Lévy-Stable random variables and the fact that all Lévy-Stable symmetric variables can be represented as the product of a symmetric and a totally skewed random variable. In the construction of these symmetric random variables we will use the Gaussian case as one of the building blocks and will see that all Lévy-Stable symmetric cases can be seen as conditionally Gaussian. Hence we will show that when log-stock or stock returns are modelled using a symmetric Lévy-Stable motion or process, the resulting option prices are weighted averages of the Black-Scholes formula with random volatility. Then we show that this result may be readily extended to cases where the log-stock follows a negatively skewed Lévy-Stable process. In other words, the model we propose is

$$\begin{aligned} \ln(S_t/S_0) &= \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s) + g(X(t)) \\ \sigma(S,t) &= f(Y(t)), \end{aligned}$$

where dW is the increment of a standard Brownian motion, $X(t)$ and $Y(t)$ follow independent Lévy-Stable process, and μ, g, f are deterministic functions.

The paper is structured as follows: Section 2 presents definitions and properties of Lévy-Stable processes. In particular we show how symmetric Lévy-Stable random variables may be 'built' as a combination of two independent Lévy-Stable random variables. Section 3 shows how option prices may be obtained when stock returns evolve according to a Lévy-Stable motion. Section 4 shows that if integrated variance is modelled as a totally skewed to the right Lévy-Stable process, option prices can be obtained for stock returns that follow a symmetric Lévy-Stable process. Section 5 shows that it is also possible to extend the results in Section 4 to obtain option prices for log-stock prices that evolve as a skewed Lévy-Stable motion or process within the Black-Scholes risk-neutral framework. Section 6 calculates option prices according to the proposed models and compares the results to those given by the Black-Scholes framework.

2 Lévy-Stable random variables

In this section we show how to obtain *any* symmetric Lévy-Stable motion as a stochastic process where the innovations are the product of two independent Lévy-Stable random variables. The only conditions we require (we will make this mathematically precise in Proposition 3) are that one of the independent random variables is symmetric and the other is totally skewed to the right. This is a simple, yet very important, result since we can choose a Gaussian random variable as one of the building blocks together with any other totally skewed random variable to 'produce' symmetric Lévy-Stable random variables. Furthermore, choosing a Gaussian random variable as one of the building blocks of a symmetric random variable will be very convenient since we will be able to relate any symmetric Lévy-Stable motion as a conditional Brownian motion, conditioned on the other building block; the totally skewed Lévy-Stable random variable.

Below we review definitions and properties of Lévy-Stable random variables; see [23], [24].

Definition 1 ***Lévy-Stable random variable.*** *Let X be a random variable. X has a Lévy-Stable distribution if for any positive numbers A, B there is a positive number C and a real number D such that*

$$AX_1 + BX_2 \stackrel{d}{=} CX + D, \tag{1}$$

where X_1 and X_2 are independent copies of X, and where $\stackrel{d}{=}$ denotes equality in distribution.

In other words, the shape of the distribution of $AX_1 + BX_2$ is the same as the distribution of X up to scale and shift.

Definition 2 ***Lévy-Stable process.*** *Let $X(t)$ be a random variable dependent on time t. Then the stochastic process $X(t)$, for $0 < t < \infty$, is a Lévy-Stable process if the finite-dimensional distribution of $X(t)$ is Lévy-Stable. The finite-dimensional distribution of a stochastic process X are the distributions of the finite-dimensional vectors $(X(t_1), \cdots, X(t_n))$, $t_1 < \cdots < t_n < \infty$.*

The characteristic function of a Lévy-Stable process is given in the following proposition.

Proposition 1 ***Characteristic Function of Lévy-Stable Process.*** *Let $X(t)$ be a Lévy-Stable process. Then the natural logarithm of its characteristic function is given in terms of certain parameters α, κ, β and m by*

$$\ln \mathbb{E}[e^{iX(t)\theta}] \equiv [\Psi(\theta)] = \begin{cases} -t\kappa^{\alpha}|\theta|^{\alpha}\left\{1 - i\beta\ sign(\theta)\tan(\alpha\pi/2)\right\} + im\theta & \textit{for } \alpha \neq 1, \\ -t\kappa|\theta|\left\{1 + \frac{2i\beta}{\pi}\ sign(\theta)\ln|\theta|\right\} + im\theta & \textit{for } \alpha = 1. \end{cases} \tag{2}$$

Note that for Lévy-Stable processes we have that $\ln \mathbb{E}[e^{iX(t)\theta}] = t\ln \mathbb{E}[e^{iX(1)\theta}]$.

If the random variable $X(1)$ belongs to a Lévy-Stable process with parameters α, κ, β, m we write $X \sim S_{\alpha}(\kappa, \beta, m)$. The parameter $\alpha \in (0, 2]$ is known as the stability index; $\kappa > 0$ is a scaling parameter; $\beta \in [-1, 1]$ is a skewness parameter and m is a location parameter.

In view of the characteristic function for Lévy-Stable processes it is straightforward to see that for the case $0 < \alpha \leq 1$ the random variable X does not have any moments, and for the case $1 < \alpha < 2$ only the first moment exists. Moreover, given the asymptotic behaviour of the tails of the distribution of a Lévy-Stable random variable, given in Property 2 below, it can be shown that the Laplace transform of X exists only when its distribution is totally skewed to the right, that is $\beta = 1$.

Property 1 *[23] Let $X \sim S_{\alpha}(\kappa, \beta, 0)$ with $0 < \alpha < 2$ and $\beta = 0$ in the case $\alpha = 1$. Then for every $0 < p < \alpha$, there is a constant $D_{\alpha,\beta}(p)$ such that*

$$\mathbb{E}[|X|^p]^{1/p} = D_{\alpha,\beta}(p)\kappa. \tag{3}$$

For a proof see .

Property 2 ***Tails of the Lévy-Stable distributions: asymptotic behaviour [23].***

Let $X \sim S_{\alpha}(\kappa, \beta, m)$ with $0 < \alpha < 2$. Then

$$\textit{as } x \to \infty \quad \mathbb{P}(X > x) \sim \begin{cases} x^{-\alpha}\frac{1+\beta}{2}\kappa^{\alpha}\frac{1-\alpha}{\Gamma(2-\alpha)\cos\alpha\pi/2} & \textit{for } \alpha \neq 1, \\ x^{-\alpha}\frac{1+\beta}{2}\kappa^{\alpha}\frac{2}{\pi} & \textit{for } \alpha = 1, \end{cases} \tag{4}$$

and

$$\textit{as } x \to -\infty \quad \mathbb{P}(X < x) \sim \begin{cases} |x|^{-\alpha}\frac{1-\beta}{2}\kappa^{\alpha}\frac{1-\alpha}{\Gamma(2-\alpha)\cos\alpha\pi/2} & \textit{for } \alpha \neq 1, \\ |x|^{-\alpha}\frac{1-\beta}{2}\kappa^{\alpha}\frac{2}{\pi} & \textit{for } \alpha = 1 \end{cases} \tag{5}$$

where the notation $a \sim b$ is used to denote $\lim_{x\to\infty} a/b = 1$.

Proposition 2 ***The Laplace Transform [23].***

The Laplace Transform $\mathbb{E}[e^{-\tau X}]$ with $\tau \geq 0$ of the Lévy-Stable variable $X \sim S_\alpha(\kappa, 1, 0)$ with $0 < \alpha \leq 2$ and scale parameter $\kappa > 0$ satisfies

$$\ln \mathbb{E}[e^{-\tau X}] = \begin{cases} -\frac{\kappa^\alpha}{\cos\frac{\pi\alpha}{2}}\tau^\alpha & \text{for } \alpha \neq 1, \\ \frac{2\kappa}{\pi}\tau \ln \tau & \text{for } \alpha = 1. \end{cases} \tag{6}$$

Lévy-Stable densities are supported on either the whole real line or a half line. The latter situation can only occur when $\alpha < 1$ and $\beta = 1$ or $\beta = -1$; in this case precise limits are given. The following lemma characterises the support of the pdf's as a function of the characteristic exponent α and the skewness β.

Lemma 1 ***Support of the probability density functions [20].*** *Let X be a Lévy-Stable random variable, $X \sim S_\alpha(\kappa, \beta, 0)$. Then the support of its pdf $f_X(x)$ is given by*

$$\mathit{Supp}(f_X(x)) = \begin{cases} [-\tan\frac{\pi\alpha}{2}, \infty) & \alpha < 1 \text{ and } \beta = 1 \\ (-\infty, \tan\frac{\pi\alpha}{2}] & \alpha < 1 \text{ and } \beta = -1 \\ (-\infty, \infty) & \text{otherwise.} \end{cases}$$

Remark 1 *At this point we note that in a financial context the plausible range for α is in the interval $1 < \alpha \leq 2$. First, for this range the first moments exist. Second, $\alpha < 1$ implies that as the distribution of the process becomes more skewed, $\beta = \pm 1$ the support will be the half real line, which clearly is not financially plausible.*

As for the Brownian motion case we can define the Lévy-Stable motion.

Definition 3 ***Standard Lévy-Stable motion.***

A stochastic process $X(t)$ is called a Standard Lévy-Stable motion if

1. *$X(0) = 0$ a.s.,*
2. *X has stationary increments, and*
3. *$X(t) - X(s) \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0)$ for any $0 \leq s < t < \infty$ and for some $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$.*

Observe that the process is Brownian motion when $\alpha = 2$ and $\beta = 0$.

Remark 2 *In the sequel we will encounter stochastic integrals of the form*

$$I(f) = \int_a^b f(s)dL(s)$$

where f is a deterministic function and dL is the increment of a standard Lévy-Stable motion. These stochastic integrals are defined in a very similar way to stochastic integrals with respect to Brownian motion. To ensure the existence of the stochastic integral $I(f)$ it will suffice, see [23], to check that

$$I(f) = \int_a^b |f(s)|^\alpha ds < \infty \qquad \textit{for } \alpha \neq 1$$

and

$$I(f) = \int_a^b |f(s) \log |f(s)||ds < \infty \qquad \textit{for } \alpha = 1.$$

It was thought that the Lévy-Stable hypothesis for stock returns or log-stock prices would not deliver reasonable prices for financial instruments. Merton [18] conjectured that if the process for price changes were a function of Lévy-Stable distributions with infinite moments, the only equilibrium value for a warrant would be the stock price itself, independent of the length to maturity. Moreover, Merton conjectured that an infinite expected future price for a stock would require an infinite discount rate to obtain finite values for the stock prices.

The existence of the Laplace transform of a totally skewed to the right Lévy-Stable random variable will enable us to prove the existence of option prices for the symmetric Lévy-Stable case as a weighted average of the classical Black-Scholes price when Brownian motion drives the conditional underlying uncertainty. First we see that any symmetric Lévy-Stable random variable can be represented as the product of a totally skewed with a symmetric Lévy-Stable variable as shown by the following proposition.

Proposition 3 ***Constructing Symmetric Variables.*** *Let $X \sim S_{\alpha'}(\kappa, 0, 0)$, $Y \sim S_{\alpha/\alpha'}((\cos \frac{\pi\alpha}{2\alpha'})^{\frac{\alpha'}{\alpha}}, 1, 0)$ with $0 < \alpha < \alpha' \leq 2$. Then if X and Y are independent, the random variable*

$$Z = Y^{1/\alpha'} X \sim S_\alpha(\kappa, 0, 0).$$

Note that we may use Brownian motion as one of the building blocks to obtain symmetric Lévy-Stable processes, see [2] and [23].

3 Option Pricing for Symmetric Lévy-Stable Processes

We now exploit the close relationship between totally skewed and symmetric Lévy-Stable random variables shown in Proposition 3. The aim is to calculate European-style option prices where the underlying log-stock or stock returns process is driven by a Lévy-Stable process with $\alpha \in (1, 2]$. As mentioned above, the GCLT indicates that the limiting distribution of the sum of many iid events is Lévy-Stable. Apart from this feature, we know that empirical data of market returns show that Brownian motion is a poor model given the rapid decay of the tails of its distribution. However, we know that Lévy-Stable distributions can accommodate thickness of tails through its shape parameter. Property 2 above shows that the smaller is the parameter α the heavier the tail. Hence, our first step is to construct a more plausible model of stock returns, from both a theoretical and empirical point of view; therefore we will use the symmetric Lévy-Stable motion to model the stochastic shocks to the stock returns process. However, on the other hand, since the Brownian motion case has been considerably studied we would like to build on these widely known results and extend them to satisfy our requirements. Therefore we will 'construct' the symmetric Lévy-Stable process with a Gaussian random variable, ie $\alpha = 2$, together with a totally skewed Lévy-Stable random variable.

3.1 Option pricing for symmetric Lévy-Stable processes

In view of Proposition 3 we know how to obtain a symmetric Lévy-Stable process from two independent Lévy-Stable random variables. Therefore, it seems natural to look at a stock returns process that is conditionally Gaussian and enquire whether we can price vanilla options.

Conjecture 1 ***Option Prices for Symmetric Lévy-Stable Processes.*** *Let $dW \sim N(0, dt)$ be independent of a totally skewed to the right Lévy-Stable process $\int_t^T Y(s)ds$.*

Now, let the price process, under the physical measure, be

$$S_T = S_t e^{(T-t)\mu + \int_t^T \sqrt{Y(s)}dW(s)}. \tag{7}$$

Let V(S,t) be the value of a European vanilla option written on the underlying stock price $S(t)$ with payoff $\Pi(S, t)$. Then the value of the financial instrument is given by

$$V(S,t) = \mathbb{E}_Y^Q \left[V_{BS}\left(S(t), t, K, \left(\frac{1}{T-t} \int_t^T Y(s)ds \right)^{1/2}, T \right) \right], \tag{8}$$

where the expected value is with respect to the random variable Y under the risk-neutral measure Q. Here $V_{BS}(S(t), t, K, \overline{Y}^{1/2}, T)$ is the Black-Scholes price with strike price K, at time t and 'volatility' $\overline{Y}^{1/2} = \left(\frac{1}{T-t} \int_t^T Y(s)ds \right)^{1/2}$.

If we assume that the distribution of $\int_t^T Y(s)ds$ is correctly specified then we can proceed to 'show' the conjecture. Before proceeding we note that since we are using Brownian motion as one of the building blocks to construct a symmetric Lévy-Stable process we have that (from Proposition 3) $\alpha' = 2$ and that is why we let the constant $\cos \frac{\pi\alpha}{2\alpha'} = \cos \frac{\pi\alpha}{4}$. Note also that $dW \sim S_2 \left(\frac{1}{\sqrt{2}} dt^{1/2}, 0, 0 \right)$, but for clarity we choose to use the more usual notation for normally distributed random variables $dW \sim N(0, dt)$.

Proof

We first ensure that the discounted stock process is, under the risk-neutral measure Q, a martingale. So that

$$\mathbb{E}^Q[e^{r(T-t)}S_T|S_t] = S_t.$$

Therefore we have that the stock process under the measure Q, (the change of measure can be performed as in [12]), is given by

$$S_T = S_t e^{r(T-t)-\frac{1}{2}\int_t^T Y(s)ds + \int_t^T \sqrt{Y(s)}dW(s)}. \tag{9}$$

The value of the option can be expressed as the expected value of the discounted payoff $\Pi(S,T)$:

$$V(S,t) \quad = \quad \mathbb{E}^Q\left[e^{-r(T-t)}\Pi(S,T)\right].$$

Now conditioning on the path of $Y(s)$, $t \le s \le T$, using iterated expectations and noting the inner expectation is given by the Black-Scholes formula we get

$$\begin{aligned} V(S,t) &= \mathbb{E}_Y^Q\left[\mathbb{E}^Q\left[e^{-r(T-t)}\Pi(S,T) \mid Y, t \le s \le T\right]\right] \\ &= \mathbb{E}_Y^Q\left[\mathrm{V_{BS}}\left(S(t), t, K, \left(\frac{1}{T-t}\int_t^T Y(s)ds\right)^{1/2}, T\right)\right]. \end{aligned}$$

Conjecture If we conjecture that the distribution of the process $\int_t^T Y(s)ds$ is such that

$$\int_t^T Y(s)ds \sim S_{\alpha/2}\left((\cos\frac{\pi\alpha}{4})^{2/\alpha}(T-t)^{\frac{2}{\alpha}}, 1, 0\right) \qquad \text{with } 1 < \alpha < 2, \tag{10}$$

we can show that the shocks to the stock process, under the physical measure, are driven by a symmetric Lévy-Stable process. We derive the characteristic function of $\int_t^T \sqrt{Y(s)}dW(s)$. First condition on the path of $Y(s)$ and then use iterated expectations to get

$$\begin{aligned} \mathbb{E}\left[e^{i\theta\int_t^T \sqrt{Y(s)}dW(s)}\right] &= \mathbb{E}_Y\left[\mathbb{E}\left[e^{i\theta\int_t^T \sqrt{Y(s)}dW(s)}|Y, t \le s \le T\right]\right] \\ &= \mathbb{E}_Y\left[e^{-\frac{1}{2}\theta^2\int_t^T Y(s)ds}\right] && (11) \\ &= e^{-\left(\frac{1}{2}\right)^{\alpha/2}|\theta|^\alpha(T-t)}. && (12) \end{aligned}$$

Hence

$$\int_t^T \sqrt{Y(s)}dW(s) \sim S_\alpha\left((1/2)^{1/2}(T-t)^{1/\alpha}, 0, 0\right),$$

ie a symmetric Lévy-Stable process.

□

Assuming that the distribution of the process $\int_t^T Y(s)ds$ is correctly specified we can interpret the above result. The intuition behind the option value is surprisingly simple: for log-stock processes that follow a symmetric Lévy-Stable process under the physical measure, option values are given by the weighted average, ie expected value, of the Black-Scholes formula where the 'volatility' is the square root of a totally skewed to the right Lévy-Stable random variable with support in the positive real line. We point out that this intuitive explanation refers to the variable $\sqrt{Y(t)}$ as the volatility, hence $\overline{Y} = \frac{1}{T-t}\int_t^T Y(s)ds$ can be seen as the 'integrated variance'. In option pricing, within the Black-Scholes framework, integrated variance is a very important component. If volatility is not constant, European option prices depend on the average variance of the underlying from the initial time t to expiry T. The following section shows that it is possible to model integrated variance as a totally skewed Lévy-Stable process and shows that option prices, where the log-stock prices follow a symmetric Lévy-Stable process, can be calculated.

4 Stochastic Volatility with Lévy-Stable Shocks

In view of Conjecture 1, and in particular the form in which the positive random variable $Y(s)$ enters the pricing equation (8), there seems to be an obvious choice to model volatility. We recall from Conjecture 1 that the form of 'average variance' is $\overline{Y} = \frac{1}{T-t}\int_t^T Y(s)ds$, where $Y(s)$ is a totally skewed to the right Lévy-Stable random variable, therefore one might be tempted to model variance, instead of volatility, as $\int_t^T \sigma^2(s)ds = \int_t^T Y(s)ds$. Hence, assuming that this is a feasible choice ie that our conjecture for the distribution of $\int_t^T Y(s)ds$ given by (10) is correct, we are saying that the SDE for the variance process could be taken as

$$\int_t^T \sigma^2(s)ds = \int_t^T dL(s), \tag{13}$$

with $\int_t^T dL(t)$ being a totally skewed to the right and positive Lévy-Stable motion. Note that, by Definition 3, we have

$$\int_t^T dL(s) \sim S_{\alpha/2}\left((T-t)^{\frac{2}{\alpha}}, 1, 0\right) \qquad \text{with } 1 < \alpha < 2.$$

Hence, at this point, we ask the question of whether a similar solution to (8) may be derived under a 'stochastic volatility' model and whether we can assume that the distribution of the integrated volatility process can be assumed to be as in (10). In other words, we ask whether we can state the same problem as in Conjecture 1 but with a second SDE driving the volatility process in the following way:

$$\begin{aligned} \ln(S_t/S_0) &= \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s) \\ \sigma(S,t) &= f(Y(t)), \end{aligned}$$

where $Y(t)$ follows a certain process that must be specified. Before addressing this question we note the following:

- First, we would like to specify a stochastic volatility model that yields a solution similar to (8) in the sense that the integrated variance, ie $\int_t^T \sigma^2(s)ds$, has a totally skewed to the right distribution so option prices are in fact those of a stock process where shocks are given by a symmetric Lévy-Stable process as in definition 3 above.
- Second, an interesting feature is that the variable $Y(s)$ is a non-negative random variable (ie $\alpha/2 < 1$); therefore we may model directly the variance process instead of volatility since it will be simpler to obtain the distribution of the integrated variance. In fact, finding the distribution of the variance process given the distribution of the volatility process is extremely hard (and the solution is not known for some classical stochastic volatility models such as the Hull-White model [11]).

Unfortunately the above choice to model integrated variance is not mathematically correct. Let us inspect the SDE (13) and explain intuitively why this is not a feasible choice. On the left-hand side we have the integrated variance $\int_t^T \sigma^2(s)ds$ which is, by construction, a continuous process. However, on the right hand side of the SDE we have the nonnegative Lévy-Stable motion $\int_t^T dL(s)$ which is by construction a purely discontinuous process.

Although the choice presented above is not correct we ask whether we can modify the SDE (13) to make it continuous and still have the integrated variance follow a totally skewed to the right Lévy-Stable process. In other words, is it possible to obtain a continuous process driven by a Lévy-Stable stochastic component? What are the sample path properties required to model integrated variance?

4.1 Sample Path Properties: Modelling Integrated Volatility

In this section we show that it is possible to specify a model for stochastic variance such that its finite-dimensional distribution is a totally skewed to the right Lévy-Stable and it possesses continuous paths. Intuitively one would like to start to understand how a purely discontinuous process such as the Lévy motion $\int_t^T dL(t)$ can be modified to obtain a continuous process. We may start by asking if introducing a suitable deterministic function of time $f(s,T)$ with $s \in \mathbb{R}^+$ in the kernel of $\int_t^T f(s,T)dL(s)$ can 'damp' the jump process and 'force' it to be continuous. The answer to this question is yes and depending on the behaviour of the kernel $f(s,T)$. For a general discussion on the path behaviour of processes of the type $\int_t^T f(s,T)dL(s)$ see [23]. In our case it will suffice to check path continuity using Kolmogorov's Continuity Condition.[1] Before stating Kolmogorov's condition we give the following definitions.

We can define a stochastic process $\{X(t), t \in T\}$, where T is a time interval, by its finite-

[1] The authors are grateful to Ben Hambly for his comments on parts of this subsection.

dimensional distributions. We can also view it as a collection of random variables and study their sample paths. One of the important concepts we introduce is that of *version*, which applies to stochastic processes in general. The process $\tilde{X}(t)$ is a version (also known as *equivalent*) of the process $X(t)$ if they have the same finite-dimensional distribution; this will be made precise below in Definition 5, but before that we provide a working definition of sample path.

Definition 4 ***Sample Path.*** *Let $X = \{X(t), t \in T\}$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$. Each element $\omega \in \Omega$ gives rise to a realisation or sample path denoted*

$$\{X(t, \omega), t \in T\}.$$

Definition 5 ***Version of a Stochastic Process.*** *Two stochastic processes $\{X(t), t \in T\}$ and $\{Y(t), t \in T\}$ are said to be versions of each other if they have the same finite-dimensional distributions, ie,*

$$\{X(t_n), n = 1, 2..., N\} \stackrel{d}{=} \{Y(t_n), n = 1, 2..., N\},$$

for any N and $t_1, \cdots, t_N \in T$; where the equality is in distribution.

Hence a version of a stochastic process is a representative of the equivalence class of all stochastic processes with a given set of finite-dimensional distributions. We must note that although the versions of the same process are defined on the same probability space they represent, in general, two different collections of measurable functions on that space but their path properties may be different. Moreover, if $\{X(t), t \in T\}$ and $\{Y(t), t \in T\}$ are defined on the same probability space and

$$\mathbb{P}(X(t) = Y(t), t \in T) = 1,$$

then obviously the two processes are versions of each other; however they might still possess very different sample paths [23].

Proposition 4 ***Kolmogorov's Continuity Condition.*** *Let $\{X(t), t \geq 0\}$ be a stochastic process obeying the bound*

$$\mathbb{E}[|X(t) - X(s)|^a] \leq D|\varphi(t) - \varphi(s)|^{1+b} \qquad \textit{for all} \qquad s, t \geq 0,$$

where a, b, and D are positive constants independent of s and t and φ is a continuous nondecreasing function. Then there exists a version $\tilde{X}(t)$ of $X(t)$ possessing continuous paths.

Remark 3 *In the sequel when a stochastic process is considered and there is an equivalent version of it satisfying Kolmogorov's condition we will always use the continuous version.*

Now we go back to our question of finding a deterministic function of time $f(s,T)$ which we take to be of the form $g(T-s)$, such that the integral representation of the process $\int_t^T g(T-s)dL(s)$ is continuous in T, ie it satisfies Kolmogorov's Continuity Condition. Since we are interested in pricing options where the underlying stochastic component is driven by a symmetric Lévy-Stable process we would like to specify a kernel $g(T-s)$ so the finite-dimensional distribution of $\int_t^T g(T-s)dL(s)$ is symmetric Lévy-Stable. There are a many such functions, hence we denote the class of such functions by $\mathbb{F}$, of which we show three examples. Two possible choices are

$$g(T-s) = T-s \qquad T \geq s \geq 0, \tag{14}$$

$$g(T-s) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-s)^n}\right) \qquad \text{for} \qquad T, s \geq 0 \text{ and } n \geq 1, \tag{15}$$

where γ is a positive constant that can be seen as a damping factor.

Initially we think of γ as a parameter that we can choose freely. As we shall see below, in Proposition 7, any particular choice of γ will still yield the desired result: to price options where returns follow a symmetric Lévy-Stable process.

Remark 4 *Moreover, we note that when $n = 1$ we get an Ornstein-Uhlenbeck-type process (OU-type). An OU-type model is the 'extension' of an OU process that instead of the shocks being driven by Brownian motion they are driven by a Lévy process, see [26]. Barndorff-Nielsen and Shephard [1] were the first to introduce OU-type stochastic volatility models driven by positive Lévy processes.*

A third choice is

$$g(T-s) = \ln(T-s+1) \qquad \text{for} \qquad T \geq s \geq 0. \tag{16}$$

Now, we continue by checking that the integral representations, for the kernels presented above,

$$\int_t^T (T-s)dL(s), \quad \int_t^T \frac{1}{\gamma}\left(1 - e^{-\gamma(T-s)^n}\right)dL(s), \quad \text{and} \int_t^T \ln(T-s+1)dL(s)$$

have versions with continuous paths. We proceed to prove it in detail for the second integral representation since the proof for the other cases are very similar. Note that in the three cases we have chosen $g(T-s)$ so that $g(0) = 0$. Intuitively as $s \to T$ the last 'jumps' of the process $L(t)$ are 'killed' leading to a continuous process.

Before proceeding we derive, since we will use this result for the proof of continuity, the distribution of the integral representation $\int_t^T \frac{1}{\gamma}\left(1 - e^{-\gamma(T-s)^n}\right)dL(s)$ which is given by the following proposition.

Proposition 5 *Let $dL \sim S_\alpha(dt^{1/\alpha}, 1, 0)$ with $0 < \alpha < 1$. Then the process*

$$X(t) = \int_0^T \frac{1}{\gamma}\left(1 - e^{-\gamma(T-s)^n}\right) dL(s) \sim S_\alpha\left(\left(\frac{1}{\gamma}\int_0^T \left|1 - e^{-\gamma(T-s)^n}\right|^\alpha ds\right)^{1/\alpha}, 1, 0\right).$$

Proof

We find the distribution of the stochastic term $\int_0^T \frac{1}{\gamma}\left(1 - e^{-\gamma(T-s)^n}\right) dL(s)$ by deriving its characteristic function. We first partition the time interval $[0, t]$ into n steps

$$0 = t_0 < t_1 < t_2 < ... < t_n = T.$$

Now we let $f(s) = f(t_k)$ for $s \in [t_k, t_{k+1})$ and use the approximation to the stochastic integral $\int_t^T f(s)dL(s) \approx \sum_{k=0}^{n-1} f(t_k)\{L(t_{k+1}) - L(t_k)\}$. We write

$$\begin{aligned} E\left[e^{i\theta \int_0^t \frac{1}{\gamma}\left(1-e^{-\gamma(T-s)^n}\right)dL(s)}\right] &\approx E\left[e^{i\theta \sum_{k=0}^{n-1} f(t_k)\{L(t_{k+1})-L(t_k)\}}\right] \\ &= \prod_{k=0}^{n-1} e^{|\theta f(t_k)|^\alpha \Delta t\{1 - i\text{sign}(\theta f(t_k))\tan(\pi\alpha/2)\}} \\ &= e^{|\theta|^\alpha \int_0^T |f(s)|^\alpha ds\{1 - i\text{sign}(\theta)\tan(\pi\alpha/2)\}}. \end{aligned}$$

Therefore, by letting the mesh of the partition go to zero,

$$\int_0^T \frac{1}{\gamma}\left(1 - e^{-\gamma(T-s)^n}\right) dL(s) \sim S_\alpha\left(\left(\frac{1}{\gamma}\int_0^T \left|1 - e^{-\gamma(t-s)^n}\right|^\alpha ds\right)^{1/\alpha}, 1, 0\right).$$

□

Now we show that the integral representation $\gamma^{-1}\int_t^T \left(1 - e^{-\gamma(T-s)^n}\right) dL(s)$ has a version with continuous paths.

Proposition 6 *Let the stochastic process $X(t)$ have the integral representation*

$$\int_t^T \frac{1}{\gamma}\left(1 - e^{-\gamma(T-s)^n}\right) dL(s)$$

where $\int_t^T dL(s)$ is a totally skewed to the right Lévy-Stable motion, i.e. $L(T) - L(t) \sim S_\alpha\left((T-t)^{1/\alpha}, 1, 0\right)$ with $0 < \alpha < 1$. Then $X(t)$ has a continuous version.

Proof

Let $t \in [s, T]$. Then according to Property 1 for any $0 < a < \alpha$ we have that the following moment exists:

$$\mathbb{E}\left[|X(T) - X(s)|^a\right] = \mathbb{E}\left[\left|\int_s^T \frac{1}{\gamma}\left(1 - e^{-\gamma(T-u)^n}\right) dL(u)\right|^a\right].$$

Now invoking Property 5 we can write the expectation as

$$\begin{aligned}\mathbb{E}\left[|X(T)-X(s)|^a\right] &= \mathbb{E}\left[\left|\left(\int_s^T \frac{1}{\gamma}\left(1-e^{-\gamma(T-u)^n}\right)^\alpha du\right)^{1/\alpha}\right|^a \phi^a\right] \\ &= \left|\left(\int_s^T \frac{1}{\gamma}\left(1-e^{-\gamma(T-u)^n}\right)^\alpha du\right)^{1/\alpha}\right|^a \mathbb{E}\left[\phi^a\right]\end{aligned}$$

where $\phi \sim S_\alpha(1,1,0)$ and recall that since $\alpha < 1$ the random variable ϕ is positive. Moreover, recall from Property 1 above that $\mathbb{E}\left[\phi^a\right] = D < \infty$ if $a < \alpha$. Now noting that for $u < t$ the function $f(u) = \frac{1}{\gamma}\left(1-e^{-\gamma(T-u)^n}\right)^\alpha$ is decreasing in u, then

$$\left|\left(\int_s^T \frac{1}{\gamma}\left(1-e^{-\gamma(T-u)^n}\right)^\alpha du\right)^{1/\alpha}\right|^a D \leq \left|\left(\frac{1}{\gamma}(T-s)\left(1-e^{-\gamma(T-s)^n}\right)^\alpha\right)^{1/\alpha}\right|^a D$$

and using the fact that $1-e^{-x} \leq x$ we have that

$$\left|\left(\frac{1}{\gamma}(T-s)\left(1-e^{-\gamma(T-s)^n}\right)^\alpha\right)^{1/\alpha}\right|^a D \leq D(T-s)^{\frac{a}{\alpha}+na}.$$

Finally, it is straightforward to see that a can be chosen so that $n > \frac{1}{a}-\frac{1}{\alpha}$, hence $a(n+1/\alpha) > 1$ for $n \geq 1$.

□

4.2 Building Blocks for the Integrated Variance

In this subsection we depict the different building blocks to obtain the integrated variance process described above. First we simulate a totally skewed to the right Lévy-Stable motion; then we get the spot variance process, by choosing an appropriate kernel; then we produce the integrated variance process. One can think of the integrated variance process as a 'smooth' version of the Lévy-Stable motion resulting from the choice of kernel $g(t-s)$. We focus on kernels of the integrated variance of the form

$$g(T-s) = \frac{1}{\gamma}\left(1-e^{-\gamma(T-s)^n}\right).$$

We look at two cases and depict them below. For both figures we have used the same Lévy-Stable motion as a building block to produce the spot and the integrated variance. In Figure 1 we used $g(T-s) = \gamma^{-1}(1-e^{-\gamma(T-s)^n})$ with $n = 1$ and $\gamma = 25$, which would yield a standard OU process. In Figure 2 we used the same kernel $g(T-s) = \gamma^{-1}(1-e^{-\gamma(T-s)^n})$ but with $n = 1.2$ and $\gamma = 25$. Note that the higher is the constant n the 'smoother' is the path of the integrated variance.

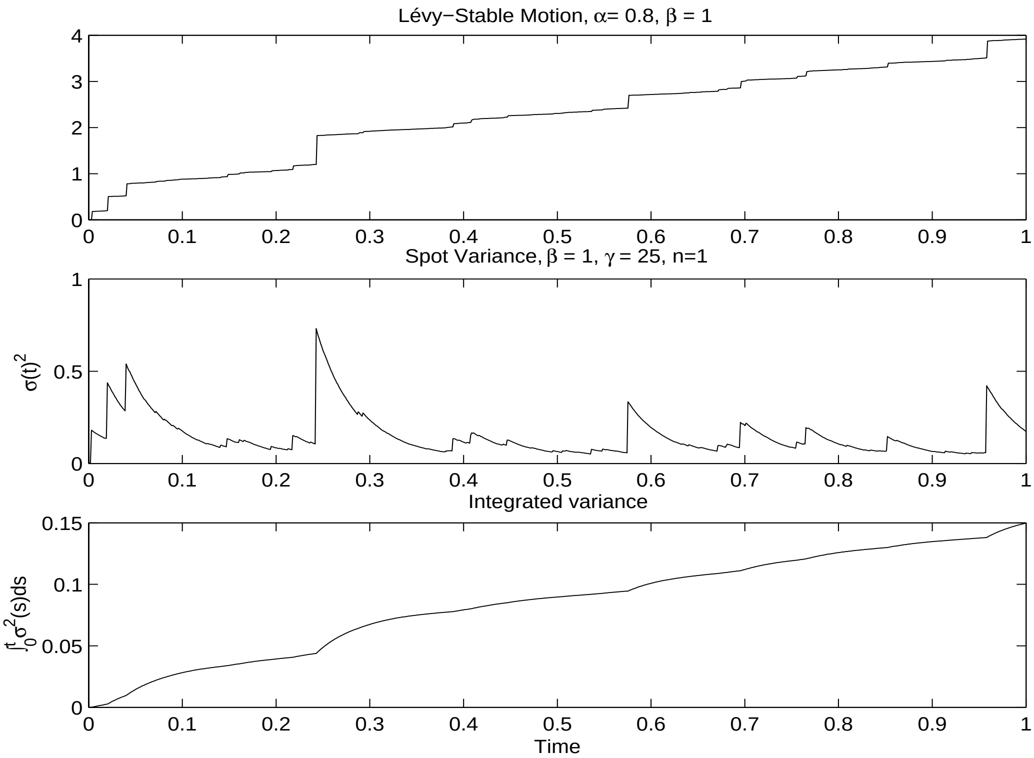

Figure 1: Simulated integrated variance with kernel $g(T-s) = 25^{-1}\left(1 - e^{-25(1-s)}\right)$.

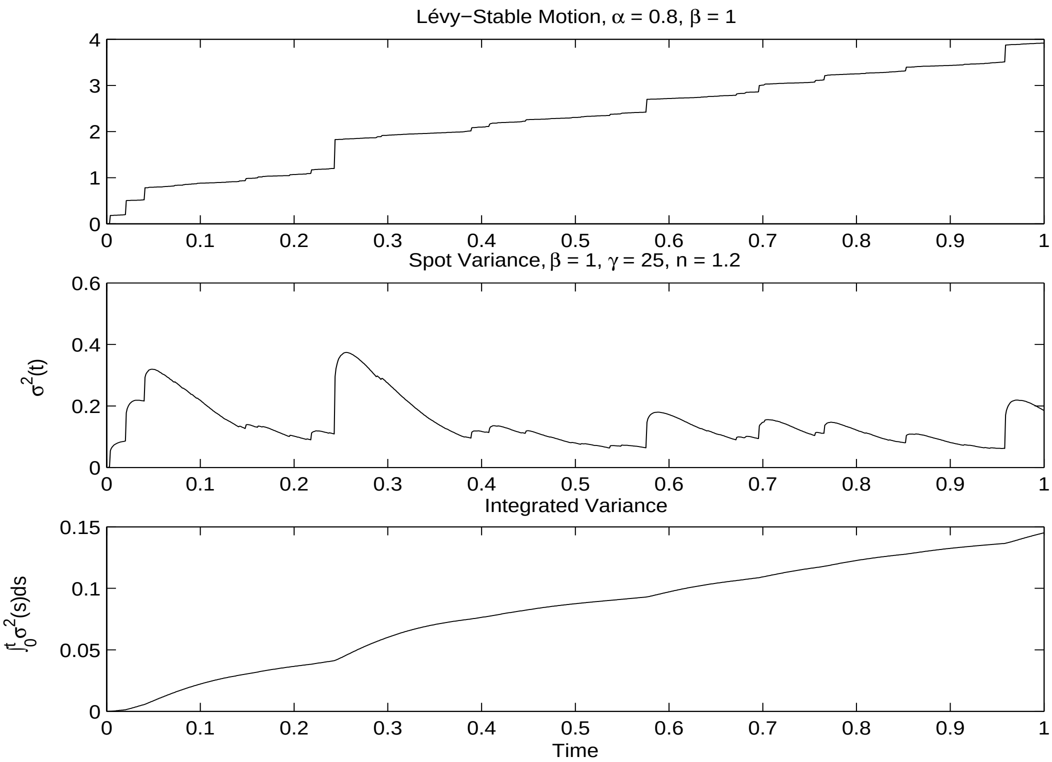

Figure 2: Simulated integrated variance with kernel $g(T-s) = 25^{-1}\left(1 - e^{-25(1-s)^{1.2}}\right)$.

4.3 Option Pricing with Lévy-Stable Volatility

The preceding subsections were devoted to finding a suitable model for stochastic volatility that would enable us to model the unconditional returns process as a symmetric Lévy-Stable process. With all the different components in place we can turn again to the question of option pricing with Lévy-Stable processes.

Before proceeding we point out that in their original work Mandelbrot and Taylor [15] proposed a model where returns are measured in terms of transaction volume. The cumulative volume, denoted by $T(t)$, followed a totally skewed Lévy-Stable process with $0 < \alpha < 1$ and the returns process was measured in terms of calendar time: $Z = W[T(t)]$ where

$$T(t+s) - T(t) \sim S_{\alpha/2}\left(s^{1/\alpha}\left(\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}, 1, 0\right)$$

and W is a standard Brownian motion. It is straightforward to show that $Z \sim S_\alpha(1/\sqrt{2}, 0, 0)$ for $0 < \alpha < 2$. In Hurst, Platen and Rachev [12], based on the Mandelbrot-Taylor model, show how to price European options when the returns of the stock process follow, under the physical measure, a symmetric Lévy-Stable motion.

Proposition 7 *Let the log-stock process follow, under the physical measure, a standard Brownian motion with integrated variance as a function of an asymmetric Lévy-Stable motion given by the following:*

$$\ln(S_T/S_t) = \int_t^T \mu(s)ds + \int_t^T \sigma(s)dW(s) \tag{17}$$

$$\int_t^T \sigma^2(s)ds = \int_t^T g(T-s)dL(s). \tag{18}$$

Here $dW(t)$ is the standard Brownian motion and

$$dL(t) \sim S_{\alpha/2}\left(2\left(\frac{1}{2}\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}\sigma_{LS}^2 dt^{2/\alpha}, 1, 0\right)$$

is a Lévy-Stable motion with $0 < \alpha < 2$, $g(T-s) \in \mathbb{F}$, and $\mu(s)$ is a deterministic function. Then option prices, under the risk-adjusted measure Q, for a European vanilla option with payoff $\Pi(S,t)$ are given by

$$V(S,t) = \mathbb{E}^Q_{\sigma(t)}\left[V_{BS}\left(S(t), t, K, \left(\frac{1}{T-t}\int_t^T \sigma^2(s)ds\right)^{1/2}, T\right)\right]. \tag{19}$$

Proof The proof is similar to that of Proposition 1. It is straightforward to show that the distribution of the shocks to the stock process is symmetric Lévy-Stable. First note that the stochastic component of the returns process is given by the random process

$$Z(T) = \int_t^T \sigma(s)dW(s). \tag{20}$$

Now we calculate the characteristic function of the random process $Z(T)$. We have

$$\mathbb{E}[e^{i\theta Z(T)}] = \mathbb{E}[e^{i\theta \int_t^T \sigma(s)dW(s)}],$$

and conditioning on the path of $\sigma(s)$ and using iterated expectations we get

$$\mathbb{E}[e^{i\theta Z(T)}] = \mathbb{E}\left[e^{-\frac{1}{2}\theta^2 \int_t^T \sigma^2(s)ds}\right].$$

Now, given that $\int_t^T \sigma^2(s)ds = \int_t^T g(T-s)dL(s)$ we write

$$\begin{aligned}\mathbb{E}[e^{i\theta Z(T)}] &= \mathbb{E}\left[e^{-\frac{1}{2}\theta^2 \int_t^T g(T-s)dL(s)}\right] \\ &= e^{-\frac{1}{2}\sigma_{LS}^\alpha \int_t^T g(T-s)^{2/\alpha}ds|\theta|^\alpha}.\end{aligned}$$

This is clearly the characteristic function of a symmetric Lévy-Stable process with index α.The result follows since the process under the risk-neutral measure Q follows

$$S_T = S_t e^{r(T-t)-\frac{1}{2}\int_t^T \sigma^2(s)ds + \int_t^T \sigma(s)dW(s)}.$$

□

Note that since the integrated variance is distributed as

$$S_{\alpha/2}\left(2\left(\frac{1}{2}\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}\sigma_{LS}^2 G(T,t)^{2/\alpha}, 1, 0\right)$$

where $G(T,t) = \int_t^T g(T-s)^{2/\alpha}ds$ and $\alpha/2 < 1$, its first moment does not exist, ie $\mathbb{E}[\int_t^T \sigma^2(s)ds] = \mathbb{E}[\int_t^T g(T-s)dL(s)] = \infty$.

Remark 5 *We also note that integrated variance can be modelled as*

$$\int_t^T \sigma^2(s)ds = h(T,t)\int_t^T g(T-s)dL(s) \tag{21}$$

where $h(T,t)$ is a deterministic function of time. In section 6 we illustrate this flexibility by choosing a plausible $h(T,t)$.

Remark 6 *In the proposition above we could have also used a model where the **returns** process followed a symmetric Lévy-Stable process and we would have obtained the same value for the option, $V(S,t)$. That is, assuming*

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma(t)dW(t) \tag{22}$$

$$\int_t^T \sigma^2(s)ds = \int_t^T g(T-s)dL(s), \tag{23}$$

where $dW(t)$ denote the increments of the standard Brownian motion and

$$dL(t) \sim S_{\alpha/2}\left(2\left(\frac{1}{2}\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}\sigma_{LS}^2 dt^{2/\alpha}, 1, 0\right)$$

is a Lévy-Stable motion with $0 < \alpha < 2$, $f(s) \in \mathbb{F}$, and r is the risk-free rate. Then, option prices, under the risk-adjusted measure Q', for a European vanilla option with payoff $\Pi(S,t)$ are given by

$$V(S,t) = \mathbb{E}_{\sigma(t)}^{Q'}\left[V_{BS}\left(S(t), t, K, \left(\frac{1}{T-t}\int_t^T \sigma^2(s)ds\right)^{1/2}, T\right)\right].$$

Note that since the integrated variance process is independent of the Brownian motion we can, using Ito's lemma, write the solution to the returns SDE (22) as

$$S_T = S_t e^{r(T-t)-\frac{1}{2}\int_t^T \sigma^2(s)ds + \int_t^T \sigma(s)dW(s)};$$

which is the same expression obtained above for the stock process under the risk-neutral measure Q.

In the introduction we mentioned, based on the GCLT, that there were two ways of modelling stock prices or stock returns. We pointed out that most of the literature models log-stock prices as a Lévy-Stable process, [16], [4], [7]. In the case presented above, when the driving Lévy-Stable process is symmetric (constructed with Brownian motion and a totally skewed Lévy-Stable process) both the log-stock price and returns follow, under the physical measure, a symmetric Lévy-Stable process. In Section 5 when we extend these results to include asymmetric Lévy-Stable processes we model log-stock prices and not returns.

As an example, we can use the approach above to derive closed-form solutions for option prices when the random shocks to the price process are distributed according to a Cauchy Lévy-Stable process, $\alpha = 1$ and $\beta = 0$.

Remark 7 ***Closed-form Solution when Returns follow a Cauchy Process.*** *Let the returns process follow, under the physical measure, a Brownian motion with volatility as a function of a totally skewed Lévy-Stable random variable, given by*

$$dS(t) = rS(t)dt + \sigma(t)S(t)dW(t) \tag{24}$$
$$\int_t^T \sigma^2(s)ds = \int_t^T g(T-s)dL(s). \tag{25}$$

Here $dW(t)$ is the standard Brownian motion, $dL(t) \sim S_{1/2}\left(dt^2, 1, 0\right)$, $g(T-s) \in \mathbb{F}$ and r denotes the risk-free rate. Then option prices, under the risk-adjusted measure Q, are given by

$$V(S,t) = \frac{T-t}{2\pi}\int_0^\infty V_{BS}(S(t), K, \overline{Y}^{1/2}, T)\frac{e^{-\frac{(T-t)^2}{2y}}}{y^{3/2}}dy,$$

where $\overline{Y} = \frac{1}{T-t}\int_t^T \sigma^2(s)ds$.

First, we express the value of a vanilla option as the discounted expected value of the terminal payoff:

$$V(S,t) = \mathbb{E}^Q\left[e^{-r(T-t)}\Pi(S,T)\right].$$

Conditioning on the volatility path and using iterated expectations we get

$$\begin{aligned} V(S,t) &= \mathbb{E}^Q\left[\mathbb{E}^Q\left[e^{-r(T-t)}\Pi(S,T) \mid \sigma(t), t \le s \le T\right]\right] \\ &= \mathbb{E}^Q[\mathrm{V_{BS}}(S(t), K, \overline{Y}^{1/2}, T)]. \end{aligned}$$

Recall that the pdf for the Lévy-Smirnov distribution, $S_{1/2}(\kappa, 1, 0)$, is given by $(\kappa/2\pi)^{1/2}\, x^{-3/2}e^{-\kappa/2x}$ with support $(0, \infty)$. Hence

$$V(S,t) = \left(\frac{T-t}{2\pi}\right)^{1/2} \int_0^\infty \mathrm{V_{BS}}(S(t), K, \overline{Y}^{1/2}, T)\frac{1}{y^{3/2}}e^{-\frac{T-t}{2y}}\, dy,$$

where $\overline{Y}^{1/2}$ is a totally skewed to the right Lévy-Stable process with $\alpha = 1/2$. Moreover, the distribution of the shocks to the returns process (24) is Cauchy, ie $\alpha = 1, \beta = 0$.

$\square$

Because the combination of a Gaussian and Lévy-Smirnov $S_{1/2}(\kappa, 1, m)$ random variables results in a Cauchy random variable $S_1(\kappa, 0, m)$. This can be seen either by calculating the characteristic function of $(dL)^{1/2}dW$ or by the convolution of their respective pdf's.

5 Option Pricing for Lévy-Stable Processes with Leverage Effect

Financial data suggests [3], [13], [6] that returns are skewed rather than symmetric. Therefore we enquire whether the results above can be extended to a more general case where the shocks to the stock process follow an asymmetric Lévy-Stable process instead of a symmetric Lévy-Stable motion or process. This section shows that this is possible when the skewness parameter $\beta \in (-1, 1)$.

In stochastic volatility models one way to introduce skewness in the log-stock process is to correlate the random shocks of the volatility process to the shocks of the stock process. It is typical in the literature to assume that the Brownian motion of the stock process, say $dW(t)$, is correlated with the Brownian motion of the volatility process, say $dZ(t)$. Thus $\mathbb{E}[dW(t)dZ(t)] = \rho dt$ and we can write $\tilde{Z}(t) = \rho W(t) + \sqrt{1-\rho^2}Z(t)$, where $\tilde{Z}(t)$ is independent of $W(t)$. The correlation parameter ρ is also known in the literature as the leverage effect and empirical studies [10] suggest that $\rho < 0$. In our case we may also include a leverage effect via a parameter ℓ to produce skewness in the stock returns. However, the notion of 'correlation' does not apply in our case because for Lévy-Stable random variables, given that moments of second and higher order do not exist, correlations do not exist. Proposition 8 shows that we can introduce negative skewness via a negative leverage effect. The financial interpretation is that in periods of high volatility prices go down and vice versa.

Proposition 8 ***Option Pricing for Asymmetric Lévy-Stable Processes.*** *Let the log-stock process follow, under the physical measure, a Brownian motion plus a leverage effect. Let the integrated volatility follow a skewed Lévy-Stable process. That is, suppose that*

$$\ln(S_T/S_t) = \int_t^T \mu(s)ds + \int_t^T \sigma(s)dW(s) + \ell \int_t^T d\tilde{L}(s) \tag{26}$$

$$\int_t^T \sigma^2(s)ds = \int_t^T g(T-s)dL(s). \tag{27}$$

Here $dW(t)$ is the standard Brownian motion independent of both $d\tilde{L}(t)$ and $dL(t)$; also

$$d\tilde{L}(t) \sim S_\alpha\left(\frac{1}{2^{1/\alpha}}\frac{\sigma_{TL}}{(1+\ell^\alpha)^{1/\alpha}}dt^{1/\alpha}, -1, 0\right)$$

and

$$dL(t) \sim S_{\alpha/2}\left(2\left(\frac{1}{2}\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}\frac{\sigma_{TL}^2}{(1+\ell^\alpha)^{2/\alpha}}dt^{2/\alpha}, 1, 0\right)$$

are two independent standard Lévy-Stable motions.[2] *Moreover, $\mu(s)$ is a deterministic function, $g(T-s) \in \mathbb{F}$ and the leverage parameter $\ell \geq 0$.*
Then option prices, under the risk-neutral measure Q, are given by

$$V(S,t) = \mathbb{E}^Q_{\tilde{L}}\left[\mathbb{E}^Q_L\left[\mathbb{E}^Q\left[V_{BS}\left(S(t)e^{\ell\int_t^T d\tilde{L}(s)}, t, K, \overline{Y}^{1/2}, T\right)\right]\tilde{L}, L|\tilde{L}\right]\right]. \tag{28}$$

Proof Let the value of a vanilla option be expressed as

$$V(S,t) = \mathbb{E}^Q[e^{-r(T-t)}\Pi(S,T)].$$

It is straightforward to see, given the independence of the integrated variance process and the Brownian motion that under the risk-neutral measure Q the stock process must follow

$$\begin{aligned} S(T) = S(t)\exp\Bigg[& r(T-t) - \frac{1}{2}\int_t^T \sigma^2(s)ds \\ & + \frac{1}{2}G(T,t)\frac{\ell^\alpha}{1+\ell^\alpha}\sigma_{TL}^\alpha \sec\frac{\pi\alpha}{2} + \int_t^T \sigma(s)dW(s) + \ell\int_t^T d\tilde{L}(s)\Bigg], \end{aligned}$$

where $G(T,t) = \int_t^T g(T-s)^{\alpha/2}ds$. By conditioning on $L(s)$ and $\tilde{L}(s)$ and using iterated expectations we get

$$\begin{aligned} V(S,t) &= \mathbb{E}^Q_{\tilde{L}}\left[\mathbb{E}^Q_L\left[\mathbb{E}^Q\left[e^{-r(T-t)}\Pi(S,T)|L,\tilde{L}\right]\tilde{L}\right]\right] \\ &= \mathbb{E}^Q_{\tilde{L}}\left[\mathbb{E}^Q_L\left[\mathrm{V_{BS}}\left(S(t)e^{\ell\int_t^T d\tilde{L}(s)}, K, \overline{Y}^{1/2}, T\right)|\tilde{L}\right]\right], \end{aligned}$$

[2]Note that $d\tilde{L}$ is totally skewed to the left and $\alpha < 2$, ie it is not restricted to be less than unity.

where $\overline{Y} = \frac{1}{T-t}\int_t^T \sigma^2(s)ds$.

□

Remark 8 *It is straightforward to verify that the shocks to the returns process above are those of a Lévy-Stable process with negative skewness* $\beta \in (-1, 0]$ *. First recall that* $G(T,t) = \int_t^T g(T-s)^{\alpha/2}ds$*. Look at the process*

$$Z(T) = \int_t^T \sigma(s)dW(s) + \ell \int_t^T d\tilde{L}(s).$$

The characteristic function of $Z(T)$ *is given by*

$$\begin{aligned}
\mathbb{E}\left[e^{i\theta Z(T)}\right] &= \mathbb{E}\left[e^{i\theta\left(\int_t^T \sigma(s)dW(s)+\ell\int_t^T d\tilde{L}(s)\right)}\right] \\
&= e^{-\frac{1}{2}G(T,t)\frac{\sigma_{TL}^{\alpha}}{1+\ell^{\alpha}}|\theta|^{\alpha}}\mathbb{E}\left[e^{i\theta\ell\int_t^T d\tilde{L}(s)}\right] \\
&= e^{-\frac{1}{2}G(T,t)\frac{\sigma_{TL}^{\alpha}}{1+\ell^{\alpha}}|\theta|^{\alpha}}e^{-\frac{1}{2}\frac{T\ell^{\alpha}}{1+\ell^{\alpha}}\sigma_{TL}^{\alpha}|\theta|^{\alpha}\left\{1-isign(\theta)\tan(\pi\alpha/2)\right\}} \\
&= e^{-\frac{1}{2}\frac{G(T,t)+T\ell^{\alpha}}{1+\ell^{\alpha}}\sigma_{LS}^{\alpha}|\theta|^{\alpha}\left\{1-\frac{-T\ell^{\alpha}}{G(T,t)+T\ell^{\alpha}}isign(\theta)\tan(\pi\alpha/2)\right\}}.
\end{aligned}$$

This is obviously the characteristic function of a skewed Lévy-Stable process with skewness parameter $\beta(T,t) = \frac{-T\ell^{\alpha}}{G(T,t)+T\ell^{\alpha}} \in (-1, 0]$*; see Proposition 1. Moreover, when* $\ell = 0$ *we obtain* $\beta = 0$ *and* $\beta \to -1$ *as* $\ell \to \infty$ *.*

Note that the integrated variance does not have a finite first moment since $\alpha/2 < 1$. However, in the case of the leverage effect $\ell\int_t^T d\tilde{L}(s)$ its first moment exists, ie $\mathbb{E}[\ell\int_t^T d\tilde{L}(s)] < \infty$ since $1 < \alpha < 2$.

Proposition 9 *It is possible to extend the results above to price European call and put options when the skewness coefficient* $\beta \in [0, 1)$*.*

Proof Using Put-Call inversion, see McCulloch [16], we have by no arbitrage that European call and put options are related in the following way

$$C(S,t;K,T,\alpha,\beta) = SKP(S^{-1},t;K^{-1},T,\alpha,-\beta).$$

□

Note that if we want to extend the previous results to model asymmetric returns we *cannot* include a leverage effect in equation (22) in the form

$$\begin{aligned}
\frac{dS(t)}{S(t)} &= \mu dt + \sigma(t)dW(t) + \ell d\tilde{L}(s) \qquad (29)\\
\int_t^T \sigma^2(s)ds &= \int_t^T g(T-s)dL(s).
\end{aligned}$$

The reason is that the solution to the SDE with leverage (29) will deliver a stock process $S(t)$ that allows negative values due to the jumps of the process $\tilde{L}(s)$.

We also point out that modelling returns using a Lévy-Stable motion is not possible since it delivers negative stock prices. For example, let the returns process that is driven by Lévy-Stable motion in the following way

$$\frac{dS}{S} = \mu dt + \varsigma dL \tag{30}$$

where μ and ς are constants and L is a Lévy-Stable motion.

Therefore the solution to the SDE (30) is given by

$$S_t = S(0)e^{L_t - \frac{1}{2}[L,L]_t^c}\Pi_{s\leq t}(1+\Delta L_s)e^{-\Delta L_s}, \tag{31}$$

where the infinite product converges and $[L, L]_t^c$ denotes the path by path continuous path of the quadratic variation $[L, L]_t$, [22]. And it is clear that in this case the stock price level can achieve negative values if jumps are such that

$$1 + \Delta L_s < 0$$

hence we cannot model returns using where the shocks exhibit jumps lower than -1.

6 Numerical results: Lévy-Stable Option Prices

In this section we show how vanilla option prices are calculated according to the above derivations. One route is to calculate the expected value of the Black-Scholes formula weighted by the stochastic volatility component and the leverage effect. Another route to price vanilla options for stock prices that follow a geometric Lévy-Stable processes is to compute the option value as an integral in Fourier Space.

We will continue to use the Black-Scholes model as a benchmark to compare the option prices obtained when the returns follow a Lévy-Stable process. We will see that our results are consistent with the findings in Hull and White [11] where the Black-Scholes model underprices in- and out-of-the-money Call option prices and overprices at-the-money options.

6.1 Option Prices for Symmetric Lévy-Stable log-Stock Prices

In this subsection we will calculate prices for vanilla options where the underlying log-stock or stock returns follow a symmetric Lévy-Stable motion. We will use Complex Fourier Transform techniques, see Lewis [14], Carr and Madan [5], to calculate the value of the options.

We start with the stock process as in Proposition 7. As pointed out in Remark 5 we can 'scale' the integrated volatility using a deterministic function. We do so in this case because it simplifies the way in which time enters the pricing equations. Hence we let the integrated variance process follow

$$\int_t^T \sigma^2(s)ds = h(T,t)\int_t^T g(T-s)dL(s),$$

where $h(T,t)$ is chosen to be:

$$h(T,t) = \frac{(T-t)^{\alpha/2}}{\int_t^T g(T-s)^{2/\alpha}ds}. \tag{32}$$

The motivation for this choice becomes clear when inspecting equation (10). Now, under the risk-neutral measure, the stock process follows

$$\begin{aligned} S_T &= S_t e^{r(T-t)-\frac{1}{2}\int_t^T \sigma^2(s)ds+\int_t^T \sigma(s)dW(s)}. &(33)\\ \int_t^T \sigma^2(s)ds &= h(T,t)\int_t^T g(T-s)dL(s), &(34)\end{aligned}$$

where $dW(t)$ is the increments of a standard Brownian motion, $dL(t) \sim S_{\alpha/2}\left(2(\frac{1}{2}\cos\frac{\pi\alpha}{4})^{2/\alpha}\sigma_{LS}^2 dt^{2/\alpha}, 1, 0\right)$ is a Lévy-Stable motion with $0<\alpha<2$, $g \in \mathbb{F}$ and r denotes the risk-free rate.

The first step we take is to calculate the characteristic function of the process

$$Z(T,t) = -\frac{1}{2}\int_t^T \sigma^2(s)ds + \int_t^T \sigma(s)dW(s).$$

Proposition 10 *The Characteristic function of $Z(T,t)$ is given by*

$$\Psi(\xi) = e^{-\frac{1}{2}\sigma_{LS}^\alpha (T-t)\left(i\xi+\xi^2\right)^{\alpha/2}}, \tag{35}$$

where $\xi = \xi_r + i\xi_i$ and $-1 \le \xi_i \le 0$. Moreover, the function $\Psi(\xi)$ is analytic in the strip $-1 < \xi_i < 0$.

Proof

The characteristic function is given by

$$\begin{aligned} \mathbb{E}^Q\left[e^{i\xi Z(T,t)}\right] &= \mathbb{E}^Q\left[e^{-\frac{1}{2}i\xi\int_t^T \sigma^2(s)ds + i\xi\int_t^T \sigma(s)dW(s)}\right] \\ &= \mathbb{E}^Q\left[e^{-\frac{1}{2}i\xi\int_t^T \sigma^2(s)ds - \frac{1}{2}\xi^2\int_t^T \sigma^2(s)ds}\right] \\ &= \mathbb{E}^Q\left[e^{-\frac{1}{2}\left(i\xi+\xi^2\right)h(T,t)\int_t^T g(T-s)dL(s)}\right] \\ &= e^{-\frac{1}{2}\sigma_{LS}^\alpha (T-t)\left(i\xi+\xi^2\right)^{\alpha/2}}. \end{aligned}$$

The last step is possible since the expected value exists, as shown below, if ξ is restricted so that $\xi_r^2 - \xi_i^2 + \xi_i \ge 0$. To show that $\mathbb{E}^Q\left[e^{-\frac{1}{2}i\xi\int_t^T \sigma^2(s)ds - \frac{1}{2}\xi^2\int_t^T \sigma^2(s)ds}\right]$ exists, we proceed in the following way:

$$\begin{aligned}\left|\mathbb{E}^Q\left[e^{-\frac{1}{2}\left(i\xi+\xi^2\right)\int_t^T g(T-s)dL(s)}\right]\right| &= \left|\mathbb{E}^Q\left[e^{-\frac{1}{2}\left(i\xi_r(1+2\xi_i)+\xi_r^2-\xi_i^2-\xi_i\right)\int_t^T g(T-s)dL(s)}\right]\right| \\ &= \mathbb{E}^Q\left[e^{-\frac{1}{2}\left(\xi_r^2-\xi_i^2-\xi_i\right)\int_t^T g(T-s)dL(s)}\right] \\ &< \infty.\end{aligned}$$

For the last step we require $\xi_r^2 - \xi_i^2 - \xi_i \geq 0$ so the expected value is finite given the tails of the distribution of $\int_t^T g(T-s)dL(s)$ as shown in Proposition 2. The region where this is true contains the strip $-1 \leq \xi_i \leq 0$. Finally, it is straightforward to observe that $\Psi(\xi)$ is analytic in the strip $-1 < \xi_i < 0$.

□

We note that if we choose $h(T,t) = 1$ the characteristic equation of $Z(T,t)$ is

$$\Psi(\xi) = e^{-\frac{1}{2}\sigma_{LS}^{\alpha}G(T,t)\left(i\xi+\xi^2\right)^{\alpha/2}},$$

where $G(T,t) = \int_t^T g(T-s)^{\alpha/2}dL(s)$.

To price call options we proceed as above and use the following expression:

$$C(x,t) = e^{x-D_0(T-t)} - \frac{1}{2\pi}e^{-r(T-t)}K\int_{i\xi_i-\infty}^{i\xi_i+\infty} e^{-i\xi z}\frac{K^{i\xi}}{\xi^2-i\xi}e^{(T-t)[\Psi(-\xi)]}d\xi \tag{36}$$

where $x = \ln S(T)$ and $0 < \xi_i < 1$ and $\Psi(-\xi)$ is the characteristic function of the process $\ln S(T)$.

6.1.1 Numerics for Symmetric Lévy-Stable log-Stock Prices

We now calculate European Style option prices when log-stock or stock returns are symmetric Lévy-Stable using (36). In order to compare these prices to those obtained using the Black-Scholes pricing formula, we have to decide how to choose the relevant parameters of the two models. In fact, the only parameter that we must carefully examine is the scaling parameter of the Lévy-Stable process; we opt for one that can be related to the standard deviation used when the classical Black-Scholes model is used. One approach is to proceed as in [12] and match a given percentile of the Normal and a symmetric Lévy-Stable distribution. For example, if we want to match the first and third quartile of a Brownian motion with standard deviation $\sigma = 0.20$to a symmetric Lévy-Stable motion $dL \sim S_\alpha(\kappa, 0, 0)$ with characteristic exponent $\alpha = 1.7$, we would require $\kappa = 0.1401$. We have chosen these parameters to compare the option prices. Moreover, in the simulations below, for illustrative purposes, we use the kernel $g(T-s) = \left(1 - e^{-(T-s)}\right)$.

Figure 3 shows the difference between European Call options when the stock returns are distributed according to a symmetric Lévy-Stable motion with annual $\sigma_{LS} = 0.1401$ and $\alpha = 1.7$ and when returns follow a Brownian motion with annual volatility $\sigma_{BS} = 0.20$.

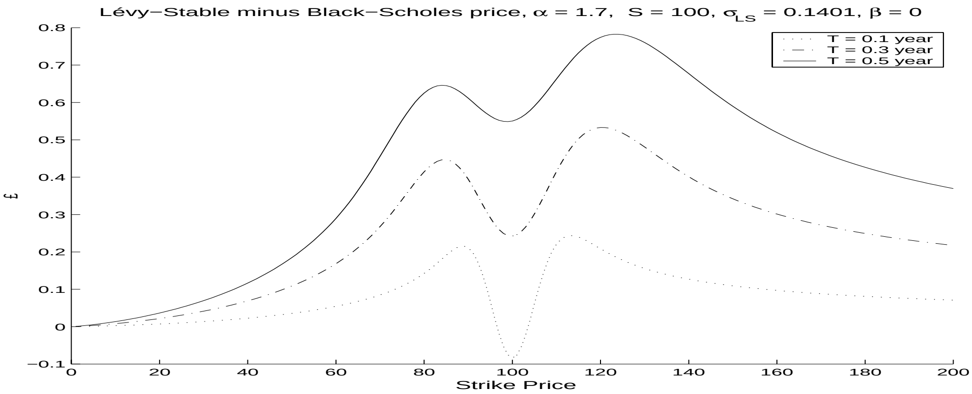

Figure 3: Difference between Lévy-Stable and Black-Scholes Call option prices for different expiry dates: $T = 0.1$ year, $T = 0.3$ year and $T = 0.5$ year. In the Black-Scholes annual volatility is $\sigma_{BS} = 20\%$ and in the symmetric Lévy-Stable case the scaling coefficient is $\sigma_{LS} = 14.01\%$ per year.

6.2 Option Prices for Asymmetric Lévy-Stable log-Stock Prices

In this subsection we obtain option prices, and implied volatilities, when there is a negative leverage effect, ie log-stock prices follow an asymmetric Lévy-Stable process. Recall that, under the risk-neutral measure Q, the stock price and variance process are given by

$$\begin{aligned} S(T) &= S(t)e^{r(T-t)-\frac{1}{2}\int_t^T \sigma^2(s)ds+\frac{1}{2}(T-t)\frac{\ell\alpha}{1+\ell^\alpha}\sigma_{LS}^\alpha \sec\frac{\pi\alpha}{2}+\int_t^T \sigma(s)dW(s)+\ell\int_t^T d\tilde{L}(s)} \\ \int_t^T \sigma^2(s)ds &= h(T,t)\int_t^T g(T-s)dL(s), \end{aligned}$$

where $h(T,t)$ is as in (32); with this choice of $h(T,t)$ the skewness parameter is $\beta = -\ell^\alpha/(1+\ell^\alpha)$.

We proceed as above and calculate the characteristic function of the process

$$Z(T,t) = -\frac{1}{2}\int_t^T \sigma^2(s)ds + \int_t^T \sigma(s)dW(s) + \ell\int_t^T d\tilde{L}(s),$$

where $dW(t)$ is the standard Brownian motion independent of both $d\tilde{L}(t)$ and $dL(t)$; $d\tilde{L}(t) \sim S_\alpha\left(\frac{1}{2^{1/\alpha}}\frac{\sigma_{TL}}{(1+\ell^\alpha)^{1/\alpha}}dt^{1/\alpha}, -1, 0\right)$ and $dL(t) \sim S_{\alpha/2}\left(2\left(\frac{1}{2}\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}\frac{\sigma_{TL}^2}{(1+\ell^\alpha)^{2/\alpha}}dt^{1/\alpha}, 1, 0\right)$ are two independent Lévy-Stable motions.

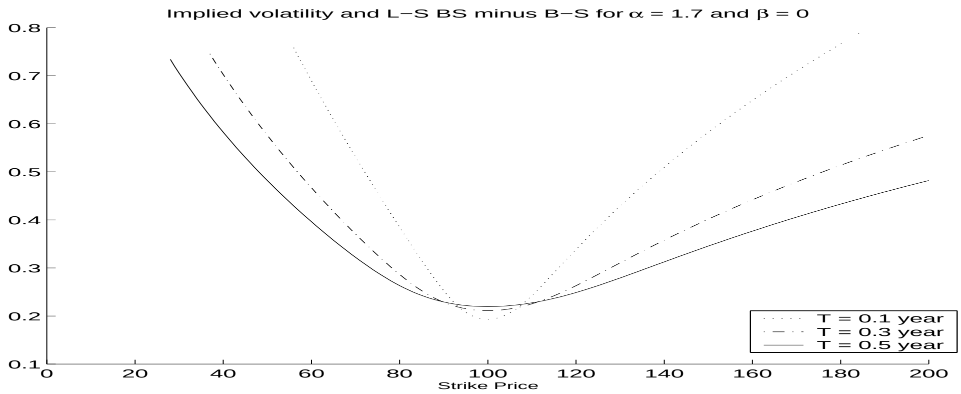

Figure 4: Black-Scholes implied volatility for the Lévy-Stable Call option prices when returns follow a symmetric Lévy-Stable motion with $\alpha = 1.7$, $\beta = 0$, $\sigma_{LS} = 14.01\%$ and three expiry dates: $T = 0.1$ year, $T = 0.3$ year and $T = 0.5$ year.

Proposition 11 *The Characteristic function of $Z(T,t)$ is given by*

$$\Psi(\xi) = e^{-\frac{1}{2}(T-t)\frac{\sigma_{LS}^{\alpha}}{1+\ell^{\alpha}}\left[\left(i\xi+\xi^2\right)^{\alpha/2}-(i\xi\ell)^{\alpha}\sec\frac{\pi\alpha}{2}\right]}, \tag{37}$$

where $-1 \leq \xi_i \leq 0$, $\xi = \xi_r + i\xi_r$. Moreover, the function $\Psi(\xi)$ is analytic in the strip $-1 < \xi_i < 0$.

Proof

The proof is very similar to the one above. It suffices to note that for $\xi_i \leq 0$

$$\begin{aligned}
\mathbb{E}^Q\left[e^{i\xi\ell\int_t^T d\tilde{L}(s)}\right] &\leq \mathbb{E}^Q\left[\left|e^{i\xi\ell\int_t^T d\tilde{L}(s)}\right|\right] \\
&= \mathbb{E}^Q\left[e^{-\xi_i\ell\int_t^T d\tilde{L}(s)}\right] \\
&< \infty.
\end{aligned}$$

Moreover, for $\xi_i < 0$ we have that $\mathbb{E}^Q\left[e^{i\xi\ell\int_t^T d\tilde{L}(s)}\right]$ is analytic, ie

$$\begin{aligned}
\frac{d}{d\xi}\mathbb{E}^Q\left[e^{i\xi\ell\int_t^T d\tilde{L}(s)}\right] &= \mathbb{E}^Q\left[i\ell\int_t^T d\tilde{L}(s)e^{i\xi\ell\int_t^T d\tilde{L}(s)}\right] \\
&< \infty.
\end{aligned}$$

Putting these results together with the results from Proposition 10 we get the desired result.

□

Remark 9 *Note that the requirement is* $-1 < \xi_i < 0$ *because* $d\tilde{L}(s)$ *is totally skewed to the left, therefore we need* $-\xi\ell > 0$.

Note again that if we let $h(T,t) = 1$ the characteristic function of $Z(T,T)$ is given by

$$\Psi(\xi) = e^{-\frac{1}{2}G(T,t)\frac{\sigma_{LS}^{\alpha}}{1+\ell^{\alpha}}\left[\left(i\xi+\xi^2\right)^{\alpha/2}-\frac{T-t}{G(T,t)}(i\xi\ell)^{\alpha}\sec\frac{\pi\alpha}{2}\right]},$$

where $G(T,t) = \int_t^T g(T-s)^{\alpha/2} ds$.

We use the same $g(T-s)$ as above and include a leverage parameter $\ell = 1$ so that returns follow a negatively skewed process $\beta = -1/2$. Figure 5 shows the difference between Lévy-Stable and Black-Scholes Call option prices for different expiry dates. In the Black-Scholes case annual volatility is $\sigma_{BS} = 0.20$ and in the asymmetric Lévy-Stable case with $\beta = -1/2$ the scaling coefficient is $\sigma_{LS} = 0.1401$ per year. Finally, Figure 6 shows the corresponding implied volatility.

7 Conclusions

The GCLT provides a very strong theoretical foundation to argue that the limiting distribution of stock return or log-stock prices follow a Lévy-Stable process. In this paper we have shown that it is possible to model stock returns and log-stock prices where the stochastic component is Lévy-Stable distributed, symmetric and negatively skewed, and European-style option prices are straightforward to calculate.

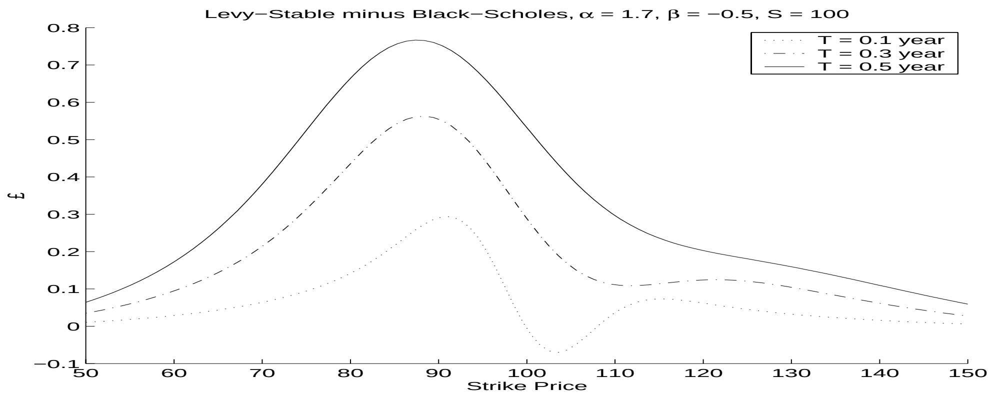

Figure 5: Difference between Lévy-Stable and Black-Scholes Call option prices for different expiry dates: $T = 0.1$ year, $T = 0.3$ year and $T = 0.5$ year. In the Black-Scholes annual volatility is $\sigma_{BS} = 0.20$ and in the asymmetric Lévy-Stable case with $\beta = -1/2$ the scaling coefficient is $\sigma_{LS} = 0.1401$ per year.

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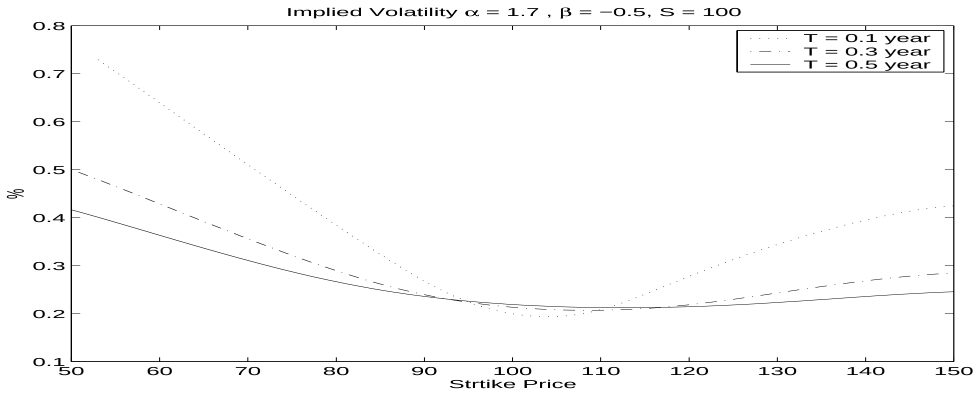

Figure 6: Black-Scholes implied volatility for the Lévy-Stable Call option prices when returns follow a symmetric Lévy-Stable motion with $\alpha = 1.7$, $\beta = -1/2$, $\sigma_{LS} = 0.1401$ and three expiry dates: $T = 0.1$ year, $T = 0.3$ year and $T = 0.5$ year.

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