Chapter 4

Lebesgue measure and integration

If you look back at what you have learned in your earlier mathematics courses, you will definitely recall a lot about area and volume — from the simple formulas for the areas of rectangles and triangles that you learned in grade school, to the quite sophisticated calculations with double and triple integrals that you had to perform in calculus class. What you have probably never seen, is a systematic theory for area and volume that unifies all the different methods and techniques.

In this chapter we shall first study such a unified theory for d-dimensional volume based on the notion of a *measure*, and then we shall use this theory to build a stronger and more flexible theory for integration. You may think of this as a reversal of previous strategies; instead of basing the calculation of volumes on integration, we shall create a theory of integration based on a more fundamental notion of volume.

The theory will cover volume in $\mathbb{R}^d$ for all $d \in \mathbb{N}$, including $d = 1$ and $d = 2$. To get a unified terminology, we shall think of the length of a set in $\mathbb{R}$ and the area of a set in $\mathbb{R}^2$ as one- and two-dimensional volume, respectively.

To get a feeling for what we are aiming for, let us assume that we want to measure the volume of subsets $A \subset \mathbb{R}^3$, and that we denote the volume of A by $\mu(A)$. What properties would we expext μ to have?

(i) $\mu(A)$ should be a nonnegative number or ∞. There are subsets of $\mathbb{R}^3$ that have an infinite volume in an intuitive sense, and we capture this intuition by the symbol ∞.

(ii) $\mu(\emptyset) = 0$. It will be convenient to assign a volume to the empty set, and the only reasonable alternative is 0.

(iii) If $A_1, A_2, \ldots, A_n, \ldots$ are disjoint (i.e. non-overlapping) sets, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$. This means that the volume of the whole is equal to the sum of the volumes of the parts.

(iv) If $A = (a_1, a_2) \times (b_1, b_2) \times (c_1, c_2)$ is a rectangular box, then $\mu(A)$ is equal to the volume of A in the traditional sence, i.e.

$$\mu(A) = (a_2 - a_1)(b_2 - b_1)(c_2 - c_1)$$

It turns out that it is impossible to measure the size of *all* subsets of A such that all these requirements are satisfied; there are sets that are simply too irregular to be measured in a good way. For this reason we shall restrict ourselves to a class of *measurable sets* which behave the way we want. The hardest part of the theory will be to decide which sets are measurable.

We shall use a two step procedure to construct our measure μ: First we shall construct an *outer measure* μ^* which will assign a size $\mu^*(A)$ to *all* subsets $A \in \mathbb{R}^3$, but which will not satisfy all the conditions (i)-(iv) above. Then we shall use μ^* to single out the class of measurable sets, and prove that if we restrict μ^* to this class, our four conditions are satisfied.

4.1 Outer measure in $\mathbb{R}^d$

The first step in our construction is to define outer measure in $\mathbb{R}^d$. The outer measure is built from rectangular boxes, and we begin by intoducing the appropriate notation and teminology.

Definition 4.1.1 *A subset A of $\mathbb{R}^d$ is called an* open box *if there are numbers $a_1^{(1)} < a_2^{(1)}$, $a_1^{(2)} < a_2^{(2)}$, ..., $a_1^{(d)} < a_2^{(d)}$ such that*

$$A = (a_1^{(1)}, a_2^{(1)}) \times (a_1^{(2)}, a_2^{(2)}) \times \ldots \times (a_1^{(d)}, a_2^{(d)})$$

In addition, we count the empty set as a rectangular box. We define the volume $|A|$ *of A to be 0 if A is the empty set, and otherwise*

$$|A| = (a_2^{(1)} - a_1^{(1)})(a_2^{(2)} - a_1^{(2)}) \cdot \ldots \cdot (a_2^{(d)} - a_1^{(d)})$$

Observe that when $d = 1, 2$ and 3, $|A|$ denotes the length, area and volume of A in the usual sense.

If $\mathcal{A} = \{A_1, A_2, \ldots, A_n, \ldots\}$ is a countable collection of open boxes, we define its *size* $|\mathcal{A}|$ by

$$|\mathcal{A}| = \sum_{k=1}^{\infty} |A_k|$$

(we may clearly have $|\mathcal{A}| = \infty$). Note that we can think of a finite collection $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ of open boxes as a countable one by putting in the empty set in the missing positions: $\mathcal{A} = \{A_1, A_2, \ldots, A_n, \emptyset, \emptyset, \ldots\}$. This is the main reason for including the empty set among the open boxes. Note also that since the boxes $A_1, A_2, \ldots$ may overlap, the size $|\mathcal{A}|$ need not be closely connected to the volume of $\bigcup_{n=1}^{\infty} A_n$.

A *covering* of a set $B \subset \mathbb{R}^d$ is a countable collection

$$\mathcal{A} = \{A_1, A_2, \dots, A_n, \dots\}$$

of open boxes such that $B \subset \bigcup_{n=1}^{\infty} A_n$. We are now ready to define outer measure.

Definition 4.1.2 *The* outer measure *of a set $B \in \mathbb{R}^d$ is defined by*

$$\mu^*(B) = \inf\{|\mathcal{A}| : \mathcal{A} \textit{ is a covering of } B \textit{ by open boxes}\}$$

The idea behind outer measure should be clear – we measure the size of B by approximating it as economically as possible from the outside by unions of open boxes. You may wonder why we use open boxes and not closed boxes

$$A = [a_1^{(1)}, a_2^{(1)}] \times [a_1^{(2)}, a_2^{(2)}] \times \dots \times [a_1^{(d)}, a_2^{(d)}]$$

in the definition above. The answer is that it does not really matter, but that open boxes are a little more convenient in some arguments. The following lemma tells us that closed boxes would have given us exactly the same result. You may want to skip the proof at the first reading.

Lemma 4.1.3 *For all $B \subset \mathbb{R}^d$,*

$$\mu^*(B) = \inf\{|\mathcal{A}| : \mathcal{A} \textit{ is a covering of } B \textit{ by } \text{closed} \textit{ boxes}\}$$

Proof: We must prove that

$$\inf\{|\mathcal{A}| : \mathcal{A} \text{ is a covering of } B \text{ by } \textit{open} \text{ boxes}\} =$$

$$= \inf\{|\mathcal{A}| : \mathcal{A} \text{ is a covering of } B \text{ by } \textit{closed} \text{ boxes}\}$$

Observe first that if $\mathcal{A}_0 = \{A_1, A_2, \dots\}$ is a covering of B by *open* boxes, we can get a covering $\mathcal{A} = \{\overline{A}_1, \overline{A}_2, \dots\}$ of B by *closed* boxes just by closing each box. Since the two coverings have the same size, this means that

$$\mu^*(B) = \inf\{|\mathcal{A}| : \mathcal{A} \text{ is a covering of } B \text{ by } \textit{open} \text{ boxes}\} \geq$$

$$\geq \inf\{|\mathcal{A}| : \mathcal{A} \text{ is a covering of } B \text{ by } \textit{closed} \text{ boxes}\}$$

To prove the opposite inequality, assume that $\epsilon > 0$ is given. If $\mathcal{A} = \{A_1, A_2, \dots\}$ is a covering of B by closed boxes, we can for each n find an open box $\tilde{A}_n$ containing A_n such that $|\tilde{A}_n| < |A_n| + \frac{\epsilon}{2^n}$. Then $\tilde{\mathcal{A}} = \{\tilde{A}_n\}$ is a covering of B by open boxes, and $|\tilde{\mathcal{A}}| < |\mathcal{A}| + \epsilon$. Since $\epsilon > 0$ is arbitrary, this shows that to any closed covering, there is an open covering arbitrarily close in size, and hence

$$\inf\{|\mathcal{A}| : \mathcal{A} \text{ is a covering of } B \text{ by } \textit{open} \text{ boxes}\} \leq$$

$$\leq \inf\{|\mathcal{A}| : \mathcal{A} \text{ is a covering of } B \text{ by } \textit{closed} \text{ boxes}\}$$

□

Here are some properties of the outer measure:

Proposition 4.1.4 *The outer measure μ^* on $\mathbb{R}^d$ satisfies:*

(i) $\mu^*(\emptyset) = 0$.

(ii) (Monotonicity) If $B \subset C$, then $\mu^*(B) \leq \mu^*(C)$.

(iii) (Subadditivity) If $\{B_n\}_{n\in\mathbb{N}}$ is a sequence of subsets of $\mathbb{R}^d$, then

$$\mu^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \mu^*(B_n)$$

(iv) For all closed boxes

$$B = [b_1^{(1)}, b_2^{(1)}] \times [b_1^{(2)}, b_2^{(2)}] \times \ldots \times [b_1^{(d)}, b_2^{(d)}]$$

we have

$$\mu^*(B) = |B| = (b_2^{(1)} - b_1^{(1)})(b_2^{(2)} - b_1^{(2)}) \cdot \ldots \cdot (b_2^{(d)} - b_1^{(d)})$$

Proof: (i) Since $\mathcal{A} = \{\emptyset, \emptyset, \emptyset, \ldots\}$ is a covering of $\emptyset$, $\mu^*(\emptyset) = 0$.

(ii) Since any covering of C is a covering of B, we have $\mu^*(B) \leq \mu^*(C)$.

(iii) If $\mu^*(B_n) = \infty$ for some $n \in \mathbb{N}$, there is nothing to prove, and we may hence assume that $\mu^*(B_n) < \infty$ for all n. Let $\epsilon > 0$ be given. For each $n \in \mathbb{N}$, we can find a covering $A_1^{(n)}, A_2^{(n)}, \ldots$ of B_n such that

$$\sum_{k=1}^{\infty} |A_k^{(n)}| < \mu^*(B_n) + \frac{\epsilon}{2^n}$$

The collection $\{A_k^{(n)}\}_{k,n\in\mathbb{N}}$ of all sets in all coverings is a countable covering of $\bigcup_{n=1}^{\infty} B_n$, and

$$\sum_{k,n\in\mathbb{N}} |A_k^{(n)}| = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} |A_k^{(n)}| \right) \leq \sum_{n=1}^{\infty} \left(\mu^*(B_n) + \frac{\epsilon}{2^n} \right) = \sum_{n=1}^{\infty} \mu^*(B_n) + \epsilon$$

(if you are unsure about these manipulation, take a look at exercise 5). This means that

$$\mu^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \mu^*(B_n) + \epsilon$$

and since ϵ is an arbitrary, positive number, we must have

$$\mu^*(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \mu^*(B_n)$$

(iv) Since we can cover B by $\mathcal{B}_\epsilon = \{B_\epsilon, \emptyset, \emptyset, \ldots\}$, where

$$B = (b_2^{(1)} + \epsilon, b_1^{(1)} - \epsilon) \times (b_2^{(2)} + \epsilon, b_1^{(2)} - \epsilon) \times \ldots \times (b_2^{(d)} + \epsilon, b_1^{(d)} - \epsilon),$$

for any $\epsilon > 0$, we se that

$$\mu^*(B) \leq |B| = (b_2^{(1)} - b_1^{(1)})(b_2^{(2)} - b_1^{(2)}) \cdot \ldots \cdot (b_2^{(d)} - b_1^{(d)})$$

The opposite inequality,

$$\mu^*(B) \geq |B| = (b_2^{(1)} - b_1^{(1)})(b_2^{(2)} - b_1^{(2)}) \cdot \ldots \cdot (b_2^{(d)} - b_1^{(d)})$$

may seem obvious, but is actually quite tricky to prove. We shall need a few lemmas to establish this and finish the proof. □

I shall carry out the remaining part of the proof of Proposition 4.1.4(iv) in the three dimensional case. The proof is exactly the same in the d-dimensional case, but the notation becomes so messy that it tends to blur the underlying ideas. Let us begin with a lemma.

Lemma 4.1.5 *Assume that the intervals (a_0, a_K), (b_0, b_N), (c_0, c_M) are partioned*

$$a_0 < a_1 < a_2 < \ldots < a_K$$

$$b_0 < b_1 < b_2 < \ldots < b_N$$

$$c_0 < c_1 < c_2 < \ldots < c_M$$

and let $\Delta a_k = a_{k+1} - a_k$, $\Delta b_n = a_{n+1} - n_n$, $\Delta c_m = c_{m+1} - c_m$. Then

$$(a_K - a_0)(b_N - b_0)(c_m - c_0) = \sum_{k,n,m} \Delta a_k \Delta b_n \Delta c_m$$

where the sum is over all triples (k, n, m) such that $0 \leq k < K$, $0 \leq n < N$, $0 \leq m < M$. In other words, if we partition the box

$$A = (a_0, a_K) \times (b_0, b_N) \times (c_0, c_M)$$

into KNM smaller boxes $B_1, B_2, \ldots, B_{KNM}$, then

$$|A| = \sum_{j=1}^{KNM} |B_j|$$

Proof: If you think geometrically, the lemma seems obvious — it just says that if you divide a big box into smaller boxes, the volume of the big box is equal to the sum of the volumes of the smaller boxes. An algebraic proof is not much harder and has the advantage of working also in higher dimensions: Note that since $a_K - a_0 = \sum_{k=0}^{K-1} \Delta a_k$, $b_N - b_0 = \sum_{n=0}^{N-1} \Delta b_n$, $c_M - c_0 = \sum_{m=0}^{M-1} \Delta c_m$, we have

$$(a_K - a_0)(b_N - b_0)(c_m - c_0) =$$

$$= \left(\sum_{k=0}^{K-1} \Delta a_k\right)\left(\sum_{n=0}^{N-1} \Delta b_n\right)\left(\sum_{m=0}^{M-1} \Delta c_m\right) =$$

$$= \sum_{k,n,m} \Delta a_k \Delta b_n \Delta c_m$$

□

The next lemma reduces the problem from countable coverings to finite ones. It is the main reason why we choose to work with open coverings.

Lemma 4.1.6 *Assume that $\mathcal{A} = \{A_1, A_2, \ldots, A_n, \ldots\}$ is a countable covering of a compact set K by open boxes. Then K is covered by a finite number $A_1, A_2, \ldots, A_n$ of elements in $\mathcal{A}$.*

Proof: Assume not, then we can for each $n \in \mathbb{N}$ find an element $x_n \in K$ which does not belong to $\bigcup_{k=1}^{n} A_k$. Since K is compact, there is a subsequence $\{x_{n_k}\}$ converging to an element $x \in K$. Since $\mathcal{A}$ is a covering of K, x must belong to an A_i. Since A_i is open, $x_{n_k} \in A_i$ for all sufficently large k. But this is impossible since $x_{n_k} \notin A_i$ when $n_k \geq i$. □

We are now ready to prove the missing inequality in Proposition 4.1.4(iv).

Lemma 4.1.7 *For all closed boxes*

$$B = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$$

we have

$$\mu^*(B) \geq |B| = (a_2 - a_1)(b_2 - b_1)(c_2 - c_1)$$

Proof: By the lemma above, it suffices to show that if $A_1, A_2, \ldots, A_n$ is a finite covering of B, then

$$|B| \leq |A_1| + |A_2| + \ldots + |A_n|$$

Let

$$A_i = (x_1^{(i)}, x_2^{(i)}) \times (y_1^{(i)}, y_2^{(i)}) \times (z_1^{(i)}, z_2^{(i)})$$

We collect all x-coordinates $x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_2^{(2)}, \ldots, x_1^{(n)}, x_2^{(n)}$ and rearrange them according to size:

$$x_0 < x_1 < x_2 < \ldots < x_I$$

Doing the same with the y- and the z-coodinates, we get partitions

$$y_0 < y_1 < y_2 < \ldots < y_J$$

$$z_0 < z_1 < z_2 < \ldots < z_K$$

Let $B_1, B_2, \ldots, B_P$ be all boxes of the form $(x_i, x_{i+1}) \times (y_j, y_{j+1}) \times (z_k, z_{k+1})$ that is contained in at least one of the sets $A_1, A_2, \ldots, A_n$. Each A_i, $1 \leq i \leq n$ is made up of a finite number of B_j's, and each B_j belongs to at least one of the A_i's. According to Lemma 4.1.5,

$$|A_i| = |B_{j_{i_1}}| + |B_{j_{i_2}}| + \ldots + |B_{j_{i_q}}|$$

where $B_{j_{i_1}}, B_{j_{i_2}}, \ldots, B_{j_{i_q}}$ are the small boxes making up A_i. If we sum over all i, we get

$$\sum_{i=1}^{n} |A_i| > \sum_{j=1}^{P} |B_j|$$

(we get an inequality since some of the B_j's belong to more than one A_i, and hence are counted twice or more on the left hand side).

On the other hand, the B_j's almost form a partition of the original box B, the only problem being that some of the B_j's stick partly outside B. If we shrink these B_j's so that they just fit inside B, we get a partition of B into even smaller boxes $C_1, C_2, \ldots, C_Q$ (some boxes may disappear when we shrink them). Using Lemma 4.1.5 again, we see that

$$|B| = \sum_{k=1}^{Q} |C_k| < \sum_{j=1}^{P} |B_j|$$

Combining the results we now have, we see that

$$|B| < \sum_{j=1}^{P} |B_j| < \sum_{i=1}^{n} |A_i|$$

and the lemma is proved. □

We have now finally established all parts of Proposition 4.1.4. and are ready to move on. The problem with the outer measure μ^* is that it fails to be countably additive: If $\{A_n\}$ is a disjoint sequence of sets, we can only guarantee that

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$$

not that

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu^*(A_n) \tag{4.1.1}$$

As it is impossible to change μ^* such that (4.1.1) holds for all disjoint sequences $\{A_n\}$ of subsets of $\mathbb{R}^d$, we shall follow a different strategy: We shall show that there is a large class $\mathcal{M}$ of subsets of $\mathbb{R}^d$ such that (4.1.1) holds for all disjoint sequences where $A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. The sets in $\mathcal{M}$ will be called measurable sets.

Exercises for Section 4.1

1. Show that all countable sets have outer measure zero.

2. Show that the x-axis has outer measure 0 in $\mathbb{R}^2$.

3. If A is a subset of $\mathbb{R}^d$ and $\mathbf{b} \in \mathbb{R}^d$, we define

$$A + \mathbf{b} = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A\}$$

   Show that $\mu^*(A + \mathbf{b}) = \mu^*(A)$.

4. If A is a subset of $\mathbb{R}^d$, define $2A = \{2\mathbf{a} \mid \mathbf{a} \in A\}$. Show that $\mu^*(2A) = 2^d \mu^*(A)$.

5. Let $\{a_{n,k}\}_{n,k\in\mathbb{N}}$ be a collection of nonnegative, real numbers, and let a be the supremum over all finite sums of distinct elements in this collection, i.e.

$$A = \sup\{\sum_{i=1}^{I} a_{n_i,k_i} \ : \ I \in \mathbb{N} \text{ and all pairs } (n_1, k_1), \ldots, (n_I, k_I) \text{ are different}\}$$

   a) Assume that $\{b_m\}_{m\in\mathbb{N}}$ is a sequence which contains each element in the set $\{a_{n,k}\}_{n,k\in\mathbb{N}}$ exactly ones. Show that $\sum_{m=1}^{\infty} b_m = a$.
   b) Show that $\sum_{n=1}^{\infty} (\sum_{k=1}^{\infty} a_{n,k}) = a$.
   c) Comment on the proof of Proposition 4.1.4(iii).

4.2 Measurable sets

We shall now begin our study of measurable sets — the sets that can be assigned a "volume" in a coherent way. The definition is rather mysterious:

Definition 4.2.1 *A subset E of $\mathbb{R}^d$ is called* measurable *if*

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A)$$

for all $A \subset \mathbb{R}^d$. The collection of all measurable sets is denoted by $\mathcal{M}$.

Although the definition above is easy to grasp, it is not easy too see why it captures the essence of the sets that are possible to measure. The best I can say is that the reason why some sets are impossible to measure, is that they have very irregular boundaries. The definition above says that a set is measurable if we can use it to split any other set in two without introducing any further irregularities, i.e. all parts of its boundary must be reasonably regular. Admittedly, this explanation is vague and not very helpful in understanding why the definition captures exactly the right notion of measurability. The best argument may simply be to show that the definition works, so let us get started.

Let us first of all make a very simple observation. Since $A = (A \cap E) \cup (A \cap E^c)$, subadditivity (recall Proposition 4.1.4(iii)) tells us that we always have

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \geq \mu^*(A)$$

Hence to prove that a set is measurable, we only need to prove that

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A)$$

Our first observation on measurable sets is simple.

Lemma 4.2.2 *If E has outer measure 0, then E is measurable. In particular, $\emptyset \in \mathcal{M}$.*

Bevis: If E has measure 0, so has $A \cap E$ since $A \cap E \subset E$. Hence

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A \cap E^c) \leq \mu^*(A)$$

for all $A \subset \mathbb{R}^d$. □

Next we have:

Proposition 4.2.3 *$\mathcal{M}$ is an algebra of sets, i.e.:*

(i) $\emptyset \in \mathcal{M}$.

(ii) *If $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$.*

(iii) *If $E_1, E_2, \ldots, E_n \in \mathcal{M}$, then $E_1 \cup E_2 \cup \ldots \cup E_n \in \mathcal{M}$.*

(iv) *If $E_1, E_2, \ldots, E_n \in \mathcal{M}$, then $E_1 \cap E_2 \cap \ldots \cap E_n \in \mathcal{M}$.*

Proof: We have already proved (i), and (ii) is obvious from the definition of measurable sets. Since $E_1 \cup E_2 \cup \ldots \cup E_n = (E_1^c \cap E_2^c \cap \ldots \cap E_n^c)^c$ by De Morgans laws, (iii) follows from (ii) and (iv). Hence it remains to prove (iv).

To prove (iv) is suffices to prove that if $E_1, E_2 \in \mathcal{M}$, then $E_1 \cap E_2 \in \mathcal{M}$ as we can then add more sets by induction. If we first use the measurability of E_1, we see that for any set $A \subset \mathbb{R}^d$

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c)$$

Using the measurability of E_2, we get

$$\mu^*(A \cap E_1) = \mu^*((A \cap E_1) \cap E_2) + \mu^*((A \cap E_1) \cap E_2^c)$$

Combining these two expressions, we have

$$\mu^*(A) = \mu^*((A \cap (E_1 \cap E_2)) + \mu^*((A \cap E_1) \cap E_2^c) + \mu^*(A \cap E_1^c)$$

Observe that (draw a picture!)

$$(A \cap E_1 \cap E_2^c) \cup (A \cap E_1^c) = A \cap (E_1 \cap E_2)^c$$

and hence

$$\mu^*(A \cap E_1 \cap E_2^c) + \mu^*(A \cap E_1^c) \geq \mu^*(A \cap (E_1 \cap E_2)^c)$$

Putting this into the expression for $\mu^*(A)$ above, we get

$$\mu^*(A) \geq \mu^*((A \cap (E_1 \cap E_2)) + \mu^*(A \cap (E_1 \cap E_2)^c)$$

which means that $E_1 \cap E_2 \in \mathcal{M}$. □

We would like to extend parts (iii) and (iv) in the proposition above to countable unions and intersection. For this we need the following lemma:

Lemma 4.2.4 *If $E_1, E_2, \ldots, E_n$ is a disjoint collection of measurable sets, then*

$$\mu^*(A \cap (E_1 \cup E_2 \cup \ldots \cup E_n)) = \mu^*(A \cap E_1) + \mu^*(A \cap E_2) + \ldots + \mu^*(A \cap E_n)$$

Proof: It suffices to prove the lemma for two sets E_1 and E_2 as we can then extend it by induction. Using the measurability of E_1, we see that

$$\mu^*(A \cap (E_1 \cup E_2)) = \mu^*((A \cap (E_1 \cup E_2)) \cap E_1) + \mu^*(A \cap (E_1 \cup E_2)) \cap E_1^c) =$$

$$= \mu^*(A \cap E_1) + \mu^*(A \cap E_2)$$

□

We can now prove that $\mathcal{M}$ is closed under countable unions.

Lemma 4.2.5 *If $A_n \in \mathcal{M}$ for each $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}$.*

Proof: Note that since $\mathcal{M}$ is an algebra,

$$E_n = A_n \cap (E_1 \cup E_2 \cup \ldots E_{n-1})^c$$

belongs to $\mathcal{M}$ for $n > 1$ (for $n = 1$, we just let $E_1 = A_1$). The new sequence $\{E_n\}$ is disjoint and have the same union as $\{A_n\}$, and hence it suffices to prove that $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{M}$, i.e.

$$\mu^*(A) \geq \mu^*\big(A \cap \bigcup_{n=1}^{\infty} E_n\big) + \mu^*\big(A \cap \big(\bigcup_{n=1}^{\infty} E_n\big)^c\big)$$

Since $\bigcup_{n=1}^{N} E_n \in \mathcal{M}$ for all $N \in \mathbb{N}$, we have:

$$\mu^*(A) = \mu^*\big(A \cap \bigcup_{n=1}^{N} E_n\big) + \mu^*\big(A \cap \big(\bigcup_{n=1}^{N} E_n\big)^c\big) \geq$$

$$\geq \sum_{n=1}^{N} \mu^*(A \cap E_n) + \mu^*\big(A \cap \big(\bigcup_{n=1}^{\infty} E_n\big)^c\big)$$

where we in the last step have used the lemma above plus the observation that $\big(\bigcup_{n=1}^{\infty} E_n\big)^c \subset \big(\bigcup_{n=1}^{N} E_n\big)^c$. Since this inequality holds for all $N \in \mathbb{N}$, we get

$$\mu^*(A) \geq \sum_{n=1}^{\infty} \mu^*(A \cap E_n) + \mu^*\big(A \cap \big(\bigcup_{n=1}^{\infty} E_n\big)^c\big)$$

By sublinearity, we have $\sum_{n=1}^{\infty} \mu^*(A \cap E_n) \geq \mu^*(\bigcup_{n=1}^{\infty}(A \cap E_n)) = \mu^*(A \cap \bigcup_{n=1}^{\infty}(E_n))$, and hence

$$\mu^*(A) \geq \mu^*\big(A \cap \bigcup_{n=1}^{\infty} E_n\big) + \mu^*\big(A \cap \big(\bigcup_{n=1}^{\infty} E_n\big)^c\big)$$

□

Let us sum up our results so far.

Theorem 4.2.6 *The measurable sets $\mathcal{M}$ form a σ-algebra, i.e.:*

(i) $\emptyset \in \mathcal{M}$

(ii) If $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$.

(iii) If $E_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$.

(iv) If $E_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} E_n \in \mathcal{M}$.

Proof: We have proved everything except (iv), which follows from (ii) and (iii) since $\bigcap_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} E_n^c\right)^c$. □

Remark: By definition, a σ-algebra is a collection of subsets satisfying (i)-(iii), but — as we have seen — point (iv) follows from the others.

There is one more thing we have to check: that M contains sufficiently many sets. So far we only know that $\mathcal{M}$ contains the sets of outer measure 0 and their complements!

In the first proof it is convenient to use *closed* coverings as in Lemma 4.1.3 to determine the outer measure.

Lemma 4.2.7 *For each i and each $a \in \mathbb{R}$, the open halfspaces*

$$H = \{(x_1, \ldots, x_i, \ldots, x_d) \in \mathbb{R}^d \ : \ x_i < a\}$$

and

$$K = \{(x_1, \ldots, x_i, \ldots, x_d) \in \mathbb{R}^d \ : \ x_i > a\}$$

are measurable.

Proof: We only prove the H-part. We have to check that for any $B \subset \mathbb{R}^d$,

$$\mu^*(B) \geq \mu^*(B \cap H) + \mu^*(B \cap H^c)$$

Given a covering $\mathcal{A} = \{A_i\}$ of B by closed boxes, we can create closed coverings $\mathcal{A}^{(1)} = \{A_i^{(1)}\}$ and $\mathcal{A}^{(1)} = \{A_i^{(2)}\}$ of $B \cap H$ and $B \cap H^c$, respectively, by putting

$$A_i^{(1)} = \{(x_1, \ldots, x_i, \ldots, x_d) \in A_i \ : \ x_i \leq a\}$$

$$A_i^{(2)} = \{(x_1, \ldots, x_i, \ldots, x_d) \in A_i \ : \ x_i \geq a\}$$

Hence

$$|\mathcal{A}| = |\mathcal{A}^{(1)}| + |\mathcal{A}^{(2)}| \geq \mu^*(B \cap H) + \mu^*(B \cap H^c)$$

and since this holds for all closed coverings $\mathcal{A}$ of B, we get

$$\mu^*(B) \geq \mu^*(B \cap H) + \mu^*(B \cap H^c)$$

□

The next step is now easy:

Lemma 4.2.8 *All open boxes are measurable.*

Proof: An open box is a finite intersection of open halfspaces. □

The next result tells us that there are many measurable sets:

Theorem 4.2.9 *All open sets in $\mathbb{R}^d$ are countable unions of open boxes. Hence all open and closed sets are measurable.*

Proof: Note first that the result for closed sets follows from the result for open sets since a closed set is the complement of an open set. To prove the theorem for open sets, let us first agree to call an open box

$$A = (a_1^{(1)}, a_2^{(1)}) \times (a_1^{(2)}, a_2^{(2)}) \times \ldots \times (a_1^{(d)}, a_2^{(d)})$$

rational if all the coordinates $a_1^{(1)}, a_2^{(1)}, a_1^{(2)}, a_2^{(2)}, \ldots, a_1^{(d)}, a_2^{(d)}$ are rational. There are only countably many rationals boxes, and hence we only need to prove that if G is an open set, then

$$G = \bigcup \{B \ : \ B \text{ is a rational box contained in } G\}$$

We leave the details to the reader. $\square$

Exercises for Section 4.2

1. Show that if $A, B \in \mathcal{M}$, then $A \setminus B \in \mathcal{M}$.
2. Explain in detail why 4.2.3(iii) follows from (ii) and (iv).
3. Carry out the induction step in the proof of Proposition 4.2.3(iv).
4. Explain the equality $(A \cap E_1 \cap E_2^c) \cup (A \cap E_1^c) = A \cap (E_1 \cap E_2)^c$ in the proof of Lemma 4.2.3.
5. Carry out the induction step in the proof of Lemma 4.2.4.
6. Explain in detail why (iv) follows from (ii) and (iii) in Theorem 4.2.6.
7. Show that all *closed* halfspaces

   $$H = \{(x_1, \ldots, x_i, \ldots, x_d) \in \mathbb{R}^d \ : \ x_i \leq a\}$$

   and

   $$K = \{(x_1, \ldots, x_i, \ldots, x_d) \in \mathbb{R}^d \ : \ x_i \geq a\}$$

   are measurable
8. Recall that if A is a subset of $\mathbb{R}^d$ and $\mathbf{b} \in \mathbb{R}^d$, then

   $$A + \mathbf{b} = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A\}$$

   Show that $A + \mathbf{b}$ is measurable if and only if A is.
9. If A is a subset of $\mathbb{R}^d$, define $2A = \{2\mathbf{a} \mid \mathbf{a} \in A\}$. Show that $2A$ is measurable if and only if A is.
10. Fill in the details in the proof of Lemma 4.2.8.
11. Complete the proof of Theorem 4.2.9.

4.3 Lebesgue measure

Having constructed the outer measure μ^* and explored its basic properties, we are now ready to define the measure μ.

Definition 4.3.1 *The* Lebesgue measure μ *is the restriction of the outer measure* μ^* *to the measurable sets, i.e. it is the function*

$$\mu : \mathcal{M} \to [0, \infty]$$

defined by

$$\mu(A) = \mu^*(A)$$

for all $A \in \mathcal{M}$*.*

Remark: Since μ and μ^* are essentially the same function, you may wonder why we have introduced a new symbol for the Lebesgue measure. The answer is that although it is not going to make much of a difference for us here, it is convenient to distinguish between the two in more theoretical studies of measurability. All you have to remember for this text, is that $\mu(A)$ and $\mu^*(A)$ are defined and equal as long as A is measurable.

We can now prove that μ has the properties we asked for at the beginning of the chapter:

Theorem 4.3.2 *The Lebesgue measure* $\mu : \mathcal{M} \to [0, \infty]$ *has the following properties:*

(i) $\mu(\emptyset) = 0$.

(ii) (Completeness) Assume that $A \in \mathcal{M}$*, and that* $\mu(A) = 0$*. Then all subset* $B \subset A$ *are measurable, and* $\mu(B) = 0$*.*

(iv) (Countable subadditivity) If $\{A_n\}_{n\in\mathbb{N}}$ *is a sequence of measurable sets, then*

$$\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

(iv) (Countable additivity) If $\{E_n\}_{n\in\mathbb{N}}$ *is a disjoint sequence of measurable sets, then*

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$$

(v) For all closed boxes

$$B = [b_1^{(1)}, b_2^{(1)}] \times [b_1^{(2)}, b_2^{(2)}] \times \ldots \times [b_1^{(d)}, b_2^{(d)}]$$

we have

$$\mu(B) = |B| = (b_2^{(1)} - b_1^{(1)})(b_2^{(2)} - b_1^{(2)}) \cdot \ldots \cdot (b_2^{(d)} - b_1^{(d)})$$

Proof: (i) and (ii) follow from Lemma 4.2.2, and (iii) follows from part (iii) of Proposisition 4.1.4 since $\mathcal{M}$ is a σ-algebra, and $\bigcup_{n=1}^{\infty} A_n$ hence is measurable. Since we know from Theorem 4.2.9 that closed boxes are measurable, part (v) follows from Proposition 4.1.4(iv).

To prove (iv), we first observe that

$$\mu(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

by (iii). To get the opposite inequality, we use Lemma 4.2.4 with $A = \mathbb{R}^d$ to see that

$$\sum_{n=1}^{N} \mu(E_n) = \mu(\bigcup_{n=1}^{N} E_n) \leq \mu(\bigcup_{n=1}^{\infty} E_n)$$

Since this holds for all $N \in \mathbb{N}$, we must have

$$\sum_{n=1}^{\infty} \mu(E_n) \leq \mu(\bigcup_{n=1}^{\infty} E_n)$$

Hence we have both inequalities, and (iii) is proved. □

In what follows, we shall often need the following simple lemma:

Lemma 4.3.3 *If C, D are measurable sets such that $C \subset D$ and $\mu(D) < \infty$, then*

$$\mu(D \setminus C) = \mu(D) - \mu(C)$$

Proof: By additivity

$$\mu(D) = \mu(C) + \mu(D \setminus C)$$

Since $\mu(D)$ is finite, so is $\mu(C)$, and it makes sense to subtract $\mu(C)$ on both sides to get

$$\mu(D \setminus C) = \mu(D) - \mu(C)$$

□

The next properties are often referred to as *continuity of measure*:

Proposition 4.3.4 *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of measurable sets.*

(i) If the sequence is increasing (i.e. $A_n \subset A_{n+1}$ for all n), then

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$$

(ii) If the sequence is decreasing (i.e. $A_n \supset A_{n+1}$ for all n), and $\mu(A_1)$ is finite, then

$$\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$$

Proof: (i) If we put $E_1 = A_1$ and $E_n = A_n \setminus A_{n-1}$ for n>1, the sequence $\{E_n\}$ is disjoint, and $\bigcup_{k=1}^n E_k = A_n$ for all N (make a drawing). Hence

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n) =$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(E_k) = \lim_{n\to\infty} \mu(\bigcup_{k=1}^{n} E_k) = \lim_{n\to\infty} \mu(A_n)$$

where we have used the additivity of μ twice.

(ii) We first observe that $\{A_1 \setminus A_n\}_{n\in\mathbb{N}}$ is an increasing sequence of sets with union $A_1 \setminus \bigcap_{n=1}^{\infty} A_n$. By part (ii), we thus have

$$\mu(A_1 \setminus \bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_1 \setminus A_n)$$

Applying Lemma 4.3.3 on both sides, we get

$$\mu(A_1) - \mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} (\mu(A_1) - \mu(A_n)) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$$

Cancelling, we have $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$, as we set out to prove. □

Remark: The finiteness condition in part (ii) may look like an unnecessary consequence of a clumsy proof, but it is actually needed. To see why, let μ be Lebesgue measure in $\mathbb{R}$, and let $A_n = [n, \infty)$. Then $\mu(A_n) = \infty$ for all n, but $\mu(\bigcap_{n=1}^{\infty} A_n) = \mu(\emptyset) = 0$. Hence $\lim_{n\to\infty} \mu(A_n) \neq \mu(\bigcap_{n=1}^{\infty} A_n)$.

Example 1: We know already that *closed* boxes have the "right" measure (Theorem 4.3.2 (iv)), but what about *open* boxes? If

$$B = (b_1^{(1)}, b_2^{(1)}) \times (b_1^{(2)}, b_2^{(2)}) \times \ldots \times (b_1^{(d)}, b_2^{(d)})$$

is an open box, let B_n be the closed box

$$B_n = \left[b_1^{(1)} + \frac{1}{n}, b_2^{(1)} - \frac{1}{n}\right] \times \left[b_1^{(2)} + \frac{1}{n}, b_2^{(2)} - \frac{1}{n}\right] \times \ldots \times \left[b_1^{(d)} + \frac{1}{n}, b_2^{(d)} - \frac{1}{n}\right]$$

obtained by moving all walls a distance $\frac{1}{n}$ inwards. By the proposition,

$$\mu(B) = \lim_{n\to\infty} \mu(B_n)$$

and since the closed boxes B_n have the "right" measure, it follows that so does the open box B.

Example 2: Let
$$K_n = [-n, n]^d$$
be the closed box centered at the origin and with edges of length $2n$. For any measurable set A, it follows from the proposition above that
$$\mu(A) = \lim_{n\to\infty} \mu(A \cap K_n)$$

We shall need one more property of measurable sets. It tells us that measurable sets can be approximated from the outside by open sets and from the inside by closed sets.

Proposition 4.3.5 *Assume that $A \subset \mathbb{R}^d$ is a measurable set. For each $\epsilon > 0$, there is an open set $G \supset A$ such that $\mu(G \setminus A) < \epsilon$, and a closed set $F \subset A$ such that $\mu(A \setminus F) < \epsilon$.*

Proof: We begin with the open sets. Assume first A has finite measure. Then there is a covering $\{B_n\}$ of A by open rectangles such that
$$\sum_{n=1}^{\infty} |B_n| < \mu(A) + \epsilon$$
Since $\mu(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} |B_n|$, we see that $G = \bigcup_{n=1}^{\infty} B_n$ is an open set such that $A \subset G$, and $\mu(G) < \mu(A) + \epsilon$. Hence
$$\mu(G \setminus A) = \mu(G) - \mu(A) < \epsilon$$
by Lemma 4.3.3.

If $\mu(A)$ is infinite, we first use the boxes K_n in Example 2 to slice A into pieces of finite measure. More precisely, we let $A_n = A \cap (K_n \setminus K_{n-1})$, and use what we have already proved to find an open set G_n such that $A_n \subset G_n$ and $\mu(G_n \setminus A_n) < \frac{\epsilon}{2^n}$. Then $G = \bigcup_{n=1}^{\infty} G_n$ is an open set which contains A, and since $G \setminus A \subset \bigcup_{n=1}^{\infty}(G_n \setminus A_n)$, we get
$$\mu(G \setminus A) \leq \sum_{n=1}^{\infty} \mu(G_n \setminus A_n) < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon,$$
proving the statement about approximation by open sets.

To prove the statement about closed sets, just note that if we apply the first part of the theorem to A^c, we get an open set $G \supset A^c$ such that $\mu(G \setminus A^c) < \epsilon$. This means that $F = G^c$ is a closed set such that $F \subset A$, and since $A \setminus F = G \setminus A^c$, we have $\mu(A \setminus F) < \epsilon$. $\square$

We have now established the basic properties of the Lebesgue measure. For the remainder of the chapter, you may forget about the construction of the measure and concentrate on the results of this section plus the properties of measurable sets summed up in theorems 4.2.6 and 4.2.9 of the previous section.

Exercises for Section 4.3

1. Explain that $A \setminus F = G \setminus A^c$ and the end of the proof of Proposition 4.3.5.
2. Show that if E_1, E_2 are measurable, then

$$\mu(E_1) + \mu(E_2) = \mu(E_1 \cup E_2) + \mu(E_1 \cap E_2)$$

3. The *symmetric difference* $A \bigtriangleup B$ of two sets A, B is defined by

$$A \bigtriangleup B = (A \setminus B) \cup (B \setminus A)$$

   A subset of $\mathbb{R}^d$ is called a $\mathcal{G}_\delta$-set if it is the intersection of countably many open sets, and it is called a $\mathcal{F}_\sigma$-set if it is union of countably many closed set.

   a) Show that if A and B are measurable, then so is $A \bigtriangleup B$.
   b) Explain why all $\mathcal{G}_\delta$- and $\mathcal{F}_\sigma$-sets are measurable.
   c) Show that if A is measurable, there is a $\mathcal{G}_\delta$-set G such that $\mu(A \bigtriangleup G) = 0$.
   d) Show that if A is measurable, there is a $\mathcal{F}_\sigma$-set F such that $\mu(A \bigtriangleup F) = 0$.

4. Assume that $A \in \mathcal{M}$ has finite measure. Show that for every $\epsilon > 0$, there is a compact set $K \subset A$ such that $\mu(A \setminus K) < \epsilon$.
5. Assume that $\{A_n\}$ is a countable sequence of measurable sets, and assume that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Show that the set

$$A = \{x \in \mathbb{R}^d \,|x \text{ belongs to infinitely many } A_n\}$$

   has measure zero.

4.4 Measurable functions

Before we turn to integration, we need to look at the functions we hope to integrate, the *measurable* functions. As functions taking the values $\pm\infty$ will occur naturally as limits of sequences of ordinary functions, we choose to include them from the beginning; hence we shall study functions

$$f : \mathbb{R}^d \to \overline{\mathbb{R}}$$

where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is the set of *extended real numbers*. Don't spend too much effort on trying to figure out what $-\infty$ and ∞ "really" are — they are just convenient symbols for describing divergence.

To some extent we may extend ordinary algebra to $\overline{\mathbb{R}}$, e.g., we shall let

$$\infty + \infty = \infty, \quad -\infty - \infty = -\infty$$

and

$$\infty \cdot \infty = \infty, \quad (-\infty) \cdot \infty = -\infty, \quad (-\infty) \cdot (-\infty) = \infty.$$

If $r \in \mathbb{R}$, we similarly let

$$\infty + r = \infty, \quad -\infty + r = -\infty$$

For products, we have to take the sign of r into account, hence

$$\infty \cdot r = \begin{cases} \infty & \text{if } r > 0 \\ \\ -\infty & \text{if } r < 0 \end{cases}$$

and similarly for $(-\infty) \cdot r$.

All the rules above are natural and intuitive. Expressions that do not have an intuitive interpretation, are usually left undefined, e.g. is $\infty - \infty$ *not* defined. There is one exception to this rule; it turns out that in measure theory (but not in other parts of mathematics!) it is convenient to define $0 \cdot \infty = \infty \cdot 0 = 0$.

Since algebraic expressions with extended real numbers are not always defined, we need to be careful and always check that our expressions make sense.

We are now ready to define measurable functions:

Definition 4.4.1 *A function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is* measurable *if*

$$f^{-1}([-\infty, r)) \in \mathcal{M}$$

for all $r \in \mathbb{R}$. In other words, the set

$$\{x \in \mathbb{R}^d \,:\, f(x) < r\}$$

must be measurable for all $r \in \mathbb{R}$.

The half-open intervals in the definition are just a convenient starting point for showing that the inverse images of more complicated sets are measurable:

Proposition 4.4.2 *If $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is measurable, then $f^{-1}(I) \in \mathcal{M}$ for all intervals $I = (s, r), I = (s, r], I = [s, r), I = [s, r]$ where $s, r \in \overline{\mathbb{R}}$. Indeed, $f^{-1}(A) \in \mathcal{M}$ for all open and closed sets A.*

Proof: We use that inverse images commute with intersections, unions and complements. First observe that for any $r \in \mathbb{R}$

$$f^{-1}\big([-\infty, r]\big) = f^{-1}\big(\bigcap_{n \in \mathbb{N}} [-\infty, r + \frac{1}{n})\big) = \bigcap_{n \in \mathbb{N}} f^{-1}\big([-\infty, r + \frac{1}{n})\big) \in \mathcal{M}$$

which shows that the closed intervals $[-\infty, r]$ are measurable. Taking complements, we see that the intervals $[s, \infty]$ and $(s, \infty]$ are measurable:

$$f^{-1}([s, \infty]) = f^{-1}([-\infty, s)^c) = \big(f^{-1}([-\infty, s))\big)^c \in \mathcal{M}$$

and

$$f^{-1}((s,\infty]) = f^{-1}([-\infty,s]^c) = \big(f^{-1}([-\infty,s])\big)^c \in \mathcal{M}$$

To show that finite intervals are measurable, we just take intersections, e.g.,

$$f^{-1}((s,r)) = f^{-1}([-\infty,r) \cap (s,\infty]) = f^{-1}([-\infty,r)) \cap f^{-1}((s,\infty]) \in \mathcal{M}$$

If A is open, we know from Theorem 4.2.9 that it is a countable union $A = \bigcup_{n\in\mathbb{N}} I_n$ of open intervals. Hence

$$f^{-1}(A) = f^{-1}\big(\bigcup_{n\in\mathbb{N}} I_n\big) = \bigcup_{n\in\mathbb{N}} f^{-1}(I_n) \in \mathcal{M}$$

Finally, if A is closed, we use that its complement is open to get

$$f^{-1}(A) = \big(f^{-1}(A^c)\big)^c \in \mathcal{M}$$

□

It is sometimes convenient to use other kinds of intervals than those in the definition to check that a function is measurable:

Proposition 4.4.3 *Consider a function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$. If either*

(i) $f^{-1}([-\infty,r]) \in \mathcal{M}$ *for all* $r \in \mathbb{R}$, *or*

(ii) $f^{-1}([r,\infty]) \in \mathcal{M}$ *for all* $r \in \mathbb{R}$, *or*

(iii) $f^{-1}((r,\infty]) \in \mathcal{M}$ *for all* $r \in \mathbb{R}$,

then f is measurable.

Proof: In either case we just have to check that $f^{-1}([-\infty,r)) \in \mathcal{M}$ for all $r \in \mathbb{R}$. This can be done by the techniques in the previous proof. The details are left to the reader. □

The next result tells us that there are many measurable functions:

Proposition 4.4.4 *All continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ are measurable.*

Proof: Since f is continuous and takes values in $\mathbb{R}$,

$$f^{-1}([-\infty,r)]) = f^{-1}((-\infty,r))$$

is an open set by Proposition 1.3.9 and thus measurable by Theorem 4.2.9. □

We shall now prove a series of results showing how we can obtain new measurable functions from old ones. These results are not very exciting, but they are necessary for the rest of the theory. Note that the functions in the next two propositions take values in $\mathbb{R}$ and not $\overline{\mathbb{R}}$.

Proposition 4.4.5 *If $f : \mathbb{R}^d \to \mathbb{R}$ is measurable, then $\phi \circ f$ is measurable for all continuous functions $\phi : \mathbb{R} \to \mathbb{R}$. In particular, f^2 is measurable.*

Proof: We have to check that

$$(\phi \circ f)^{-1}((-\infty, r)) = f^{-1}(\phi^{-1}((-\infty, r)))$$

is measurable. Since ϕ is contiunuous, $\phi^{-1}((-\infty, r))$ is open, and consequently $f^{-1}(\phi^{-1}((-\infty, r)))$ is measurable by Proposition 4.4.2. To see that f^2 is measurable, apply the first part of the theorem to the function $\phi(x) = x^2$. □

Proposition 4.4.6 *If the functions $f, g :\to \mathbb{R}$ are measurable, so are $f + g$, $f - g$, and fg.*

Proof: To prove that $f + g$ is measurable, observe first that $f + g < r$ means that $f < r - g$. Since the rational numbers are dense, it follows that there is a rational number q such that $f < q < r - g$. Hence

$$(f + g)^{-1}([-\infty, r)) = \{x \in \mathbb{R}^d \,|\, (f + g) < r) =$$

$$\bigcup_{q \in \mathbb{Q}} \Big(\{x \in \mathbb{R}^d \,|\, f(x) < q\} \cap \{x \in \mathbb{R}^d \,|\, g < r - q\} \Big)$$

which is measurable since $\mathbb{Q}$ is countable and a countabe union of measurable sets is measurable. A similar argument proves that $f - g$ is measurable.

To prove that fg is measurable, note that by Proposition 4.4.5 and what we have already proved, f^2, g^2, and $(f + g)^2$ are measurable, and hence

$$fg = \frac{1}{2}\left((f + g)^2 - f^2 - g^2\right)$$

is measurable (check the details). □

We would often like to apply the result above to functions that take values in the extended real numbers, but the problem is that the expressions need not make sense. As we shall mainly be interested in functions that are finite except on a set of measure zero, there is a way out of the problem. Let us start with the terminology.

Definition 4.4.7 *We say that a measurable function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is* finite almost everywhere *if the set $\{x \in \mathbb{R}^d : f(x) = \pm\infty\}$ has measure zero. We say that two measurable functions $f, g : \mathbb{R}^d \to \overline{\mathbb{R}}$ are* equal almost everywhere *if the set $\{x \in \mathbb{R}^d : f(x) \neq g(x)\}$ has measure zero. We usually abbreviate "almost everywhere" by "a.e.".*

If the measurable functions f and g are finite a.e., we can modify them to get measurable functions f' and g' which take values in $\mathbb{R}$ and are equal a.e. to f and g, respectively (see exercise 11). By the proposition above, $f' + g'$, $f' - g'$ and $f'g'$ are measurable, and for many purposes they are good representatives for $f + g$, $f - g$ and fg.

Let us finally see what happens to limits of sequences.

Proposition 4.4.8 *If $\{f_n\}$ is a sequence of measurable functions, then $\sup_{n\in\mathbb{N}} f_n(x)$, $\inf_{n\in\mathbb{N}} f_n(x)$, $\limsup_{n\to\infty} f_n(x)$ and $\liminf_{n\to\infty} f_n(x)$ are measurable. If the sequence converges pointwise, then $\lim_{n\to\infty} f_n(x)$ is a measurable function.*

Proof: To see that $f(x) = \sup_{n\in\mathbb{N}} f_n(x)$ is measurable, we use Proposition 4.4.3(iii). For any $r \in \mathbb{R}$

$$f^{-1}((r,\infty)) = \{x \in \mathbb{R}^d : \sup_{n\in\mathbb{N}} f_n(x) > r\} =$$

$$= \bigcup_{n\in\mathbb{N}} \{\mathbf{x} \in \mathbb{R}^d \ : \ f_n(x) > r\} = \bigcup_{n\in\mathbb{N}} f_n^{-1}((r,\infty]) \in \mathcal{M}$$

and hence f is measurable by Propostion 4.4.3(iii). The argument for $\inf_{n\in\mathbb{N}} f_n(x)$ is similar.

To show that $\limsup_{n\to\infty} f_n(x)$ is measurable, first observe that the functions

$$g_k(x) = \sup_{n\geq k} f_n(x)$$

are measurable by what we have already shown. Since

$$\limsup_{n\to\infty} f_n(x) = \inf_{k\in\mathbb{N}} g_k(x)\big),$$

the measurability of $\limsup_{n\to\infty} f_n(x)$ follows. A similar argument holds for $\liminf_{n\to\infty} f_n(x)$. If the sequence converges pointwise, then $\lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x)$ and is hence measurable. □

Let us sum up what we have done so far in this chapter. We have constructed the Lebesgue measure μ which assigns a d-dimensional volume to a large class of subset of $\mathbb{R}^d$, and we have explored the basic properties of a class of measurable functions which are closely connected to the Lebesgue measure. In the following sections we shall combine the two to create a theory of integration which is stronger and more flexible than the one you are used to.

Exercises for Section 4.4

1. Show that if $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is measurable, the sets $f^{-1}(\{\infty\})$ and $f^{-1}(\{-\infty\})$ are measurable.
2. Complete the proof of Proposition 4.4.2 by showing that f^{-1} of the intervals $(-\infty, r)$, $(-\infty, r]$, $[r, \infty)$, (r, ∞), $(-\infty, \infty)$, where $r \in \mathbb{R}$, are measurable.
3. Prove Proposition 4.4.3.
4. Show that if $f_1, f_2, \ldots, f_n$ are measurable functions with values in $\mathbb{R}$, then $f_1 + f_2 + \cdots + f_n$ and $f_1 f_2 \cdot \ldots \cdot f_n$ are measurable.
5. The *indicator function* of a set $A \subset \mathbb{R}$ is defined by
$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \\ 0 & \text{otherwise} \end{cases}$$
   a) Show that $\mathbf{1}_A$ is a measurable function if and only if $A \in \mathcal{M}$.
   b) A *simple* function is a function $f : \mathbb{R}^d \to \mathbb{R}$ of the form
$$f(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}(x)$$
   where $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $A_1, A_2, \ldots, A_n \in \mathcal{M}$. Show that all simple functions are measurable.
6. Let $\{E_n\}$ be a disjoint sequence of measurable sets such that $\bigcup_{n=1}^{\infty} E_n = \mathbb{R}^d$, and let $\{f_n\}$ be a sequence of measurable functions. Show that the function defined by
$$f(x) = f_n(x) \text{ when } x \in E_n$$
   is measurable.
7. Fill in the details of the proof of the fg part of Proposition 4.4.6. You may want to prove first that if $h : \mathbb{R}^d \to \mathbb{R}$ is measurable, then so is $\frac{h}{2}$.
8. Prove the inf- and the lim inf-part of Proposition 4.4.8.
9. Let us write $f \sim g$ to denote that f and g are two measurable functions which are equal a.e.. Show that $\sim$ is an equivalence relation, i.e.:
   (i) $f \sim f$
   (ii) If $f \sim g$, then $g \sim f$.
   (iii) If $f \sim g$ and $g \sim h$, then $f \sim h$.
10. Show that if $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is measurable and $g : \mathbb{R}^d \to \overline{\mathbb{R}}$ equals f almost everywhere, then g is measurable.
11. Assume that $f : \mathbb{R}^d \to \mathbb{R}$ is finite a.e. Define a new function $f' : \mathbb{R}^d \to \mathbb{R}$ by
$$f'(x) = \begin{cases} f(x) & \text{if } f(x) \text{ is finite} \\ \\ 0 & \text{otherwise} \end{cases}$$
   Show that f' is measurable and equal to f a.e.
12. A sequence $\{f_n\}$ of measurable functions is said to *converge almost everywhere* to f if there is a set A of measure 0 such that $f_n(x) \to f(x)$ for all $x \notin A$. Show that f is measurable.

4.5 Integration of simple functions

If A is a subset of $\mathbb{R}^d$, we define its *indicator function* by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \\ 0 & \text{otherwise} \end{cases}$$

The indicator function is measuable if and only if A is measurable.

A measurable function $f : \mathbb{R}^d \to \mathbb{R}$ is called a *simple* function if it takes only finitely many different values $a_1, a_2, \dots, a_n$. We may then write

$$f(x) = \sum_{i=1}^{n} a_1 \mathbf{1}_{A_i}(x)$$

where the sets $A_i = \{x \in \mathbb{R}^d \mid f(x) = a_i\}$ are disjoint and measurable. Note that if one of the a_i's is zero, the term does not contribute to the sum, and it is occasionally convenient to drop it.

If we instead start with measurable sets $B_1, B_2, \dots, B_m$ and real numbers $b_1, b_2, \dots, b_m$, then

$$g(x) = \sum_{i=1}^{m} b_i \mathbf{1}_{B_i}(x)$$

is measurable and takes only finitely many values, and hence is a simple function. The difference between f and g is that the sets $A_1, A_2, \dots, A_n$ in f are disjoint with union $\mathbb{R}^d$, and that the numbers $a_1, a_2, \dots, a_n$ are distinct. The same need not be the case for g. We say that the simple function f is on *standard form*, while g is not.

You may think of a simple function as a generalized step function. The difference is that step functions are constant on intervals (in $\mathbb{R}$), rectangles (in $\mathbb{R}^2$), or boxes (in higher dimensions), while simple functions need only be constant on much more complicated (but still measurable) sets.

We can now define the integral of a nonnegative simple function.

Definition 4.5.1 *Assume that*

$$f(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}(x)$$

is a nonnegative simple function on standard form. Then the (Lebesgue) integral *of f is defined by*

$$\int f \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$$

Recall that we are using the convention that $0 \cdot \infty = 0$, and hence $a_i \mu(A_i) = 0$ if $a_i = 0$ and $\mu(A_i) = \infty$.

Note that the integral of a simple function is

$$\int \mathbf{1}_A \, d\mu = \mu(A)$$

To see that the definition is reasonable, assume that you are in $\mathbb{R}^2$. Since $\mu(A_i)$ measures the area of the set A_i, the product $a_i\mu(A_i)$ measures in an intuitive way the volume of the solid with base A_i and height a_i.

We need to know that the formula in the definition also holds when the simple function is not on standard form. The first step is the following, simple lemma

Lemma 4.5.2 *If*

$$g(x) = \sum_{j=1}^{m} b_j \mathbf{1}_{B_j}(x)$$

is a nonnegative simple function where the B_j's are disjoint and $\mathbb{R}^d = \bigcup_{j=1}^m B_j$, then

$$\int g \, d\mu = \sum_{j=1}^{n} b_j \mu(B_j)$$

Proof: The problem is that the values $b_1, b_2, \ldots, b_m$ need not be distinct, but this is easily fixed: If $c_1, c_2, \ldots, c_k$ are the distinct values taken by g, let $b_{i,1}$, $b_{i,2}$,. . . ,b_{i,n_i} be the b_j's that are equal to c_i, and let $C_i = B_{i,1} \cup B_{i,2} \cup \ldots \cup B_{i,n_i}$. Then $\mu(C_i) = \mu(B_{i,1}) + \mu(B_{i,2}) + \ldots + \mu(B_{i,n_i})$, and hence

$$\sum_{j=1}^{n} b_j \mu(B_j) = \sum_{i=1}^{k} c_i \mu(C_i)$$

Since $g(x) = \sum_{i=1}^k c_i \mathbf{1}_{C_i}(x)$ is the standard form representation of g, we have

$$\int g \, d\mu = \sum_{j=1}^{n} c_i \mu(C_i)$$

and the lemma is proved □

The next step is also easy:

Proposition 4.5.3 *Assume that f and g are two nonnegative simple functions, and let c be a nonnnegative, real number. Then*

(i) $\int cf \, d\mu = c \int f \, d\mu$

(ii) $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

Proof: (i) is left to the reader. To prove (ii), let

$$f(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}(x)$$

$$g(x) = \sum_{j=1}^{n} b_j \mathbf{1}_{B_j}(x)$$

be standard form representations of f and g, and define $C_{i,j} = A_i \cap B_j$. By the lemma above

$$\int f \, d\mu = \sum_{i,j} a_i \mu(C_{i,j})$$

and

$$\int g \, d\mu = \sum_{i,j} b_j \mu(C_{i,j})$$

and also

$$\int (f+g) \, d\mu = \sum_{i,j} (a_i + b_j) \mu(C_{i,j})$$

since the value of $f + g$ on $C_{i,j}$ is $a_i + b_j$ □

We can now easily prove that the formula in Definition 4.5.1 holds for all positive representations of step functions:

Corollary 4.5.4 *If $f(x) = \sum_{n=1} a_i \mathbf{1}_{A_i}(x)$ is a step function with $a_i \geq 0$ for all i, then*

$$\int f \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$$

Proof: By the Proposition

$$\int f \, d\mu = \int \sum_{i=1}^{n} a_i \mathbf{1}_{A_i} \, d\mu = \sum_{i=1}^{n} \int a_i \mathbf{1}_{A_i} \, d\mu = \sum_{i=1}^{n} a_i \int \mathbf{1}_{A_i} \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$$

□

We need to prove yet another, almost obvious result. We write $g \leq f$ to say that $g(x) \leq f(x)$ for all x.

Proposition 4.5.5 *Assume that f and g are two nonnegative simple functions. If $g \leq f$, then*

$$\int g \, d\mu \leq \int f \, d\mu$$

Proof: We use the same trick as in the proof of Proposition 4.5.3: Let

$$f(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}(x)$$

$$g(x) = \sum_{j=1}^{n} b_j \mathbf{1}_{B_j}(x)$$

be standard form representations of f and g, and define $C_{i,j} = A_i \cap B_j$. Then

$$\int f\,d\mu = \sum_{i,j} a_i \mu(C_{i,j}) \geq \sum_{i,j} b_j \mu(C_{i,j}) = \int g\,d\mu$$

□

We shall end this section with a key result on limits of integrals, but first we need some notation. Observe that if $f = \sum_{i=1}^{n} a_n \mathbf{1}_{A_n}$ is a simple function and B is a measurable set, then $\mathbf{1}_B f = \sum_{i=1}^{n} a_n \mathbf{1}_{A_n \cap B}$ is also a measurable function. We shall write

$$\int_B f\,d\mu = \int \mathbf{1}_B f\,d\mu$$

and call this the *integral of f over B*. The lemma below may seem obvious, but it is the key to many later results.

Lemma 4.5.6 *Assume that B is a measurable set, b a positive real number, and $\{f_n\}$ an increasing sequence of nonnegative simple functions such that $\lim_{n\to\infty} f_n(x) \geq b$ for all $x \in B$. Then $\lim_{n\to\infty} \int_B f_n\,d\mu \geq b\mu(B)$.*

Proof: Let a be any positive number less than b, and define

$$A_n = \{x \in B \mid f_n(x) \geq a\}$$

Since $f_n(x) \uparrow b$ for all $x \in B$, we see that the sequence $\{A_n\}$ is increasing and that

$$B = \bigcup_{n=1}^{\infty} A_n$$

By continuity of measure (Proposition 4.3.4(i)), $\mu(B) = \lim_{n\to\infty} \mu(A_n)$, and hence for any positive number m less that $\mu(B)$, we can find an $N \in \mathbb{N}$ such that $\mu(A_n) > m$ when $n \geq N$. Since $f_n \geq a$ on A_n, we thus have

$$\int_B f_n\,d\mu \geq \int_{A_n} a\,d\mu = am$$

whenever $n \geq N$. Since this holds for any number a less than b and any number m less than $\mu(B)$, we must have $\lim_{n\to\infty} \int_B f_n\,d\mu \geq b\mu(B)$ □

To get the result we need, we extend the lemma to simple functions:

Proposition 4.5.7 *Let g be a nonnegative simple function and assume that $\{f_n\}$ is an increasing sequence of nonnegative simple functions such that $\lim_{n\to\infty} f_n(x) \geq g(x)$ for all x. Then*

$$\lim_{n\to\infty} \int f_n \, d\mu \geq \int g \, d\mu$$

Proof: Let $g(x) = \sum_{i=1}^{m} b_i \mathbf{1}_{B_1}(x)$ be the standard form of g. If any of the b_i's is zero, we may just drop that term in the sum, so that we from now on assume that all the b_i's are nonzero. By Corollary 4.5.3(ii), we have

$$\int_{B_1 \cup B_2 \cup \ldots \cup B_m} f_n \, d\mu = \int_{B_1} f_n \, d\mu + \int_{B_2} f_n \, d\mu + \ldots + \int_{B_m} f_n \, d\mu$$

By the lemma, $\lim_{n\to\infty} \int_{B_i} f_n \, d\mu \geq b_i \mu(B_i)$, and hence

$$\lim_{n\to\infty} \int f_n \, d\mu \geq \lim_{n\to\infty} \int_{B_1 \cup B_2 \cup \ldots \cup B_m} f_n \, d\mu \geq \sum_{i=1}^{m} b_i \mu(B_i) = \int g \, d\mu$$

□

We are now ready to extend the Lebesgue integral to all positive, measurable functions. This will be the topic of the next section.

Exercises for Section 4.5

1. Show that if f is a measurable function, then the *level set*

$$A_a = \{x \in \mathbb{R}^d \mid f(x) = a\}$$

   is measurable for all $a \in \overline{\mathbb{R}}$.

2. Check that according to Definition 4.5.1, $\int \mathbf{1}_A \, \mathbf{d}\mu = \mu(A)$ for all $A \in \mathcal{M}$.

3. Prove part (i) of Proposition 4.5.3.

4. Show that if $f_1, f_2, \ldots, f_n$ are simple functions, then so are

$$h(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

   and

$$h(x) = \min\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

5. Let $A = \mathbb{Q} \cap [0, 1]$. This function is not integrable in the Riemann sense. What is $\int \mathbf{1}_A \, d\mu$?

4.6 Integrals of nonnegative functions

We are now ready to define the integral of a general, nonnegative, measurable function.

Definition 4.6.1 *If $f : \mathbb{R}^d \to [0, \infty]$ is measurable, we define*

$$\int f \, d\mu = \sup\{\int g \, d\mu \,|\, g \text{ is a nonnegative simple function, } g \leq f\}$$

Remark: Note that if f is a simple function, we now have two definitions of $\int f \, d\mu$; the original one in Definition 4.5.1 and a new one in the definition above. It follows from Proposition 4.5.5 that the two definitions agree.

The definition above is natural, but also quite abstract, and we shall work toward a reformulation that is often easier to handle.

Proposition 4.6.2 *Let $f : \mathbb{R}^d \to [0, \infty]$ be a measurable function, and assume that $\{h_n\}$ is an increasing sequence of simple functions converging pointwise to f. Then*

$$\lim_{n\to\infty} \int h_n \, d\mu = \int f \, d\mu$$

Proof: Since the sequence $\{\int h_n \, d\mu\}$ is increasing by Proposition 4.5.5, the limit clearly exists (it may be ∞), and since $\int h_n \, d\mu \leq \int f \, d\mu$ for all n, we must have

$$\lim_{n\to\infty} \int h_n \, d\mu \leq \int f \, d\mu$$

To get the opposite inequality, it suffices to show that

$$\lim_{n\to\infty} \int h_n \, d\mu \geq \int g \, d\mu$$

for each simple function $g \leq f$, but this follows from Proposition 4.5.7. □

The proposition above would lose much of its power if there weren't any increasing sequences of simple functions converging to f. The next result tells us that there always are. Pay attention to the argument, it is a key to why the theory works.

Proposition 4.6.3 *If $f : \mathbb{R}^d \to [0, \infty)$ is measurable, there is an increasing sequence $\{h_n\}$ of simple functions converging pointwise to f. Moreover, for each n either $f_n(x) - \frac{1}{2^n} < h_n(x) \leq f_n(x)$ or $h_n(x) = 2^n$*

Proof: To construct the simple function h_n, we cut the interval $[0, 2^n)$ into half-open subintervals of length $\frac{1}{2^n}$, i.e. intervals

$$I_k = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$$

where $0 \leq k < 2^{2n}$, and then let

$$A_k = f^{-1}(I_k)$$

We now define

$$h_n(x) = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n}\mathbf{1}_{A_k}(x) + 2^n\mathbf{1}_{\{x \mid f(x)\geq 2^n\}}$$

By definition, h_n is a simple function no greater than f. Since the intervals get narrower and narrower and cover more and more of $[0,\infty)$, it is easy to see that h_n converges pointwise to f. To see why the sequence increases, note that each time we increase n by one, we split each of the former intervals I_k in two, and this will cause the new step function to equal the old one for some x's and jump one step upwards for others (make a drawing).

The last statement follows directly from the construction. □

Remark: You should compare the partitions in the proof above to the partitions you have seen in earlier treatments of integration. When we integrate a function of one variable in calculus, we partition an interval $[a,b]$ on the x-axis and use this partition to approximate the original function by a step function. In the proof above, we instead partitioned the y-axis into intervals and used this partition to approximate the original function by a simple function. The difference is that the latter approach gives us much better control over what is going one; the partition controls the oscillations of the function. The price we have to pay, it that we get simple functions instead of step functions, and to use simple functions for integration, we need measure theory.

Let us combine the last two results in a handy corollary:

Corollary 4.6.4 *If $f : \mathbb{R}^d \to [0,\infty)$ is measurable, there is an increasing sequence $\{h_n\}$ of simple functions converging pointwise to f, and*

$$\int f\,d\mu = \lim_{n\to\infty} \int h_n\,d\mu$$

Let us take a look at some properties of the integral.

Proposition 4.6.5 *Assume that $f, g : \mathbb{R}^d \to [0,\infty]$ are measurable functions and that c is a nonnegative, real number. Then:*

(i) $\int cf\,d\mu = c\int f\,d\mu$.

(ii) $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$.

(iii) *If $g \leq f$, then* $\int g\,d\mu \leq \int f\,d\mu$.

Proof: (iii) is immediate from the definition, and (i) is left to the reader. To prove (ii), let $\{f_n\}$ and $\{g_n\}$ be to increasing sequence of simple functions converging to f and g, respectively. Then $\{f_n+g_n\}$ is an increasing sequence of simple functions converging to $f+g$, and

$$\int (f+g)\,d\mu = \lim_{n\to\infty} \int (f_n+g_n)\,d\mu = \lim_{n\to\infty} \left(\int f_n\,d\mu + \int g_n\,d\mu \right) =$$

$$= \lim_{n\to\infty} \int f_n\,d\mu + \lim_{n\to\infty} \int g_n\,d\mu = \int f\,d\mu + \int g\,d\mu$$

□

One of the big advantages of Lebesgue integration over traditional Riemann integration, is that the Lebesgue integral is much better behaved with respect to limits. The next result is our first example:

Theorem 4.6.6 (Monotone Convergence Theorem) *If $\{f_n\}$ is an increasing sequence of nonnegative, measurable functions such that $f(x) = \lim_{n\to\infty} f_n(x)$ for all x, then*

$$\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu$$

In other words,

$$\lim_{n\to\infty} \int f_n\,d\mu = \int \lim_{n\to\infty} f_n\,d\mu$$

Proof: We know from Proposition 4.4.8 that f is measurable, and hence the integral $\int f\,d\mu$ is defined. Since $f_n \leq f$, we have $\int f_n\,d\mu \leq \int f\,d\mu$ for all n, and hence

$$\lim_{n\to\infty} \int f_n\,d\mu \leq \int f\,d\mu$$

To prove the opposite inequality, we approximate each f_n by simple functions as in the proof of Proposition 4.6.3; in fact, let h_n be the n-th approximation to f_n. Assume that we can prove that the sequence $\{h_n\}$ converges to f; then

$$\lim_{n\to\infty} \int h_n\,d\mu = \int f\,d\mu$$

by Proposition 4.6.2. Since $f_n \geq h_n$, this would give us the desired inequality

$$\lim_{n\to\infty} \int f_n\,d\mu \geq \int f\,d\mu$$

It remains to show that $h_n(x) \to f(x)$ for all x. From Proposition 4.6.3 we know that for all n, either $f_n(x) - \frac{1}{2^n} < h_n(x) \leq f_n(x)$ or $h_n(x) = 2^n$. If $h_n(x) = 2^n$ for infinitely many n, then $h_n(x)$ goes to ∞, and hence to $f(x)$. If $h_n(x)$ is *not* equal to 2^n for infinitely many n, then we eventually have

$f_n(x) - \frac{1}{2^n} < h_n(x) \leq f_n(x)$, and hence $h_n(x)$ converges to $f(x)$ since $f_n(x)$ does. □

We would really have liked the formula

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu \tag{4.6.1}$$

above to hold in general, but as the following example shows, this is not the case.

Example 1: Let $f_n = \mathbf{1}_{[n,n+1]}$. Then $\lim_{n\to\infty} f_n(x) = 0$ for all x, but $\int f_n \, d\mu = 1$. Hence

$$\lim_{n\to\infty} \int f_n \, d\mu = 1$$

but

$$\int \lim_{n\to\infty} f_n \, d\mu = 0$$

There are many results in measure theory giving conditions for (4.6.1) to hold, but there is no ultimate theorem covering all others. There is, however, a simple inequality that always holds.

Theorem 4.6.7 (Fatou's Lemma) *Assume that $\{f_n\}$ is a sequence of non-negative, measurable functions. Then*

$$\liminf_{n\to\infty} \int f_n \, d\mu \geq \int \liminf_{n\to\infty} f_n \, d\mu$$

Proof: Let $g_k(x) = \inf_{k\geq n} f_n(x)$. Then $\{g_k\}$ is an increasing sequence of measurable functions, and by the Monotone Convergence Theorem

$$\lim_{k\to\infty} \int g_k \, d\mu = \int \lim_{k\to\infty} g_k \, d\mu = \int \liminf_{n\to\infty} f_n \, d\mu$$

where we have used the definition of $\liminf$ in the last step. Since $f_k \geq g_k$, we have $\int f_k \, d\mu \geq \int g_k \, d\mu$, and hence

$$\liminf_{k\to\infty} \int f_k \, d\mu \geq \lim_{k\to\infty} \int g_k \, d\mu = \int \liminf_{n\to\infty} f_n \, d\mu$$

and the result is proved. □

Fatou's Lemma is often a useful tool in establishing more sophisticated results, see Exercise 14 for a typical example.

Just as for simple functions, we define integrals over measurable subsets A of $\mathbb{R}^d$ by the formula

$$\int_A f\,d\mu = \int \mathbf{1}_A f\,d\mu$$

So far we have allowed our integrals to be infinite, but we are mainly interested in situations where $\int f\,d\mu$ is finite:

Definition 4.6.8 *A function* $f : \mathbb{R}^d \to [0,\infty]$ *is said to be* integrable *if it is measurable and* $\int f\,d\mu < \infty$.

Exercises for Section 4.6

1. Assume $f : \mathbb{R}^d \to [0,\infty]$ is a nonnegative simple function. Show that the two definitions of $\int f\,d\mu$ given in Definitions 4.5.1 and 4.6.1 coincide.
2. Prove Proposition 4.6.5(i).
3. Show that if $f : \mathbb{R}^d \to [0,\infty]$ is measurable, then
$$\mu(\{x \in \mathbb{R}^d \mid f(x) \geq a\}) \leq \frac{1}{a}\int f\,d\mu$$
for all positive, real numbers a.
4. In this problem, $f, g : \mathbb{R}^d \to [0,\infty]$ are measurable functions.
   a) Show that $\int f\,d\mu = 0$ if and only if $f = 0$ a.e.
   b) Show that if $f = g$ a.e., then $\int f\,d\mu = \int g\,d\mu$.
   c) Show that if $\int_E f\,d\mu = \int_E g\,d\mu$ for all measurable sets E, then $f = g$ a.e.
5. In this problem, $f : \mathbb{R}^d \to [0,\infty]$ is a measurable function and A, B are measurable sets.
   a) Show that $\int_A f\,d\mu \leq \int f\,d\mu$
   b) Show that if A, B are disjoint, then $\int_{A\cup B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu$.
   c) Show that in general $\int_{A\cup B} f\,d\mu + \int_{A\cap B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu$.
6. Show that if $f : \mathbb{R}^d \to [0,\infty]$ is integrable, then f is finite a.e.
7. Let $f : \mathbb{R} \to \mathbb{R}$ be the function
$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ & \\ 0 & \text{otherwise} \end{cases}$$
and for each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be the function
$$f_n(x) = \begin{cases} 1 & \text{if } x = \frac{p}{q} \text{ where } p \in \mathbb{Z},\ q \in \mathbb{N}, q \leq n \\ & \\ 0 & \text{otherwise} \end{cases}$$
   a) Show that $\{f_n(x)\}$ is an increasing sequence converging to $f(x)$ for all $x \in \mathbb{R}$.

b) Show that each f_n is Riemann integrable over $[0,1]$ with $\int_0^1 f_n(x)\,dx = 0$ (this is integration as taught in calculus courses).

c) Show that f is *not* Riemann integrable over $[0,1]$.

d) Show that the one-dimensional Lebesgue integral $\int_{[0,1]} f\,d\mu$ exists and find its value.

8. a) Let $\{u_n\}$ be a sequence of positive, measurable functions. Show that

$$\int \sum_{n=1}^{\infty} u_n\,d\mu = \sum_{n=1}^{\infty} \int u_n\,d\mu$$

b) Assume that f is a nonnnegative, measurable function and that $\{B_n\}$ is a disjoint sequence of measurable sets with union B. Show that

$$\int_B f\,d\mu = \sum_{n=1}^{\infty} \int_{B_n} f\,d\mu$$

9. Assume that f is a nonnegative, measurable function and that $\{A_n\}$ is an increasing sequence of measurable sets with union A. Show that

$$\int_A f\,d\mu = \lim_{n\to\infty} \int_{A_n} f\,d\mu$$

10. Show the following generalization of the Monotone Convergence Theorem: If $\{f_n\}$ is an increasing sequence of nonnegative, measurable functions such that $f(x) = \lim_{n\to\infty} f_n(x)$ *almost everywhere.* (i.e. for all x outside a set N of measure zero), then

$$\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu$$

11. Find a decreasing sequence $\{f_n\}$ of measurable functions $f_n : \mathbb{R} \to [0,\infty)$ converging pointwise to zero such that $\lim_{n\to\infty} \int f_n\,d\mu \neq 0$

12. Assume that $f : \mathbb{R}^d \to [0,\infty]$ is a measurable function, and that $\{f_n\}$ is a sequence of measurable functions converging pointwise to f. Show that if $f_n \leq f$ for all n,

$$\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu$$

13. Assume that $\{f_n\}$ is a sequence of nonnegative functions converging pointwise to f. Show that if

$$\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu < \infty,$$

then

$$\lim_{n\to\infty} \int_E f_n\,d\mu = \int_E f\,d\mu$$

for all measurable $E \subset \mathbb{R}^d$.

14. Assume that $g : \mathbb{R}^d \to [0, \infty]$ is an *integrable* function, and that $\{f_n\}$ is a sequence of nonnegative, measurable functions converging pointwise to a function f. Show that if $f_n \leq g$ for all n, then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

Hint: Apply Fatou's Lemma to both sequences $\{f_n\}$ and $\{g - f_n\}$.

4.7 Integrable functions

So far we only know how to integrate nonnegative functions, but it is not difficult to extend the theory to general functions. Given a function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$, we first observe that $f = f_+ - f_-$, where f_+ and f_- are the nonnegative functions

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

Note that f_+ and f_- are measurable if f is.

Definition 4.7.1 *A function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is called* integrable *if it is measurable, and $f+$ and f_- are integrable. We define the* (Lebesgue) integral *of f by*

$$\int f \, d\mu = \int f_+ \, d\mu - \int f_- \, d\mu$$

The next lemma gives a useful characterization of integrable functions. The proof is left to the reader.

Lemma 4.7.2 *A measurable function f is integrable if and only if its absolute value $|f|$ is integrable, i.e. if and only if $\int |f| \, d\mu < \infty$.*

The next lemma is a useful technical tool:

Lemma 4.7.3 *Assume that $g : \mathbb{R}^d \to [0, \infty]$ and $h : \mathbb{R}^d \to [0, \infty]$ are two integrable, nonnegative functions, and that $f(x) = g(x) - h(x)$ at all points where the difference is defined. Then f is integrable and*

$$\int f \, d\mu = \int g \, d\mu - \int h \, d\mu$$

Proof: Note that since g and h are integrable, they are finite a.e., and hence $f = g - h$ a.e. Modifying g and h on a set of measure zero (this will not change their integrals), we may assume that $f(x) = g(x) - h(x)$ for all x. Since $|f(x)| = |g(x) - h(x)| \leq |g(x)| + |h(x)|$, it follows from the lemma above that f is integrable.

As

$$f(x) = f_+(x) - f_-(x) = g(x) - h(x)$$

we have

$$f_+(x) + h(x) = g(x) + f_-(x)$$

where we on both sides have sums of nonnegative functions. By Proposition 4.6.5(ii), we get

$$\int f_+ \, d\mu + \int h \, d\mu = \int g \, d\mu + \int f_- \, d\mu$$

Rearranging the integrals (they are all finite), we get

$$\int f \, d\mu = \int f_+ \, d\mu - \int f_- \, d\mu = \int g \, d\mu - \int h \, d\mu$$

and the lemma is proved. □

We are now ready to prove that the integral behaves the way we expect:

Proposition 4.7.4 *Assume that $f, g : \mathbb{R}^d \to \overline{\mathbb{R}}$ are integrable functions, and that c is a constant. Then $f + g$ and cf are integrable, and*

(i) $\int cf \, d\mu = c \int f \, d\mu$.

(ii) $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$.

(iii) If $g \leq f$, *then* $\int g \, d\mu \leq \int f \, d\mu$.

Proof: (i) is left to the reader (treat positive and negative c's separately). To prove (ii), first note that since f and g are integrable, the sum $f(x) + g(x)$ is defined a.e., and by changing f and g on a set of measure zero (this doesn't change their integrals), we may assume that $f(x) + g(x)$ i defined everywhere. Since

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|,$$

$f + g$ is integrable. Obviously,

$$f + g = (f_+ - f_-) + (g_+ - g_-) = (f_+ + g_+) - (f_- + g_-)$$

and hence by the lemma above and Proposition 4.6.5(ii)

$$\int (f + g) \, d\mu = \int (f_+ + g_+) \, d\mu - \int (f_- + g_-) \, d\mu =$$

$$= \int f_+ \, d\mu + \int g_+ \, d\mu - \int f_- \, d\mu - \int g_- \, d\mu =$$
$$= \int f_+ \, d\mu - \int f_- \, d\mu + \int g_+ \, d\mu - \int g_- \, d\mu =$$
$$= \int f \, d\mu + \int g \, d\mu$$

To prove (iii), note that $f - g$ is a nonnegative function and hence by (i) and (ii):

$$\int f \, d\mu - \int g \, d\mu = \int f \, d\mu + \int (-1) g \, d\mu = \int (f - g) \, d\mu \geq 0$$

Consequently, $\int f \, d\mu \geq \int g \, d\mu$ and the proposition is proved. □

We can now extend our limit theorems to general, integrable functions. The following result is probably the most useful of all limit theorems for integrals as it quite strong and at the same time easy to use. It tells us that if a convergent sequence of functions is dominated by an integrable function, then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu$$

Theorem 4.7.5 (Lebesgue's Dominated Convergence Theorem) *Assume that $g : \mathbb{R}^d \to \overline{\mathbb{R}}$ is a nonnegative, integrable function and that $\{f_n\}$ is a sequence of measurable functions converging pointwise to f. If $|f_n| \leq g$ for all n, then*

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

Proof: First observe that since $|f| \leq g$, f is integrable. Next note that since $\{g - f_n\}$ and $\{g + f_n\}$ are two sequences of nonnegative measurable functions, Fatou's Lemma gives:

$$\liminf_{n\to\infty} \int (g - f_n) \, d\mu \geq \int \liminf_{n\to\infty} (g - f_n) \, d\mu = \int (g - f) \, d\mu = \int g \, d\mu - \int f \, d\mu$$

and

$$\liminf_{n\to\infty} \int (g + f_n) \, d\mu \geq \int \liminf_{n\to\infty} (g + f_n) \, d\mu = \int (g + f) \, d\mu = \int g \, d\mu + \int f \, d\mu$$

On the other hand,

$$\liminf_{n\to\infty} \int (g - f_n) \, d\mu = \int g \, d\mu - \limsup_{n\to\infty} \int f_n \, d\mu$$

and

$$\liminf_{n\to\infty} \int (g + f_n) \, d\mu = \int g \, d\mu + \liminf_{n\to\infty} \int f_n \, d\mu$$

Combining the two expressions for $\liminf_{n\to\infty} \int (g - f_n)\, d\mu$, we see that

$$\int g\, d\mu - \limsup_{n\to\infty} \int f_n\, d\mu \geq \int g\, d\mu - \int f\, d\mu$$

and hence

$$\limsup_{n\to\infty} \int f_n\, d\mu \leq \int f\, d\mu$$

Combining the two expressions for $\liminf_{n\to\infty} \int (g + f_n)\, d\mu$, we similarly get

$$\liminf_{n\to\infty} \int f_n\, d\mu \geq \int f\, d\mu$$

Hence

$$\limsup_{n\to\infty} \int f_n\, d\mu \leq \int f\, d\mu \leq \liminf_{n\to\infty} f_n\, d\mu$$

which means that $\lim_{n\to\infty} \int f_n\, d\mu = \int f\, d\mu$. The theorem is proved. □

Remark: It is easy to check that we can relax the conditions above somewhat: If $f_n(x)$ converges to $f(x)$ a.e., and $|f_n(x) \leq g(x)$ fails on a set of measure zero, the conclusion still holds (see Exercise 8 for the precise statement).

Let us take a look at a typical application of the theorem:

Proposition 4.7.6 *Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, and assume that there is an integrable function $g : \mathbb{R} \to [0, \infty]$ such that $|f(x, y)| \leq g(y)$ for all $x, y \in \mathbb{R}$. Then the function*

$$h(x) = \int f(x, y)\, d\mu(y)$$

is continuous (the expression $\int f(x, y)\, d\mu(y)$ means that we for each fixed x integrate $f(x, y)$ as a function of y).

Proof: According to Proposition 1.2.5 it suffices to prove that if $\{a_n\}$ is a sequence converging to a point a, then $h(a_n)$ converges to $h(a)$. Observe that

$$h(a_n) = \int f(a_n, y)\, d\mu(y)$$

and

$$h(a) = \int f(a, y)\, d\mu(y)$$

Observe also that since f is continuous, $f(a_n, y) \to f(a, y)$ for all y. Hence $\{f(a_n, y)\}$ is a sequence of functions which is dominated by the integrable function g and which converges pointwise to $f(a, y)$. By Lebesgue's Dominated Convergence Theorem,

$$\lim_{n\to\infty} h(a_n) = \lim_{n\to\infty} \int f(a_n, y)\, d\mu = \int f(a, y)\, d\mu = h(a)$$

and the proposition is proved. □

As before, we define $\int_A f\, d\mu = \int f\mathbf{1}_A\, d\mu$ for measurable sets A. We say that f *is integrable over* A if $f\mathbf{1}_A$ is integrable.

Exercises to Section 4.7

1. Show that if f is measurable, so are f_+ and f_-.

2. Show that if an integrable function f is zero a.e., then $\int f\, d\mu = 0$.

3. Prove Lemma 4.7.1.

4. Prove Proposition 4.7.4(i). You may want to treat positive and negative c's separately.

5. Assume that $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is a measurable function.

   a) Show that if f is integrable over a measurable set A, and A_n is an increasing sequence of measurable sets with union A, then

   $$\lim_{n\to\infty} \int_{A_n} f\, d\mu = \int_A f\, d\mu$$

   b) Assume that $\{B_n\}$ is a decreasing sequence of measurable sets with intersection B. Show that if f is integrable over B_1, then

   $$\lim_{n\to\infty} \int_{B_n} f\, d\mu = \int_B f\, d\mu$$

6. Show that if $f : \mathbb{R}^d \to \mathbb{R}$ is integrable over a measurable set A, and A_n is a disjoint sequence of measurable sets with union A, then

   $$\int_A f\, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f\, d\mu$$

7. Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be a measurable function, and define

   $$A_n = \{x \in \mathbb{R}^d \mid f(x) \geq n\}$$

   Show that

   $$\lim_{n\to\infty} \int_{A_n} f\, d\mu = 0$$

8. Prove the following slight extension of the Dominated Convergence Theorem:

   Theorem: Assume that $g : \mathbb{R}^d \to \overline{\mathbb{R}}$ is a nonnegative, integrable function and that $\{f_n\}$ is a sequence of measurable functions converging a.e. to f. If $|f_n(x)| \leq g(x)$ a.e. for each n, then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

9. Assume that $g : \mathbb{R}^2 \to \mathbb{R}$ is continuous and that $y \to g(x, y)$ is integrable for each x. Assume also that the partial derivative $\frac{\partial g}{\partial x}(x, y)$ exists for all x and y, and that there is an integrable function $h : \mathbb{R} \to [0, \infty]$ such that

$$\left| \frac{\partial g}{\partial x}(x, y) \right| \leq h(y)$$

   for all x, y. Then the function

$$f(x) = \int g(x, y) \, d\mu(y)$$

   is differentiable at all points x and

$$f'(x) = \int \frac{\partial g}{\partial x}(x, y) \, d\mu(y)$$

4.8 $L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$

In this final section we shall connect integration theory to the theory of normed spaces in Chapter 3. Recall from Definition 3.5.2 that a norm on a real vector space V is a function $\|\cdot\| : V \to [0, \infty)$ satisfying

(i) $\|\mathbf{u}\| \geq 0$ with equality if and only if $\mathbf{u} = 0$.

(ii) $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$ for all $\alpha \in \mathbb{R}$ and all $\mathbf{u} \in V$.

(iii) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ for all $\mathbf{u}, \mathbf{v} \in V$.

Let us now put

$$\mathcal{L}^1(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \overline{\mathbb{R}} \, : \, f \text{ is integrable}\}$$

and define $\|\cdot\|_1 : \mathcal{L}^1(\mathbb{R}^d) \to [0, \infty)$ by

$$\|f\|_1 = \int |f| \, d\mu$$

It is not hard to see that $\|\cdot\|_1$ satisfies the three axioms above with one exception; $\|f\|_1$ may be zero even when f is not zero — actually $\|f\|_1 = 0$ if and only if $f = 0$ a.e.

The usual way to fix this is to consider two functions f and g to be equal if they are equal almost everywhere. To be more precise, let us write $f \sim g$ if f and g are equal a.e., and define the *equivalence class* of f to be the set

$$[f] = \{g \in \mathcal{L}^1(\mathbb{R}^d) \,|\, g \sim f\}$$

Note that two such equivalence classes $[f]$ and $[g]$ are either equal (if f equals g a.e.) or disjoint (if f is not equal to g a.e.). If we let $L^1(\mathbb{R}^d)$ be the collection of all equivalence classes, we can organize $L^1(\mathbb{R}^d)$ as a normed vector space by defining

$$\alpha[f] = [\alpha f] \quad \text{and} \quad [f] + [g] = [f+g] \quad \text{and} \quad |[f]|_1 = \|f\|_1$$

The advantage of the space $(L^1(\mathbb{R}^d), |\cdot|_1)$ compared to $(\mathcal{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ is that it is a normed space where all the theorems we have proved about such spaces apply — the disadvantage is that the elements are no longer functions, but equivalence classes of functions. In practice, there is very little difference between $(L^1(\mathbb{R}^d), |\cdot|_1)$ and $(\mathcal{L}^1(\mathbb{R}^d), \|\cdot\|_1)$, and mathematicians tend to blur the distinction between the two spaces: they pretend to work in $L^1(\mathbb{R}^d)$, but still consider the elements as functions. We shall follow this practice here; it is totally harmless as long as you remember that whenever we talk about an element of $L^1(\mathbb{R}^d)$ as a function, we are really choosing a representative from an equivalence class (Exercise 3 gives a more thorough and systematic treatment of $L^1(\mathbb{R}^d)$).

The most important fact about $(L^1(\mathbb{R}^d), |\cdot|_1)$ is that it is complete. In many ways, this is the most impressive success of the theory of Lebesgue integration: We have seen in previous chapters how important completeness is, and it is a great advantage to work with a theory of integration where the space of integrable functions is naturally complete. Before we turn to the proof, you may want to remind yourself of Proposition 3.5.5 which shall be our main tool.

Theorem 4.8.1 $(L^1(\mathbb{R}^d), |\cdot|_1)$ *is complete.*

Proof: Assume that $\{u_n\}$ is a sequence of functions in $L^1(\mathbb{R}^d)$ such that the series $\sum_{n=1}^\infty |u_n|_1$ converges. According to Proposition 3.5.5, it suffices to show that the series $\sum_{n=1}^\infty u_n(x)$ must converge in $L^1(\mathbb{R}^d)$. Observe that

$$\infty > \sum_{n=1}^\infty |u_n|_1 = \lim_{N\to\infty} \sum_{n=1}^N |u_n|_1 = \lim_{N\to\infty} \sum_{n=1}^N \int |u_n|\, d\mu =$$

$$= \lim_{N\to\infty} \int \sum_{n=1}^N |u_n|\, d\mu = \int \lim_{N\to\infty} \sum_{n=1}^N |u_n|\, d\mu = \int \sum_{n=1}^\infty |u_n|\, d\mu$$

where we have used the Monotone Convergence Theorem to move the limit inside the integral sign. This means that the function

$$g(x) = \sum_{n=1}^{\infty} |u_n(x)|$$

is integrable. We shall use g as the dominating function in the Dominated Convergence Theorem.

Let us first observe that since $g(x) = \sum_{n=1}^{\infty} |u_n(x)|$ is integrable, the series converges a.e. Hence the sequence $\sum_{n=1}^{\infty} u_n(x)$ (without the absolute values) converges absolutely a.e., and hence it converges a.e. in the ordinary sense. Let $f(x) = \sum_{n=1}^{\infty} u_n(x)$ (put $f(x) = 0$ on the null set where the series does not converge). It remains to prove that the series converges in L^1-sense, i.e. that $|f - \sum_{n=1}^{N} u_n|_1 \to 0$ as $N \to \infty$. By definition of f, $\lim_{N\to\infty} \left(f(x) - \sum_{n=1}^{N} u_n(x) \right) = 0$ a.e. Since $|f(x) - \sum_{n=1}^{N} u_n(x)| = |\sum_{n=N+1}^{\infty} u_n(x)| \leq g(x)$ a.e., it follows from Dominated Convergence Theorem (actually from the slight extension in Exercise 4.7.8) that

$$|f - \sum_{n=1}^{N} u_n|_1 = \int |f - \sum_{n=1}^{N} u_n| \, d\mu \to 0$$

The theorem is proved. □

Let us take a brief look at another space of the same kind. Let

$$\mathcal{L}^2(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \overline{\mathbb{R}} \, : \, |f|^2 \text{ is integrable}\}$$

and define $\|\cdot\|_2 : \mathcal{L}^2 \to [0, \infty)$ by

$$\|f\|_2 = \left(\int |f|^2 \, d\mu \right)^{\frac{1}{2}}$$

It turns out (see Exercise 4) that $\mathcal{L}^2(\mathbb{R}^d)$ is a vector space, and that $\|\cdot\|$ is a norm on $\mathcal{L}^2(\mathbb{R}^d)$, except that $\|f\|_2 = 0$ if $f = 0$ a.e. If we consider functions as equal if they are equal a.e., we can turn $(\mathcal{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ into a normed space $(L^2(\mathbb{R}^d), |\cdot|_2)$ just as we did with $\mathcal{L}^1(\mathbb{R}^d)$. One of the advantages of this space, is that it is an inner product space with inner product

$$\langle f, g \rangle = \int f g \, d\mu$$

By almost exactly the same argument as for $L^1(\mathbb{R}^d)$, one may prove:

Theorem 4.8.2 *$(L^2(\mathbb{R}^d), |\cdot|_2)$ is complete.*

Let me finally mention that $L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ are just two representatives of a whole family of spaces. For any $p \in [1, \infty)$, we may let

$$\mathcal{L}^p(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \overline{\mathbb{R}} \, : \, |f|^p \text{ is integrable}\}$$

and define $\| \cdot \|_p : \mathcal{L}^p \to [0, \infty)$ by

$$\|f\|_2 = \left(\int |f|^2 \, d\mu \right)^{\frac{1}{p}}$$

Proceeding as before, we get complete, normed spaces $(L^p(\mathbb{R}^d), | \cdot |_p)$.

Exercises for Section 4.8

1. Show that $\mathcal{L}^1(\mathbb{R}^d)$ is a vector space. Since the set of *all* functions from $\mathbb{R}^d$ to $\mathbb{R}$ is a vector space, it suffices to show that $\mathcal{L}^1(\mathbb{R}^d)$ is a subspace, i.e. that cf and $f + g$ are in $\mathcal{L}^1(\mathbb{R}^d)$ whenever $f, g \in \mathcal{L}^1(\mathbb{R}^d)$ and $c \in \mathbb{R}$.

2. Show that $\| \cdot \|_1$ satisfies the following conditions:

   (i) $\|f\|_1 \geq 0$ for all f, and $\|\mathbf{0}\|_1 = 0$ (here $\mathbf{0}$ is the function that is constant 0).

   (ii) $\|cf\|_1 = |c|\|f\|_1$ for all $f \in \mathcal{L}^1(\mathbb{R}^d)$ and all $c \in \mathbb{R}$.

   (iii) $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$ for all $f, g \in \mathcal{L}^1(\mathbb{R}^d)$

   This means that $\| \cdot \|_1$ is a *seminorm.*

3 If $f, g \in \mathcal{L}^1(\mathbb{R}^d)$, we write $f \sim g$ if $f = g$ a.e. Recall that the equivalence class $[f]$ of f is defined by

$$[f] = \{g \in \mathcal{L}(\mathbb{R}^d) \, : \, g \sim f\}$$

   a) Show that two equivalence classes $[f]$ and $[g]$ are either equal or disjoint.

   b) Show that if $f \sim f'$ and $g \sim g'$, then $f + g \sim f' + g'$. Show also that $cf \sim cf'$ for all $c \in \mathbb{R}$.

   c) Show that if $f \sim g$, then $\|f - g\|_1 = 0$ and $\|f\|_1 = \|g\|_1$.

   d) Show that the set $L^1(\mathbb{R}^d)$ of all equivalence classes is a normed space if we define scalar multiplication, addition and norm by:

      (i) $c[f] = [cf]$ for all $c \in \mathbb{R}$, $f \in \mathcal{L}^1(\mathbb{R}^d)$.

      (ii) $[f] + [g] = [f + g]$ for all $f, g \in \mathcal{L}^1(\mathbb{R}^d)$

      (iii) $|[f]|_1 = \|f\|_1$ for all $f \in \mathcal{L}^1(\mathbb{R}^d)$.

      Why do we need to establish the results in (i), (ii), and (iii) before we can make these definitions?

4. a) Show that $\mathcal{L}^2(\mathbb{R}^d)$ is a vector space. Since the set of *all* functions from $\mathbb{R}^d$ to $\mathbb{R}$ is a vector space, it suffices to show that $\mathcal{L}^2(\mathbb{R}^d)$ is a subspace, i.e. that cf and $f + g$ are in $\mathcal{L}^2(\mathbb{R}^d)$ whenever $f, g \in \mathcal{L}^2(\mathbb{R}^d)$ and $c \in \mathbb{R}$. (To show that $f + g \in \mathcal{L}^2(\mathbb{R}^d)$, you may want to use that $|a + b|^2 \leq 2|a|^2 + 2|b|^2$ for all real numbers a, b).

b) Show that if $f, g \in \mathcal{L}^2(\mathbb{R}^d)$, then fg is integrable. (You may want to use the identity $|fg| = \frac{1}{2}((|f|+|g|)^2 - |f|^2 - |g|^2)$.

c) Show that the *semi inner product*

$$\langle f, g\rangle = \int fg\,d\mu$$

on $\mathcal{L}^2(\mathbb{R}^d)$ satisfies:

(i) $\langle f, g\rangle = \langle g, f\rangle$ for all $f, g \in \mathcal{L}^2(\mathbb{R}^d)$.

(ii) $\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$ for all $f, g, h \in \mathcal{L}^2(\mathbb{R}^d)$.

(iii) $\langle cf, g\rangle = c\langle f, g\rangle$ for all $c \in \mathbb{R}$, $f, g \in \mathcal{L}^2(\mathbb{R}^d)$.

(iv) For all $f \in \mathcal{L}^2(\mathbb{R}^d)$, $\langle f, f\rangle \geq 0$ with equality if $f = \mathbf{0}$ (here $\mathbf{0}$ is the function that is constant 0).

Show also that $\langle f, f\rangle = 0$ if and only if $f = 0$ a.e.

e) Assume that $f, f', g, g' \in \mathcal{L}^2(\mathbb{R}^d)$, and that $f = f'$, $g = g'$ a.e. Show that $\langle f, g\rangle = \langle f', g'\rangle$

5. Show that $(L^2(\mathbb{R}^d), |\cdot|_2)$ is complete by modifying the proof of Theorem 4.8.1.