Hahn-Banach theorems

The Hahn-Banach theorem (H-B theorem, for short), in its various forms, is without doubt the most important theorem in Convexity.

Zorn's Lemma. Let us recall the so-called Zorn's Lemma which is equivalent to the Axiom of Choice, usually assumed to be true in Mathematical Analysis. (The Axion of Choice is independent of usual basic axioms of the Set Theory).

A *partially ordered set* is a set E equipped with a binary relation "$\leq$" which is reflexive, transitive and antisymmetric, but not necessarily defined for each couple of elements of E. A set $C \subset E$ is a *chain* (or a linearly ordered set) if every two elements of C are comparable by "$\leq$", that is, for each $x, y \in C$ we have either $x \leq y$ or $y \leq x$. An *upper bound* of a set $A \subset X$ is an element of $z \in X$ such that $x \leq z$ for each $x \in A$. A *maximal element* of A is an $a_0 \in A$ such that $a = a_0$ whenever $a \in A, a_0 \leq a$ (in other words, A contains no element which is strictly greater that a_0). Notice that a maximal element of A is not necessarily an upper bound for A (there can be elements incomparable with it) and not necessarily unique.

Now, we are ready to state the **Zorn's Lemma**: *Let E be a partially ordered set. If every chain in E has an upper bound, then E contains a maximal element.*

Hahn-Banach extension theorems. Recall that $p\colon X \to \mathbb{R} \cup \{+\infty\}$ is *sublinear* if $p(0) = 0$, $p(tx) = tp(x)$ and $p(x + y) \leq p(x) + p(y)$ whenever $t > 0$ and $x, y \in X$.

Theorem 0.1 (Algebraic H-B Extension Theorem)**.** *Let L be a subspace of a vector space X. Let $p\colon X \to \mathbb{R} \cup \{+\infty\}$ be sublinear. Suppose that*

$$L + \operatorname{dom}(p) = X \tag{1}$$

(this condition is automatically satisfied if p is real-valued!).
If $\ell\colon L \to \mathbb{R}$ is a linear functional such that

$$\ell(u) \leq p(u) \quad \textit{whenever } u \in L,$$

then ℓ can be extended to a linear functional $\hat{\ell}\colon X \to \mathbb{R}$ such that $\hat{\ell} \leq p$ on X.

Proof. (a) First, let us prove that ℓ can be extended to one dimension more, that is, to a subspace $M = \operatorname{span}(L \cup \{v_0\})$ where $v_0 \in X \setminus L$. Each element of M is of the form $u + tv_0$ with $u \in L$ and $t \in \mathbb{R}$, and the needed extension should satisfy $\hat{\ell}(u + tv_0) = \ell(u) + t\hat{\ell}(v_0)$. So, we have to choose the real number $\alpha := \hat{\ell}(v_0)$ so that

$$\ell(u) + \alpha t \leq p(u + tv_0) \qquad (u \in L,\ t \in \mathbb{R}). \tag{2}$$

Note that (2) is satisfied for $t = 0$. Thus, considering separately $t > 0$ and $t = -s < 0$, we see that α has to satisfy the inequalitites

$$\begin{aligned} \alpha &\leq \inf\left\{\tfrac{p(u+tv_0)-\ell(u)}{t} \,:\, t > 0,\ u \in L\right\} =: b\,, \\ \alpha &\geq \sup\left\{\tfrac{\ell(u)-p(u-sv_0)}{s} \,:\, s > 0,\ u \in L\right\} =: a\,. \end{aligned} \tag{3}$$

Observe that positive homogeneity of p, ℓ implies

$$a = \sup\{\ell(w) - p(w - v_0) : w \in L\}\,, \quad b = \inf\{p(z + v_0) - \ell(z) : z \in L\}\,.$$

We claim that $a \leq b$. To show this, it is sufficient to show that

$$\ell(w) - p(w - v_0) \leq p(z + v_0) - \ell(z) \qquad (w, z \in L)\,, \tag{4}$$

which is equivalent to

$$\ell(w+z) \le p(w-v_0) + p(z+v_0)\,.$$

But this is easy:

$$\ell(w+z) \le p(w+z) = p((w-v_0)+(z+v_0)) \le p(w-v_0) + p(z+v_0)\,.$$

Now, by our assumption (1), $\pm v_0 \in L + \mathrm{dom}(p)$, and hence there exist $w_0, z_0 \in L$ such that both $w_0 - v_0$ and $z_0 + v_0$ belong to $\mathrm{dom}(p)$. Then (4) implies that $a \le p(z_0+v_0) - \ell(z_0) < +\infty$ and $b \ge \ell(w_0) - p(w_0 - v_0) > -\infty$. Hence there exists $\alpha \in \mathbb{R}$ such that $a \le \alpha \le b$, that is, (3) is satisfied.

(b) Now, the case when L has a finite codimension in X follows easily by repeated application of (a).

(c) For the proof of the general case, we shall need the Zorn's Lemma. Let E be the set of all couples (M,h) where M is a subspace of X containing L, and $h\colon M \to \mathbb{R}$ is a linear extension of ℓ. This set is nonempty (since it contains the couple (L,ℓ)) and is partially ordered by the following relation "$\le$":

$$(M_1,h_1) \le (M_2,h_2) \quad \overset{\text{def}}{\Longleftrightarrow} \quad M_1 \subset M_2 \text{ and } h_2|_{M_1} = h_1.$$

Let $\mathcal{C} \subset E$ be a (nonempty) chain. Then $N := \bigcup_{M\in\mathcal{C}} M$ is a subspace containing L, and there exists a unique (necessarily linear) functional $g\colon N \to \mathbb{R}$ that extends each h such that $(M,h) \in \mathcal{C}$. Thus (N,g) is an upper bound for $\mathcal{C}$. By Zorn's Lemma, E has a maximal element $(\hat{L},\hat{\ell})$. If $\hat{L}$ is a proper subset of X, we can apply the first part of this proof to extend it one dimension more, but this contradicts maximality. Thus we must have $\hat{L}$, and we are done. □

Corollary 0.2 (Norm-preserving extension)**.** *Let L be a subspace of a normed space X. Then every functional $\ell \in L^*$ has an extension $x^* \in X^*$ such that $\|x^*\|_{X^*} = \|\ell\|_{L^*}$.*

Proof. The functional $p(x) = \|\ell\|\,\|x\|$ is sublinear on X, and $\ell \le p$ on L. By Theorem 0.1, there exists a linear extension $\hat{\ell}\colon X \to \mathbb{R}$ of ℓ, such that $\hat{\ell} \le p$. Then

$$\|\ell\| \le \|\hat{\ell}\| = \sup_{\|x\|=1} \hat{\ell}(x) \le \sup_{\|x\|=1} p(x) = \|\ell\|\,.$$

Thus $\hat{\ell} \in X^*$ and $\|\hat{\ell}\| = \|\ell\|$, hence we can put $x^* = \hat{\ell}$. □

Recall that $\|x^*\| = \sup_{\|x\|=1} x^*(x)$; it is well-known that the "sup" is not necessarily attained. The next corollary shows that we have the symmetric formula $\|x\| = \max_{\|x^*\|=1} x^*(x)$ with, in addition, "max" instead of only "sup".

Corollary 0.3. *Let X be a normed space, $x_0 \in X$. Then*

$$\|x_0\| = \max_{\substack{x^*\in X^*\\ \|x^*\|=1}} x^*(x_0)\,.$$

Proof. The formula is trivial for $x_0 = 0$. Let $x_0 \ne 0$. Consider the one-dimensional subspace $L = \mathrm{span}\{x_0\} = \mathbb{R}x_0$, and define $\ell\colon L \to \mathbb{R}$ by $\ell(tx_0) = t\|x_0\|$. Then $\ell \in L^*$. By Corollary 0.2, there exists a norm-preserving extension $x^* \in X^*$ of ℓ.

Then $x^*(x_0) = \ell(x_0) = \|x_0\|$. Now, the formula follows from the fact that, for any $\|y^*\| = 1$, we have $y^*(x_0) \le \|x_0\|$. □

Corollary 0.4. *Let X be a normed space. Then X^* separates the points of X, i.e.,*

$$\forall x, y \in X,\, x \neq y \;\; \exists x^* \in X^* :\; x^*(x) \neq x^*(y)\,.$$

Proof. By linearity, it suffices to show that if $v \neq 0$ then there exists $x^* \in X^*$ such that $x^*(v) \neq 0$. But this follows immediately from Corollary 0.3. □

Notice that the proof of the following theorem is almost the same as that of Corollary 0.2: we just use the Minkowski functional of A in place of the norm (which is the Minkowski functional of the unit ball).

Theorem 0.5. *Let L be a subspace of a t.v.s. X, and $A \subset X$ a convex absorbing set. Let $\ell\colon L \to \mathbb{R}$ be a linear functional, and $m \in \mathbb{R}$ be such that $\ell(y) \le m$ on $A \cap L$. Then there exists a linear extension $\hat{\ell}\colon X \to \mathbb{R}$ of ℓ such that $\hat{\ell} \le m$ on A. If, moreover, X is a t.v.s. and A is a neighborhood of 0, then $\hat{\ell}$ is continuous.*

Proof. Let p_A be the Minkowski functional of A. Then p_A is sublinear and real-valued, and $\{p_A < 1\} \subset A \subset \{p_A \le 1\}$. We claim that

$$\ell(y) \le m\, p_A(y) \quad \text{whenever } y \in L. \tag{5}$$

Indeed, for each $\alpha > p_A(y)$, we have $\frac{u}{\alpha} \in A$ and hence $\ell(y) = \alpha\ell(\frac{y}{\alpha}) \le m\alpha$. Pass to the limit $\alpha \searrow p_A(y)$.

Now, by Theorem 0.1, there exists a linear extension $\hat{\ell}\colon X \to \mathbb{R}$ of ℓ, such that $\hat{\ell} \le m p_A$. Hence $\hat{\ell} \le m$ on A. The last part of the statement follows immediately from the fact that a linear functional (on a t.v.s.) is continuous if and only if it is bounded above on a neighborhood of 0. □

Hahn-Banach separation theorems. Recall that a *hyperplane* in a vector space X is a set of the form $H = \ell^{-1}(\alpha)$ where $\alpha \in \mathbb{R}$ and $\ell\colon X \to \mathbb{R}$ is a nontrivial linear functional. Let us consider the corresponding two *algebraically closed halfspaces*

$$H_+ := \{\ell(x) \ge \alpha\}\,, \qquad H_- = \{\ell(x) \le \alpha\}$$

determined by H. We shall say that H_+ is *opposite* to H_-, and vice-versa. We already know that, if X is also a topological vector space, then the hyperplane H is closed if and only if the functional ℓ is continuous.

Recall also that a point x belongs to the *algebraic interior* of a set $E \subset X$ (notation $x \in \text{a-int}(E)$) if $x \in \text{int}_L(E \cap L)$ for every line $L \subset X$ containing x.

Definition 0.6. Let A, B be nonempty sets in a vector space X. We say that a nontrivial linear functional $\ell\colon X \to \mathbb{R}$:

(a) *separates A and B* if if either $\sup \ell(A) \le \inf \ell(B)$ or $\sup \ell(B) \le \inf \ell(A)$ (equivalently, in the above notation, if A and B are contained in opposite algebraically closed halfspaces determined by a hyperplane $H = \ell^{-1}(\alpha)$).

(b) *strongly separates A and B* if either $\sup \ell(A) < \inf \ell(B)$ or $\sup \ell(B) < \inf \ell(A)$. (In this case, A, B can be "separated by a strip between two distinct parallel hyperplanes given by ℓ".)

We shall say that A, B *can be (strongly) separated by a hyperplane* if there exists a nontrivial linear functional ℓ on X that (strongly) separates A, B.

Observation 0.7. *Let A be a subset of a vector space X, such that* $\text{a-int}(A) \neq \emptyset$*. Let $\ell \colon X \to \mathbb{R}$ be a nontrivial linear functional, $\alpha \in \mathbb{R}$. The following assertions are equivalent:*

(i) $A \subset \{\ell \leq \alpha\}$*;*
(ii) $\text{a-int}(A) \subset \{\ell \leq \alpha\}$*;*
(iii) $\text{a-int}(A) \subset \{\ell < \alpha\}$*.*

Proof. The implication $(i) \Rightarrow (ii)$ is obvious.
$(ii) \Rightarrow (iii)$. If (iii) is false, there exists $a \in \text{a-int}(A)$ such that $\ell(a) = \alpha$. Take a vector $v \in X$ such that $\ell(v) > 0$. There exists $t > 0$ with $a + tv \in A$, but then necessarily $a + \frac{t}{2}v \in \text{a-int}(A)$ and $\ell(a + \frac{t}{2}v) > \alpha$; thus (ii) is false.
$(iii) \Rightarrow (i)$. Assume that (iii) holds while (i) is false. Fix $a \in \text{a-int}(A)$ and $x \in A$ with $\ell(x) > \alpha$. Then the relatively open segment (a, x) is contained in $\text{a-int}(A)$ and intersects the hyperplane $\ell^{-1}(\alpha)$, which contradicts (iii). □

Remark 0.8. Let two sets A, B be separated by a linear functional ℓ. Then, for each $v \in X$, the sets $A + v$ and $B + v$ are separated by ℓ. (And an analogous statement holds with "strongly separated" instead of "separated".)

Let us start with a weak version of a separation theorem, in which we separate a point from a convex set.

Proposition 0.9. *Let C be a convex set in a vector space X. If* $\text{a-int}(C) \neq \emptyset$ *and $x_0 \in X \setminus \text{a-int}(C)$, then $\{x_0\}$ and C can be separated by a hyperplane. If, moreover, X is a t.v.s. and* $\text{int}(C) \neq \emptyset$*, then $\{x_0\}$ and C can be separated by a closed hyperplane.*

Proof. By Remark 0.8, we can suppose that $0 \in \text{a-int}(C)$. In this case, C is absorbing and $x_0 \neq 0$. Consider the subspace $L = \mathbb{R}x_0$. It is easy to see that the algebraic interior of the interval $L \cap C$ contains 0 but not x_0 (otherwise x_0 would belong to $\text{a-int}(C)$). Thus the linear functional $\ell \colon L \to \mathbb{R}$, $\ell(tx_0) = t$, satisfies $\ell(y) \leq \ell(x_0)$ whenever $y \in C \cap L$. By Theorem 0.5, there exists a linear extension $\hat{\ell} \colon X \to \mathbb{R}$ of ℓ such that $\sup \hat{\ell}(C) \leq \ell(x_0)$, which is what we needed. Moreover, in the "t.v.s. case", if we start with $0 \in \text{int}(C)$, the functional $\hat{\ell}$ is also continuous. □

Theorem 0.10 (H-B Separation Theorem)**.** *Let A and B be two nonempty convex sets in a vector space X. If* $\text{a-int}(A) \neq \emptyset$ *and* $B \cap \text{a-int}(A) = \emptyset$*, then A, B can be separated by a hyperplane. If, moreover, X is a t.v.s.,* $\text{int}(A) \neq \emptyset$ *and* $B \cap \text{int}(A) = \emptyset$*, then A and B can be separated by a closed hyperplane.*

Proof. The set $A_0 = \text{a-int}(A)$ is convex, nonempty and disjoint from B. Then the convex set $C = A_0 - B$ has a nonempty algebraic interior and does not contain 0. By Proposition 0.9, there exists a nontrivial linear functional ℓ on X such that $\sup \ell(C) \leq \ell(0) = 0$. It follows that

$$\ell(a) \leq \ell(b) \qquad \text{whenever } a \in A_0,\ b \in B.$$

Thus $\sup \ell(A_0) \leq \inf \ell(B)$. By Observation 0.7, $\sup \ell(A) \leq \inf \ell(B)$. The proof of the second part proceedes in the same way, with "int" in place of "a-int". □

Remark 0.11. Let A, B be as in the previous theorem. If a nontrivial linear functional ℓ separates A and B, then it "separates strictly" a-int(A) (or int(A)) and B in the following sense:

$$\ell(a) < \inf \ell(B) \qquad \text{whenever } a \in \text{a-int}(A) \text{ (or } a \in \text{int}(A)\text{, respectively).}$$

This follows immediately from Observation 0.7.

Hahn-Banach strong separation theorem. Let us start with the following technical lemma which can be viewed as a generalization of the fact that any two disjoint closed sets, one of which is compact, have a positive distance.

Lemma 0.12. *Let A and B be two nonempty disjoint closed sets such that A is compact. Then there exists a neighborhood V of 0 such that $(A+V) \cap B = \emptyset$.*

Proof. For each $a \in A$, fix $U_a \in \mathcal{U}(0)$ such that $(a + U_a) \cap B = \emptyset$, and an open $V_a \in \mathcal{U}(0)$ with $V_a + V_a \subset U_a$. Since the sets $a + V_a$ $(a \in A)$ form an open covering of the compact set A, there exists $a_1, \ldots, a_n \in A$ such that $A \subset \bigcup_{i=1}^n (a_n + V_{a_i})$. The set

$$V = \bigcap_{i=1}^n V_{a_i}$$

is a neighborhood of 0, and $A + V \subset \bigcup_{i=1}^n (a_i + V_{a_i} + V) \subset \bigcup_{i=1}^n (a_i + V_{a_i} + V_{a_i}) \subset \bigcup_{i=1}^n (a_i + U_{a_i})$. Then $(A+V) \cap B \subset \bigcup_{i=1}^n (a_i + U_{a_i}) \cap B = \emptyset$. □

Recall that a topological vector space is *locally convex* if it admits a base of neighborhoods of 0 consisting of convex sets.

Theorem 0.13 (H-B Strong Separation Theorem)**.** *Let X be a locally convex t.v.s., and $A, B \subset X$ two disjoint nonempty closed convex sets one of which is compact. Then A and B can be strongly separated by a closed hyperplane.*

Proof. By Lemma 0.12, there exists an open convex $V \in \mathcal{U}(0)$ such that $(A+V) \cap B = \emptyset$. Since the set $A + V$ is open and convex, we can apply Theorem 0.10 to get $x^* \in X^* \setminus \{0\}$ such that $\sup x^*(A+V) \leq \inf x^*(B)$. Since $A \subset A + V = \text{int}(A+V)$, Observation 0.7 implies that

$$x^*(a) < \inf x^*(B) \qquad \text{whenever } a \in A.$$

Thus $\sup x^*(A) = \max x^*(A) < \inf x^*(B)$. □

Corollary 0.14. *Let C be a nonempty closed convex set in a locally convex t.v.s. X, and $x_0 \in X \setminus C$. Then x_0 and C can be strongly separated by a closed hyperplane.*

Separation of functions. Let us denote by $X^\sharp$ the algebraic dual of a vector space X, that is

$$X^\sharp = \{\ell : \ \ell \text{ is a linear functional on } X\}\,.$$

It is an easy exercise to show that $(X \times \mathbb{R})^\sharp$ is algebraically isomorphic with $X^\sharp \times \mathbb{R}$ and the isomorphism

$$(X \times \mathbb{R})^\sharp \ni \Phi \longleftrightarrow (\ell, \alpha) \in X^\sharp \times \mathbb{R}$$

is given by the formula

$$\Phi(x,t) = \ell(x) + \alpha t\,.$$

In the same way, for topological vector spaces, we have an algebraic isomorphism of $(X \times \mathbb{R})^*$ and $X^* \times \mathbb{R}$. (This isomorphism is even topological in all reasonable situations, for instance, for normed spaces.)

Lemma 0.15. *Let X be a vector space, and $\Phi \in (X \times \mathbb{R})^\sharp$. If Φ separates two vertically situated points (in the sense that, for some $x_0 \in X$ and $t_1, t_2 \in \mathbb{R}$, $\Phi(x_0, t_1) \neq \Phi(x_0, t_2)$), then each hyperplane of the type $\Phi^{-1}(\beta)$ is the graph of an affine function on X. If, moreover, X is a t.v.s. and $\Phi \in (X \times \mathbb{R})^*$, then the affine function is continuous.*

Proof. Let $\ell \in X^\sharp$ and $\alpha \in \mathbb{R}$ be such that

$$\Phi(x,t) = \ell(x) + \alpha t \qquad (x \in X,\ t \in \mathbb{R}).$$

Since $0 \neq \Phi(x_0, t_1) - \Phi(x_0, t_2) = \alpha(t_1 - t_2)$, we have $\alpha \neq 0$. Then the following conditions are clearly equivalent:

- $(x,t) \in \Phi^{-1}(\beta)$;
- $\ell(x) + \alpha t = \beta$;
- $t = -\frac{1}{\alpha}\ell(x) + \frac{\beta}{\alpha}$;
- $(x,t) \in \text{graph}(a)$, where $a(x) = -\frac{1}{\alpha}\ell(x) + \frac{\beta}{\alpha}$.

The proof is complete since a is an affine function which is continuous whenever Φ is. □

Theorem 0.16 (Separation of functions)**.** *Let X be a real vector space, $f\colon X \to (-\infty, +\infty]$ a convex function, and $g\colon X \to [-\infty, +\infty)$ a concave function such that $g \leq f$. Assume that*

$$\text{a-int}(\text{dom}(f)) \cap \text{dom}(g) \neq \emptyset\,.$$

Then there exists an affine function $a\colon X \to \mathbb{R}$ such that $g \leq a \leq f$. If, moreover, f is continuous at some point of $\text{dom}(f)$*, then also a is continuous.*

Proof. The epigraph

$$\text{epi}(f) = \{(x,t) \in X \times \mathbb{R} : t \geq f(x)\}$$

and the hypograph

$$\text{hypo}(g) = \{(x,t) \in X \times \mathbb{R} : t \leq g(x)\}$$

are convex sets in $X \times \mathbb{R}$. Clearly, the algebraic interior of $\text{epi}(f)$ is contained in the "strict epigraph" $\{(x,t) : t > f(x)\}$. Thus

$$\text{a-int}(\text{epi}(f)) \cap \text{hypo}(g) \neq \emptyset\,.$$

Let us show that a-int(epi(f)) $\neq \emptyset$. Fix $x_0 \in$ a-int(dom(f)) and $t_0 > f(x_0)$. We claim that $(x_0, t_0) \in$ a-int(epi(f)). To show this, we have to show that, for each $(v, s) \in X \times \mathbb{R}$, there exists $\lambda > 0$ such that $(x_0, t_0) + \lambda(v, s) = (x_0 + \lambda v, t_0 + \lambda s) \in$ epi(f), that is,

$$t_0 + \lambda s \geq f(x_0 + \lambda v)\,.$$

This is trivial for $v = 0$ from the choice of (x_0, t_0). For $v \neq 0$, it follows from the continuity of the (convex) function $\lambda \mapsto f(x_0 + \lambda v)$ on a neighborhood of 0.

By the H-B Separation Theorem (Theorem 0.10), there exists a nonzero $\Phi \in (X \times \mathbb{R})^\sharp$ such that $\sup \Phi(\mathrm{epi}(f)) \leq \inf \Phi(\mathrm{hypo}(g))$. By Observation 0.7, $\Phi(x_0, t_0) < \inf \Phi(\mathrm{hypo}(g)) \leq \Phi(x_0, g(x_0))$, which means that Φ separates two vertically situated points (x_0, t_0) and $(x_0, g(x_0))$. By Lemma 0.15, a separating hyperplane $\Phi^{-1}(\alpha)$ (with $\sup \Phi(\mathrm{epi}(f)) \leq \alpha \leq \inf \Phi(\mathrm{hypo}(g))$) is the graph of an affine function $a\colon X \to \mathbb{R}$, which necessarily satisfies $g \leq a \leq f$.

In the second part of the statement, a is continuous since it is bounded above by f which is bounded above on an open set. □

Remark 0.17. Notice that the assumption that f is continuous at some point of dom(f) implies that dom(f) has a nonempty interior, and hence int(dom(f)) = a-int(dom(f)).

As an example of an application of the above theorem, we present the following useful corollary.

Corollary 0.18. *Let X be a normed space, $f\colon X \to (-\infty, +\infty]$ a convex function, $A \subset X$ an affine set, and $a\colon A \to \mathbb{R}$ a continuous affine function such that $a \leq f$ on A. Assume that*

$$\mathrm{int}(\mathrm{dom}(f)) \cap A \neq \emptyset\,,$$

and either f is continuous at some point of dom(f)*, or f is l.s.c. and X is a Banach space. Then there exists a continuous affine extension $\hat{a}\colon X \to \mathbb{R}$ of a, such that $\hat{a} \leq f$.*

Proof. The function $g\colon X \to [-\infty, +\infty)$, defined by

$$g(x) = \begin{cases} a(x) & \text{for } x \in A \\ -\infty & \text{otherwise} \end{cases}\,,$$

is concave and satisfies $g \leq f$. Apply Theorem 0.16 to get an affine function $b\colon X \to \mathbb{R}$ such that $b|_A = a$ and $g \leq b \leq f$. Using Remark 0.17 and known facts about continuity of convex functions on open convex sets, it is easy to see that our assumptions imply that f is continuous on int(dom(f)). Thus b is continuous. Since $a \leq b$ on A, there exists a constant $\beta \geq 0$ such that $a = (b|_A) - \beta$. Consequently, we can put $\hat{a} = b - \beta$. □