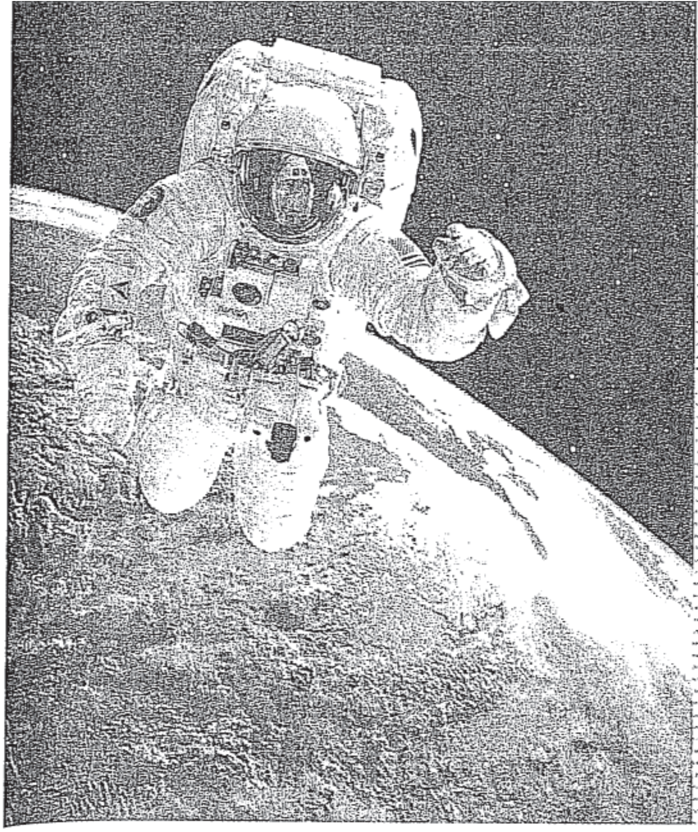


PART C Fourier Analysis. Partial Differential Equations (PDEs)

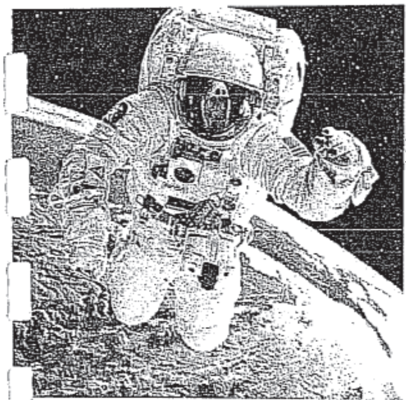
Fourier analysis concerns **periodic phenomena,** as they occur quite frequently in engineering and elsewhere—think of rotating parts of machines, alternating electric currents, or the motion of planets. Related periodic functions may be complicated. This situation poses the important practical task of representing these complicated functions in terms of simple periodic functions, namely, cosines and sines. These representations will be infinite series, called **Fourier series.**[1]

The creation of these series was one of the most path-breaking events in applied mathematics, and we mention that it also had considerable influence on mathematics as a whole, on the concept of a function, on integration theory, on convergence theory for series, and so on (see Ref. [GR7] in App. 1).

Chapter 11 is concerned mainly with Fourier series. However, the underlying ideas can also be extended to *nonperiodic* phenomena. This leads to *Fourier integrals* and *transforms.* A common name for the whole area is **Fourier analysis.**

Chapter 12 deals with the most important **partial differential equations (PDEs)** of physics and engineering. This is the area in which Fourier analysis has its most basic applications, related to boundary and initial value problems of mechanics, heat flow, electrostatics, and other fields.

[1]JEAN-BAPTISTE JOSEPH FOURIER (1768–1830), French physicist and mathematician, lived and taught in Paris, accompanied Napoléon in the Egyptian War, and was later made prefect of Grenoble. The beginnings on Fourier series can be found in works by Euler and by Daniel Bernoulli, but it was Fourier who employed them in a systematic and general manner in his main work, *Théorie analytique de la chaleur* (*Analytic Theory of Heat,* Paris, 1822), in which he developed the theory of heat conduction (heat equation; see Sec. 12.5), making these series a most important tool in applied mathematics.



CHAPTER 11

Fourier Series, Integrals, and Transforms

Fourier series (Sec. 11.1) are infinite series designed to represent general periodic functions in terms of simple ones, namely, cosines and sines. They constitute a very important tool, in particular in solving problems that involve ODEs and PDEs.

In this chapter we discuss Fourier series and their engineering use from a practical point of view, in connection with ODEs and with the approximation of periodic functions. Application to PDEs follows in Chap. 12.

The *theory* of Fourier series is complicated, but we shall see that the *application* of these series is rather simple. Fourier series are in a certain sense more universal than the familiar Taylor series in calculus because many *discontinuous* periodic functions of practical interest can be developed in Fourier series but, of course, do not have Taylor series representations.

In the last sections (11.7–11.9) we consider **Fourier integrals** and **Fourier transforms,** which extend the ideas and techniques of Fourier series to nonperiodic functions and have basic applications to PDEs (to be shown in the next chapter).

Prerequisite: Elementary integral calculus (needed for Fourier coefficients)
Sections that may be omitted in a shorter course: 11.4–11.9
References and Answers to Problems: App. 1 Part C, App. 2.

11.1 Fourier Series

Fourier series are the basic tool for representing periodic functions, which play an important role in applications. A function $f(x)$ is called a **periodic function** if $f(x)$ is defined for all real x (perhaps except at some points, such as $x = \pm\pi/2, \pm 3\pi/2, \cdots$ for $\tan x$) and if there is some positive number p, called a **period** of $f(x)$, such that

$$f(x + p) = f(x) \qquad \text{for all } x. \tag{1}$$

The graph of such a function is obtained by periodic repetition of its graph in any interval of length p (Fig. 255).

Familiar periodic functions are the cosine and sine functions. Examples of functions that are not periodic are x, x^2, x^3, e^x, $\cosh x$, and $\ln x$, to mention just a few.

If $f(x)$ has period p, it also has the period $2p$ because (1) implies $f(x + 2p) = f([x + p] + p) = f(x + p) = f(x)$, etc.; thus for any integer $n = 1, 2, 3, \cdots$,

$$f(x + np) = f(x) \qquad \text{for all } x. \tag{2}$$

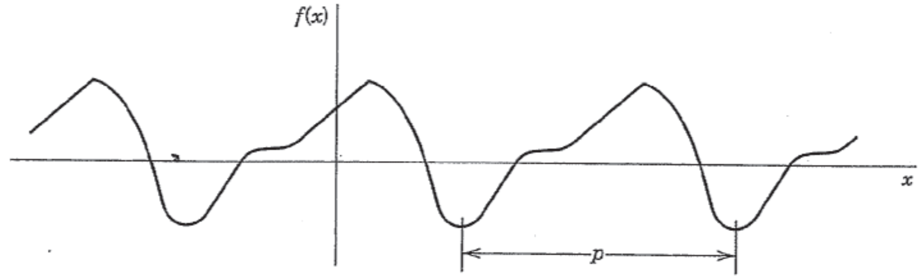


Fig. 255. Periodic function

Furthermore if $f(x)$ and $g(x)$ have period p, then $af(x) + bg(x)$ with any constants a and b also has the period p.

Our problem in the first few sections of this chapter will be the representation of various ***functions f(x) of period 2π*** in terms of the simple functions

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \cdots, \quad \cos nx, \quad \sin nx, \cdots. \tag{3}$$

All these functions have the period 2π. They form the so-called **trigonometric system.** Figure 256 shows the first few of them (except for the constant 1, which is periodic with any period).

The series to be obtained will be a **trigonometric series,** that is, a series of the form

$$\begin{aligned} &a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots \\ &= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \end{aligned} \tag{4}$$

$a_0, a_1, b_1, a_2, b_2, \cdots$ are constants, called the **coefficients** of the series. We see that each term has the period 2π. Hence *if the coefficients are such that the series converges, its sum will be a function of period* 2π.

It can be shown that if the series on the left side of (4) converges, then inserting parentheses on the right gives a series that converges and has the same sum as the series on the left. This justifies the equality in (4).

Now suppose that $f(x)$ is a given function of period 2π and is such that it can be **represented** by a series (4), that is, (4) converges and, moreover, has the sum $f(x)$. Then, using the equality sign, we write

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{5}$$

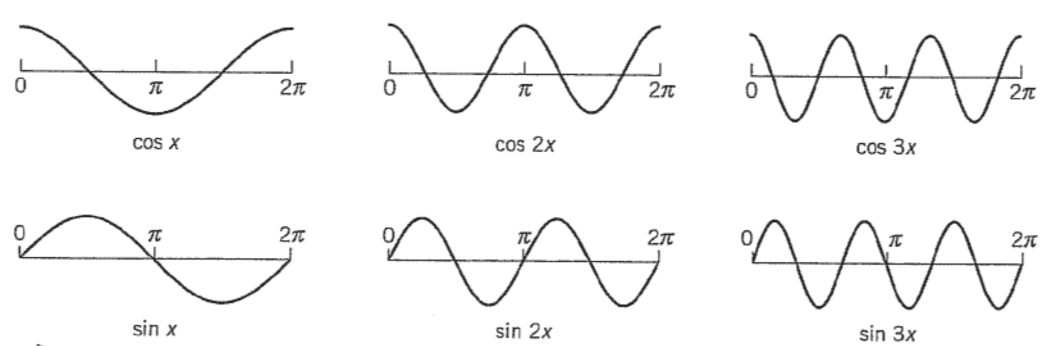


Fig. 256. Cosine and sine functions having the period 2π

and call (5) the **Fourier series** of $f(x)$. We shall prove that in this case the coefficients of (5) are the so-called **Fourier coefficients** of $f(x)$, given by the **Euler formulas**

$$(6)\quad \begin{aligned} &\text{(a)} & a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &\text{(b)} & a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx & n = 1, 2, \cdots \\ &\text{(c)} & b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx & n = 1, 2, \cdots. \end{aligned}$$

The name "Fourier series" is sometimes also used in the exceptional case that (5) with coefficients (6) does not converge or does not have the sum $f(x)$—this may happen but is merely of theoretical interest. (For Euler see footnote 4 in Sec. 2.5.)

A Basic Example

Before we derive the Euler formulas (6), let us become familiar with the application of (5) and (6) in the case of an important example. Since your work for other functions will be quite similar, try to fully understand every detail of the integrations, which because of the n involved differ somewhat from what you have practiced in calculus. Do not just routinely use your software, but make observations: How are continuous functions (cosines and sines) able to represent a given discontinuous function? How does the quality of the approximation increase if you take more and more terms of the series? Why are the approximating functions, called the **partial sums** of the series, always zero at 0 and π? Why is the factor $1/n$ (obtained in the integration) important?

EXAMPLE 1 Periodic Rectangular Wave (Fig. 257a)

Find the Fourier coefficients of the periodic function $f(x)$ in Fig. 257a. The formula is

$$(7)\qquad f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$

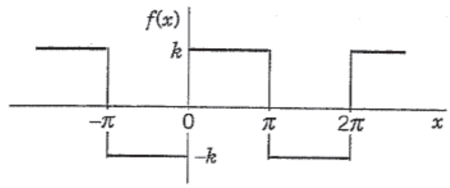
Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc. (The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$.)

Solution. From (6a) we obtain $a_0 = 0$. This can also be seen without integration, since the area under the curve of $f(x)$ between $-\pi$ and π is zero. From (6b),

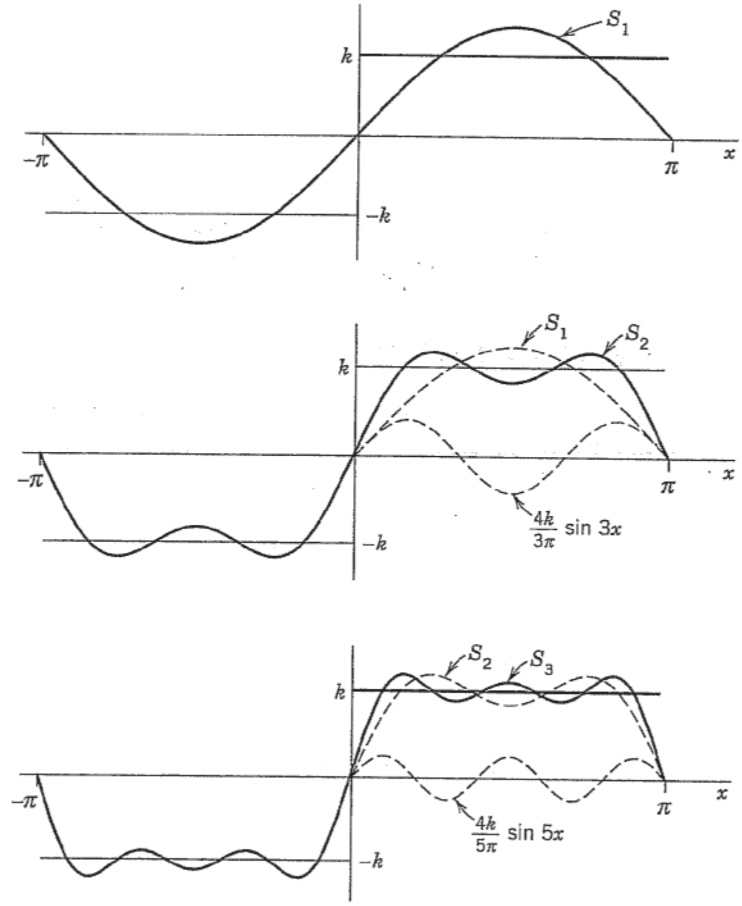
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right] \\ &= \frac{1}{\pi}\left[-k\,\frac{\sin nx}{n}\bigg|_{-\pi}^{0} + k\,\frac{\sin nx}{n}\bigg|_{0}^{\pi}\right] = 0 \end{aligned}$$

because $\sin nx = 0$ at $-\pi$, 0, and π for all $n = 1, 2, \cdots$. Similarly, from (6c) we obtain

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right] \\ &= \frac{1}{\pi}\left[k\,\frac{\cos nx}{n}\bigg|_{-\pi}^{0} - k\,\frac{\cos nx}{n}\bigg|_{0}^{\pi}\right]. \end{aligned}$$



(a) The given function $f(x)$ (Periodic rectangular wave)



(b) The first three partial sums of the corresponding Fourier series

Fig. 257. Eample 1

Since $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, this yields

$$b_n = \frac{k}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right] = \frac{2k}{n\pi}(1 - \cos n\pi).$$

Now, $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, etc.; in general,

$$\cos n\pi = \begin{cases} -1 & \text{for odd } n, \\ 1 & \text{for even } n, \end{cases} \qquad \text{and thus} \qquad 1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n. \end{cases}$$

Hence the Fourier coefficients b_n of our function are

$$b_1 = \frac{4k}{\pi}, \qquad b_2 = 0, \qquad b_3 = \frac{4k}{3\pi}, \qquad b_4 = 0, \qquad b_5 = \frac{4k}{5\pi}, \cdots.$$

Since the a_n are zero, the Fourier series of $f(x)$ is

$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right). \tag{8}$$

The partial sums are

$$S_1 = \frac{4k}{\pi}\sin x, \qquad S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right), \qquad \text{etc.,}$$

Their graphs in Fig. 257 seem to indicate that the series is convergent and has the sum $f(x)$, the given function. We notice that at $x = 0$ and $x = \pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the limits $-k$ and k of our function, at these points.

Furthermore, assuming that $f(x)$ is the sum of the series and setting $x = \pi/2$, we have

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right).$$

thus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}.$$

This is a famous result obtained by Leibniz in 1673 from geometric considerations. It illustrates that the values of various series with constant terms can be obtained by evaluating Fourier series at specific points. ■

Derivation of the Euler Formulas (6)

The key to the Euler formulas (6) is the **orthogonality** of (3), a concept of basic importance, as follows.

THEOREM 1

Orthogonality of the Trigonometric System (3)

The trigonometric system (3) *is orthogonal on the interval* $-\pi \leqq x \leqq \pi$ (hence also on $0 \leqq x \leqq 2\pi$ or any other interval of length 2π because of periodicity); *that is, the integral of the product of any two functions in* (3) *over that interval is* 0, *so that for any integers n and m,*

$$\text{(a)} \quad \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \qquad (n \neq m)$$

$$\text{(9)} \quad \text{(b)} \quad \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \qquad (n \neq m)$$

$$\text{(c)} \quad \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \qquad (n \neq m \text{ or } n = m).$$

PROOF This follows simply by transforming the integrands trigonometrically from products into sums. In (9a) and (9b), by (11) in App. A3.1,

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos (n+m)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos (n-m)x \, dx$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos (n-m)x \, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos (n+m)x \, dx.$$

Since $m \neq n$ (integer!), the integrals on the right are all 0. Similarly, in (9c), for all integer m and n (without exception; do you see why?)

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin (n + m)x \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin (n - m)x \, dx = 0 + 0. \quad \blacksquare$$

Application of Theorem 1 to the Fourier Series (5)

We prove (6a). Integrating on both sides of (5) from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] dx.$$

We now assume that termwise integration is allowed. (We shall say in the proof of Theorem 2 when this is true.) Then we obtain

$$\int_{-\pi}^{\pi} f(x) \, dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \, dx \right).$$

The first term on the right equals $2\pi a_0$. Integration shows that all the other integrals are 0. Hence division by 2π gives (6a).

We prove (6b). Multiplying (5) on both sides by $\cos mx$ with any ***fixed*** positive integer m and integrating from $-\pi$ to π, we have

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \cos mx \, dx. \tag{10}$$

We now integrate term by term. Then on the right we obtain an integral of $a_0 \cos mx$, which is 0; an integral of $a_n \cos nx \cos mx$, which is $a_m\pi$ for $n = m$ and 0 for $n \neq m$ by (9a); and an integral of $b_n \sin nx \cos mx$, which is 0 for all n and m by (9c). Hence the right side of (10) equals $a_m\pi$. Division by π gives (6b) (with m instead of n).

We finally prove (6c). Multiplying (5) on both sides by $\sin mx$ with any ***fixed*** positive integer m and integrating from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \sin mx \, dx. \tag{11}$$

Integrating term by term, we obtain on the right an integral of $a_0 \sin mx$, which is 0; an integral of $a_n \cos nx \sin mx$, which is 0 by (9c); and an integral of $b_n \sin nx \sin mx$, which is $b_m\pi$ if $n = m$ and 0 if $n \neq m$, by (9b). This implies (6c) (with n denoted by m). This completes the proof of the Euler formulas (6) for the Fourier coefficients. $\blacksquare$

Convergence and Sum of a Fourier Series

The class of functions that can be represented by Fourier series is surprisingly large and general. Sufficient conditions valid in most applications are as follows.

THEOREM 2

Representation by a Fourier Series

Let $f(x)$ be periodic with period 2π and piecewise continuous (see Sec. 6.1) *in the interval $-\pi \leqq x \leqq \pi$. Furthermore, let $f(x)$ have a left-hand derivative and a right-hand derivative at each point of that interval. Then the Fourier series* (5) *of $f(x)$* [with coefficients (6)] *converges. Its sum is $f(x)$, except at points x_0 where $f(x)$ is discontinuous. There the sum of the series is the average of the left- and right-hand limits*[2] *of $f(x)$ at x_0.*

PROOF We prove convergence in Theorem 2. We prove convergence for a continuous function $f(x)$ having continuous first and second derivatives. Integrating (6b) by parts, we obtain

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \left.\frac{f(x)\sin nx}{n\pi}\right|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx.$$

The first term on the right is zero. Another integration by parts gives

$$a_n = \left.\frac{f'(x)\cos nx}{n^2\pi}\right|_{-\pi}^{\pi} - \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cos nx\,dx.$$

The first term on the right is zero because of the periodicity and continuity of $f'(x)$. Since f'' is continuous in the interval of integration, we have

$$|f''(x)| < M$$

for an appropriate constant M. Furthermore, $|\cos nx| \leqq 1$. It follows that

$$|a_n| = \frac{1}{n^2\pi}\left|\int_{-\pi}^{\pi} f''(x)\cos nx\,dx\right| < \frac{1}{n^2\pi}\int_{-\pi}^{\pi} M\,dx = \frac{2M}{n^2}.$$

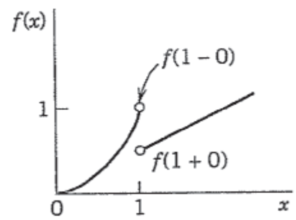


Fig. 258. Left- and right-hand limits

$f(1-0) = 1,$

$f(1+0) = \frac{1}{2}$

of the function

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x/2 & \end{cases}$$

[2] The **left-hand limit** of $f(x)$ at x_0 is defined as the limit of $f(x)$ as x approaches x_0 from the left and is commonly denoted by $f(x_0 - 0)$. Thus

$$f(x_0 - 0) = \lim_{h\to 0} f(x_0 - h) \text{ as } h \to 0 \text{ through positive values.}$$

The **right-hand limit** is denoted by $f(x_0 + 0)$ and

$$f(x_0 + 0) = \lim_{h\to 0} f(x_0 + h) \text{ as } h \to 0 \text{ through positive values.}$$

The **left-** and **right-hand derivatives** of $f(x)$ at x_0 are defined as the limits of

$$\frac{f(x_0 - h) - f(x_0 - 0)}{-h} \quad \text{and} \quad \frac{f(x_0 + h) - f(x_0 + 0)}{h},$$

respectively, as $h \to 0$ through positive values. Of course if $f(x)$ is continuous at x_0, the last term in both numerators is simply $f(x_0)$.

Similarly, $|b_n| < 2\,M/n^2$ for all n. Hence the absolute value of each term of the Fourier series of $f(x)$ is at most equal to the corresponding term of the series

$$|a_0| + 2M\left(1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \cdots\right)$$

which is convergent. Hence that Fourier series converges and the proof is complete. (Readers already familiar with uniform convergence will see that, by the Weierstrass test in Sec. 15.5, under our present assumptions the Fourier series converges uniformly, and our derivation of (6) by integrating term by term is then justified by Theorem 3 of Sec. 15.5.)

The proof of convergence in the case of a piecewise continuous function $f(x)$ and the proof that under the assumptions in the theorem the Fourier series (5) with coefficients (6) represents $f(x)$ are substantially more complicated; see, for instance, Ref. [C12]. ■

EXAMPLE 2 **Convergence at a Jump as Indicated in Theorem 2**

The rectangular wave in Example 1 has a jump at $x = 0$. Its left-hand limit there is $-k$ and its right-hand limit is k (Fig. 257). Hence the average of these limits is 0. The Fourier series (8) of the wave does indeed converge to this value when $x = 0$ because then all its terms are 0. Similarly for the other jumps. This is in agreement with Theorem 2. ■

Summary. A Fourier series of a given function $f(x)$ of period 2π is a series of the form (5) with coefficients given by the Euler formulas (6). Theorem 2 gives conditions that are sufficient for this series to converge and at each x to have the value $f(x)$, except at discontinuities of $f(x)$, where the series equals the arithmetic mean of the left-hand and right-hand limits of $f(x)$ at that point.

PROBLEM SET 11.1

1. **(Calculus review)** Review integration techniques for integrals as they are likely to arise from the Euler formulas, for instance, definite integrals of $x \cos nx$, $x^2 \sin nx$, $e^{-2x} \cos nx$, etc.

2–3 FUNDAMENTAL PERIOD

The *fundamental period* is the smallest positive period. Find it for

2. $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$

3. $\cos nx$, $\sin nx$, $\cos \dfrac{2\pi x}{k}$, $\sin \dfrac{2\pi x}{k}$, $\cos \dfrac{2\pi n x}{k}$, $\sin \dfrac{2\pi n x}{k}$

4. Show that $f = const$ is periodic with any period but has no fundamental period.

5. If $f(x)$ and $g(x)$ have period p, show that $h(x) = af(x) + bg(x)$ (a, b, constant) has the period p. Thus all functions of period p form a **vector space**.

6. **(Change of scale)** If $f(x)$ has period p, show that $f(ax)$, $a \neq 0$, and $f(x/b)$, $b \neq 0$, are periodic functions of x of periods p/a and bp, respectively. Give examples.

7–12 GRAPHS OF 2π-PERIODIC FUNCTIONS

Sketch or graph $f(x)$, of period 2π, which for $-\pi < x < \pi$ is given as follows.

7. $f(x) = x$

8. $f(x) = e^{-|x|}$

9. $f(x) = \pi - |x|$

10. $f(x) = |\sin 2x|$

11. $f(x) = \begin{cases} -x^3 & \text{if } -\pi < x < 0 \\ x^3 & \text{if } \quad 0 < x < \pi \end{cases}$

12. $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ \cos \frac{1}{2}x & \text{if } \quad 0 < x < \pi \end{cases}$

13–24 FOURIER SERIES

Showing the details of your work, find the Fourier series of the given $f(x)$, which is assumed to have the period 2π. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$.

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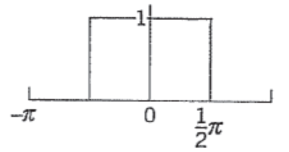

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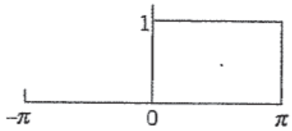

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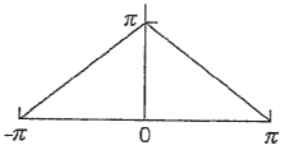

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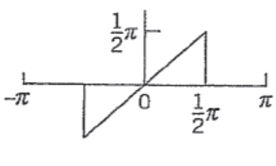

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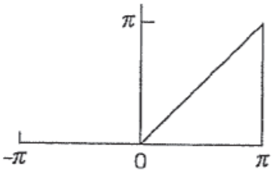

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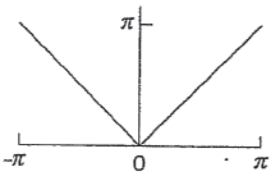

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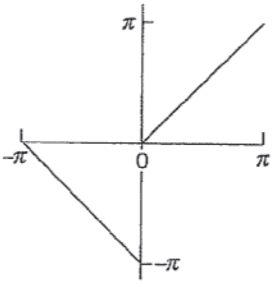
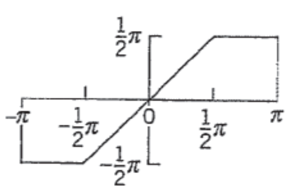


20.



21. $f(x) = x^2 \ (-\pi < x < \pi)$

22. $f(x) = x^2 \ (0 < x < 2\pi)$

23. $f(x) = \begin{cases} x^2 & \text{if } -\frac{1}{2}\pi < x < \frac{1}{2}\pi \\ \frac{1}{4}\pi^2 & \text{if } \frac{1}{2}\pi < x < \frac{3}{2}\pi \end{cases}$

24. $f(x) = \begin{cases} -4x & \text{if } -\pi < x < 0 \\ 4x & \text{if } 0 < x < \pi \end{cases}$

25. **(Discontinuities)** Verify the last statement in Theorem 2 for the discontinuities of $f(x)$ in Prob. 13.

26. **CAS EXPERIMENT. Graphing.** Write a program for graphing partial sums of the following series. Guess from the graph what $f(x)$ the series may represent. Confirm or disprove your guess by using the Euler formulas.

(a) $2(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots)$
$\quad - 2(\frac{1}{2}\sin 2x + \frac{1}{4}\sin 4x + \frac{1}{6}\sin 6x \cdots)$

(b) $\frac{1}{2} + \frac{4}{\pi^2}(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots)$

(c) $\frac{2}{3}\pi^2 + 4(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + - \cdots)$

27. **CAS EXPERIMENT. Order of Fourier Coefficients.** The order seems to be $1/n$ if f is discontinous, and $1/n^2$ if f is continuous but $f' = df/dx$ is discontinuous, $1/n^3$ if f and f' are continuous but f'' is discontinuous, etc. Try to verify this for examples. Try to prove it by integrating the Euler formulas by parts. What is the practical significance of this?

28. **PROJECT. Euler Formulas in Terms of Jumps Without Integration.** Show that for a function whose third derivative is identically zero,

$$a_n = \frac{1}{n\pi}\left[-\sum j_s \sin nx_s - \frac{1}{n}\sum j_s' \cos nx_s + \frac{1}{n^2}\sum j_s'' \sin nx_s\right]$$

$$b_n = \frac{1}{n\pi}\left[\sum j_s \cos nx_s - \frac{1}{n}\sum j_s' \sin nx_s - \frac{1}{n^2}\sum j_s'' \cos nx_s\right]$$

where $n = 1, 2, \cdots$ and we sum over all the jumps j_s, j_s', j_s'' of f, f', f', respectively, located at x_s.

29. Apply the formulas in Project 28 to the function in Prob. 21 and compare the results.

30. **CAS EXPERIMENT. Orthogonality.** Integrate and graph the integral of the product $\cos mx \cos nx$ (with various integer m and n of your choice) from $-a$ to a as a function of a and conclude orthogonality of $\cos mx$ and $\cos nx$ ($m \neq n$) for $a = \pi$ from the graph. For what m and n will you get orthogonality for $a = \pi/2$, $\pi/3$, $\pi/4$? Other a? Extend the experiment to $\cos mx \sin nx$ and $\sin mx \sin nx$.

11.2 Functions of Any Period $p = 2L$

The functions considered so far had period 2π, for the simplicity of the formulas. Of course, periodic functions in applications will generally have other periods. However, we now show that the transition from period $p = 2\pi$ to a period $2L$ is quite simple. The notation $p = 2L$ is practical because L will be the length of a violin string (Sec. 12.2) or the length of a rod in heat conduction (Sec. 12.5), and so on.

The idea is simply to find and use a ***change of scale*** that gives from a function $g(v)$ of period 2π a function of period $2L$. Now from (5) and (6) in the last section with $g(v)$ instead of $f(x)$ we have the Fourier series

$$g(v) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv) \tag{1}$$

with coefficients

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v)\, dv \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \cos nv\, dv \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \sin nv\, dv. \end{aligned} \tag{2}$$

We can now write the change of scale as $v = kx$ with k such that the old period $v = 2\pi$ gives for the new variable x the new period $x = 2L$. Thus, $2\pi = k2L$. Hence $k = \pi/L$ and

$$v = kx = \pi x/L. \tag{3}$$

This implies $dv = (\pi/L)\, dx$, which upon substitution into (2) cancels $1/2\pi$ and $1/\pi$ and gives instead the factors $1/2L$ and $1/L$. Writing

$$g(v) = f(x), \tag{4}$$

we thus obtain from (1) the **Fourier series** of the function $f(x)$ of period $2L$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \tag{5}$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas**

$$\begin{aligned} &\text{(a)} \quad a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \\ &\text{(b)} \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots \\ &\text{(c)} \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots. \end{aligned} \tag{6}$$

Just as in Sec. 11.1, we continue to call (5) with any coefficients a **trigonometric series.** And we can integrate from 0 to $2L$ or over any other interval of length $p = 2L$.

EXAMPLE 1 Periodic Rectangular Wave

Find the Fourier series of the function (Fig. 259)

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } \quad 1 < x < 2 \end{cases} \qquad p = 2L = 4, \quad L = 2.$$

Solution. From (6a) we obtain $a_0 = k/2$ (verify!). From (6b) we obtain

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}.$$

Thus $a_n = 0$ if n is even and

$$a_n = 2k/n\pi \quad \text{if} \quad n = 1, 5, 9, \cdots, \qquad a_n = -2k/n\pi \quad \text{if} \quad n = 3, 7, 11, \cdots.$$

From (6c) we find that $b_n = 0$ for $n = 1, 2, \cdots$. Hence the Fourier series is

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right).$$

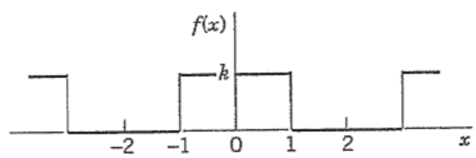


Fig. 259. Example 1

EXAMPLE 2 Periodic Rectangular Wave

Find the Fourier series of the function (Fig. 260)

$$f(x) = \begin{cases} -k & \text{if } -2 < x < 0 \\ k & \text{if } \quad 0 < x < 2 \end{cases} \qquad p = 2L = 4, \qquad L = 2.$$

Solution. $a_0 = 0$ from (6a). From (6b), with $1/L = 1/2$,

$$\begin{aligned} a_n &= \frac{1}{2}\left[\int_{-2}^{0}(-k)\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} k\cos\frac{n\pi x}{2}\,dx\right] \\ &= \frac{1}{2}\left[-\frac{2k}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{0} + \frac{2k}{n\pi}\sin\frac{n\pi x}{2}\Big|_{0}^{2}\right] = 0, \end{aligned}$$

so that the Fourier series has no cosine terms. From (6c),

$$\begin{aligned} b_n &= \frac{1}{2}\left[\frac{2k}{n\pi}\cos\frac{n\pi x}{2}\Big|_{-2}^{0} - \frac{2k}{n\pi}\cos\frac{n\pi x}{2}\Big|_{0}^{2}\right] \\ &= \frac{k}{n\pi}(1 - \cos n\pi - \cos n\pi + 1) = \begin{cases} 4k/n\pi & \text{if } n = 1, 3, \cdots \\ 0 & \text{if } n = 2, 4, \cdots. \end{cases} \end{aligned}$$

Hence the Fourier series of $f(x)$ is

$$f(x) = \frac{4k}{\pi}\left(\sin\frac{\pi}{2}x + \frac{1}{3}\sin\frac{3\pi}{2}x + \frac{1}{5}\sin\frac{5\pi}{2}x + \cdots\right).$$

It is interesting that we could have derived this from (8) in Sec. 11.1, namely, by the scale change (3). Indeed, writing v instead of x, we have in (8), Sec. 11.1,

$$\frac{4k}{\pi}\left(\sin v + \frac{1}{3}\sin 3v + \frac{1}{5}\sin 5v + \cdots\right).$$

Since the period 2π in v corresponds to $2L = 4$, we have $k = \pi/L = \pi/2$ and $v = kx = \pi x/2$ in (3); hence we obtain the Fourier series of $f(x)$, as before. ▪

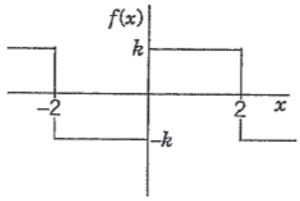


Fig. 260. Example 2

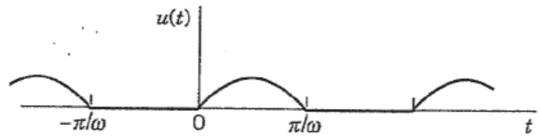


Fig. 261. Half-wave rectifier

EXAMPLE 3 Half-Wave Rectifier

A sinusoidal voltage $E \sin \omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave (Fig. 261). Find the Fourier series of the resulting periodic function

$$u(t) = \begin{cases} 0 & \text{if } -L < t < 0, \\ E\sin\omega t & \text{if } \quad 0 < t < L \end{cases} \qquad p = 2L = \frac{2\pi}{\omega}, \qquad L = \frac{\pi}{\omega}.$$

Solution. Since $u = 0$ when $-L < t < 0$, we obtain from (6a), with t instead of x,

$$a_0 = \frac{\omega}{2\pi}\int_0^{\pi/\omega} E\sin\omega t\,dt = \frac{E}{\pi}$$

and from (6b), by using formula (11) in App. A3.1 with $x = \omega t$ and $y = n\omega t$,

$$a_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\sin\omega t\cos n\omega t\,dt = \frac{\omega E}{2\pi}\int_0^{\pi/\omega}[\sin(1+n)\omega t + \sin(1-n)\omega t]\,dt.$$

If $n = 1$, the integral on the right is zero, and if $n = 2, 3, \cdots$, we readily obtain

$$a_n = \frac{\omega E}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$$

$$= \frac{E}{2\pi}\left(\frac{-\cos(1+n)\pi + 1}{1+n} + \frac{-\cos(1-n)\pi + 1}{1-n}\right).$$

If n is odd, this is equal to zero, and for even n we have

$$a_n = \frac{E}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right) = -\frac{2E}{(n-1)(n+1)\pi} \qquad (n = 2, 4, \cdots).$$

In a similar fashion we find from (6c) that $b_1 = E/2$ and $b_n = 0$ for $n = 2, 3, \cdots$. Consequently,

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t + \cdots\right).$$ ▪

PROBLEM SET 11.2

1–11 FOURIER SERIES FOR PERIOD $p = 2L$

Find the Fourier series of the function $f(x)$, of period $p = 2L$, and sketch or graph the first three partial sums. (Show the details of your work.)

1. $f(x) = -1\ (-2 < x < 0),\ f(x) = 1\ (0 < x < 2),\ p = 4$
2. $f(x) = 0\ (-2 < x < 0),\ f(x) = 4\ (0 < x < 2),\ p = 4$
3. $f(x) = x^2 \quad (-1 < x < 1), \quad p = 2$
4. $f(x) = \pi x^3/2 \quad (-1 < x < 1), \quad p = 2$
5. $f(x) = \sin \pi x \quad (0 < x < 1), \quad p = 1$
6. $f(x) = \cos \pi x \quad (-\frac{1}{2} < x < \frac{1}{2}), \quad p = 1$
7. $f(x) = |x| \quad (-1 < x < 1), \quad p = 2$
8. $f(x) = \begin{cases} 1 + x \text{ if } -1 < x < 0 \\ 1 - x \text{ if } \quad 0 < x < 1, \end{cases} \quad p = 2$
9. $f(x) = 1 - x^2 \quad (-1 < x < 1), \quad p = 2$
10. $f(x) = 0\ (-2 < x < 0),\ f(x) = x\ (0 < x < 2),\ p = 4$
11. $f(x) = -x \quad (-1 < x < 0), \quad f(x) = x \quad (0 < x < 1)$, $f(x) = 1 \quad (1 < x < 3), \quad p = 4$
12. **(Rectifier)** Find the Fourier series of the function obtained by passing the voltage $v(t) = V_0 \cos 100\pi t$ through a half-wave rectifier.
13. Show that the familiar identities $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$ and $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$ can be interpreted as Fourier series expansions. Develop $\cos^4 x$.
14. Obtain the series in Prob. 7 from that in Prob. 8.
15. Obtain the series in Prob. 6 from that in Prob. 5.
16. Obtain the series in Prob. 3 from that in Prob. 21 of Problem Set 11.1.
17. Using Prob. 3, show that $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + - \cdots = \frac{1}{12}\pi^2$.
18. Show that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{1}{6}\pi^2$.
19. **CAS PROJECT. Fourier Series of $2L$-Periodic Functions. (a)** Write a program for obtaining partial sums of a Fourier series (1).

    (b) Apply the program to Probs. 2–5, graphing the first few partial sums of each of the four series on common axes. Choose the first five or more partial sums until they approximate the given function reasonably well. Compare and comment.
20. **CAS EXPERIMENT. Gibbs Phenomenon.** The partial sums $s_n(x)$ of a Fourier series show oscillations near a discontinuity point. These oscillations do not disappear as n increases but instead become sharp "spikes." They were explained mathematically by J. W. Gibbs[3]. Graph $s_n(x)$ in Prob. 10. When $n = 50$, say, you will see those oscillations quite distinctly. Consider other Fourier series of your choice in a similar way. Compare.

11.3 Even and Odd Functions. Half-Range Expansions

The function in Example 1, Sec. 11.2, is *even,* and its Fourier series has only *cosine* terms. The function in Example 2, Sec. 11.2, is *odd*, and its Fourier series has only *sine* terms.

Recall that g is **even** if $g(-x) = g(x)$, so that its graph is symmetric with respect to the vertical axis (Fig. 262). A function h is **odd** if $h(-x) = -h(x)$ (Fig. 263).

Now the cosine terms in the Fourier series (5), Sec. 11.2, are even and the sine terms are odd. So it should not be a surprise that an even function is given by a series of cosine terms and an odd function by a series of sine terms. Indeed, the following holds.

[3]JOSIAH WILLARD GIBBS (1839–1903), American mathematician, professor of mathematical physics at Yale from 1871 on, one of the founders of vector calculus [another being O. Heaviside (see Sec. 6.1)], mathematical thermodynamics, and statistical mechanics. His work was of great importance to the development of mathematical physics.

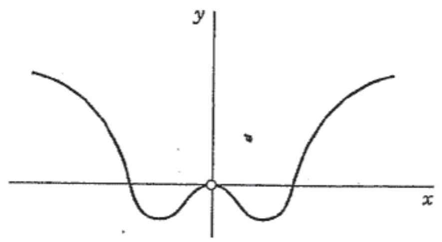


Fig. 262. Even function

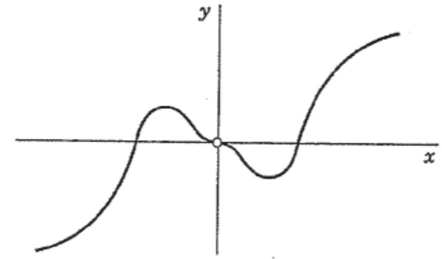


Fig. 263. Odd function

THEOREM 1

Fourier Cosine Series, Fourier Sine Series

The Fourier series of an ***even*** *function of period 2L is a* **"Fourier cosine series"**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \qquad (f \text{ even}) \tag{1}$$

with coefficients (note: integration from 0 to L only!)

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, \cdots. \tag{2}$$

The Fourier series of an ***odd*** *function of period 2L is a* **"Fourier sine series"**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \qquad (f \text{ odd}) \tag{3}$$

with coefficients

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{4}$$

PROOF Since the definite integral of a function gives the area under the curve of the function between the limits of integration, we have

$$\int_{-L}^{L} g(x)\, dx = 2 \int_0^L g(x)\, dx \qquad \text{for even } g$$

$$\int_{-L}^{L} h(x)\, dx = 0 \qquad \text{for odd } h$$

as is obvious from the graphs of g and h. (Give a formal proof.) Now let f be even. Then (6a), Sec. 11.2, gives a_0 in (2). Also, the integrand in (6b), Sec. 11.2, is even (a product of even functions is even), so that (6b) gives a_n in (2). Furthermore, the integrand in (6c), Sec. 11.2, is the even f times the odd sine, so that the integrand (the product) is odd, the integral is zero, and there are no sine terms in (1).

Similarly, if f is odd, the integrals for a_0 and a_n in (6a) and (6b), Sec. 11.2, are zero, f times the sine in (6c) is even, (6c) implies (4), and there are no cosine terms in (3). ■

The Case of Period 2π. If $L = \pi$, then $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ (f even) with coefficients

$$(2^*) \qquad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \qquad n = 1, 2, \cdots$$

and $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ (f odd) with coefficients

$$(4^*) \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx, \qquad n = 1, 2, \cdots.$$

For instance, $f(x)$ in Example 1, Sec. 11.1, is odd and is represented by a Fourier sine series.

Further simplifications result from the following property, whose very simple proof is left to the student.

THEOREM 2

Sum and Scalar Multiple

The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

EXAMPLE 1

Rectangular Pulse

The function $f^*(x)$ in Fig. 264 is the sum of the function $f(x)$ in Example 1 of Sec 11.1 and the constant k. Hence, from that example and Theorem 2 we conclude that

$$f^*(x) = k + \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$ ■

EXAMPLE 2

Half-Wave Rectifier

The function $u(t)$ in Example 3 of Sec. 11.2 has a Fourier cosine series plus a single term $v(t) = (E/2)\sin \omega t$. We conclude from this and Theorem 2 that $u(t) - v(t)$ must be an even function. Verify this graphically. (See Fig. 265.) ■

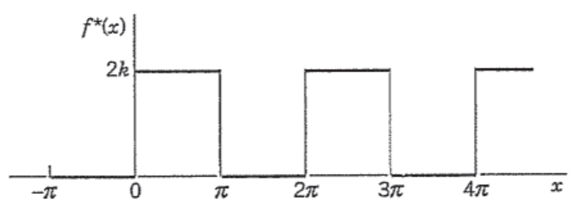


Fig. 264. Example 1

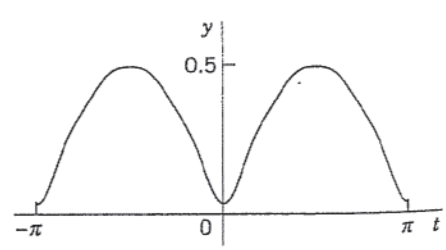


Fig. 265. $u(t) - v(t)$ with $E = 1$, $\omega = 1$

EXAMPLE 3 **Sawtooth Wave**

Find the Fourier series of the function (Fig. 266)

$$f(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$

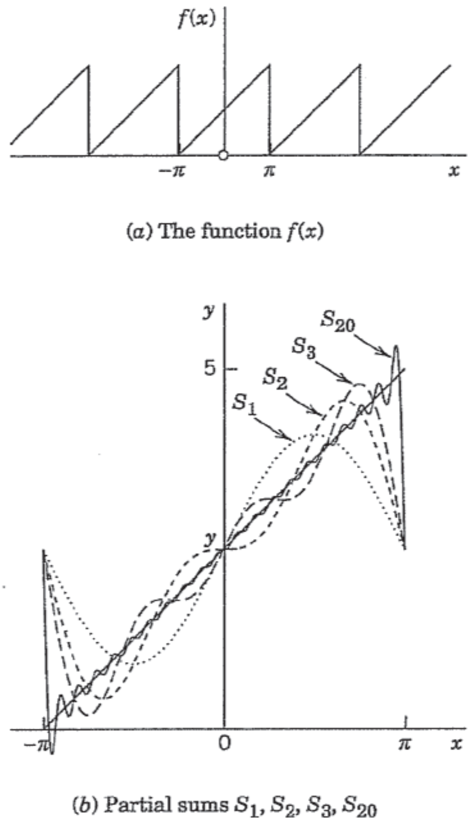


Fig. 266. Example 3

Solution. We have $f = f_1 + f_2$, where $f_1 = x$ and $f_2 = \pi$. The Fourier coefficients of f_2 are zero, except for the first one (the constant term), which is π. Hence, by Theorem 2, the Fourier coefficients a_n, b_n are those of f_1, except for a_0, which is π. Since f_1 is odd, $a_n = 0$ for $n = 1, 2, \cdots$, and

$$b_n = \frac{2}{\pi} \int_0^{\pi} f_1(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx.$$

Integrating by parts, we obtain

$$b_n = \frac{2}{\pi} \left[\frac{-x \cos nx}{n} \Bigg|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \, dx \right] = -\frac{2}{n} \cos n\pi.$$

Hence $b_1 = 2$, $b_2 = -2/2$, $b_3 = 2/3$, $b_4 = -2/4$, $\cdots$, and the Fourier series of $f(x)$ is

$$f(x) = \pi + 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - + \cdots \right).$$

Half-Range Expansions

Half-range expansions are Fourier series. The idea is simple and useful. Figure 267 explains it. We want to represent $f(x)$ in Fig. 267a by a Fourier series, where $f(x)$ may be the shape of a distorted violin string or the temperature in a metal bar of length L, for example. (Corresponding problems will be discussed in Chap. 12.) Now comes the idea.

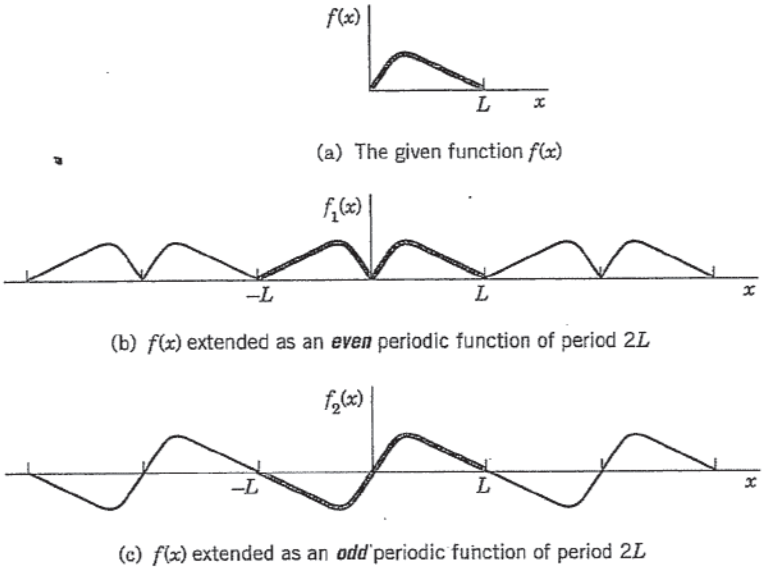


Fig. 267. (a) Function $f(x)$ given on an interval $0 \leqq x \leqq L$

(b) **Even extension** to the full "range" (interval) $-L \leqq x \leqq L$ (heavy curve) and the periodic extension of period $2L$ to the x-axis

(c) **Odd extension** to $-L \leqq x \leqq L$ (heavy curve) and the periodic extension of period $2L$ to the x-axis

We could extend $f(x)$ as a function of period L and develop the extended function into a Fourier series. But this series would in general contain *both* cosine *and* sine terms. We can do better and get simpler series. Indeed, for our given f we can calculate Fourier coefficients from (2) or from (4) in Theorem 1. And we have a choice and can take what seems more practical. If we use (2), we get (1). This is the **even periodic extension** f_1 of f in Fig. 267b. If we choose (4) instead, we get (3), the **odd periodic extension** f_2 of f in Fig. 267c.

Both extensions have period $2L$. This motivates the name **half-range expansions:** f is given (and of physical interest) only on half the range, half the interval of periodicity of length $2L$.

Let us illustrate these ideas with an example that we shall also need in Chap. 12.

EXAMPLE 4 "Triangle" and Its Half-Range Expansions

Find the two half-range expansions of the function (Fig. 268)

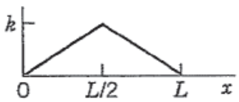


Fig. 268. The given function in Example 4

$$f(x) = \begin{cases} \dfrac{2k}{L}x & \text{if } 0 < x < \dfrac{L}{2} \\[2ex] \dfrac{2k}{L}(L - x) & \text{if } \dfrac{L}{2} < x < L. \end{cases}$$

Solution. *(a) Even periodic extension.* From (2) we obtain

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} x\, dx + \frac{2k}{L}\int_{L/2}^{L} (L - x)\, dx\right] = \frac{k}{2},$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x \cos\frac{n\pi}{L}x\, dx + \frac{2k}{L}\int_{L/2}^{L} (L - x)\cos\frac{n\pi}{L}x\, dx\right].$$

We consider a_n. For the first integral we obtain by integration by parts

$$\int_0^{L/2} x \cos \frac{n\pi}{L} x \, dx = \frac{Lx}{n\pi} \sin \frac{n\pi}{L} x \Big|_0^{L/2} - \frac{L}{n\pi} \int_0^{L/2} \sin \frac{n\pi}{L} x \, dx$$

$$= \frac{L^2}{2n\pi} \sin \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} \left(\cos \frac{n\pi}{2} - 1 \right).$$

Similarly, for the second integral we obtain

$$\int_{L/2}^{L} (L - x) \cos \frac{n\pi}{L} x \, dx = \frac{L}{n\pi} (L - x) \sin \frac{n\pi}{L} x \Big|_{L/2}^{L} + \frac{L}{n\pi} \int_{L/2}^{L} \sin \frac{n\pi}{L} x \, dx$$

$$= \left(0 - \frac{L}{n\pi} \left(L - \frac{L}{2} \right) \sin \frac{n\pi}{2} \right) - \frac{L^2}{n^2\pi^2} \left(\cos n\pi - \cos \frac{n\pi}{2} \right).$$

We insert these two results into the formula for a_n. The sine terms cancel and so does a factor L^2. This gives

$$a_n = \frac{4k}{n^2\pi^2} \left(2 \cos \frac{n\pi}{2} - \cos n\pi - 1 \right).$$

Thus,

$$a_2 = -16k/(2^2\pi^2), \qquad a_6 = -16k/(6^2\pi^2), \qquad a_{10} = -16k/(10^2\pi^2), \cdots$$

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \cdots$. Hence the first half-range expansion of $f(x)$ is (Fig. 269a)

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2} \left(\frac{1}{2^2} \cos \frac{2\pi}{L} x + \frac{1}{6^2} \cos \frac{6\pi}{L} x + \cdots \right).$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$.

(b) ***Odd periodic extension.*** Similarly, from (4) we obtain

$$b_n = \frac{8k}{n^2\pi^2} \sin \frac{n\pi}{2}. \tag{5}$$

Hence the other half-range expansion of $f(x)$ is (Fig. 269b)

$$f(x) = \frac{8k}{\pi^2} \left(\frac{1}{1^2} \sin \frac{\pi}{L} x - \frac{1}{3^2} \sin \frac{3\pi}{L} x + \frac{1}{5^2} \sin \frac{5\pi}{L} x - + \cdots \right).$$

This series represents the odd periodic extension of $f(x)$, of period $2L$.

Basic applications of these results will be shown in Secs. 12.3 and 12.5.

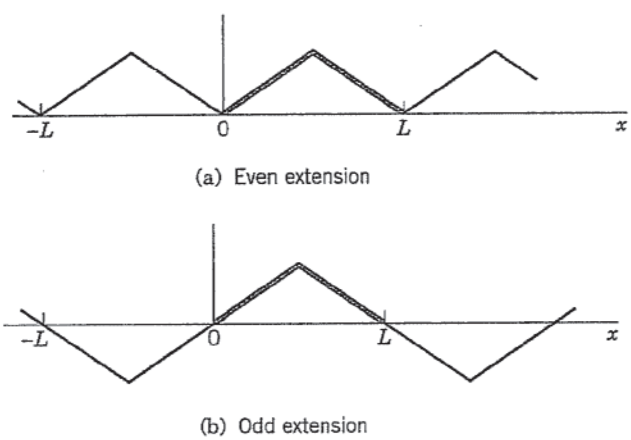


Fig. 269. Periodic extensions of $f(x)$ in Example 4

PROBLEM SET 11.3

1–9 EVEN AND ODD FUNCTIONS

e the following functions even, odd, or neither even nor odd?

. $|x|$, $x^2 \sin nx$, $x + x^2$, $e^{-|x|}$, $\ln x$, $x \cosh x$

. $\sin(x^2)$, $\sin^2 x$, $x \sinh x$, $|x^3|$, $e^{\pi x}$, xe^x, $\tan 2x$, $x/(1 + x^2)$

Are the following functions, which are assumed to be iodic of period 2π, even, odd, or neither even nor odd?

. $f(x) = x^3 \quad (-\pi < x < \pi)$

4. $f(x) = x^2 \quad (-\pi/2 < x < 3\pi/2)$

. $f(x) = e^{-4x} \quad (-\pi < x < \pi)$

. $f(x) = x^3 \sin x \quad (-\pi < x < \pi)$

7. $f(x) = x|x| - x^3 \quad (-\pi < x < \pi)$

. $f(x) = 1 - x + x^3 - x^5 \quad (-\pi < x < \pi)$

. $f(x) = 1/(1 + x^2)$ if $-\pi < x < 0$, $f(x) = -1/(1 + x^2)$ if $0 < x < \pi$

. **PROJECT. Even and Odd Functions.** **(a)** Are the following expressions even or odd? Sums and products of even functions and of odd functions. Products of even times odd functions. Absolute values of odd functions. $f(x) + f(-x)$ and $f(x) - f(-x)$ for arbitrary $f(x)$.

(b) Write e^{kx}, $1/(1 - x)$, $\sin(x + k)$, $\cosh(x + k)$ as sums of an even and an odd function.

(c) Find all functions that are both even and odd.

(d) Is $\cos^3 x$ even or odd? $\sin^3 x$? Find the Fourier series of these functions. Do you recognize familiar identities?

–16 FOURIER SERIES OF EVEN AND ODD FUNCTIONS

the given function even or odd? Find its Fourier series. Sketch or graph the function and some partial sums. (Show the details of your work.)

. $f(x) = \pi - |x| \quad (-\pi < x < \pi)$

12. $f(x) = 2x|x| \quad (-1 < x < 1)$

13. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

14. $f(x) = \begin{cases} \pi e^{-x} & \text{if } -\pi < x < 0 \\ \pi e^{x} & \text{if } 0 < x < \pi \end{cases}$

15. $f(x) = \begin{cases} 2 & \text{if } -2 < x < 0 \\ 0 & \text{if } 0 < x < 2 \end{cases}$

16. $f(x) = \begin{cases} 1 - \frac{1}{2}|x| & \text{if } -2 < x < 2 \\ 0 & \text{if } 2 < x < 6 \end{cases} \quad (p = 8)$

17–25 HALF-RANGE EXPANSIONS

Find (a) the Fourier cosine series, (b) the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. (Show the details of your work.)

17. $f(x) = 1 \quad (0 < x < 2)$

18. $f(x) = x \quad (0 < x < \frac{1}{2})$

19. $f(x) = 2 - x \quad (0 < x < 2)$

20. $f(x) = \begin{cases} 0 & (0 < x < 2) \\ 1 & (2 < x < 4) \end{cases}$

21. $f(x) = \begin{cases} 1 & (0 < x < 1) \\ 2 & (1 < x < 2) \end{cases}$

22. $f(x) = \begin{cases} x & (0 < x < \pi/2) \\ \pi/2 & (\pi/2 < x < \pi) \end{cases}$

23. $f(x) = x \quad (0 < x < L)$

24. $f(x) = x^2 \quad (0 < x < L)$

25. $f(x) = \pi - x \quad (0 < x < \pi)$

26. Illustrate the formulas in the proof of Theorem 1 with examples. Prove the formulas.

11.4 Complex Fourier Series. *Optional*

In this optional section we show that the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

can be written in complex form, which sometimes simplifies calculations (see Example 1, on page 498). This complex form can be obtained because in complex, the exponential function e^{it} and $\cos t$ and $\sin t$ are related by the basic **Euler formula** (see (11) in Sec. 2.2)

$$e^{it} = \cos t + i \sin t. \qquad \text{Thus} \qquad e^{-it} = \cos t - i \sin t. \tag{2}$$

Conversely, by adding and subtracting these two formulas, we obtain

$$\text{(3)} \qquad \text{(a)} \quad \cos t = \frac{1}{2}(e^{it} + e^{-it}), \qquad \text{(b)} \quad \sin t = \frac{1}{2i}(e^{it} - e^{-it}).$$

From (3), using $1/i = -i$ in $\sin t$ and setting $t = nx$ in both formulas, we get

$$\begin{aligned} a_n \cos nx + b_n \sin nx &= \frac{1}{2}\, a_n(e^{inx} + e^{-inx}) + \frac{1}{2i}\, b_n(e^{inx} - e^{-inx}) \\ &= \frac{1}{2}\,(a_n - ib_n)e^{inx} + \frac{1}{2}\,(a_n + ib_n)e^{-inx}. \end{aligned}$$

We insert this into (1). Writing $a_0 = c_0$, $\frac{1}{2}(a_n - ib_n) = c_n$, and $\frac{1}{2}(a_n + ib_n) = k_n$, we get from (1)

$$\text{(4)} \qquad f(x) = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + k_n e^{-inx}).$$

The coefficients $c_1, c_2, \cdots$, and $k_1, k_2, \cdots$ are obtained from (6b), (6c) in Sec. 11.1 and then (2) above with $t = nx$,

$$\text{(5)} \qquad \begin{aligned} c_n &= \frac{1}{2}\,(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i \sin nx)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\, dx \\ k_n &= \frac{1}{2}\,(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx + i \sin nx)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\, dx. \end{aligned}$$

Finally, we can combine (5) into a single formula by the trick of writing $k_n = c_{-n}$. Then (4), (5), and $c_0 = a_0$ in (6a) of Sec. 11.1 give (summation from $-\infty$!)

$$\text{(6)} \qquad \begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx}, \\ c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\, dx, \qquad n = 0, \pm 1, \pm 2, \cdots. \end{aligned}$$

This is the so-called *complex form of the Fourier series* or, more briefly, the **complex Fourier series,** of $f(x)$. The c_n are called the **complex Fourier coefficients** of $f(x)$.

For a function of period $2L$ our reasoning gives the **complex Fourier series**

$$\text{(7)} \qquad \begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\, dx, \qquad n = 0, \pm 1, \pm 2, \cdots. \end{aligned}$$

EXAMPLE 1 **Complex Fourier Series**

Find the complex Fourier series of $f(x) = e^x$ if $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$ and obtain from it the usual Fourier series.

Solution. Since $\sin n\pi = 0$ for integer n, we have

$$e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = \cos n\pi = (-1)^n.$$

With this we obtain from (6) by integration

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-inx}\, dx = \frac{1}{2\pi} \frac{1}{1 - in} e^{x - inx} \bigg|_{x=-\pi}^{\pi} = \frac{1}{2\pi} \frac{1}{1 - in} (e^{\pi} - e^{-\pi})(-1)^n.$$

On the right,

$$\frac{1}{1 - in} = \frac{1 + in}{(1 - in)(1 + in)} = \frac{1 + in}{1 + n^2} \qquad \text{and} \qquad e^{\pi} - e^{-\pi} = 2 \sinh \pi.$$

Hence the complex Fourier series is

$$(8) \qquad e^x = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{1 + in}{1 + n^2} e^{inx} \qquad (-\pi < x < \pi).$$

From this let us derive the real Fourier series. Using (2) with $t = nx$ and $i^2 = -1$, we have in (8)

$$(1 + in)e^{inx} = (1 + in)(\cos nx + i \sin nx) = (\cos nx - n \sin nx) + i(n \cos nx + \sin nx).$$

Now (8) also has a corresponding term with $-n$ instead of n. Since $\cos(-nx) = \cos nx$ and $\sin(-nx) = -\sin nx$, we obtain in this term

$$(1 - in)e^{-inx} = (1 - in)(\cos nx - i \sin nx) = (\cos nx - n \sin nx) - i(n \cos nx + \sin nx).$$

If we add these two expressions, the imaginary parts cancel. Hence their sum is

$$2(\cos nx - n \sin nx), \qquad n = 1, 2, \cdots.$$

For $n = 0$ we get 1 (not 2) because there is only one term. Hence the real Fourier series is

$$(9) \qquad e^x = \frac{2 \sinh \pi}{\pi} \left[\frac{1}{2} - \frac{1}{1 + 1^2} (\cos x - \sin x) + \frac{1}{1 + 2^2} (\cos 2x - 2 \sin 2x) - + \cdots \right].$$

In Fig. 270 the poor approximation near the jumps at $\pm\pi$ is a case of the Gibbs phenomenon (see CAS Experiment 20 in Problem Set 11.2). ■

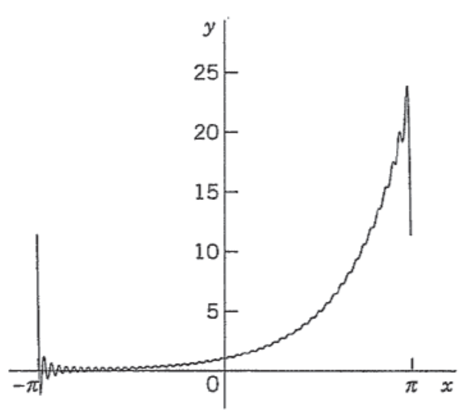


Fig. 270. Partial sum of (9), terms from $n = 0$ to 50

PROBLEM SET 11.4

1. **(Calculus review)** Review complex numbers.
2. **(Even and odd functions)** Show that the complex Fourier coefficients of an even function are real and those of an odd function are pure imaginary.
3. **(Fourier coefficients)** Show that $a_0 = c_0,\ a_n = c_n + c_{-n},\ b_n = i(c_n - c_{-n})$.
4. Verify the calculations in Example 1.
5. Find further terms in (9) and graph partial sums with your CAS.
6. Obtain the real series in Example 1 directly from the Euler formulas in Sec. 11.

7–13 COMPLEX FOURIER SERIES

Find the complex Fourier series of the following functions. (Show the details of your work.)

7. $f(x) = -1$ if $-\pi < x < 0$, $f(x) = 1$ if $0 < x < \pi$
8. Convert the series in Prob. 7 to real form.
9. $f(x) = x \quad (-\pi < x < \pi)$
10. Convert the series in Prob. 9 to real form.
11. $f(x) = x^2 \quad (-\pi < x < \pi)$
12. Convert the series in Prob. 11 to real form.
13. $f(x) = x \quad (0 < x < 2\pi)$
14. **PROJECT. Complex Fourier Coefficients.** It is very interesting that the c_n in (6) can be derived directly by a method similar to that for a_n and b_n in Sec. 11.1. For this, multiply the series in (6) by e^{-imx} with fixed integer m, and integrate termwise from $-\pi$ to π on both sides (allowed, for instance, in the case of uniform convergence) to get

$$\int_{-\pi}^{\pi} f(x)e^{-imx}\,dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx.$$

Show that the integral on the right equals 2π when $n = m$ and 0 when $n \neq m$ [use (3b)], so that you get the coefficient formula in (6).

11.5 Forced Oscillations

Fourier series have important applications in connection with ODEs and PDEs. We show this for a basic problem modeled by an ODE. Various applications to PDEs will follow in Chap. 12. This will show the enormous usefulness of Euler's and Fourier's ingenious idea of splitting up periodic functions into the simplest ones possible.

From Sec. 2.8 we know that forced oscillations of a body of mass m on a spring of modulus k are governed by the ODE

$$my'' + cy' + ky = r(t) \tag{1}$$

where $y = y(t)$ is the displacement from rest, c the damping constant, k the spring constant (spring modulus), and $r(t)$ the external force depending on time t. Figure 271 shows the model and Fig. 272 its electrical analog, an RLC-circuit governed by

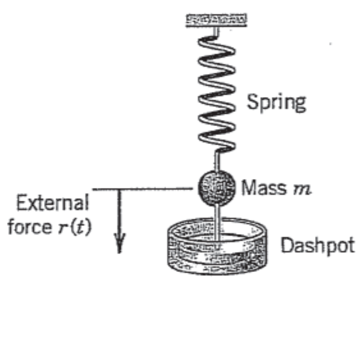


Fig. 271. Vibrating system under consideration

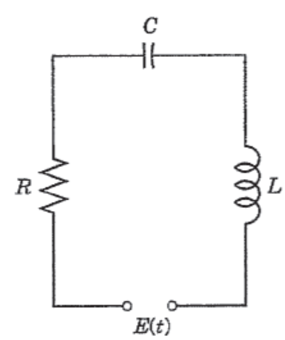


Fig. 272. Electrical analog of the system in Fig. 271 (*RLC*-circuit)

$$LI'' + RI' + \frac{1}{C} I = E'(t) \qquad \text{(Sec. 2.9)}. \tag{1*}$$

We consider (1). If $r(t)$ is a sine or cosine function and if there is damping ($c > 0$), then the steady-state solution is a harmonic oscillation with frequency equal to that of $r(t)$. However, if $r(t)$ is not a pure sine or cosine function but is any other periodic function, then the steady-state solution will be a superposition of harmonic oscillations with frequencies equal to that of $r(t)$ and integer multiples of the latter. And if one of these frequencies is close to the (practical) resonant frequency of the vibrating system (see Sec. 2.8), then the corresponding oscillation may be the dominant part of the response of the system to the external force. This is what the use of Fourier series will show us. Of course, this is quite surprising to an observer unfamiliar with Fourier series, which are highly important in the study of vibrating systems and resonance. Let us discuss the entire situation in terms of a typical example.

EXAMPLE 1 Forced Oscillations under a Nonsinusoidal Periodic Driving Force

In (1), let $m = 1$ (gm), $c = 0.05$ (gm/sec), and $k = 25$ (gm/sec^2), so that (1) becomes

$$y'' + 0.05y' + 25y = r(t) \tag{2}$$

where $r(t)$ is measured in gm · cm/sec^2. Let (Fig. 273)

$$r(t) = \begin{cases} t + \dfrac{\pi}{2} & \text{if } -\pi < t < 0, \\ -t + \dfrac{\pi}{2} & \text{if } \quad 0 < t < \pi, \end{cases} \qquad r(t + 2\pi) = r(t).$$

Find the steady-state solution $y(t)$.

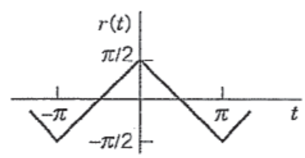


Fig. 273. Force in Example 1

Solution. We represent $r(t)$ by a Fourier series, finding

$$r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right) \tag{3}$$

(take the answer to Prob. 11 in Problem Set 11.3 minus $\frac{1}{2}\pi$ and write t for x). Then we consider the ODE

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \qquad (n = 1, 3, \cdots) \tag{4}$$

whose right side is a single term of the series (3). From Sec. 2.8 we know that the steady-state solution $y_n(t)$ of (4) is of the form

$$y_n = A_n \cos nt + B_n \sin nt. \tag{5}$$

By substituting this into (4) we find that

$$(6)\qquad A_n = \frac{4(25 - n^2)}{n^2 \pi D_n}, \qquad B_n = \frac{0.2}{n\pi D_n}, \qquad \text{where} \qquad D_n = (25 - n^2)^2 + (0.05n)^2.$$

Since the ODE (2) is linear, we may expect the steady-state solution to be

$$(7)\qquad y = y_1 + y_3 + y_5 + \cdots$$

where y_n is given by (5) and (6). In fact, this follows readily by substituting (7) into (2) and using the Fourier series of $r(t)$, provided that termwise differentiation of (7) is permissible. (Readers already familiar with the notion of uniform convergence [Sec. 15.5] may prove that (7) may be differentiated term by term.)

From (6) we find that the amplitude of (5) is (a factor $\sqrt{D_n}$ cancels out)

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2 \pi \sqrt{D_n}}.$$

Numeric values are

$$C_1 = 0.0531$$
$$C_3 = 0.0088$$
$$C_5 = 0.2037$$
$$C_7 = 0.0011$$
$$C_9 = 0.0003.$$

Figure 274 shows the input (multiplied by 0.1) and the output. For $n = 5$ the quantity D_n is very small, the denominator of C_5 is small, and C_5 is so large that y_5 is the dominating term in (7). Hence the output is almost a harmonic oscillation of five times the frequency of the driving force, a little distorted due to the term y_1, whose amplitude is about 25% of that of y_5. You could make the situation still more extreme by decreasing the damping constant c. Try it. ■

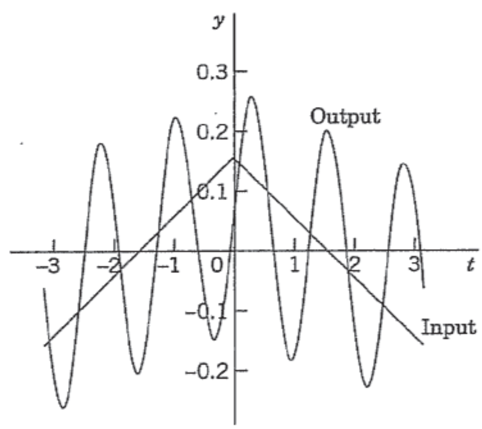


Fig. 274. Input and steady-state output in Example 1

PROBLEM SET 11.5

1. **(Coefficients)** Derive the formula for C_n from A_n and B_n.
2. **(Spring constant)** What would happen to the amplitudes C_n in Example 1 (and thus to the form of the vibration) if we changed the spring constant to the value 9? If we took a stiffer spring with $k = 81$? First guess.
3. **(Damping)** In Example 1 change c to 0.02 and discuss how this changes the output.
4. **(Input)** What would happen in Example 1 if we replaced $r(t)$ with its derivative (the rectangular wave)? What is the ratio of the new C_n to the old ones?

5–11 **GENERAL SOLUTION**

Find a general solution of the ODE $y'' + \omega^2 y = r(t)$ with $r(t)$ as given. (Show the details of your work.)

5. $r(t) = \cos \omega t,\ \omega = 0.5, 0.8, 1.1, 1.5, 5.0, 10.0$

6. $r(t) = \cos \omega_1 t + \cos \omega_2 t \quad (\omega^2 \neq \omega_1{}^2,\ \omega_2{}^2)$

7. $r(t) = \sum_{n=1}^{N} a_n \cos nt, \quad |\omega| \neq 1, 2, \cdots, N$

8. $r(t) = \sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \frac{1}{7} \sin 7t$

9. $r(t) = \begin{cases} t + \pi & \text{if } -\pi < t < 0 \\ -t + \pi & \text{if } 0 < t < \pi \end{cases}$

and $r(t + 2\pi) = r(t),\ |\omega| \neq 0, 1, 3, \cdots$

10. $r(t) = \begin{cases} t & \text{if } -\pi/2 < t < \pi/2 \\ \pi - t & \text{if } \pi/2 < t < 3\pi/2 \end{cases}$

and $r(t + 2\pi) = r(t), \quad |\omega| \neq 1, 3, 5, \cdots$

11. $r(t) = \frac{\pi}{4} |\sin t|$ if $-\pi < t < \pi$ and

$r(t + 2\pi) = r(t), \quad |\omega| \neq 0, 2, 4, \cdots$

12. **(CAS Program)** Write a program for solving the ODE just considered and for jointly graphing input and output of an initial value problem involving that ODE. Apply the program to Probs. 5 and 9 with initial values of your choice.

13. **(Sign of coefficients)** Some A_n in Example 1 are positive and some negative. Is this physically understandable?

14–17 **STEADY-STATE DAMPED OSCILLATIONS**

Find the steady-state oscillation of $y'' + cy' + y = r(t)$ with $c > 0$ and $r(t)$ as given. (Show the details of your work.)

14. $r(t) = a_n \cos nt$

15. $r(t) = \sin 3t$

16. $r(t) = \begin{cases} \pi t & \text{if } -\pi/2 < t < \pi/2 \\ \pi(\pi - t) & \text{if } \pi/2 < t < 3\pi/2 \end{cases}$

and $r(t + 2\pi) = r(t)$

17. $r(t) = \sum_{n=1}^{N} b_n \sin nt$

18. **CAS EXPERIMENT. Maximum Output Term.** Graph and discuss outputs of $y'' + cy' + ky = r(t)$ with $r(t)$ as in Example 1 for various c and k with emphasis on the maximum C_n and its ratio to the second largest $|C_n|$.

19–20 ***RLC*-CIRCUIT**

Find the steady-state current $I(t)$ in the *RLC*-circuit in Fig. 272, where $R = 100\ \Omega$, $L = 10$ H, $C = 10^{-2}$ F and $E(t)$ V as follows and periodic with period 2π. Sketch or graph the first four partial sums. Note that the coefficients of the solution decrease rapidly.

19. $E(t) = 200t(\pi^2 - t^2)\ (-\pi < t < \pi)$

20. $E(t) = \begin{cases} 100(\pi t + t^2) & \text{if } -\pi < t < 0 \\ 100(\pi t - t^2) & \text{if } 0 < t < \pi \end{cases}$

11.6 Approximation by Trigonometric Polynomials

Fourier series play a prominent role in differential equations. Another field in which they have major applications is **approximation theory**, which concerns the approximation of functions by other (usually simpler) functions. In connection with Fourier series the idea is as follows.

Let $f(x)$ be a function on the interval $-\pi \leqq x \leqq \pi$ that can be represented on this interval by a Fourier series. Then the ***N*th partial sum** of the series

$$f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \tag{1}$$

is an approximation of the given $f(x)$. It is natural to ask whether (1) is the "best" approximation of f by a **trigonometric polynomial of degree *N***, that is, by a function of the form

$$F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \qquad (N \text{ fixed}) \tag{2}$$

where "best" means that the "error" of the approximation is as small as possible.

Of course, we must first define what we mean by the **error** E of such an approximation. We could choose the maximum of $|f - F|$. But in connection with Fourier series it is better to choose a definition that measures the goodness of agreement between f and F *on the whole interval* $-\pi \leqq x \leqq \pi$. This seems preferable, in particular if f has jumps: F in Fig. 275 is a good overall approximation of f, but the maximum of $|f - F|$ (more precisely, the *supremum*) is large (it equals at least half the jump of f at x_0). We choose

$$E = \int_{-\pi}^{\pi} (f - F)^2 \, dx, \tag{3}$$

This is called the **square error** of F relative to the function f on the interval $-\pi \leqq x \leqq \pi$. Clearly, $E \geqq 0$.

N being fixed, we want to determine the coefficients in (2) such that E is minimum. Since $(f - F)^2 = f^2 - 2fF + F^2$, we have

$$E = \int_{-\pi}^{\pi} f^2 \, dx - 2 \int_{-\pi}^{\pi} fF \, dx + \int_{-\pi}^{\pi} F^2 \, dx. \tag{4}$$

We square (2), insert it into the last integral in (4), and evaluate the occurring integrals. This gives integrals of $\cos^2 nx$ and $\sin^2 nx$ ($n \geqq 1$), which equal π, and integrals of $\cos nx$, $\sin nx$, and $(\cos nx)(\sin mx)$, which are zero (just as in Sec. 11.1). Thus

$$\int_{-\pi}^{\pi} F^2 \, dx = \int_{-\pi}^{\pi} \left[A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \right]^2 dx$$

$$= \pi(2A_0^2 + A_1^2 + \cdots + A_N^2 + B_1^2 + \cdots + B_N^2).$$

We now insert (2) into the integral of fF in (4). This gives integrals of $f \cos nx$ as well as $f \sin nx$, just as in Euler's formulas, Sec. 11.1, for a_n and b_n (each multiplied by A_n or B_n). Hence

$$\int_{-\pi}^{\pi} fF \, dx = \pi(2A_0 a_0 + A_1 a_1 + \cdots + A_N a_N + B_1 b_1 + \cdots + B_N b_N).$$

With these expressions, (4) becomes

$$\begin{aligned} E = \int_{-\pi}^{\pi} f^2 \, dx &- 2\pi \left[2A_0 a_0 + \sum_{n=1}^{N} (A_n a_n + B_n b_n) \right] \\ &+ \pi \left[2A_0^2 + \sum_{n=1}^{N} (A_n^2 + B_n^2) \right]. \end{aligned} \tag{5}$$

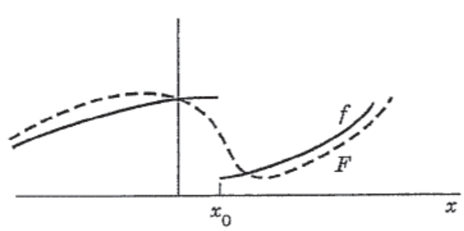


Fig. 275. Error of approximation

We now take $A_n = a_n$ and $B_n = b_n$ in (2). Then in (5) the second line cancels half of the integral-free expression in the first line. Hence for this choice of the coefficients of F the square error, call it E^*, is

$$E^* = \int_{-\pi}^{\pi} f^2\, dx - \pi \left[2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right]. \tag{6}$$

We finally subtract (6) from (5). Then the integrals drop out and we get terms $A_n^2 - 2A_na_n + a_n^2 = (A_n - a_n)^2$ and similar terms $(B_n - b_n)^2$:

$$E - E^* = \pi \left\{ 2(A_0 - a_0)^2 + \sum_{n=1}^{N} \left[(A_n - a_n)^2 + (B_n - b_n)^2 \right] \right\}.$$

Since the sum of squares of real numbers on the right cannot be negative,

$$E - E^* \geqq 0, \qquad \text{thus} \qquad E \geqq E^*,$$

and $E = E^*$ if and only if $A_0 = a_0, \cdots, B_N = b_N$. This proves the following fundamental minimum property of the partial sums of Fourier series.

THEOREM 1

Minimum Square Error

The square error of F in (2) (with fixed N) relative to f on the interval $-\pi \leqq x \leqq \pi$ is minimum if and only if the coefficients of F in (2) are the Fourier coefficients of f. This minimum value E^* is given by (6).

From (6) we see that E^* cannot increase as N increases, but may decrease. Hence *with increasing N the partial sums of the Fourier series of f yield better and better approximations to f*, considered from the viewpoint of the square error.

Since $E^* \geqq 0$ and (6) holds for every N, we obtain from (6) the important **Bessel's inequality**

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leqq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2\, dx \tag{7}$$

for the Fourier coefficients of any function f for which integral on the right exists. (For F. W. Bessel see Sec. 5.5.)

It can be shown (see [C12] in App. 1) that for such a function f, **Parseval's theorem** holds; that is, formula (7) holds with the equality sign, so that it becomes **Parseval's identity**[4]

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2\, dx. \tag{8}$$

[4]MARC ANTOINE PARSEVAL (1755–1836), French mathematician. A physical interpretation of the identity follows in the next section.

EXAMPLE 1 Minimum Square Error for the Sawtooth Wave

Compute the minimum square error E^* of $F(x)$ with $N = 1, 2, \cdots, 10, 20, \cdots, 100$ and 1000 relative to

$$f(x) = x + \pi \qquad (-\pi < x < \pi)$$

on the interval $-\pi \leqq x \leqq \pi$.

Solution. $F(x) = \pi + 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots + \frac{(-1)^{N+1}}{N}\sin Nx)$ by Example 3 in Sec. 11.3. From this and (6),

$$E^* = \int_{-\pi}^{\pi} (x+\pi)^2\,dx - \pi\left(2\pi^2 + 4\sum_{n=1}^{N}\frac{1}{n^2}\right).$$

Numeric values are:

N	E^*	N	E^*	N	E^*	N	E^*
1	8.1045	6	1.9295	20	0.6129	70	0.1782
2	4.9629	7	1.6730	30	0.4120	80	0.1561
3	3.5666	8	1.4767	40	0.3103	90	0.1389
4	2.7812	9	1.3216	50	0.2488	100	0.1250
5	2.2786	10	1.1959	60	0.2077	1000	0.0126

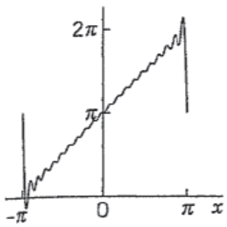


Fig. 276. F with $N = 20$ in Example 1

$F = S_1, S_2, S_3$ are shown in Fig. 266 in Sec. 11.3, and $F = S_{20}$ is shown in Fig. 276. Although $|f(x) - F(x)|$ is large at $\pm\pi$ (how large?), where f is discontinuous, F approximates f quite well on the whole interval, except near $\pm\pi$, where "waves" remain owing to the Gibbs phenomenon (see CAS Experiment 20 in Problem Set 11.2).

Can you think of functions f for which E^* decreases more quickly with increasing N? ■

This is the end of our discussion of Fourier series, which has emphasized the practical aspects of these series, as needed in applications. In the last three sections of this chapter we show how ideas and techniques in Fourier series can be extended to ***nonperiodic*** functions.

PROBLEM SET 11.6

1–9 MINIMUM SQUARE ERROR

Find the trigonometric polynomial $F(x)$ of the form (2) for which the square error with respect to the given $f(x)$ on the interval $-\pi \leqq x \leqq \pi$ is minimum, and compute the minimum value for $N = 1, 2, \cdots, 5$ (or also for larger values if you have a CAS).

1. $f(x) = x \ (-\pi < x < \pi)$
2. $f(x) = x^2 \ (-\pi < x < \pi)$
3. $f(x) = |x| \ (-\pi < x < \pi)$
4. $f(x) = x^3 \ (-\pi < x < \pi)$
5. $f(x) = |\sin x| \ (-\pi < x < \pi)$
6. $f(x) = e^{-|x|} \ (-\pi < x < \pi)$
7. $f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$
8. $f(x) = \begin{cases} x & \text{if } -\frac{1}{2}\pi < x < \frac{1}{2}\pi \\ 0 & \text{if } \frac{1}{2}\pi < x < \frac{3}{2}\pi \end{cases}$
9. $f(x) = x(x + \pi)$ if $-\pi < x < 0$, $f(x) = x(-x + \pi)$ if $0 < x < \pi$

10. CAS EXPERIMENT. Size and Decrease of E^*. Compare the size of the minimum square error E^* for functions of your choice. Find experimentally the factors on which the decrease of E^* with N depends. For each function considered find the smallest N such that $E^* < 0.1$.

11. (Monotonicity) Show that the minimum square error (6) is a monotone decreasing function of N. How can you use this in practice?

12–16 PARSEVAL'S IDENTITY

Using Parseval's identity, prove that the series have the indicated sums. Compute the first few partial sums to see that the convergence is rapid.

12. $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96} = 1.01467\,8032$

(Use Prob. 15 in Sec. 11.1.)

13. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} = 1.23370\,0550$

(Use Prob. 13 in Sec. 11.1.)

14. $\frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \cdots$

$= \frac{\pi^2}{16} - \frac{1}{2} = 0.11685\,0275$

(Use Prob. 5, this set.)

15. $1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} = 1.08232\,3234$

(Use Prob. 21 in Sec. 11.1.)

16. $1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots = \frac{\pi^6}{960} = 1.00144\,7078$

(Use Prob. 9, this set.)

11.7 Fourier Integral

Fourier series are powerful tools for problems involving functions that are periodic or are of interest on a finite interval only. Sections 11.3 and 11.5 first illustrated this, and various further applications follow in Chap. 12. Since, of course, many problems involve functions that are **nonperiodic** *and are of interest on the whole x-axis,* we ask what can be done to extend the method of Fourier series to such functions. This idea will lead to "Fourier integrals."

In Example 1 we start from a special function f_L of period $2L$ and see what happens to its Fourier series if we let $L \to \infty$. Then we do the same for an *arbitrary* function f_L of period $2L$. This will motivate and suggest the main result of this section, which is an integral representation given in Theorem 1 (below).

EXAMPLE 1 Rectangular Wave

Consider the periodic rectangular wave $f_L(x)$ of period $2L > 2$ given by

$$f_L(x) = \begin{cases} 0 & \text{if} \quad -L < x < -1 \\ 1 & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < L. \end{cases}$$

The left part of Fig. 277 shows this function for $2L = 4, 8, 16$ as well as the nonperiodic function $f(x)$, which we obtain from f_L if we let $L \to \infty$,

$$f(x) = \lim_{L\to\infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We now explore what happens to the Fourier coefficients of f_L as L increases. Since f_L is even, $b_n = 0$ for all n. For a_n the Euler formulas (6), Sec. 11.2, give

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}, \qquad a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{1} \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\,\frac{\sin(n\pi/L)}{n\pi/L}.$$

This sequence of Fourier coefficients is called the **amplitude spectrum** of f_L because $|a_n|$ is the maximum amplitude of the wave $a_n \cos(n\pi x/L)$. Figure 277 shows this spectrum for the periods $2L = 4, 8, 16$. We see that for increasing L these amplitudes become more and more dense on the positive w_n-axis, where $w_n = n\pi/L$. Indeed, for $2L = 4, 8, 16$ we have 1, 3, 7 amplitudes per "half-wave" of the function $(2 \sin w_n)/(Lw_n)$ (dashed in the figure). Hence for $2L = 2^k$ we have $2^{k-1} - 1$ amplitudes per half-wave, so that these amplitudes will eventually be everywhere dense on the positive w_n-axis (and will decrease to zero).

The outcome of this example gives an intuitive impression of what about to expect if we turn from our special function to an arbitrary one, as we shall do next. ∎

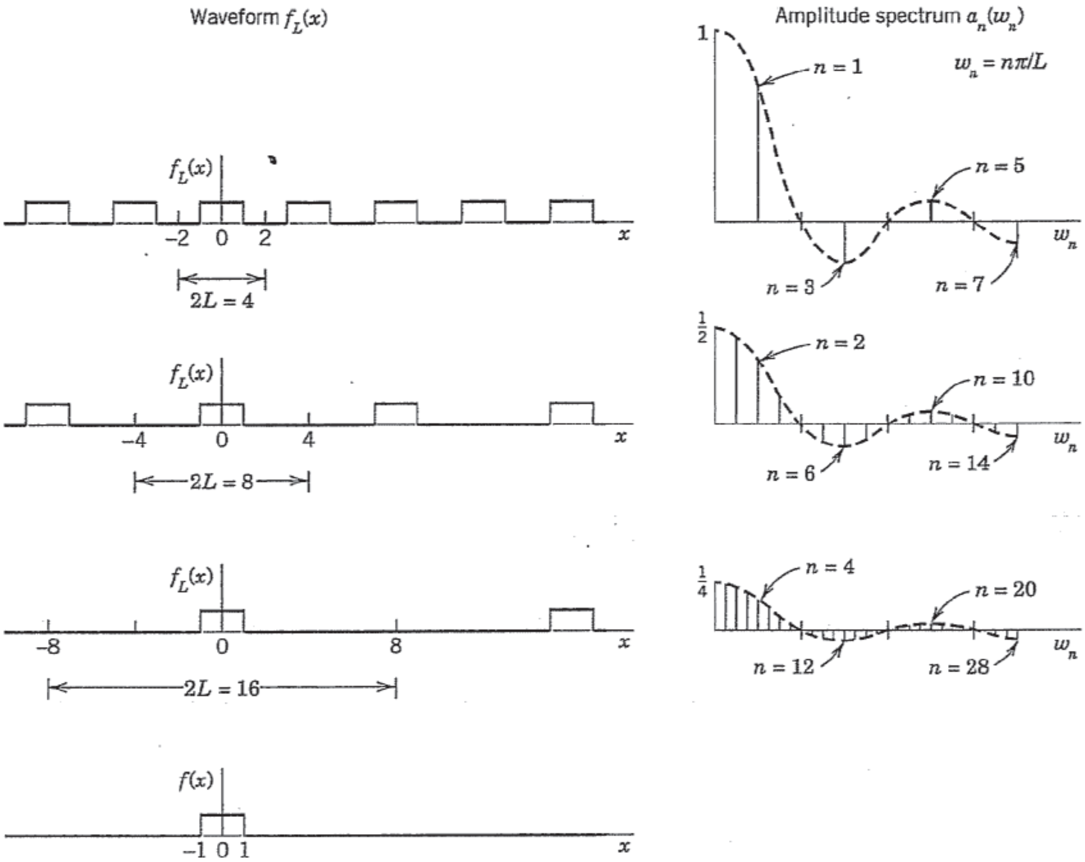


Fig. 277. Waveforms and amplitude spectra in Example 1

From Fourier Series to Fourier Integral

We now consider any periodic function $f_L(x)$ of period $2L$ that can be represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \qquad w_n = \frac{n\pi}{L}$$

and find out what happens if we let $L \to \infty$. Together with Example 1 the present calculation will suggest that we should expect an integral (instead of a series) involving $\cos wx$ and $\sin wx$ with w no longer restricted to integer multiples $w = w_n = n\pi/L$ of π/L but taking *all* values. We shall also see what form such an integral might have.

If we insert a_n and b_n from the Euler formulas (6), Sec. 11.2, and denote the variable of integration by v, the Fourier series of $f_L(x)$ becomes

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v\,dv + \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v\,dv\right].$$

We now set

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}.$$

Then $1/L = \Delta w/\pi$, and we may write the Fourier series in the form

$$(1) \quad f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos w_n x)\,\Delta w \int_{-L}^{L} f_L(v)\cos w_n v\,dv + (\sin w_n x)\,\Delta w \int_{-L}^{L} f_L(v)\sin w_n v\,dv\right].$$

This representation is valid for any fixed L, arbitrarily large, but finite.

We now let $L \to \infty$ and assume that the resulting nonperiodic function

$$f(x) = \lim_{L\to\infty} f_L(x)$$

is **absolutely integrable** on the x-axis; that is, the following (finite!) limits exist:

$$(2) \quad \lim_{a\to-\infty}\int_{a}^{0}|f(x)|\,dx + \lim_{b\to\infty}\int_{0}^{b}|f(x)|\,dx \quad \left(\text{written } \int_{-\infty}^{\infty}|f(x)|\,dx\right).$$

Then $1/L \to 0$, and the value of the first term on the right side of (1) approaches zero. Also $\Delta w = \pi/L \to 0$ and it seems ***plausible*** that the infinite series in (1) becomes an integral from 0 to ∞, which represents $f(x)$, namely,

$$(3) \quad f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv\,dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv\,dv\right] dw.$$

If we introduce the notations

$$(4) \quad A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv, \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$$

we can write this in the form

$$(5) \quad f(x) = \int_{0}^{\infty}[A(w)\cos wx + B(w)\sin wx]\,dw.$$

This is called a representation of $f(x)$ by a **Fourier integral.**

It is clear that our naive approach merely ***suggests*** the representation (5), but by no means establishes it; in fact, the limit of the series in (1) as Δw approaches zero is not the definition of the integral (3). Sufficient conditions for the validity of (5) are as follows.

THEOREM 1

Fourier Integral

If $f(x)$ is piecewise continuous (see Sec. 6.1) *in every finite interval and has a right-hand derivative and a left-hand derivative at every point* (see Sec 11.1) *and if the integral* (2) *exists, then $f(x)$ can be represented by a Fourier integral* (5) *with A and B given by* (4). *At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point* (see Sec. 11.1). (Proof in Ref. [C12]; see App. 1.)

Applications of Fourier Integrals

The main application of Fourier integrals is in solving ODEs and PDEs, as we shall see for PDEs in Sec. 12.6. However, we can also use Fourier integrals in integration and in discussing functions defined by integrals, as the next examples (2 and 3) illustrate.

EXAMPLE 2 Single Pulse, Sine Integral

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases} \qquad \text{(Fig. 278).}$$

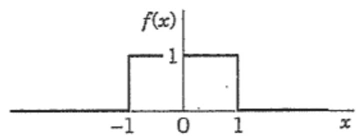


Fig. 278. Example 2

Solution. From (4) we obtain

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv = \frac{1}{\pi} \int_{-1}^{1} \cos wv \, dv = \frac{\sin wv}{\pi w} \Big|_{-1}^{1} = \frac{2 \sin w}{\pi w}$$

$$B(w) = \frac{1}{\pi} \int_{-1}^{1} \sin wv \, dv = 0$$

and (5) gives the answer

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\cos wx \sin w}{w} \, dw. \tag{6}$$

The average of the left- and right-hand limits of $f(x)$ at $x = 1$ is equal to $(1 + 0)/2$, that is, $1/2$.

Furthermore, from (6) and Theorem 1 we obtain (multiply by $\pi/2$)

$$\int_0^{\infty} \frac{\cos wx \sin w}{w} \, dw = \begin{cases} \pi/2 & \text{if} \quad 0 \leqq x < 1, \\ \pi/4 & \text{if} \quad x = 1, \\ 0 & \text{if} \quad x > 1. \end{cases} \tag{7}$$

We mention that this integral is called **Dirichlet's discontinous factor.** (For P. L. Dirichlet see Sec. 10.8.)

The case $x = 0$ is of particular interest. If $x = 0$, then (7) gives

$$\int_0^{\infty} \frac{\sin w}{w} \, dw = \frac{\pi}{2}. \tag{8*}$$

We see that this integral is the limit of the so-called **sine integral**

$$\text{Si}(u) = \int_0^{u} \frac{\sin w}{w} \, dw \tag{8}$$

as $u \to \infty$. The graphs of $\text{Si}(u)$ and of the integrand are shown in Fig. 279.

In the case of a Fourier series the graphs of the partial sums are approximation curves of the curve of the periodic function represented by the series. Similarly, in the case of the Fourier integral (5), approximations are obtained by replacing ∞ by numbers a. Hence the integral

$$\frac{2}{\pi} \int_0^{a} \frac{\cos wx \sin w}{w} \, dw \tag{9}$$

approximates the right side in (6) and therefore $f(x)$.

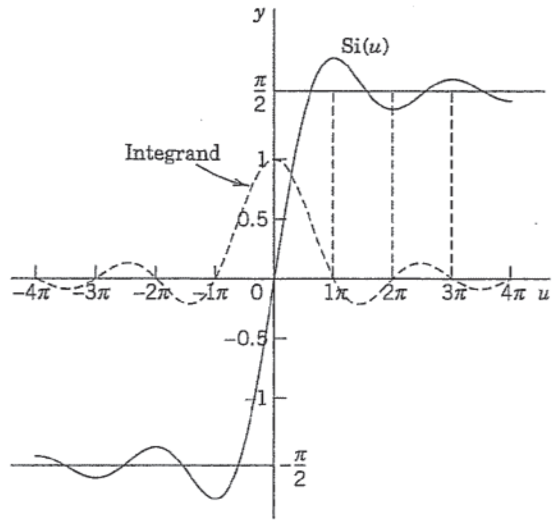


Fig. 279. Sine integral Si(*u*) and integrand

Figure 280 shows oscillations near the points of discontinuity of $f(x)$. We might expect that these oscillations disappear as a approaches infinity. But this is not true; with increasing a, they are shifted closer to the points $x = \pm 1$. This unexpected behavior, which also occurs in connection with Fourier series, is known as the **Gibbs phenomenon.** (See also Problem Set 11.2.) We can explain it by representing (9) in terms of sine integrals as follows. Using (11) in App. A3.1, we have

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_0^a \frac{\sin(w + wx)}{w}\,dw + \frac{1}{\pi}\int_0^a \frac{\sin(w - wx)}{w}\,dw.$$

In the first integral on the right we set $w + wx = t$. Then $dw/w = dt/t$, and $0 \leqq w \leqq a$ corresponds to $0 \leqq t \leqq (x+1)a$. In the last integral we set $w - wx = -t$. Then $dw/w = dt/t$, and $0 \leqq w \leqq a$ corresponds to $0 \leqq t \leqq (x-1)a$. Since $\sin(-t) = -\sin t$, we thus obtain

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_0^{(x+1)a} \frac{\sin t}{t}\,dt - \frac{1}{\pi}\int_0^{(x-1)a} \frac{\sin t}{t}\,dt.$$

From this and (8) we see that our integral (9) equals

$$\frac{1}{\pi}\,\mathrm{Si}(a[x+1]) - \frac{1}{\pi}\,\mathrm{Si}(a[x-1])$$

and the oscillations in Fig. 280 result from those in Fig. 279. The increase of a amounts to a transformation of the scale on the axis and causes the shift of the oscillations (the waves) toward the points of discontinuity -1 and 1.

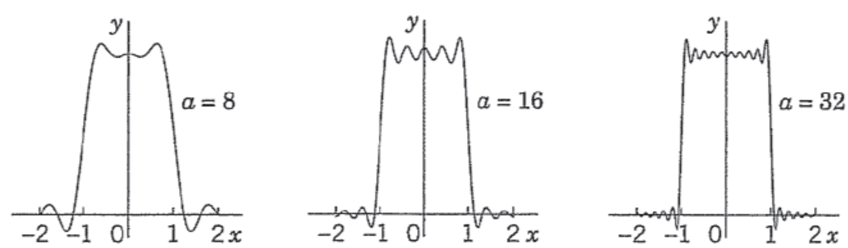


Fig. 280. The integral (9) for a = 8, 16, and 32

Fourier Cosine Integral and Fourier Sine Integral

For an even or odd function the Fourier integral becomes simpler. Just as in the case of Fourier series (Sec. 11.3), this is of practical interest in saving work and avoiding errors. The simplifications follow immediately from the formulas just obtained.

Indeed, if $f(x)$ is an ***even*** function, then $B(w) = 0$ in (4) and

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\, dv. \tag{10}$$

The Fourier integral (5) then reduces to the **Fourier cosine integral**

$$f(x) = \int_0^\infty A(w)\cos wx\, dw \qquad (f \text{ even}). \tag{11}$$

Similarly, if $f(x)$ is ***odd,*** then in (4) we have $A(w) = 0$ and

$$B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin wv\, dv. \tag{12}$$

The Fourier integral (5) then reduces to the **Fourier sine integral**

$$f(x) = \int_0^\infty B(w)\sin wx\, dw \qquad (f \text{ odd}). \tag{13}$$

Evaluation of Integrals

Earlier in this section we pointed out that the main application of the Fourier integral is in differential equations but that Fourier integral representations also help in evaluating certain integrals. To see this, we show the method for an important case, the Laplace integrals.

EXAMPLE 3 Laplace Integrals

We shall derive the Fourier cosine and Fourier sine integrals of $f(x) = e^{-kx}$, where $x > 0$ and $k > 0$ (Fig. 281). The result will be used to evaluate the so-called Laplace integrals.

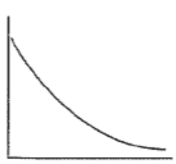


Fig. 281. $f(x)$ in Example 3

Solution. (a) From (10) we have $A(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv\, dv$. Now, by integration by parts,

$$\int e^{-kv}\cos wv\, dv = -\frac{k}{k^2+w^2}\, e^{-kv}\left(-\frac{w}{k}\sin wv + \cos wv\right).$$

If $v = 0$, the expression on the right equals $-k/(k^2 + w^2)$. If v approaches infinity, that expression approaches zero because of the exponential factor. Thus

$$A(w) = \frac{2k/\pi}{k^2+w^2}. \tag{14}$$

By substituting this into (11) we thus obtain the Fourier cosine integral representation

$$f(x) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}\, dw \qquad (x > 0,\quad k > 0).$$

From this representation we see that

$$\int_0^\infty \frac{\cos wx}{k^2 + w^2}\, dw = \frac{\pi}{2k} e^{-kx} \qquad (x > 0, \quad k > 0). \tag{15}$$

(b) Similarly, from (12) we have $B(w) = \frac{2}{\pi}\int_0^\infty e^{-kv} \sin wv\, dv$. By integration by parts,

$$\int e^{-kv} \sin wv\, dv = -\frac{w}{k^2 + w^2} e^{-kv}\left(\frac{k}{w}\sin wv + \cos wv\right).$$

This equals $-w/(k^2 + w^2)$ if $v = 0$, and approaches 0 as $v \to \infty$. Thus

$$B(w) = \frac{2w/\pi}{k^2 + w^2}. \tag{16}$$

From (13) we thus obtain the Fourier sine integral representation

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{w \sin wx}{k^2 + w^2}\, dw.$$

From this we see that

$$\int_0^\infty \frac{w \sin wx}{k^2 + w^2}\, dw = \frac{\pi}{2} e^{-kx} \qquad (x > 0, \quad k > 0). \tag{17}$$

The integrals (15) and (17) are called the **Laplace integrals.**

PROBLEM SET 11.7

1–6 EVALUATION OF INTEGRALS

Show that the given integral represents the indicated function. *Hint.* Use (5), (11), or (13); the integral tells you which one, and its value tells you what function to consider. (Show the details of your work.)

1. $$\int_0^\infty \frac{\cos xw + w \sin xw}{1 + w^2}\, dw = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

2. $$\int_0^\infty \frac{\sin w - w \cos w}{w^2} \sin xw\, dw = \begin{cases} \pi x/2 & \text{if } 0 < x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

3. $$\int_0^\infty \frac{\cos xw}{1 + w^2}\, dw = \frac{\pi}{2} e^{-x} \text{ if } x > 0$$

4. $$\int_0^\infty \frac{\sin w}{w} \cos xw\, dw = \begin{cases} \pi/2 & \text{if } 0 \leqq x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

5. $$\int_0^\infty \frac{\cos(\pi w/2)}{1 - w^2} \cos xw\, dw = \begin{cases} \frac{\pi}{2}\cos x & \text{if } 0 < |x| < \pi/2 \\ 0 & \text{if } |x| \geqq \pi/2 \end{cases}$$

6. $$\int_0^\infty \frac{\sin \pi w \sin xw}{1 - w^2}\, dw = \begin{cases} \frac{\pi}{2}\sin x & \text{if } 0 \leqq x \leqq \pi \\ 0 & \text{if } x > \pi \end{cases}$$

7–12 FOURIER COSINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (11).

7. $$f(x) = \begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$$

8. $f(x) = \begin{cases} x^2 & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases}$

9. $f(x) = \begin{cases} x & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$

10. $f(x) = \begin{cases} x/2 & \text{if} \quad 0 < x < 1 \\ 1 - x/2 & \text{if} \quad 1 < x < 2 \\ 0 & \text{if} \quad x > 2 \end{cases}$

11. $f(x) = \begin{cases} \sin x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$

12. $f(x) = \begin{cases} e^{-x} & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases}$

13. CAS EXPERIMENT. Approximate Fourier Cosine Integrals. Graph the integrals in Prob. 7, 9, and 11 as functions of x. Graph approximations obtained by replacing ∞ with finite upper limits of your choice. Compare the quality of the approximations. Write a short report on your empirical results and observations.

14–19 FOURIER SINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (13).

14. $f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases}$

15. $f(x) = \begin{cases} \sin x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$

16. $f(x) = \begin{cases} 1 - x^2 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$

17. $f(x) = \begin{cases} \pi - x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$

18. $f(x) = \begin{cases} \cos x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$

19. $f(x) = \begin{cases} a - x & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases}$

20. PROJECT. Properties of Fourier Integrals
(a) Fourier cosine integral. Show that (11) implies

(a1) $$f(ax) = \frac{1}{a}\int_0^\infty A\left(\frac{w}{a}\right)\cos xw\, dw$$

$(a > 0)$ *(Scale change)*

(a2) $$xf(x) = \int_0^\infty B^*(w)\sin xw\, dw,$$

$$B^* = -\frac{dA}{dw}, \qquad A \text{ as in (10)}$$

(a3) $$x^2 f(x) = \int_0^\infty A^*(w)\cos xw\, dw,$$

$$A^* = -\frac{d^2A}{dw^2}.$$

(b) Solve Prob. 8 by applying (a3) to the result of Prob. 7.

(c) Verify (a2) for $f(x) = 1$ if $0 < x < a$ and $f(x) = 0$ if $x > a$.

(d) Fourier sine integral. Find formulas for the Fourier sine integral similar to those in (a).

11.8 Fourier Cosine and Sine Transforms

An **integral transform** is a transformation in the form of an integral that produces from given functions new functions depending on a different variable. These transformations are of interest mainly as tools for solving ODEs, PDEs, and integral equations, and they often also help in handling and applying special functions. The **Laplace transform** (Chap. 6) is of this kind and is by far the most important integral transform in engineering.

The next in order of importance are Fourier transforms. We shall see that these transforms can be obtained from the Fourier integral in Sec. 11.7 in a rather simple fashion. In this section we consider two of them, which are real, and in the next section a third one that is complex.

Fourier Cosine Transform

For an ***even*** function $f(x)$, the Fourier integral is the Fourier cosine integral

(1) (a) $f(x) = \int_0^\infty A(w) \cos wx \, dw,$ where (b) $A(w) = \frac{2}{\pi}\int_0^\infty f(v) \cos wv \, dv$

[see (10), (11), Sec. 11.7]. We now set $A(w) = \sqrt{2/\pi}\, \hat{f}_c(w)$, where c suggests "cosine." Then from (1b), writing $v = x$, we have

(2)
$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

and from (1a),

(3)
$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw.$$

ATTENTION! In (2) we integrate with respect to x and in (3) with respect to w. Formula (2) gives from $f(x)$ a new function $\hat{f}_c(w)$, called the **Fourier cosine transform** of $f(x)$. Formula (3) gives us back $f(x)$ from $\hat{f}_c(w)$, and we therefore call $f(x)$ the **inverse Fourier cosine transform** of $\hat{f}_c(w)$.

The process of obtaining the transform $\hat{f}_c$ from a given f is also called the **Fourier cosine transform** or the *Fourier cosine transform method.*

Fourier Sine Transform

Similarly, for an ***odd*** function $f(x)$, the Fourier integral is the Fourier sine integral [see (12), (13), Sec. 11.7]

(4) (a) $f(x) = \int_0^\infty B(w) \sin wx \, dw,$ where (b) $B(w) = \frac{2}{\pi}\int_0^\infty f(v) \sin wv \, dv.$

We now set $B(w) = \sqrt{2/\pi}\, \hat{f}_s(w)$, where s suggests "sine." Then from (4b), writing $v = x$, we have

(5)
$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx.$$

This is called the **Fourier sine transform** of $f(x)$. Similarly, from (4a) we have

(6)
$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dw.$$

This is called the **inverse Fourier sine transform** of $\hat{f}_s(w)$. The process of obtaining $\hat{f}_s(w)$ from $f(x)$ is also called the **Fourier sine transform** or the *Fourier sine transform method.*

Other notations are

$$\mathcal{F}_c(f) = \hat{f}_c, \qquad \mathcal{F}_s(f) = \hat{f}_s$$

and $\mathcal{F}_c^{-1}$ and $\mathcal{F}_s^{-1}$ for the inverses of $\mathcal{F}_c$ and $\mathcal{F}_s$, respectively.

EXAMPLE 1 **Fourier Cosine and Fourier Sine Transforms**

Find the Fourier cosine and Fourier sine transforms of the function

$$f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases} \qquad \text{(Fig. 282).}$$

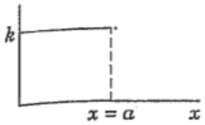


Fig. 282. $f(x)$ in Example 1

Solution. From the definitions (2) and (5) we obtain by integration

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\, k \int_0^a \cos wx\, dx = \sqrt{\frac{2}{\pi}}\, k \left(\frac{\sin aw}{w}\right)$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\, k \int_0^a \sin wx\, dx = \sqrt{\frac{2}{\pi}}\, k \left(\frac{1 - \cos aw}{w}\right).$$

This agrees with formulas 1 in the first two tables in Sec. 11.10 (where $k = 1$).
Note that for $f(x) = k = const$ $(0 < x < \infty)$, these transforms do not exist. (Why?) ■

EXAMPLE 2 **Fourier Cosine Transform of the Exponential Function**

Find $\mathcal{F}_c(e^{-x})$.

Solution. By integration by parts and recursion,

$$\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos wx\, dx = \sqrt{\frac{2}{\pi}}\, \frac{e^{-x}}{1 + w^2}(-\cos wx + w \sin wx)\Bigg|_0^\infty = \frac{\sqrt{2/\pi}}{1 + w^2}.$$

This agrees with formula 3 in Table I, Sec. 11.10, with $a = 1$. See also the next example. ■

What did we do to introduce the two integral transforms under consideration? Actually not much: We changed the notations A and B to get a "symmetric" distribution of the constant $2/\pi$ in the original formulas (10)–(13), Sec. 11.7. This redistribution is a standard convenience, but it is not essential. One could do without it.

What have we gained? We show next that these transforms have operational properties that permit them to convert differentiations into algebraic operations (just as the Laplace transform does). This is the key to their application in solving differential equations.

Linearity, Transforms of Derivatives

If $f(x)$ is absolutely integrable (see Sec. 11.7) on the positive x-axis and piecewise continuous (see Sec. 6.1) on every finite interval, then the Fourier cosine and sine transforms of f exist.

Furthermore, if f and g have Fourier cosine and sine transforms, so does $af + bg$ for any constants a and b, and by (2),

$$\begin{aligned} \mathcal{F}_c(af + bg) &= \sqrt{\frac{2}{\pi}} \int_0^\infty [af(x) + bg(x)] \cos wx\, dx \\ &= a\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx\, dx + b\sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos wx\, dx. \end{aligned}$$

The right side is $a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$. Similarly for $\mathcal{F}_s$, by (5). This shows that the Fourier cosine and sine transforms are **linear operations,**

$$\text{(7)} \quad \begin{aligned} &\text{(a)} \quad \mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g), \\ &\text{(b)} \quad \mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g). \end{aligned}$$

THEOREM 1

Cosine and Sine Transforms of Derivatives

Let $f(x)$ be continuous and absolutely integrable on the x-axis, let $f'(x)$ be piecewise continuous on every finite interval, and let let $f(x) \to 0$ *as* $x \to \infty$. *Then*

$$\text{(8)} \quad \begin{aligned} &\text{(a)} \quad \mathcal{F}_c\{f'(x)\} = w\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f(0), \\ &\text{(b)} \quad \mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\}. \end{aligned}$$

PROOF This follows from the definitions by integration by parts, namely,

$$\begin{aligned} \mathcal{F}_c\{f'(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos wx \, dx \\ &= \sqrt{\frac{2}{\pi}} \left[f(x) \cos wx \Big|_0^\infty + w \int_0^\infty f(x) \sin wx \, dx \right] \\ &= -\sqrt{\frac{2}{\pi}}\, f(0) + w\mathcal{F}_s\{f(x)\}; \end{aligned}$$

and similarly,

$$\begin{aligned} \mathcal{F}_s\{f'(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin wx \, dx \\ &= \sqrt{\frac{2}{\pi}} \left[f(x) \sin wx \Big|_0^\infty - w \int_0^\infty f(x) \cos wx \, dx \right] \\ &= 0 - w\mathcal{F}_c\{f(x)\}. \end{aligned}$$

■

Formula (8a) with f' instead of f gives (when f', f'' satisfy the respective assumptions for f, f' in Theorem 1)

$$\mathcal{F}_c\{f''(x)\} = w\mathcal{F}_s\{f'(x)\} - \sqrt{\frac{2}{\pi}}\, f'(0);$$

hence by (8b)

$$\text{(9a)} \quad \mathcal{F}_c\{f''(x)\} = -w^2\mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f'(0).$$

Similarly,

$$\text{(9b)} \quad \mathcal{F}_s\{f''(x)\} = -w^2\mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}}\, wf(0).$$

A basic application of (9) to PDEs will be given in Sec. 12.6. For the time being we show how (9) can be used for deriving transforms.

EXAMPLE 3 An Application of the Operational Formula (9)

Find the Fourier cosine transform $\mathcal{F}_c(e^{-ax})$ of $f(x) = e^{-ax}$, where $a > 0$.

Solution. By differentiation, $(e^{-ax})'' = a^2e^{-ax}$; thus

$$a^2f(x) = f''(x).$$

From this, (9a), and the linearity (7a),

$$\begin{aligned} a^2\mathcal{F}_c(f) &= \mathcal{F}_c(f'') \\ &= -w^2\mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}}\, f'(0) \\ &= -w^2\mathcal{F}_c(f) + a\sqrt{\frac{2}{\pi}}\,. \end{aligned}$$

Hence

$$(a^2 + w^2)\mathcal{F}_c(f) = a\sqrt{2/\pi}.$$

The *answer* is (see Table I, Sec. 11.10)

$$\mathcal{F}_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2 + w^2}\right) \qquad (a > 0).$$

Tables of Fourier cosine and sine transforms are included in Sec. 11.10.

PROBLEM SET 11.8

1–10 FOURIER COSINE TRANSFORM

1. Let $f(x) = -1$ if $0 < x < 1$, $f(x) = 1$ if $1 < x < 2$, $f(x) = 0$ if $x > 2$. Find $\hat{f}_c(w)$.
2. Let $f(x) = x$ if $0 < x < k$, $f(x) = 0$ if $x > k$. Find $\hat{f}_c(w)$.
3. Derive formula 3 in Table 1 of Sec. 11.10 by integration.
4. Find the inverse Fourier cosine transform $f(x)$ from the answer to Prob. 1. *Hint.* Use Prob. 4 in Sec. 11.7.
5. Obtain $\mathcal{F}_c^{-1}(1/(1 + w^2))$ from Prob. 3 in Sec. 11.7.
6. Obtain $\mathcal{F}_c^{-1}(e^{-w})$ by integration.
7. Find $\mathcal{F}_c((1 - x^2)^{-1} \cos(\pi x/2))$. *Hint.* Use Prob. 5 in Sec. 11.7.
8. Let $f(x) = x^2$ if $0 < x < 1$ and 0 if $x > 1$. Find $\mathcal{F}_c(f)$.
9. Does the Fourier cosine transform of $x^{-1} \sin x$ exist? Of $x^{-1} \cos x$? Give reasons.
10. $f(x) = 1$ $(0 < x < \infty)$ has no Fourier cosine or sine transform. Give reasons.

11–20 FOURIER SINE TRANSFORM

11. Find $\mathcal{F}_s(e^{-\pi x})$ by integration.
12. Find the answer to Prob. 11 from (9b).
13. Obtain formula 8 in Table II of Sec. 11.11 from (8b) and a suitable formula in Table I.
14. Let $f(x) = \sin x$ if $0 < x < \pi$ and 0 if $x > \pi$. Find $\mathcal{F}_s(f)$. Compare with Prob. 6 in Sec. 11.7. Comment.
15. In Table II of Sec. 11.10 obtain formula 2 from formula 4, using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ [(30) in App. 3.1].
16. Show that $\mathcal{F}_s(x^{-1/2}) = w^{-1/2}$ by setting $wx = t^2$ and using $S(\infty) = \sqrt{\pi/8}$ in (38) of App. 3.1.
17. Obtain $\mathcal{F}_s(e^{-ax})$ from (8a) and formula 3 in Table I of Sec. 11.10.
18. Show that $\mathcal{F}_s(x^{-3/2}) = 2w^{1/2}$. *Hint.* Set $wx = t^2$, integrate by parts, and use $C(\infty) = \sqrt{\pi/8}$ in (38) of App. 3.1.
19. **(Scale change)** Using the notation of (5), show that $f(ax)$ has the Fourier sine transform $(1/a)\hat{f}_s(w/a)$.
20. **WRITING PROJECT. Obtaining Fourier Cosine and Sine Transforms.** Write a short report on ways of obtaining these transforms, giving illustrations with examples of your own.

11.9 Fourier Transform. Discrete and Fast Fourier Transforms

The two transforms in the last section are real. We now consider a third one, called the **Fourier transform,** which is complex. We shall obtain this transform from the complex Fourier integral, which we explain first.

Complex Form of the Fourier Integral

The (real) Fourier integral is [see (4), (5), Sec. 11.7]

$$f(x) = \int_0^\infty [A(w) \cos wx + B(w) \sin wx]\, dw$$

where

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv\, dv, \qquad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv\, dv.$$

Substituting A and B into the integral for f, we have

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(v)\, [\cos wv \cos wx + \sin wv \sin wx]\, dv\, dw.$$

By the addition formula for the cosine [(6) in App. A3.1] the expression in the brackets $[\cdots]$ equals $\cos(wv - wx)$ or, since the cosine is even, $\cos(wx - wv)$. We thus obtain

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^{\infty} f(v) \cos(wx - wv)\, dv \right] dw. \tag{1*}$$

The integral in brackets is an *even* function of w, call it $F(w)$, because $\cos(wx - wv)$ is an even function of w, the function f does not depend on w, and we integrate with respect to v (not w). Hence the integral of $F(w)$ from $w = 0$ to ∞ is $1/2$ times the integral of $F(w)$ from $-\infty$ to ∞. Thus (note the change of the integration limit!)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) \cos(wx - wv)\, dv \right] dw. \tag{1}$$

We claim that the integral of the form (1) with sin instead of cos is zero:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) \sin(wx - wv)\, dv \right] dw = 0. \tag{2}$$

This is true since $\sin(wx - wv)$ is an odd function of w, which makes the integral in brackets an odd function of w, call it $G(w)$. Hence the integral of $G(w)$ from $-\infty$ to ∞ is zero, as claimed.

We now take the integrand of (1) plus i $(= \sqrt{-1})$ times the integrand of (2) and use the **Euler formula** [(11) in Sec. 2.2]

$$e^{ix} = \cos x + i \sin x. \tag{3}$$

Taking $wx - wv$ instead of x in (3) and multiplying by $f(v)$ gives

$$f(v)\cos(wx - wv) + if(v)\sin(wx - wv) = f(v)e^{i(wx-wv)}.$$

Hence the result of adding (1) plus i times (2), called the **complex Fourier integral,** is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)e^{iw(x-v)}\,dv\,dw \qquad (i = \sqrt{-1}). \tag{4}$$

It is now only a very short step to our present goal, the Fourier transform.

Fourier Transform and Its Inverse

Writing the exponential function in (4) as a product of exponential functions, we have

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-iwv}\,dv\right]e^{iwx}\,dw. \tag{5}$$

The expression in brackets is a function of w, is denoted by $\hat{f}(w)$, and is called the **Fourier transform** of f; writing $v = x$, we have

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx. \tag{6}$$

With this, (5) becomes

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}\,dw \tag{7}$$

and is called the **inverse Fourier transform** of $\hat{f}(w)$.

Another notation for the Fourier transform is

$$\hat{f} = \mathcal{F}(f),$$

so that

$$f = \mathcal{F}^{-1}(\hat{f}).$$

The process of obtaining the Fourier transform $\mathcal{F}(f) = \hat{f}$ from a given f is also called the **Fourier transform** or the *Fourier transform method.*

Conditions sufficient for the existence of the Fourier transform (involving concepts defined in Secs. 6.1 and 11.7) are as follows, as we state without proof.

THEOREM 1

Existence of the Fourier Transform

If $f(x)$ is absolutely integrable on the x-axis and piecewise continuous on every finite interval, then the Fourier transform $\hat{f}(w)$ of $f(x)$ given by (6) exists.

EXAMPLE 1 **Fourier Transform**

Find the Fourier transform of $f(x) = 1$ if $|x| < 1$ and $f(x) = 0$ otherwise.

Solution. Using (6) and integrating, we obtain

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-iwx}}{-iw} \bigg|_{-1}^{1} = \frac{1}{-iw\sqrt{2\pi}} (e^{-iw} - e^{iw}).$$

As in (3) we have $e^{iw} = \cos w + i \sin w$, $e^{-iw} = \cos w - i \sin w$, and by subtraction

$$e^{iw} - e^{-iw} = 2i \sin w.$$

Substituting this in the previous formula on the right, we see that i drops out and we obtain the answer

$$\hat{f}(w) = \sqrt{\frac{\pi}{2}}\, \frac{\sin w}{w}.$$

EXAMPLE 2 **Fourier Transform**

Find the Fourier transform $\mathcal{F}(e^{-ax})$ of $f(x) = e^{-ax}$ if $x > 0$ and $f(x) = 0$ if $x < 0$; here $a > 0$.

Solution. From the definition (6) we obtain by integration

$$\mathcal{F}(e^{-ax}) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, \frac{e^{-(a+iw)x}}{-(a+iw)} \bigg|_{x=0}^{\infty} = \frac{1}{\sqrt{2\pi}\,(a+iw)}.$$

This proves formula 5 of Table III in Sec. 11.10.

Physical Interpretation: Spectrum

The nature of the representation (7) of $f(x)$ becomes clear if we think of it as a superposition of sinusoidal oscillations of all possible frequencies, called a **spectral representatio**
This name is suggested by optics, where light is such a superposition of colo (frequencies). In (7), the **"spectral density"** $\hat{f}(w)$ measures the intensity of $f(x)$ in the frequency interval between w and $w + \Delta w$ (Δw small, fixed). We claim that in connectic with vibrations, the integral

$$\int_{-\infty}^{\infty} |\hat{f}(w)|^2\, dw$$

can be interpreted as the **total energy** of the physical system. Hence an integral of $|\hat{f}(w)|^2$ from a to b gives the contribution of the frequencies w between a and b to the total energ

To make this plausible, we begin with a mechanical system giving a single frequenc namely, the harmonic oscillator (mass on a spring, Sec. 2.4)

$$my'' + ky = 0.$$

Here we denote time t by x. Multiplication by y' gives $my'y'' + ky'y = 0$. By integration

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}ky^2 = E_0 = const$$

where $v = y'$ is the velocity. The first term is the kinetic energy, the second the potenti energy, and E_0 the total energy of the system. Now a general solution is (use (3) ... Sec. 11.4 with $t = x$)

$$y = a_1 \cos w_0 x + b_1 \sin w_0 x = c_1 e^{iw_0 x} + c_{-1} e^{-iw_0 x}, \qquad {w_0}^2 = k/m$$

where $c_1 = (a_1 - ib_1)/2$, $c_{-1} = \bar{c}_1 = (a_1 + ib_1)/2$. We write simply $A = c_1 e^{iw_0 x}$, $B = c_{-1} e^{-iw_0 x}$. Then $y = A + B$. By differentiation, $v = y' = A' + B' = iw_0(A - B)$. Substitution of v and y on the left side of the equation for E_0 gives

$$E_0 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}ky^2 = \tfrac{1}{2}m(iw_0)^2(A - B)^2 + \tfrac{1}{2}k(A + B)^2.$$

Here ${w_0}^2 = k/m$, as just stated; hence $m{w_0}^2 = k$. Also $i^2 = -1$, so that

$$E_0 = \tfrac{1}{2}k[-(A - B)^2 + (A + B)^2] = 2kAB = 2kc_1 e^{iw_0 x} c_{-1} e^{-iw_0 x} = 2kc_1 c_{-1} = 2k|c_1|^2.$$

Hence *the energy is proportional to the square of the amplitude* $|c_1|$.

As the next step, if a more complicated system leads to a periodic solution $y = f(x)$ that can be represented by a Fourier series, then instead of the single energy term $|c_1|^2$ we get a series of squares $|c_n|^2$ of Fourier coefficients c_n given by (6), Sec. 11.4. In this case we have a **"discrete spectrum"** (or **"point spectrum"**) consisting of countably many isolated frequencies (infinitely many, in general), the corresponding $|c_n|^2$ being the contributions to the total energy.

Finally, a system whose solution can be represented by an integral (7) leads to the above integral for the energy, as is plausible from the cases just discussed.

Linearity. Fourier Transform of Derivatives

New transforms can be obtained from given ones by

THEOREM 2

Linearity of the Fourier Transform

The Fourier transform is a **linear operation;** *that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants a and b, the Fourier transform of $af + bg$ exists, and*

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g). \tag{8}$$

PROOF This is true because integration is a linear operation, so that (6) gives

$$\begin{aligned}\mathcal{F}\{af(x) + bg(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{-iwx}\, dx \\ &= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-iwx}\, dx \\ &= a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}.\end{aligned}$$

In applying the Fourier transform to differential equations, the key property is that differentiation of functions corresponds to multiplication of transforms by iw:

THEOREM 3

Fourier Transform of the Derivative of $f(x)$

Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

$$\mathcal{F}\{f'(x)\} = iw\mathcal{F}\{f(x)\}. \tag{9}$$

PROOF From the definition of the Fourier transform we have

$$\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)e^{-iwx}\, dx.$$

Integrating by parts, we obtain

$$\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \left[f(x)e^{-iwx} \Big|_{-\infty}^{\infty} - (-iw) \int_{-\infty}^{\infty} f(x)e^{-iwx}\, dx \right].$$

Since $f(x) \to 0$ as $|x| \to \infty$, the desired result follows, namely,

$$\mathcal{F}\{f'(x)\} = 0 + iw\mathcal{F}\{f(x)\}.$$

Two successive applications of (9) give

$$\mathcal{F}(f'') = iw\mathcal{F}(f') = (iw)^2\mathcal{F}(f).$$

Since $(iw)^2 = -w^2$, we have for the transform of the second derivative of f

$$\mathcal{F}\{f''(x)\} = -w^2\mathcal{F}\{f(x)\}. \tag{10}$$

Similarly for higher derivatives.

An application of (10) to differential equations will be given in Sec. 12.6. For the time being we show how (9) can be used to derive transforms.

EXAMPLE 3 **Application of the Operational Formula (9)**

Find the Fourier transform of xe^{-x^2} from Table III, Sec 11.10.

Solution. We use (9). By formula 9 in Table III.

$$\begin{aligned}
\mathcal{F}(xe^{-x^2}) &= \mathcal{F}\left\{-\frac{1}{2}\left(e^{-x^2}\right)'\right\} \\
&= -\frac{1}{2}\,\mathcal{F}\left\{\left(e^{-x^2}\right)'\right\} \\
&= -\frac{1}{2}\, iw\mathcal{F}(e^{-x^2}) \\
&= -\frac{1}{2}\, iw\, \frac{1}{\sqrt{2}}\, e^{-w^2/4} \\
&= -\frac{iw}{2\sqrt{2}}\, e^{-w^2/4}.
\end{aligned}$$

Convolution

The **convolution** $f * g$ of functions f and g is defined by

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)\,dp = \int_{-\infty}^{\infty} f(x-p)g(p)\,dp. \tag{11}$$

The purpose is the same as in the case of Laplace transforms (Sec. 6.5): taking the convolution of two functions and then taking the transform of the convolution is the same as multiplying the transforms of these functions (and multiplying them by $\sqrt{2\pi}$):

THEOREM 4

Convolution Theorem

Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded, and absolutely integrable on the x-axis. Then

$$\mathcal{F}(f * g) = \sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(g). \tag{12}$$

PROOF By the definition,

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(x-p)\,dp\, e^{-iwx}\,dx.$$

An interchange of the order of integration gives

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(x-p)e^{-iwx}\,dx\,dp.$$

Instead of x we now take $x - p = q$ as a new variable of integration. Then $x = p + q$ and

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(q)e^{-iw(p+q)}\,dq\,dp.$$

This double integral can be written as a product of two integrals and gives the desired result

$$\begin{aligned}\mathcal{F}(f * g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)e^{-iwp}\,dp \int_{-\infty}^{\infty} g(q)e^{-iwq}\,dq \\ &= \frac{1}{\sqrt{2\pi}} [\sqrt{2\pi}\,\mathcal{F}(f)][\sqrt{2\pi}\,\mathcal{F}(g)] = \sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(g).\end{aligned}$$ ∎

By taking the inverse Fourier transform on both sides of (12), writing $\hat{f} = \mathcal{F}(f)$ and $\hat{g} = \mathcal{F}(g)$ as before, and noting that $\sqrt{2\pi}$ and $1/\sqrt{2\pi}$ in (12) and (7) cancel each other, we obtain

$$(f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(w)\hat{g}(w)e^{iwx}\,dw, \tag{13}$$

a formula that will help us in solving partial differential equations (Sec. 12.6).

Discrete Fourier Transform (DFT), Fast Fourier Transform (FFT)

In using Fourier series, Fourier transforms, and trigonometric approximations (Sec. 11.6) we have to assume that a function $f(x)$, to be developed or transformed, is given on some interval, over which we integrate in the Euler formulas, etc. Now very often a function $f(x)$ is given only in terms of values at finitely many points, and one is interested in extending Fourier analysis to this case. The main application of such a "discrete Fourier analysis" concerns large amounts of equally spaced data, as they occur in telecommunication, time series analysis, and various simulation problems. In these situations, dealing with sampled values rather than with functions, we can replace the Fourier transform by the so-called **discrete Fourier transform (DFT)** as follows.

Let $f(x)$ be periodic, for simplicity of period 2π. We assume that N measurements of $f(x)$ are taken over the interval $0 \leqq x \leqq 2\pi$ at regularly spaced points

$$x_k = \frac{2\pi k}{N}, \qquad k = 0, 1, \cdots, N-1. \tag{14}$$

We also say that $f(x)$ is being **sampled** at these points. We now want to determine a **complex trigonometric polynomial**

$$q(x) = \sum_{n=0}^{N-1} c_n e^{inx_k} \tag{15}$$

that **interpolates** $f(x)$ at the nodes (14), that is, $q(x_k) = f(x_k)$, written out, with f_k denoting $f(x_k)$,

$$f_k = f(x_k) = q(x_k) = \sum_{n=0}^{N-1} c_n e^{inx_k}, \qquad k = 0, 1, \cdots, N-1. \tag{16}$$

Hence we must determine the coefficients $c_0, \cdots, c_{N-1}$ such that (16) holds. We do this by an idea similar to that in Sec. 11.1 for deriving the Fourier coefficients by using the orthogonality of the trigonometric system. Instead of integrals we now take sums. Namely, we multiply (16) by e^{-imx_k} (note the minus!) and sum over k from 0 to $N-1$. Then we interchange the order of the two summations and insert x_k from (14). This gives

$$\sum_{k=0}^{N-1} f_k e^{-imx_k} = \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} c_n e^{i(n-m)x_k} = \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} e^{i(n-m)2\pi k/N}. \tag{17}$$

Now

$$e^{i(n-m)2\pi k/N} = \left[e^{i(n-m)2\pi/N}\right]^k.$$

We donote $[\cdots]$ by r. For $n = m$ we have $r = e^0 = 1$. The sum of *these* terms over k equals N, the number of these terms. For $n \neq m$ we have $r \neq 1$ and by the formula for a geometric sum [(6) in Sec. 15.1 with $q = r$ and $n = N - 1$]

$$\sum_{k=0}^{N-1} r^k = \frac{1 - r^N}{1 - r} = 0$$

because $r^N = 1$; indeed, since k, m, and n are integers,

$$r^N = e^{i(n-m)2\pi k} = \cos 2\pi k(n-m) + i \sin 2\pi k(n-m) = 1 + 0 = 1.$$

This shows that the right side of (17) equals $c_m N$. Writing n for m and dividing by N, we thus obtain the desired coefficient formula

$$\textbf{(18*)} \qquad c_n = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-inx_k} \qquad f_k = f(x_k), \quad n = 0, 1, \cdots, N-1.$$

Since computation of the c_n (by the fast Fourier transform, below) involves successive halfing of the problem size N, it is practical to drop the factor $1/N$ from c_n and define the **discrete Fourier transform** of the given signal $\mathbf{f} = [f_0 \quad \cdots \quad f_{N-1}]^T$ to be the vector $\hat{\mathbf{f}} = [\hat{f}_0 \quad \cdots \quad \hat{f}_{N-1}]$ with components

$$\textbf{(18)} \qquad \hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-inx_k}, \qquad f_k = f(x_k), \quad n = 0, \cdots, N-1.$$

This is the frequency spectrum of the signal.

In vector notation, $\hat{\mathbf{f}} = \mathbf{F}_N \mathbf{f}$, where the $N \times N$ **Fourier matrix** $\mathbf{F}_N = [e_{nk}]$ has the entries [given in (18)]

$$(19) \qquad e_{nk} = e^{-inx_k} = e^{-2\pi ink/N} = w^{nk}, \qquad w = w_N = e^{-2\pi i/N},$$

where $n, k = 0, \cdots, N-1$.

EXAMPLE 4 Discrete Fourier Transform (DFT). Sample of $N = 4$ Values

Let $N = 4$ measurements (sample values) be given. Then $w = e^{-2\pi i/N} = e^{-\pi i/2} = -i$ and thus $w^{nk} = (-i)^{nk}$. Let the sample values be, say $\mathbf{f} = [0 \quad 1 \quad 4 \quad 9]^T$. Then by (18) and (19),

$$(20) \qquad \hat{\mathbf{f}} = \mathbf{F}_4\mathbf{f} = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix}.$$

From the first matrix in (20) it is easy to infer what $\mathbf{F}_N$ looks like for arbitrary N, which in practice may be 1000 or more, for reasons given below. ■

From the DFT (the frequency spectrum) $\hat{\mathbf{f}} = \mathbf{F}_N\mathbf{f}$ we can recreate the given signal $\mathbf{f} = \mathbf{F}_N^{-1}\mathbf{f}$, as we shall now prove. Here $\mathbf{F}_N$ and its complex conjugate $\overline{\mathbf{F}}_N = \frac{1}{N}[\overline{w}^{nk}]$ satisfy

$$(21a) \qquad \overline{\mathbf{F}}_N\mathbf{F}_N = \mathbf{F}_N\overline{\mathbf{F}}_N = N\mathbf{I}$$

where $\mathbf{I}$ is the $N \times N$ unit matrix; hence $\mathbf{F}_N$ has the inverse

$$(21b) \qquad \mathbf{F}_N^{-1} = \frac{1}{N}\overline{\mathbf{F}}_N.$$

PROOF We prove (21). By the multiplication rule (row times column) the product matrix $\mathbf{G}_N = \overline{\mathbf{F}}_N \mathbf{F}_N = [g_{jk}]$ in (21a) has the entries $g_{jk} =$ *Row j of* $\overline{\mathbf{F}}_N$ *times Column k of* $\mathbf{F}_N$. That is, writing $W = \overline{w}^j w^k$, we prove that

$$g_{jk} = (\overline{w}^j w^k)^0 + (\overline{w}^j w^k)^1 + \cdots + (\overline{w}^j w^k)^{N-1}$$

$$= W^0 + W^1 + \cdots + W^{N-1} = \begin{cases} 0 & \text{if} \quad j \neq k \\ N & \text{if} \quad j = k. \end{cases}$$

Indeed, when $j = k$, then $\overline{w}^k w^k = (\overline{w}w)^k = (e^{2\pi i/N} e^{-2\pi i/N})^k = 1^k = 1$, so that the sum of *these* N terms equals N; these are the diagonal entries of $\mathbf{G}_N$. Also, when $j \neq k$, then $W \neq 1$ and we have a geometric sum (whose value is given by (6) in Sec. 15.1 with $q = W$ and $n = N - 1$)

$$W^0 + W^1 + \cdots + W^{N-1} = \frac{1 - W^N}{1 - W} = 0$$

because $W^N = (\overline{w}^j w^k)^N = (e^{2\pi i})^j (e^{-2\pi i})^k = 1^j \cdot 1^k = 1$. ∎

We have seen that $\hat{\mathbf{f}}$ is the frequency spectrum of the signal $f(x)$. Thus the components $\hat{f}_n$ of $\hat{\mathbf{f}}$ give a resolution of the 2π-periodic function $f(x)$ into simple (complex) harmonics. Here one should use only n's that are much smaller than $N/2$, to avoid **aliasing.** By this we mean the effect caused by sampling at too few (equally spaced) points, so that, for instance, in a motion picture, rotating wheels appear as rotating too slowly or even in the wrong sense. Hence in applications, N is usually large. But this poses a problem. Eq. (18) requires $O(N)$ operations for any particular n, hence $O(N^2)$ operations for, say, all $n < N/2$. Thus, already for 1000 sample points the straightforward calculation would involve millions of operations. However, this difficulty can be overcome by the so called **fast Fourier transform (FFT),** for which codes are readily available (e.g. in Maple). The FFT is a computational method for the DFT that needs only $O(N) \log_2 N$ operations instead of $O(N^2)$. It makes the DFT a practical tool for large N. Here one chooses $N = 2^p$ (p integer) and uses the special form of the Fourier matrix to break down the given problem into smaller problems. For instance, when $N = 1000$, those operations are reduced by a factor $1000/\log_2 1000 \approx 100$.

The breakdown produces two problems of size $M = N/2$. This breakdown is possible because for $N = 2M$ we have in (19)

$$w_N{}^2 = w_{2M}{}^2 = (e^{-2\pi i/N})^2 = e^{-4\pi i/(2M)} = e^{-2\pi i/M} = w_M.$$

The given vector $\mathbf{f} = [f_0 \quad \cdots \quad f_{N-1}]^\mathsf{T}$ is split into two vectors with M components each, namely, $\mathbf{f}_{\text{ev}} = [f_0 \quad f_2 \quad \cdots \quad f_{N-2}]^\mathsf{T}$ containing the even components of $\mathbf{f}$, and $\mathbf{f}_{\text{od}} = [f_1 \quad f_3 \quad \cdots \quad f_{N-1}]^\mathsf{T}$ containing the odd components of $\mathbf{f}$. For $\mathbf{f}_{\text{ev}}$ and $\mathbf{f}_{\text{od}}$ we determine the DFTs

$$\hat{\mathbf{f}}_{\text{ev}} = [\hat{f}_{\text{ev},0} \quad \hat{f}_{\text{ev},2} \quad \cdots \quad \hat{f}_{\text{ev},N-2}]^\mathsf{T} = \mathbf{F}_M \mathbf{f}_{\text{ev}}$$

and

$$\hat{\mathbf{f}}_{\text{od}} = [\hat{f}_{\text{od},1} \quad \hat{f}_{\text{od},3} \quad \cdots \quad \hat{f}_{\text{od},N-1}]^\mathsf{T} = \mathbf{F}_M \mathbf{f}_{\text{od}}$$

involving the same $M \times M$ matrix $\mathbf{F}_M$. From these vectors we obtain the components of the DFT of the given vector $\mathbf{f}$ by the formulas

$$\text{(22)} \quad \begin{aligned} &\text{(a)} \quad \hat{f}_n = \hat{f}_{\text{ev},n} + w_N{}^n \hat{f}_{\text{od},n} && n = 0, \cdots, M-1 \\ &\text{(b)} \quad \hat{f}_{n+M} = \hat{f}_{\text{ev},n} - w_N{}^n \hat{f}_{\text{od},n} && n = 0, \cdots, M-1. \end{aligned}$$

For $N = 2^p$ this breakdown can be repeated $p - 1$ times in order to finally arrive at $N/2$ problems of size 2 each, so that the number of multiplications is reduced as indicated above.

We show the reduction from $N = 4$ to $M = N/2 = 2$ and then prove (22).

EXAMPLE 5 Fast Fourier Transform (FFT). Sample of $N = 4$ Values

When $N = 4$, then $w = w_N = -i$ as in Example 4 and $M = N/2 = 2$, hence $w = w_M = e^{-2\pi i/2} = e^{-\pi i} = -1$. Consequently,

$$\hat{\mathbf{f}}_{\text{ev}} = \begin{bmatrix} \hat{f}_0 \\ \hat{f}_2 \end{bmatrix} = \mathbf{F}_2\mathbf{f}_{\text{ev}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_0 + f_2 \\ f_0 - f_2 \end{bmatrix}$$

$$\hat{\mathbf{f}}_{\text{od}} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_3 \end{bmatrix} = \mathbf{F}_2\mathbf{f}_{\text{od}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_1 + f_3 \\ f_1 - f_3 \end{bmatrix}.$$

From this and (22a) we obtain

$$\hat{f}_0 = \hat{f}_{\text{ev},0} + w_N^{\;0}\hat{f}_{\text{od},0} = (f_0 + f_2) + (f_1 + f_3) = f_0 + f_1 + f_2 + f_3$$
$$\hat{f}_1 = \hat{f}_{\text{ev},1} + w_N^{\;1}\hat{f}_{\text{od},1} = (f_0 - f_2) - i(f_1 + f_3) = f_0 - if_1 - f_2 + if_3.$$

Similarly, by (22b),

$$\hat{f}_2 = \hat{f}_{\text{ev},0} - w_N^{\;0}\hat{f}_{\text{od},0} = (f_0 + f_2) - (f_1 + f_3) = f_0 - f_1 + f_2 - f_3$$
$$\hat{f}_3 = \hat{f}_{\text{ev},1} - w_N^{\;1}\hat{f}_{\text{od},1} = (f_0 - f_2) - (-i)(f_1 - f_3) = f_0 + if_1 - f_2 - if_3.$$

This agrees with Example 4, as can be seen by replacing 0, 1, 4, 9 with f_0, f_1, f_2, f_3. ■

We prove (22). From (18) and (19) we have for the components of the DFT

$$\hat{f}_n = \sum_{k=0}^{N-1} w_N^{\;kn} f_k.$$

Splitting into two sums of $M = N/2$ terms each gives

$$\hat{f}_n = \sum_{k=0}^{M-1} w_N^{\;2kn} f_{2k} + \sum_{k=0}^{M-1} w_N^{\;(2k+1)n} f_{2k+1}.$$

We now use $w_N^{\;2} = w_M$ and pull out $w_N^{\;n}$ from under the second sum, obtaining

$$\hat{f}_n = \sum_{k=0}^{M-1} w_M^{\;kn} f_{\text{ev},k} + w_N^{\;n} \sum_{k=0}^{M-1} w_M^{\;kn} f_{\text{od},k}. \tag{23}$$

The two sums are $f_{\text{ev},n}$ and $f_{\text{od},n}$, the components of the "half-size" transforms $\mathbf{F}\,\mathbf{f}_{\text{ev}}$ and $\mathbf{F}\,\mathbf{f}_{\text{od}}$.

Formula (22a) is the same as (23). In (22b) we have $n + M$ instead of n. This causes a sign change in (23), namely $-w_N^{\;n}$ before the second sum because

$$w_N^{\;M} = e^{-2\pi iM/N} = e^{-2\pi i/2} = e^{-\pi i} = -1.$$

This gives the minus in (22b) and completes the proof. ■

PROBLEM SET 11.9

1. (Review) Show that $1/i = -i$, $e^{ix} + e^{-ix} = 2\cos x$, $e^{ix} - e^{-ix} = 2i \sin x$.

2–9 FOURIER TRANSFORMS BY INTEGRATION

Find the Fourier transform of $f(x)$ (without using Table III in Sec. 11.10). Show the details.

2. $f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \quad (k > 0) \\ 0 & \text{if } x > 0 \end{cases}$

3. $f(x) = \begin{cases} k & \text{if } 0 < x < b \\ 0 & \text{otherwise} \end{cases}$

4. $f(x) = \begin{cases} e^{2ix} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

5. $f(x) = \begin{cases} k & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

6. $f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

7. $f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

8. $f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$

9. $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

OTHER METHODS

10. Find the Fourier transform of $f(x) = xe^{-x}$ if $x > 0$ and 0 if $x < 0$ from formula 5 in Table III and (9) in the text. *Hint:* Consider xe^{-x} and e^{-x}.

11. Obtain $\mathcal{F}(e^{-x^2/2})$ from formula 9 in Table III.

12. Obtain formula 7 in Table III from formula 8.

13. Obtain formula 1 in Table III from formula 2.

14. TEAM PROJECT. Shifting. (a) Show that if $f(x)$ has a Fourier transform, so does $f(x - a)$, and $\mathcal{F}\{f(x - a)\} = e^{-iwa}\mathcal{F}\{f(x)\}$.

(b) Using (a), obtain formula 1 in Table III, Sec. 11.10, from formula 2.

(c) Shifting on the w-Axis. Show that if $\hat{f}(w)$ is the Fourier transform of $f(x)$, then $\hat{f}(w - a)$ is the Fourier transform of $e^{iax}f(x)$.

(d) Using (c), obtain formula 7 in Table III from 1 and formula 8 from 2.

11.10 Tables of Transforms

Table I. Fourier Cosine Transforms

See (2) in Sec. 11.8.

	$f(x)$	$\hat{f}_c(w) = \mathcal{F}_c(f)$
1	$\begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin aw}{w}$
2	x^{a-1} $(0 < a < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(a)}{w^a} \cos \frac{a\pi}{2}$ ($\Gamma(a)$ see App. A3.1.)
3	e^{-ax} $(a > 0)$	$\sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + w^2}\right)$
4	$e^{-x^2/2}$	$e^{-w^2/2}$
5	e^{-ax^2} $(a > 0)$	$\frac{1}{\sqrt{2a}} e^{-w^2/(4a)}$
6	$x^n e^{-ax}$ $(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{n!}{(a^2 + w^2)^{n+1}} \operatorname{Re} (a + iw)^{n+1}$ Re = Real part
7	$\begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\sqrt{2\pi}} \left[\frac{\sin a(1 - w)}{1 - w} + \frac{\sin a(1 + w)}{1 + w}\right]$
8	$\cos (ax^2)$ $(a > 0)$	$\frac{1}{\sqrt{2a}} \cos \left(\frac{w^2}{4a} - \frac{\pi}{4}\right)$
9	$\sin (ax^2)$ $(a > 0)$	$\frac{1}{\sqrt{2a}} \cos \left(\frac{w^2}{4a} + \frac{\pi}{4}\right)$
10	$\frac{\sin ax}{x}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}} (1 - u(w - a))$ (See Sec. 6.3.)
11	$\frac{e^{-x} \sin x}{x}$	$\frac{1}{\sqrt{2\pi}} \arctan \frac{2}{w^2}$
12	$J_0(ax)$ $(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{a^2 - w^2}} (1 - u(w - a))$ (See Secs. 5.5, 6.3.)

Table II. Fourier Sine Transforms

See (5) in Sec. 11.8.

	$f(x)$	$\hat{f}_s(w) = \mathcal{F}_s(f)$
1	$\begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos aw}{w} \right]$
2	$1/\sqrt{x}$	$1/\sqrt{w}$
3	$1/x^{3/2}$	$2\sqrt{w}$
4	$x^{a-1} \quad (0 < a < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(a)}{w^a} \sin \frac{a\pi}{2}$ ($\Gamma(a)$ see App. A3.1.)
5	$e^{-ax} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \left(\frac{w}{a^2 + w^2} \right)$
6	$\frac{e^{-ax}}{x} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \arctan \frac{w}{a}$
7	$x^n e^{-ax} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{n!}{(a^2 + w^2)^{n+1}} \operatorname{Im} (a + iw)^{n+1}$ Im = Imaginary part
8	$xe^{-x^2/2}$	$we^{-w^2/2}$
9	$xe^{-ax^2} \quad (a > 0)$	$\frac{w}{(2a)^{3/2}} e^{-w^2/4a}$
10	$\begin{cases} \sin x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\sqrt{2\pi}} \left[\frac{\sin a(1 - w)}{1 - w} - \frac{\sin a(1 + w)}{1 + w} \right]$
11	$\frac{\cos ax}{x} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} u(w - a)$ (See Sec. 6.3.)
12	$\arctan \frac{2a}{x} \quad (a > 0)$	$\sqrt{2\pi} \frac{\sinh aw}{w} e^{-aw}$

Table III. Fourier Transforms

See (6) in Sec. 11.9.

	$f(x)$	$\hat{f}(w) = \mathcal{F}(f)$
1	$\begin{cases} 1 & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin bw}{w}$
2	$\begin{cases} 1 & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{e^{-ibw} - e^{-icw}}{iw\sqrt{2\pi}}$
3	$\dfrac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\dfrac{\pi}{2}}\,\dfrac{e^{-a\lvert w\rvert}}{a}$
4	$\begin{cases} x & \text{if } 0 < x < b \\ 2x - b & \text{if } b < x < 2b \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{-1 + 2e^{ibw} - e^{-2ibw}}{\sqrt{2\pi}\, w^2}$
5	$\begin{cases} e^{-ax} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad (a > 0)$	$\dfrac{1}{\sqrt{2\pi}(a + iw)}$
6	$\begin{cases} e^{ax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{e^{(a-iw)c} - e^{(a-iw)b}}{\sqrt{2\pi}(a - iw)}$
7	$\begin{cases} e^{iax} & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin b(w - a)}{w - a}$
8	$\begin{cases} e^{iax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{i}{\sqrt{2\pi}}\,\dfrac{e^{ib(a-w)} - e^{ic(a-w)}}{a - w}$
9	$e^{-ax^2} \quad (a > 0)$	$\dfrac{1}{\sqrt{2a}}\, e^{-w^2/4a}$
10	$\dfrac{\sin ax}{x} \quad (a > 0)$	$\sqrt{\dfrac{\pi}{2}}$ if $\lvert w\rvert < a$; 0 if $\lvert w\rvert > a$

CHAPTER 11 REVIEW QUESTIONS AND PROBLEMS

1. What is a Fourier series? A Fourier sine series? A half-range expansion?
2. Can a discontinuous function have a Fourier series? A Taylor series? Explain.
3. Why did we start with period 2π? How did we proceed to functions of any period p?
4. What is the trigonometric system? Its main property by which we obtained the Euler formulas?
5. What do you know about the convergence of a Fourier series?
6. What is the Gibbs phenomenon?
7. What is approximation by trigonometric polynomials? The minimum square error?
8. What is remarkable about the response of a vibrating system to an *arbitrary* periodic force?
9. What do you know about the Fourier integral? Its applications?
10. What is the Fourier sine transform? Give examples.

11–20 FOURIER SERIES

Find the Fourier series of $f(x)$ as given over one period. Sketch $f(x)$. (Show the details of your work.)

11. $f(x) = \begin{cases} -k & \text{if } -1 < x < 0 \\ k & \text{if } 0 < x < 1 \end{cases}$

12. $f(x) = \begin{cases} 0 & \text{if } -\pi/2 < x < \pi/2 \\ 1 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

13. $f(x) = x \quad (-2\pi < x < 2\pi)$

14. $f(x) = |x| \quad (-2 < x < 2)$

15. $f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 2 - x & \text{if } 1 < x < 3 \end{cases}$

16. $f(x) = \begin{cases} -1 - x & \text{if } -1 < x < 0 \\ 1 - x & \text{if } 0 < x < 1 \end{cases}$

17. $f(x) = |\sin 8\pi x| \ (-1/8 < x < 1/8)$

18. $f(x) = e^x \ (-\pi < x < \pi)$

19. $f(x) = x^2 \ (-\pi/2 < x < \pi/2)$

20. $f(x) = x \ (0 < x < 2\pi)$

21–23 Using the answers to suitable odd-numbered problems, find the sum of

21. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$

22. $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots$

23. $1 + \frac{1}{9} + \frac{1}{25} + \cdots$

24. **(Parseval's identity)** Obtain the result of Prob. 23 by applying Parseval's identity to Prob. 12.
25. What are the sum of the cosine terms and the sum of the sine terms in a Fourier series whose sum is $f(x)$? Give two examples.
26. **(Half-range expansion)** Find the half-range sine series of $f(x) = 0$ if $0 < x < \pi/2$, $f(x) = 1$ if $\pi/2 < x < \pi$. Compare with Prob. 12.
27. **(Half-range cosine series)** Find the half-range cosine series of $f(x) = x \ (0 < x < 2\pi)$. Compare with Prob. 20.

28–29 MINIMUM SQUARE ERROR

Compute the minimum square errors for the trigonometric polynomials of degree $N = 1, \cdots, 8$:

28. For $f(x)$ in Prob. 12.
29. For $f(x) = x \ (-\pi < x < \pi)$.

30–31 GENERAL SOLUTION

Solve $y'' + \omega^2 y = r(t)$, where $|\omega| \neq 0, 1, 2, \cdots, r(t)$ is 2π-periodic and:

30. $r(t) = t(\pi^2 - t^2) \quad (-\pi < t < \pi)$
31. $r(t) = t^2 \quad (-\pi < t < \pi)$

32–37 FOURIER INTEGRALS AND TRANSFORMS

Sketch the given function and represent it as indicated. If you have a CAS, graph approximate curves obtained by replacing ∞ with finite limits; also look for Gibbs phenomena.

32. $f(x) = 1$ if $1 < x < 2$ and 0 otherwise, by a Fourier integral
33. $f(x) = x$ if $0 < x < 1$ and 0 otherwise, by a Fourier integral

34. $f(x) = 1 + x/2$ if $-2 < x < 0$, $f(x) = 1 - x/2$ if $0 < x < 2$, $f(x) = 0$ otherwise, by a Fourier cosine integral

35. $f(x) = -1 - x/2$ if $-2 < x < 0$, $f(x) = 1 - x/2$ if $0 < x < 2$, $f(x) = 0$ otherwise, by a Fourier sine integral

36. $f(x) = -4 + x^2$ if $-2 < x < 0$, $f(x) = 4 - x^2$ if $0 < x < 2$, $f(x) = 0$ otherwise, by a Fourier sine integral

37. $f(x) = 4 - x^2$ if $-2 < x < 2$, $f(x) = 0$ otherwise, by a Fourier cosine integral

38. Find the Fourier transform of $f(x) = k$ if $a < x < b$, $f(x) = 0$ otherwise.

39. Find the Fourier cosine transform of $f(x) = e^{-2x}$ if $x > 0$, $f(x) = 0$ if $x < 0$.

40. Find $\mathcal{F}_c(e^{-2x})$ and $\mathcal{F}_s(e^{-2x})$ by formulas involving second derivatives.

SUMMARY OF CHAPTER 11

Fourier Series, Integrals, Transforms

Fourier series concern **periodic functions** $f(x)$ of period $p = 2L$, that is, by definition $f(x + p) = f(x)$ for all x and some fixed $p > 0$; thus, $f(x + np) = f(x)$ for any integer n. These series are of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x\right) \qquad (1) \quad \text{(Sec. 11.2)}$$

with coefficients, called the **Fourier coefficients** of $f(x)$, given by the Euler formulas (Sec. 11.2)

$$(2) \qquad a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx, \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$

where $n = 1, 2, \cdots$. For period 2π we simply have (Sec. 11.1)

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (1^*)$$

with the *Fourier coefficients* of $f(x)$ (Sec. 11.1)

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

Fourier series are fundamental in connection with periodic phenomena, particularly in models involving differential equations (Sec. 11.5, Chap. 12). If $f(x)$ is even $[f(-x) = f(x)]$ or odd $[f(-x) = -f(x)]$, they reduce to **Fourier cosine** or **Fourier sine series,** respectively (Sec. 11.3). If $f(x)$ is given for $0 \leqq x \leqq L$ only, it has two **half-range expansions** of period $2L$, namely, a cosine and a sine series (Sec. 11.3).

The set of cosine and sine functions in (1) is called the **trigonometric system.** Its most basic property is its **orthogonality** on an interval of length $2L$; that is, for all integers m and $n \neq m$ we have

$$\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx = 0, \qquad \int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx = 0$$

and for all integers m and n,

$$\int_{-L}^{L} \cos \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx = 0.$$

This orthogonality was crucial in deriving the Euler formulas (2).

Partial sums of Fourier series minimize the **square error** (Sec. 11.6).

Ideas and techniques of Fourier series extend to nonperiodic functions $f(x)$ defined on the entire real line; this leads to the **Fourier integral**

$$f(x) = \int_{0}^{\infty} [A(w) \cos wx + B(w) \sin wx]\, dw \qquad \text{(Sec. 11.7)} \tag{3}$$

where

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv\, dv, \qquad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv\, dv \tag{4}$$

or, in complex form (Sec. 11.9),

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx}\, dw \qquad (i = \sqrt{-1}) \tag{5}$$

where

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx. \tag{6}$$

Formula (6) transforms $f(x)$ into its **Fourier transform** $\hat{f}(w)$, and (5) is the inverse transform.

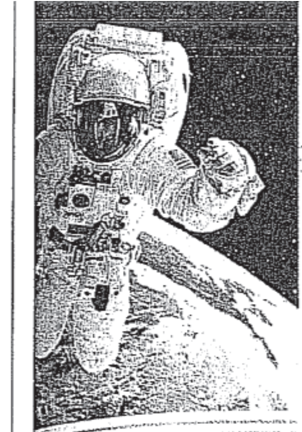
Related to this are the **Fourier cosine transform** (Sec. 11.8)

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos wx\, dx \tag{7}$$

and the **Fourier sine transform** (Sec. 11.8)

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin wx\, dx. \tag{8}$$

The **discrete Fourier transform (DFT)** and a practical method of computing it, called the **fast Fourier transform (FFT),** are discussed in Sec. 11.9.



CHAPTER 12

Partial Differential Equations (PDEs)

PDEs are models of various physical and geometrical problems, arising when the unknown functions (the solutions) depend on two or more variables, usually on time t and one or several space variables. It is fair to say that only the simplest physical systems can be modeled by ODEs, whereas most problems in dynamics, elasticity, heat transfer, electromagnetic theory, and quantum mechanics require PDEs. Indeed, the range of applications of PDEs is enormous, compared to that of ODEs.

In this chapter we concentrate on the most important PDEs of applied mathematics, the wave equations governing the vibrating string (Sec. 12.2) and the vibrating membrane (Sec. 12.7), the heat equation (Sec. 12.5), and the Laplace equation (Secs. 12.5, 12.10). We derive these PDEs from physics and consider methods for solving **initial and boundary value problems,** that is, methods of obtaining solutions satisfying conditions that are given by the physical situation.

In Secs. 12.6 and 12.11 we show that PDEs can also be solved by Fourier and Laplace transform methods.

COMMENT. *Numerics for PDEs* is explained in Secs. 21.4–21.7.

Prerequisites: Linear ODEs (Chap. 2), Fourier series (Chap. 11)
Sections that may be omitted in a shorter course: 12.6, 12.9–12.11
References and Answers to Problems: App. 1 Part C, App. 2

12.1 Basic Concepts

A **partial differential equation (PDE)** is an equation involving one or more partial derivatives of an (unknown) function, call it u, that depends on two or more variables, often time t and one or several variables in space. The order of the highest derivative is called the **order** of the PDE. As for ODEs, second-order PDEs will be the most important ones in applications.

Just as for ordinary differential equations (ODEs) we say that a PDE is **linear** if it is of the first degree in the unknown function u and its partial derivatives. Otherwise we call it **nonlinear.** Thus, all the equations in Example 1 on p. 536 are linear. We call a *linear* PDE **homogeneous** if each of its terms contains either u or one of its partial derivatives. Otherwise we call the equation **nonhomogeneous.** Thus, (4) in Example 1 (with f not identically zero) is nonhomogeneous, whereas the other equations are homogeneous.

EXAMPLE 1 **Important Second-Order PDEs**

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ *One-dimensional wave equation*

(2) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ *One-dimensional heat equation*

(3) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ *Two-dimensional Laplace equation*

(4) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$ *Two-dimensional Poisson equation*

(5) $$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$ *Two-dimensional wave equation*

(6) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$ *Three-dimensional Laplace equation*

Here c is a positive constant, t is time, x, y, z are Cartesian coordinates, and *dimension* is the number of these coordinates in the equation. ■

A **solution** of a PDE in some region R of the space of the independent variables is a function that has all the partial derivatives appearing in the PDE in some domain D (definition in Sec. 9.6) containing R, and satisfies the PDE everywhere in R.

Often one merely requires that the function is continuous on the boundary of R, has those derivatives in the interior of R, and satisfies the PDE in the interior of R. Letting R lie in D simplifies the situation regarding derivatives on the boundary of R, which is then the same on the boundary as it is in the interior of R.

In general, the totality of solutions of a PDE is very large. For example, the functions

(7) $$u = x^2 - y^2, \qquad u = e^x \cos y, \qquad u = \sin x \cosh y, \qquad u = \ln(x^2 + y^2)$$

which are entirely different from each other, are solutions of (3), as you may verify. We shall see later that the unique solution of a PDE corresponding to a given physical problem will be obtained by the use of **additional conditions** arising from the problem. For instance, this may be the condition that the solution u assume given values on the boundary of the region R (**"boundary conditions"**). Or, when time t is one of the variables, u (or $u_t = \partial u/\partial t$ or both) may be prescribed at $t = 0$ (**"initial conditions"**).

We know that if an ODE is linear and homogeneous, then from known solutions we can obtain further solutions by superposition. For PDEs the situation is quite similar:

THEOREM 1 **Fundamental Theorem on Superposition**

If u_1 and u_2 are solutions of a ***homogeneous linear*** *PDE in some region R, then*

$$u = c_1 u_1 + c_2 u_2$$

with any constants c_1 and c_2 is also a solution of that PDE in the region R.

The simple proof of this important theorem is quite similar to that of Theorem 1 in Sec. 2.1 and is left to the student.

Verification of solutions in Probs. 14–25 proceeds as for ODEs. Problems 1–12 concern PDEs solvable like ODEs. To help the student with them, we consider two typical examples.

EXAMPLE 2 Solving $u_{xx} - u = 0$ Like an ODE

Find solutions u of the PDE $u_{xx} - u = 0$ depending on x and y.

Solution. Since no y-derivatives occur, we can solve this PDE like $u'' - u = 0$. In Sec. 2.2 we would have obtained $u = Ae^{x} + Be^{-x}$ with constant A and B. Here A and B may be functions of y, so that the answer is

$$u(x, y) = A(y)e^{x} + B(y)e^{-x}$$

with arbitrary functions A and B. We thus have a great variety of solutions. Check the result by differentiation. ■

EXAMPLE 3 Solving $u_{xy} = -u_x$ Like an ODE

Find solutions $u = u(x, y)$ of this PDE.

Solution. Setting $u_x = p$, we have $p_y = -p$, $p_y/p = -1$, $\ln p = -y + \tilde{c}(x)$, $p = c(x)e^{-y}$ and by integration with respect to x,

$$u(x, y) = f(x)e^{-y} + g(y) \qquad \text{where} \qquad f(x) = \int c(x)\,dx;$$

here, $f(x)$ and $g(y)$ are arbitrary. ■

PROBLEM SET 12.1

1–12 PDEs SOLVABLE AS ODEs

This happens if a PDE involves derivatives with respect to one variable only (or can be transformed to such a form), so that the other variable(s) can be treated as parameter(s). Solve for $u = u(x, y)$:

1. $u_{yy} + 16u = 0$

2. $u_{xx} = u$

3. $u_{yy} = 0$

4. $u_y + 2yu = 0$

5. $u_y + u = e^{xy}$

6. $u_{xx} = 4y^2u$

7. $u_y = (\cosh x)yu$

8. $u_y = 2xyu$

9. $y^2u_{yy} + 2yu_y - 2u = 0$

10. $u_{yy} = 4xu_y$

11. $u_{xy} = u_x$

12. $u_{yy} + 10u_y + 25u = e^{-5y}$

13. (Fundamental Theorem) Prove Fundamental Theorem 1 for second-order PDEs in two and three independent variables.

14–25 VERIFICATION OF SOLUTIONS

Verify (by substitution) that the given function is a solution of the indicated PDE. Sketch or graph the solution as a surface in space.

14–17 Wave Equation (1) with suitable c

14. $u = 4x^2 + t^2$

15. $u = \sin 8x \cos 2t$

16. $u = \sin 3x \sin 18t$

17. $u = \sin kx \cos kct$

18–21 Heat Equation (2) with suitable c

18. $u = e^{-2kt} \cos 8x$

19. $u = e^{-\pi^2 t} \sin 4x$

20. $u = e^{-4\omega^2 t} \sin \omega x$

21. $u = e^{-\omega^2 c^2 t} \cos \omega x$

22–25 Laplace Equation (3)

22. u in (7) in the text

23. $u = \cos 2y \sinh 2x$

24. $u = \arctan(y/x)$

25. $u = e^{x^2 - y^2} \sin 2xy$

26. TEAM PROJECT. Verification of Solutions

(a) Wave equation. Verify that $u(x, t) = v(x + ct) + w(x - ct)$ with any twice differentiable functions v and w satisfies (1).

(b) Poisson equation. Verify that each u satisfies (4) with $f(x, y)$ as indicated.

$$u = x^4 + y^4 \qquad f = 12(x^2 + y^2)$$

$$u = \cos x \sin y \qquad f = -2 \cos x \sin y$$

$$u = y/x \qquad f = 2y/x^3$$

(c) Laplace equation. Verify that $u = 1/\sqrt{x^2 + y^2 + z^2}$ satisfies (6) and $u = \ln(x^2 + y^2)$ satisfies (3). Is $u = 1/\sqrt{x^2 + y^2}$ a solution of (3)? Of what Poisson equation?

(d) Verify that u with any (sufficiently often differentiable) v and w satisfies the given PDE.

$u = v(x) + w(y)$ $\qquad u_{xy} = 0$

$u = v(x)w(y)$ $\qquad uu_{xy} = u_x u_y$

$u = v(x + 3t) + w(x - 3t)$ $\qquad u_{tt} = 9u_{xx}$

27. (Boundary value problem) Verify that the function $u(x, y) = a \ln(x^2 + y^2) + b$ satisfies Laplace's equation (3) and determine a and b so that u satisfies the boundary conditions $u = 110$ on the circle $x^2 + y^2 = 1$ and $u = 0$ on the circle $x^2 + y^2 = 100$.

28–30 SYSTEMS OF PDEs

Solve

28. $u_x = 0, \; u_y = 0$

29. $u_{xx} = 0, \; u_{xy} = 0$

30. $u_{xx} = 0, \; u_{yy} = 0$

12.2 Modeling: Vibrating String, Wave Equation

As a first important PDE let us derive the equation modeling small transverse vibrations of an elastic string, such as a violin string. We place the string along the x-axis, stretch it to length L, and fasten it at the ends $x = 0$ and $x = L$. We then distort the string, and at some instant, call it $t = 0$, we release it and allow it to vibrate. The problem is to determine the vibrations of the string, that is, to find its deflection $u(x, t)$ at any point x and at any time $t > 0$; see Fig. 283.

$u(x, t)$ will be the solution of a PDE that is the model of our physical system to be derived. This PDE should not be too complicated, so that we can solve it. Reasonable simplifying assumptions (just as for ODEs modeling vibrations in Chap. 2) are as follows.

Physical Assumptions

1. The mass of the string per unit length is constant ("homogeneous string"). The string is perfectly elastic and does not offer any resistance to bending.
2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.
3. The string performs small transverse motions in a vertical plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value.

Under these assumptions we may expect solutions $u(x, t)$ that describe the physical reality sufficiently well.

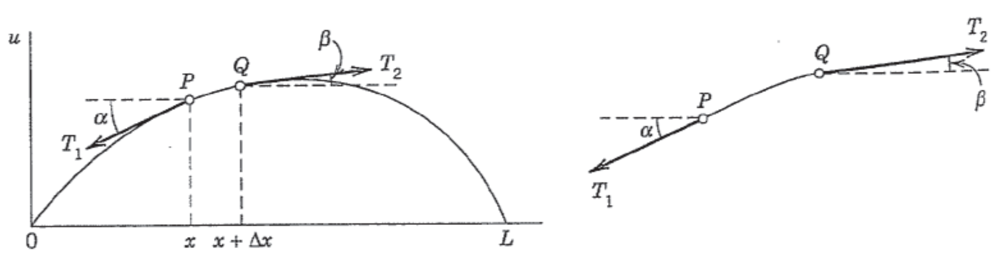


Fig. 283. Deflected string at fixed time t. Explanation on p. 539

Derivation of the PDE of the Model ("Wave Equation") from Forces

The model of the vibrating string will consist of a PDE ("wave equation") and additional conditions. To obtain the PDE, we consider the ***forces acting on a small portion of the string*** (Fig. 283). This method is typical of modeling in mechanics and elsewhere.

Since the string offers no resistance to bending, the tension is tangential to the curve of the string at each point. Let T_1 and T_2 be the tension at the endpoints P and Q of that portion. Since the points of the string move vertically, there is no motion in the horizontal direction. Hence the horizontal components of the tension must be constant. Using the notation shown in Fig. 283, we thus obtain

$$T_1 \cos \alpha = T_2 \cos \beta = T = \textit{const.} \tag{1}$$

In the vertical direction we have two forces, namely, the vertical components $-T_1 \sin \alpha$ and $T_2 \sin \beta$ of T_1 and T_2; here the minus sign appears because the component at P is directed downward. By **Newton's second law** the resultant of these two forces is equal to the mass $\rho \, \Delta x$ of the portion times the acceleration $\partial^2 u/\partial t^2$, evaluated at some point between x and $x + \Delta x$; here ρ is the mass of the undeflected string per unit length, and Δx is the length of the portion of the undeflected string. (Δ is generally used to denote small quantities; this has nothing to do with the Laplacian ∇^2, which is sometimes also denoted by Δ.) Hence

$$T_2 \sin \beta - T_1 \sin \alpha = \rho \, \Delta x \, \frac{\partial^2 u}{\partial t^2} .$$

Using (1), we can divide this by $T_2 \cos \beta = T_1 \cos \alpha = T$, obtaining

$$\frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha} = \tan \beta - \tan \alpha = \frac{\rho \, \Delta x}{T} \frac{\partial^2 u}{\partial t^2} . \tag{2}$$

Now $\tan \alpha$ and $\tan \beta$ are the slopes of the string at x and $x + \Delta x$:

$$\tan \alpha = \left(\frac{\partial u}{\partial x}\right)\bigg|_x \qquad \text{and} \qquad \tan \beta = \left(\frac{\partial u}{\partial x}\right)\bigg|_{x+\Delta x} .$$

Here we have to write *partial* derivatives because u depends also on time t. Dividing (2) by Δx, we thus have

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)\bigg|_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)\bigg|_x\right] = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2} .$$

If we let Δx approach zero, we obtain the linear PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{T}{\rho} . \tag{3}$$

This is called the **one-dimensional wave equation.** We see that it is homogeneous and of the second order. The physical constant T/ρ is denoted by c^2 (instead of c) to indicate that this constant is *positive,* a fact that will be essential to the form of the solutions. "One-dimensional" means that the equation involves only one space variable, x. In the next section we shall complete setting up the model and then show how to solve it by a general method that is probably the most important one for PDEs in engineering mathematics.

12.3 Solution by Separating Variables. Use of Fourier Series

The model of a vibrating elastic string (a violin string, for instance) consists of the **one-dimensional wave equation**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c^2 = \frac{T}{\rho} \tag{1}$$

for the unknown deflection $u(x, t)$ of the string, a PDE that we have just obtained, and some ***additional conditions,*** which we shall now derive.

Since the string is fastened at the ends $x = 0$ and $x = L$ (see Sec. 12.2), we have the two **boundary conditions**

$$\text{(a)}\quad u(0, t) = 0, \qquad \text{(b)}\quad u(L, t) = 0 \qquad \text{for all } t. \tag{2}$$

Furthermore, the form of the motion of the string will depend on its *initial deflection* (deflection at time $t = 0$), call it $f(x)$, and on its *initial velocity* (velocity at $t = 0$), call it $g(x)$. We thus have the two **initial conditions**

$$\text{(a)}\quad u(x, 0) = f(x), \qquad \text{(b)}\quad u_t(x, 0) = g(x) \qquad (0 \leqq x \leqq L) \tag{3}$$

where $u_t = \partial u/\partial t$. We now have to find a solution of the PDE (1) satisfying the conditions (2) and (3). This will be the solution of our problem. We shall do this in three steps, as follows.

Step 1. By the **"method of separating variables"** or *product method,* setting $u(x, t) = F(x)G(t)$, we obtain from (1) two ODEs, one for $F(x)$ and the other one for $G(t)$.

Step 2. We determine solutions of these ODEs that satisfy the boundary conditions (2).

Step 3. Finally, using **Fourier series,** we compose the solutions gained in Step 2 to obtain a solution of (1) satisfying both (2) and (3), that is, the solution of our model of the vibrating string.

Step 1. Two ODEs from the Wave Equation (1)

In the **method of separating variables,** or *product method,* we determine solutions of the wave equation (1) of the form

$$u(x, t) = F(x)G(t) \tag{4}$$

which are a product of two functions, each depending only on one of the variables x and t. This is a powerful general method that has various applications in engineering mathematics, as we shall see in this chapter. Differentiating (4), we obtain

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = F''G$$

where dots denote derivatives with respect to t and primes derivatives with respect to x. By inserting this into the wave equation (1) we have

$$F\ddot{G} = c^2 F''G.$$

Dividing by c^2FG and simplifying gives

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F}.$$

The variables are now separated, the left side depending only on t and the right side only on x. Hence both sides must be constant because if they were variable, then changing t or x would affect only one side, leaving the other unaltered. Thus, say,

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k.$$

Multiplying by the denominators gives immediately two ***ordinary*** DEs

(5) $$F'' - kF = 0$$

and

(6) $$\ddot{G} - c^2 kG = 0.$$

Here, the **separation constant** k is still arbitrary.

Step 2. Satisfying the Boundary Conditions (2)

We now determine solutions F and G of (5) and (6) so that $u = FG$ satisfies the boundary conditions (2), that is,

(7) $$u(0, t) = F(0)G(t) = 0, \qquad u(L, t) = F(L)G(t) = 0 \qquad \text{for all } t.$$

We first solve (5). If $G \equiv 0$, then $u = FG \equiv 0$, which is of no interest. Hence $G \not\equiv 0$ and then by (7),

(8) $$\text{(a)} \quad F(0) = 0, \qquad \text{(b)} \quad F(L) = 0.$$

We show that k must be negative. For $k = 0$ the general solution of (5) is $F = ax + b$, and from (8) we obtain $a = b = 0$, so that $F \equiv 0$ and $u = FG \equiv 0$, which is of no interest. For positive $k = \mu^2$ a general solution of (5) is

$$F = Ae^{\mu x} + Be^{-\mu x}$$

and from (8) we obtain $F \equiv 0$ as before (verify!). Hence we are left with the possibility of choosing k negative, say, $k = -p^2$. Then (5) becomes $F'' + p^2F = 0$ and has as a general solution

$$F(x) = A \cos px + B \sin px.$$

From this and (8) we have

$$F(0) = A = 0 \qquad \text{and then} \qquad F(L) = B \sin pL = 0.$$

We must take $B \neq 0$ since otherwise $F \equiv 0$. Hence $\sin pL = 0$. Thus

$$\text{(9)} \qquad pL = n\pi, \qquad \text{so that} \qquad p = \frac{n\pi}{L} \qquad (n \text{ integer}).$$

Setting $B = 1$, we thus obtain infinitely many solutions $F(x) = F_n(x)$, where

$$\text{(10)} \qquad F_n(x) = \sin \frac{n\pi}{L}x \qquad (n = 1, 2, \cdots).$$

These solutions satisfy (8). [For negative integer n we obtain essentially the same solutions, except for a minus sign, because $\sin(-\alpha) = -\sin \alpha$.]

We now solve (6) with $k = -p^2 = -(n\pi/L)^2$ resulting from (9), that is,

$$\text{(11*)} \qquad \ddot{G} + \lambda_n{}^2 G = 0 \qquad \text{where} \qquad \lambda_n = cp = \frac{cn\pi}{L}.$$

A general solution is

$$G_n(t) = B_n \cos \lambda_n t + B_n{}^* \sin \lambda_n t.$$

Hence solutions of (1) satisfying (2) are $u_n(x, t) = F_n(x)G_n(t) = G_n(t)F_n(x)$, written out

$$\textbf{(11)} \qquad u_n(x, t) = (B_n \cos \lambda_n t + B_n{}^* \sin \lambda_n t) \sin \frac{n\pi}{L}x \qquad (n = 1, 2, \cdots).$$

These functions are called the **eigenfunctions,** or *characteristic functions,* and the values $\lambda_n = cn\pi/L$ are called the **eigenvalues,** or *characteristic values,* of the vibrating string. The set $\{\lambda_1, \lambda_2, \cdots\}$ is called the **spectrum.**

Discussion of Eigenfunctions. We see that each u_n represents a harmonic motion having the **frequency** $\lambda_n/2\pi = cn/2L$ cycles per unit time. This motion is called the nth **normal mode** of the string. The first normal mode is known as the *fundamental mode* ($n = 1$), and the others are known as *overtones;* musically they give the octave, octave plus fifth, etc. Since in (11)

$$\sin \frac{n\pi x}{L} = 0 \qquad \text{at} \qquad x = \frac{L}{n}, \frac{2L}{n}, \cdots, \frac{n-1}{n}L,$$

the nth normal mode has $n - 1$ **nodes,** that is, points of the string that do not move (in addition to the fixed endpoints); see Fig. 284.

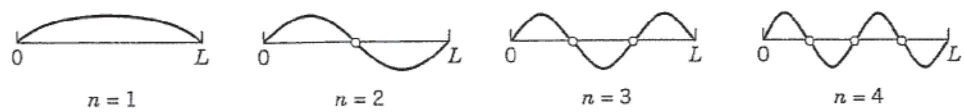


Fig. 284. Normal modes of the vibrating string

Figure 285 shows the second normal mode for various values of t. At any instant the string has the form of a sine wave. When the left part of the string is moving down, the other half is moving up, and conversely. For the other modes the situation is similar.

Tuning is done by changing the tension T. Our formula for the frequency $\lambda_n/2\pi = cn/2L$ of u_n with $c = \sqrt{T/\rho}$ [see (3), Sec. 12.2] confirms that effect because it shows that the frequency is proportional to the tension. T cannot be increased indefinitely, but can you see what to do to get a string with a high fundamental mode? (Think of both L and ρ.) Why is a violin smaller than a double-bass?

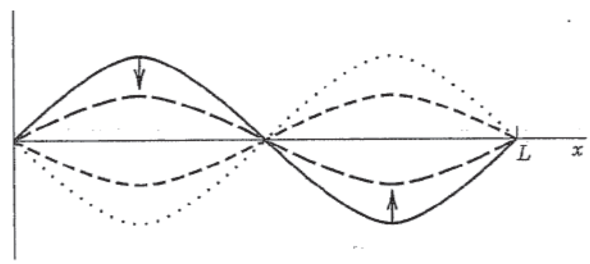


Fig. 285. Second normal mode for various values of t

Step 3. Solution of the Entire Problem. Fourier Series

The eigenfunctions (11) satisfy the wave equation (1) and the boundary conditions (2) (string fixed at the ends). A single u_n will generally not satisfy the initial conditions (3). But since the wave equation (1) is linear and homogeneous, it follows from Fundamental Theorem 1 in Sec. 12.1 that the sum of finitely many solutions u_n is a solution of (1). To obtain a solution that also satisfies the initial conditions (3), we consider the infinite series (with $\lambda_n = cn\pi/L$ as before)

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x. \tag{12}$$

Satisfying Initial Condition (3a) (Given Initial Displacement). From (12) and (3a) we obtain

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x). \tag{13}$$

Hence we must choose the B_n's so that $u(x, 0)$ becomes the **Fourier sine series** of $f(x)$. Thus, by (4) in Sec. 11.3,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, \cdots. \tag{14}$$

Satisfying Initial Condition (3b) (Given Initial Velocity). Similarly, by differentiating (12) with respect to t and using (3b), we obtain

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty} (-B_n\lambda_n \sin \lambda_n t + B_n{}^*\lambda_n \cos \lambda_n t) \sin \frac{n\pi x}{L}\right]_{t=0}$$

$$= \sum_{n=1}^{\infty} B_n{}^*\lambda_n \sin \frac{n\pi x}{L} = g(x).$$

Hence we must choose the $B_n{}^*$'s so that for $t = 0$ the derivative $\partial u/\partial t$ becomes the Fourier sine series of $g(x)$. Thus, again by (4) in Sec. 11.3,

$$B_n{}^*\lambda_n = \frac{2}{L}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx.$$

Since $\lambda_n = cn\pi/L$, we obtain by division

$$\text{(15)} \qquad B_n{}^* = \frac{2}{cn\pi}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, \cdots.$$

Result. Our discussion shows that $u(x, t)$ given by (12) with coefficients (14) and (15) is a solution of (1) that satisfies all the conditions in (2) and (3), provided the series (12) converges and so do the series obtained by differentiating (12) twice termwise with respect to x and t and have the sums $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial t^2$, respectively, which are continuous.

Solution (12) Established. According to our derivation the solution (12) is at first a purely formal expression, but we shall now establish it. For the sake of simplicity we consider only the case when the initial velocity $g(x)$ is identically zero. Then the $B_n{}^*$ are zero, and (12) reduces to

$$\text{(16)} \qquad u(x, t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi x}{L}, \qquad \lambda_n = \frac{cn\pi}{L}.$$

It is possible to ***sum this series,*** that is, to write the result in a closed or finite form. For this purpose we use the formula [see (11), App. A3.1]

$$\cos \frac{cn\pi}{L} t \sin \frac{n\pi}{L} x = \frac{1}{2}\left[\sin\left\{\frac{n\pi}{L}(x - ct)\right\} + \sin\left\{\frac{n\pi}{L}(x + ct)\right\}\right].$$

Consequently, we may write (16) in the form

$$u(x, t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x - ct)\right\} + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x + ct)\right\}.$$

These two series are those obtained by substituting $x - ct$ and $x + ct$, respectively, for the variable x in the Fourier sine series (13) for $f(x)$. Thus

$$\text{(17)} \qquad u(x, t) = \tfrac{1}{2}[f^*(x - ct) + f^*(x + ct)]$$

where f^* is the odd periodic extension of f with the period $2L$ (Fig. 286). Since the initial deflection $f(x)$ is continuous on the interval $0 \leqq x \leqq L$ and zero at the endpoints, it follows from (17) that $u(x, t)$ is a continuous function of both variables x and t for all values of the variables. By differentiating (17) we see that $u(x, t)$ is a solution of (1), provided $f(x)$ is twice differentiable on the interval $0 < x < L$, and has one-sided second derivatives at $x = 0$ and $x = L$, which are zero. Under these conditions $u(x, t)$ is established as a solution of (1), satisfying (2) and (3) with $g(x) \equiv 0$. ■

Fig. 286. Odd periodic extension of $f(x)$

Generalized Solution. If $f'(x)$ and $f''(x)$ are merely piecewise continuous (see Sec. 6.1), or if those one-sided derivatives are not zero, then for each t there will be finitely many values of x at which the second derivatives of u appearing in (1) do not exist. Except at these points the wave equation will still be satisfied. We may then regard $u(x, t)$ as a **"generalized solution,"** as it is called, that is, as a solution in a broader sense. For instance, a triangular initial deflection as in Example 1 (below) leads to a generalized solution.

Physical Interpretation of the Solution (17). The graph of $f^*(x - ct)$ is obtained from the graph of $f^*(x)$ by shifting the latter ct units to the right (Fig. 287). This means that $f^*(x - ct)$ $(c > 0)$ represents a wave that is traveling to the right as t increases. Similarly, $f^*(x + ct)$ represents a wave that is traveling to the left, and $u(x, t)$ is the superposition of these two waves.

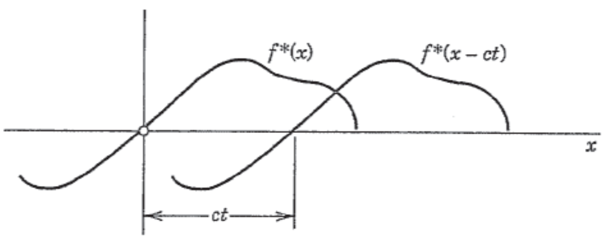


Fig. 287. Interpretation of (17)

EXAMPLE 1 Vibrating String if the Initial Deflection Is Triangular

Find the solution of the wave equation (1) corresponding to the triangular initial deflection

$$f(x) = \begin{cases} \dfrac{2k}{L}x & \text{if} \quad 0 < x < \dfrac{L}{2} \\ \dfrac{2k}{L}(L - x) & \text{if} \quad \dfrac{L}{2} < x < L \end{cases}$$

and initial velocity zero. (Figure 288 shows $f(x) = u(x, 0)$ at the top.)

Solution. Since $g(x) \equiv 0$, we have $B_n^* = 0$ in (12), and from Example 4 in Sec. 11.3 we see that the B_n are given by (5), Sec. 11.3. Thus (12) takes the form

$$u(x, t) = \frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi}{L}x\cos\frac{\pi c}{L}t - \frac{1}{3^2}\sin\frac{3\pi}{L}x\cos\frac{3\pi c}{L}t + - \cdots\right].$$

For graphing the solution we may use $u(x, 0) = f(x)$ and the above interpretation of the two functions in the representation (17). This leads to the graph shown in Fig. 288. ■

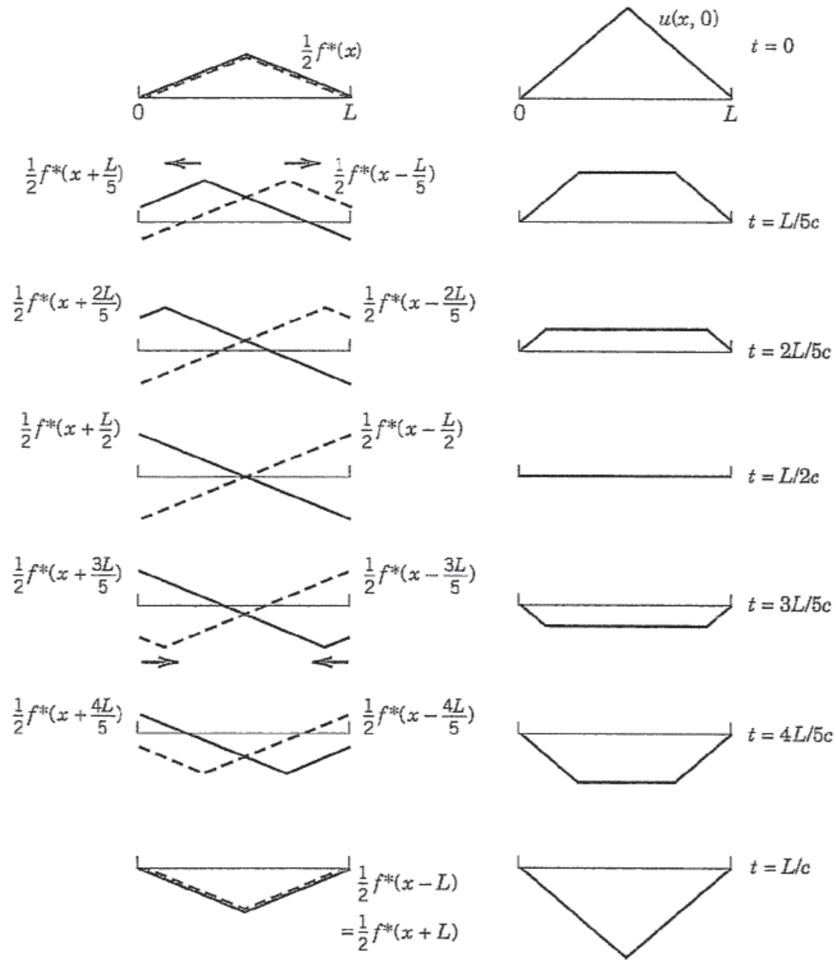


Fig. 288. Solution $u(x, t)$ in Example 1 for various values of t (right part of the figure) obtained as the superposition of a wave traveling to the right (dashed) and a wave traveling to the left (left part of the figure)

PROBLEM SET 12.3

1–10 DEFLECTION OF THE STRING

Find $u(x, t)$ for the string of length $L = 1$ and $c^2 = 1$ when the initial velocity is zero and the initial deflection with small k (say, 0.01) is as follows. Sketch or graph $u(x, t)$ as in Fig. 288.

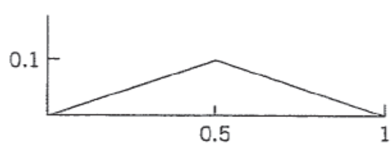
1. $k \sin 2\pi x$

2. $k(\sin \pi x - \frac{1}{3}\sin 3\pi x)$

3. $kx(1 - x)$

4. $kx(1 - x^2)$

5.



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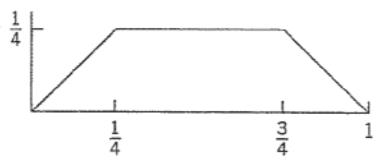

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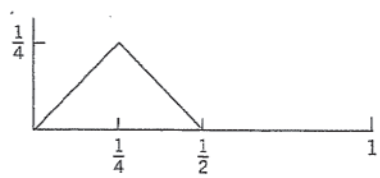

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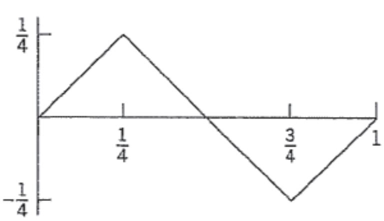

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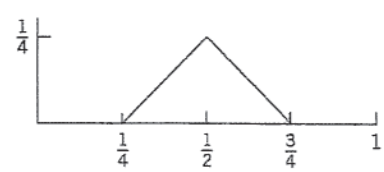
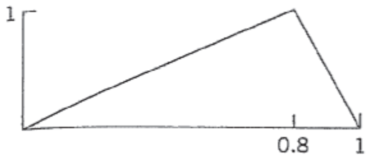


10. 

11. **(Frequency)** How does the frequency of the fundamental mode of the vibrating string depend on the length of the string? On the mass per unit length? What happens to the string if we double the tension? Why is a contrabass larger than a violin?

12. **(Nonzero initial velocity)** Find the deflection $u(x, t)$ of the string of length $L = \pi$ and $c^2 = 1$ for zero initial displacement and "triangular" initial velocity $u_t(x, 0) = 0.01x$ if $0 \leqq x \leqq \frac{1}{2}\pi$, $u_t(x, 0) = 0.01(\pi - x)$ if $\frac{1}{2}\pi \leqq x \leqq \pi$. (Initial conditions with $u_t(x, 0) \neq 0$ are hard to realize experimentally.)

13. **CAS PROJECT. Graphing Normal Modes.** Write a program for graphing u_n with $L = \pi$ and c^2 of your choice similarly as in Fig. 284. Apply the program to u_2, u_3, u_4. Also graph these solutions as surfaces over the xt-plane. Explain the connection between these two kinds of graphs.

14. **TEAM PROJECT. Forced Vibrations of an Elastic String.** Show the following.

(a) Substitution of

$$u(x, t) = \sum_{n=1}^{\infty} G_n(t) \sin \frac{n\pi x}{L} \tag{17}$$

(L = length of the string) into the wave equation (1) governing free vibrations leads to [see (10*)]

$$\ddot{G}_n + \lambda_n^2 G = 0, \qquad \lambda_n = \frac{cn\pi}{L}. \tag{18}$$

(b) Forced vibrations of the string under an external force $P(x, t)$ per unit length acting normal to the string are governed by the PDE

$$u_{tt} = c^2 u_{xx} + \frac{P}{\rho}. \tag{19}$$

(c) For a sinusoidal force $P = A\rho \sin \omega t$ we obtain

$$\frac{P}{\rho} = A \sin \omega t = \sum_{n=1}^{\infty} k_n(t) \sin \frac{n\pi x}{L},$$

$$k_n(t) = \begin{cases} (4A/n\pi) \sin \omega t & (n \text{ odd}) \\ 0 & (n \text{ even}). \end{cases} \tag{20}$$

Substituting (17) and (20) into (19) gives

$$\ddot{G}_n + \lambda_n^2 G_n = \frac{2A}{n\pi}(1 - \cos n\pi) \sin \omega t.$$

If $\lambda_n^2 \neq \omega^2$, the solution is

$$G_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t + \frac{2A(1 - \cos n\pi)}{n\pi(\lambda_n^2 - \omega^2)} \sin \omega t.$$

Determine B_n and B_n^* so that u satisfies the initial conditions $u(x, 0) = f(x)$, $u_t(x, 0) = 0$.

(d) (Resonance) Show that if $\lambda_n = \omega$, then

$$G_n(t) = B_n \cos \omega t + B_n^* \sin \omega t - \frac{A}{n\pi\omega}(1 - \cos n\pi) t \cos \omega t.$$

(e) (Reduction of boundary conditions) Show that a problem (1)–(3) with more complicated boundary conditions $u(0, t) = 0$, $u(L, t) = h(t)$, can be reduced to a problem for a new function v satisfying conditions $v(0, t) = v(L, t) = 0$, $v(x, 0) = f_1(x)$, $v_t(x, 0) = g_1(x)$ but a nonhomogeneous wave equation. *Hint:* Set $u = v + w$ and determine w suitably.

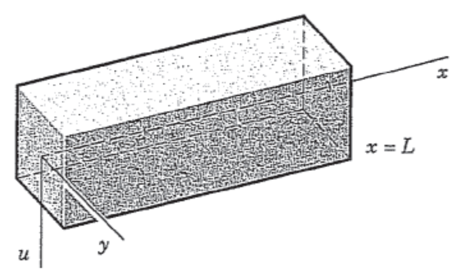


Fig. 289. Elastic beam

15–20 SEPARATION OF A FOURTH-ORDER PDE. VIBRATING BEAM

By the principles used in modeling the string it can be shown that small free vertical vibrations of a uniform elastic beam (Fig. 289) are modeled by the fourth-order PDE

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4} \qquad \text{(Ref. [C11])} \tag{21}$$

where $c^2 = EI/\rho A$ (E = Young's modulus of elasticity, I = moment of intertia of the cross section with respect to the y-axis in the figure, ρ = density, A = cross-sectional area). (*Bending* of a beam under a load is discussed in Sec. 3.3.)

15. Substituting $u = F(x)G(t)$ into (21), show that

$$F^{(4)}/F = -\ddot{G}/c^2 G = \beta^4 = const,$$

$$F(x) = A \cos \beta x + B \sin \beta x + C \cosh \beta x + D \sinh \beta x,$$

$$G(t) = a \cos c\beta^2 t + b \sin c\beta^2 t.$$

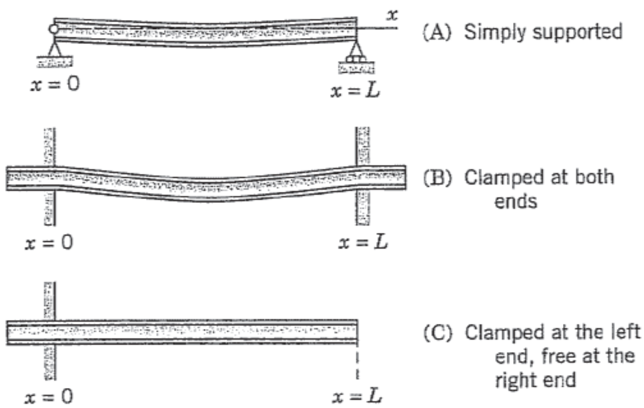


Fig. 290. Supports of a beam

16. **(Simply supported beam in Fig. 290A)** Find solutions $u_n = F_n(x)G_n(t)$ of (21) corresponding to zero initial velocity and satisfying the boundary conditions (see Fig. 290A)

$$u(0, t) = 0, \; u(L, t) = 0$$
(ends simply supported for all times t),

$$u_{xx}(0, t) = 0, \; u_{xx}(L, t) = 0$$
(zero moments, hence zero curvature, at the ends).

17. Find the solution of (21) that satisfies the conditions in Prob. 16 as well as the initial condition

$$u(x, 0) = f(x) = x(L - x).$$

18. Compare the results of Probs. 17 and 3. What is the basic difference between the frequencies of the normal modes of the vibrating string and the vibrating beam?

19. **(Clamped beam in Fig. 290B)** What are the boundary conditions for the clamped beam in Fig. 290B? Show that F in Prob. 15 satisfies these conditions if βL is a solution of the equation

$$\cosh \beta L \cos \beta L = 1. \tag{22}$$

Determine approximate solutions of (22), for instance, graphically from the intersections of the curves of $\cos \beta L$ and $1/\cosh \beta L$.

20. **(Clamped-free beam in Fig. 290C)** If the beam is clamped at the left and free at the right (Fig. 290C), the boundary conditions are

$$u(0, t) = 0, \qquad u_x(0, t) = 0,$$
$$u_{xx}(L, t) = 0, \qquad u_{xxx}(L, t) = 0.$$

Show that F in Prob. 15 satisfies these conditions if βL is a solution of the equation

$$\cosh \beta L \cos \beta L = -1. \tag{23}$$

Find approximate solutions of (18).

12.4 D'Alembert's Solution of the Wave Equation. Characteristics

It is interesting that the solution (17), Sec. 12.3, of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{T}{\rho}, \tag{1}$$

can be immediately obtained by transforming (1) in a suitable way, namely, by introducing the new independent variables

$$v = x + ct, \qquad w = x - ct. \tag{2}$$

Then u becomes a function of v and w. The derivatives in (1) can now be expressed in terms of derivatives with respect to v and w by the use of the chain rule in Sec. 9.6. Denoting partial derivatives by subscripts, we see from (2) that $v_x = 1$ and $w_x = 1$. For simplicity let us denote $u(x, t)$, as a function of v and w, by the same letter u. Then

$$u_x = u_v v_x + u_w w_x = u_v + u_w.$$

We now apply the chain rule to the right side of this equation. We assume that all the partial derivatives involved are continuous, so that $u_{wv} = u_{vw}$. Since $v_x = 1$ and $w_x = 1$, we obtain

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = u_{vv} + 2u_{vw} + u_{ww}.$$

Transforming the other derivative in (1) by the same procedure, we find

$$u_{tt} = c^2(u_{vv} - 2u_{vw} + u_{ww}).$$

By inserting these two results in (1) we get (see footnote 2 in App. A3.2)

$$u_{vw} \equiv \frac{\partial^2 u}{\partial w\, \partial v} = 0. \tag{3}$$

The point of the present method is that (3) can be readily solved by two successive integrations, first with respect to w and then with respect to v. This gives

$$\frac{\partial u}{\partial v} = h(v) \qquad \text{and} \qquad u = \int h(v)\, dv + \psi(w).$$

Here $h(v)$ and $\psi(w)$ are arbitrary functions of v and w, respectively. Since the integral is a function of v, say, $\phi(v)$, the solution is of the form $u = \phi(v) + \psi(w)$. In terms of x and t, by (2), we thus have

$$u(x, t) = \phi(x + ct) + \psi(x - ct). \tag{4}$$

This is known as **d'Alembert's solution**[1] of the wave equation (1).

Its derivation was much more elegant than the method in Sec. 12.3, but d'Alembert's method is special, whereas the use of Fourier series applies to various equations, as we shall see.

D'Alembert's Solution Satisfying the Initial Conditions

$$\text{(a)}\quad u(x, 0) = f(x), \qquad \text{(b)}\quad u_t(x, 0) = g(x). \tag{5}$$

These are the same as (3) in Sec. 12.3. By differentiating (4) we have

$$u_t(x, t) = c\phi'(x + ct) - c\psi'(x - ct) \tag{6}$$

[1]JEAN LE ROND D'ALEMBERT (1717–1783), French mathematician, also known for his important work in mechanics.

We mention that the general theory of PDEs provides a systematic way for finding the transformation (2) that simplifies (1). See Ref. [C8] in App. 1.

where primes denote derivatives with respect to the *entire* arguments $x + ct$ and $x - ct$, respectively, and the minus sign comes from the chain rule. From (4)–(6) we have

$$u(x, 0) = \phi(x) + \psi(x) = f(x), \tag{7}$$

$$u_t(x, 0) = c\phi'(x) - c\psi'(x) = g(x). \tag{8}$$

Dividing (8) by c and integrating with respect to x, we obtain

$$\phi(x) - \psi(x) = k(x_0) + \frac{1}{c}\int_{x_0}^{x} g(s)\, ds, \qquad k(x_0) = \phi(x_0) - \psi(x_0). \tag{9}$$

If we add this to (7), then ψ drops out and division by 2 gives

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\, ds + \frac{1}{2} k(x_0). \tag{10}$$

Similarly, subtraction of (9) from (7) and division by 2 gives

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\, ds - \frac{1}{2} k(x_0). \tag{11}$$

In (10) we replace x by $x + ct$; we then get an integral from x_0 to $x + ct$. In (11) we replace x by $x - ct$ and get minus an integral from x_0 to $x - ct$ or plus an integral from $x - ct$ to x_0. Hence addition of $\phi(x + ct)$ and $\psi(x - ct)$ gives $u(x, t)$ [see (4)] in the form

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds. \tag{12}$$

If the initial velocity is zero, we see that this reduces to

$$u(x, t) = \tfrac{1}{2}[f(x + ct) + f(x - ct)], \tag{13}$$

in agreement with (17) in Sec. 12.3. You may show that because of the boundary conditions (2) in that section the function f must be odd and must have the period $2L$.

Our result shows that the two initial conditions [the functions $f(x)$ and $g(x)$ in (5)] determine the solution uniquely.

The solution of the wave equation by the Laplace transform method will be shown in Sec. 12.11.

Characteristics. Types and Normal Forms of PDEs

The idea of d'Alembert's solution is just a special instance of the **method of characteristics.** This concerns PDEs of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y) \tag{14}$$

(as well as PDEs in more than two variables). Equation (14) is called **quasilinear** because it is linear in the highest derivatives (but may be arbitrary otherwise). There are three types of PDEs (14), depending on the discriminant $AC - B^2$, as follows.

Type	Defining Condition	Example in Sec. 12.1
Hyperbolic	$AC - B^2 < 0$	Wave equation (1)
Parabolic	$AC - B^2 = 0$	Heat equation (2)
Elliptic	$AC - B^2 > 0$	Laplace equation (3)

Note that (1) and (2) in Sec. 12.1 involve t, but to have y as in (14), we set $y = ct$ in (1), obtaining $u_{tt} - c^2u_{xx} = c^2(u_{yy} - u_{xx}) = 0$. And in (2) we set $y = c^2t$, so that $u_t - c^2u_{xx} = c^2(u_y - u_{xx})$.

A, B, C may be functions of x, y, so that a PDE may be **of mixed type,** that is, of different type in different regions of the xy-plane. An important mixed-type PDE is the **Tricomi equation** (see Prob. 10).

Transformation of (14) to Normal Form. The normal forms of (14) and the corresponding transformations depend on the type of the PDE. They are obtained by solving the **characteristic equation** of (14), which is the ODE

$$Ay'^2 - 2By' + C = 0 \tag{15}$$

where $y' = dy/dx$ (note $-2B$, not $+2B$). The solutions of (15) are called the **characteristics** of (14), and we write them in the form $\Phi(x, y) = const$ and $\Psi(x, y) = const$. Then the transformations giving new variables v, w instead of x, y and the normal forms of (14) are as follows.

Type	New Variables		Normal Form
Hyperbolic	$v = \Phi$	$w = \Psi$	$u_{vw} = F_1$
Parabolic	$v = x$	$w = \Phi = \Psi$	$u_{ww} = F_2$
Elliptic	$v = \frac{1}{2}(\Phi + \Psi)$	$w = \dfrac{1}{2i}(\Phi - \Psi)$	$u_{vv} + u_{ww} = F_3$

Here, $\Phi = \Phi(x, y)$, $\Psi = \Psi(x, y)$, $F_1 = F_1(v, w, u, u_v, u_w)$, etc., and we denote u as function of v, w again by u, for simplicity. We see that the normal form of a hyperbolic PDE is as in d'Alembert's solution. In the parabolic case we get just one family of solutions $\Phi = \Psi$. In the elliptic case, $i = \sqrt{-1}$, and the characteristics are complex and are of minor interest. For derivation, see Ref. [GR3] in App. 1.

EXAMPLE 1 D'Alembert's Solution Obtained Systematically

The theory of characteristics gives d'Alembert's solution in a systematic fashion. To see this, we write the wave equation $u_{tt} - c^2u_{xx} = 0$ in the form (14) by setting $y = ct$. By the chain rule, $u_t = u_y y_t = cu_y$ and $u_{tt} = c^2u_{yy}$. Division by c^2 gives $u_{xx} - u_{yy} = 0$, as stated before. Hence the characteristic equation is $y'^2 - 1 = (y' + 1)(y' - 1) = 0$. The two families of solutions (characteristics) are $\Phi(x, y) = y + x = const$ and $\Psi(x, y) = y - x = const$. This gives the new variables $v = \Phi = y + x = ct + x$ and $w = \Psi = y - x = ct - x$ and d'Alembert's solution $u = f_1(x + ct) + f_2(x - ct)$. ∎

PROBLEM SET 12.4

1. Show that c is the speed of each of the two waves given by (4).
2. Show that because of the boundary conditions (2), Sec. 12.3, the function f in (13) of this section must be odd and of period $2L$.
3. If a steel wire 2 m in length weighs 0.9 nt (about 0.20 lb) and is stretched by a tensile force of 300 nt (about 67.4 lb), what is the corresponding speed of transverse waves?
4. What are the frequencies of the eigenfunctions in Prob. 3?
5. **Longitudinal Vibrations of an Elastic Bar or Rod.** These vibrations in the direction of the x-axis are modeled by the wave equation $u_{tt} = c^2 u_{xx}$, $c^2 = E/\rho$ (see Tolstov [C9], p. 275). If the rod is fastened at one end, $x = 0$, and free at the other, $x = L$, we have $u(0, t) = 0$ and $u_x(L, t) = 0$. Show that the motion corresponding to initial displacement $u(x, 0) = f(x)$ and initial velocity zero is

$$u = \sum_{n=0}^{\infty} A_n \sin p_n x \cos p_n ct,$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin p_n x \, dx, \qquad p_n = \frac{(2n+1)\pi}{2L}.$$

6–9 GRAPHING SOLUTIONS

Using (13), sketch or graph a figure (similar to Fig. 288 in Sec. 12.3) of the deflection $u(x, t)$ of a vibrating string (length $L = 1$, ends fixed, $c = 1$) starting with initial velocity 0 and initial deflection (k small, say, $k = 0.01$).

6. $f(x) = k \sin \pi x$
7. $f(x) = k(1 - \cos 2\pi x)$
8. $f(x) = kx(1 - x)$
9. $f(x) = k(x - x^3)$

10. **(Tricomi and Airy equations[2])** Show that the *Tricomi equation* $yu_{xx} + u_{yy} = 0$ is of mixed type. Obtain the **Airy equation** $G'' - yG = 0$ from the Tricomi equation by separation. (For solutions, see p. 446 of Ref. [GR1] listed in App. 1.)

11–20 NORMAL FORMS

Find the type, transform to normal form, and solve. (Show the details of your work.)

11. $u_{xy} - u_{yy} = 0$
12. $u_{xx} - 2u_{xy} + u_{yy} = 0$
13. $u_{xx} + 9u_{yy} = 0$
14. $u_{xx} + u_{xy} - 2u_{yy} = 0$
15. $u_{xx} + 2u_{xy} + u_{yy} = 0$
16. $xu_{xy} - yu_{yy} = 0$
17. $u_{xx} - 4u_{xy} + 4u_{yy} = 0$
18. $u_{xx} + 2u_{xy} + 5u_{yy} = 0$
19. $xu_{xx} - yu_{xy} = 0$
20. $u_{xx} - 4u_{xy} + 3u_{yy} = 0$

12.5 Heat Equation: Solution by Fourier Series

From the wave equation we now turn to the next "big" PDE, the **heat equation**

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u, \qquad c^2 = \frac{K}{\sigma\rho},$$

which gives the temperature $u(x, y, z, t)$ in a body of homogeneous material. Here c^2 is the thermal diffusivity, K the thermal conductivity, σ the specific heat, and ρ the density of the material of the body. $\nabla^2 u$ is the Laplacian of u, and with respect to Cartesian coordinates x, y, z,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

The heat equation was derived in Sec. 10.8. It is also called the **diffusion equation.**

As an important application, let us first consider the temperature in a long thin metal bar or wire of constant cross section and homogeneous material, which is oriented along the x-axis (Fig. 291) and is perfectly insulated laterally, so that heat flows in the x-direction

[2]SIR GEORGE BIDELL AIRY (1801–1892), English mathematician, known for his work in elasticity. FRANCESCO TRICOMI (1897–1978), Italian mathematician, who worked in integral equations.

Fig. 291. Bar under consideration

only. Then u depends only on x and time t, and the heat equation becomes the **one-dimensional heat equation**

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

This seems to differ only very little from the wave equation, which has a term u_{tt} instead of u_t, but we shall see that this will make the solutions of (1) behave quite differently from those of the wave equation.

We shall solve (1) for some important types of boundary and initial conditions. We begin with the case in which the ends $x = 0$ and $x = L$ of the bar are kept at temperature zero, so that we have the **boundary conditions**

$$u(0, t) = 0, \qquad u(L, t) = 0 \qquad \text{for all } t. \tag{2}$$

Furthermore, the initial temperature in the bar at time $t = 0$ is given, say, $f(x)$, so that we have the **initial condition**

$$u(x, 0) = f(x) \qquad [f(x) \text{ given}]. \tag{3}$$

Here we must have $f(0) = 0$ and $f(L) = 0$ because of (2).

We shall determine a solution $u(x, t)$ of (1) satisfying (2) and (3)—one initial condition will be enough, as opposed to two initial conditions for the wave equation. Technically, our method will parallel that for the wave equation in Sec. 12.3: a separation of variables, followed by the use of Fourier series. You may find a step-by-step comparison worthwhile.

***Step 1.* Two ODEs from the heat equation (1).** Substitution of a product $u(x, t) = F(x)G(t)$ into (1) gives $F\dot{G} = c^2 F''G$ with $\dot{G} = dG/dt$ and $F'' = d^2F/dx^2$. To separate the variables, we divide by c^2FG, obtaining

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F}. \tag{4}$$

The left side depends only on t and the right side only on x, so that both sides must equal a constant k (as in Sec. 12.3). You may show that for $k = 0$ or $k > 0$ the only solution $u = FG$ satisfying (2) is $u \equiv 0$. For negative $k = -p^2$ we have from (4)

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = -p^2.$$

Multiplication by the denominators gives immediately the two ODEs

$$F'' + p^2 F = 0 \tag{5}$$

and

$$\dot{G} + c^2p^2G = 0. \tag{6}$$

Step 2. **Satisfying the boundary conditions (2).** We first solve (5). A general solution is

$$F(x) = A \cos px + B \sin px. \tag{7}$$

From the boundary conditions (2) it follows that

$$u(0, t) = F(0)G(t) = 0 \qquad \text{and} \qquad u(L, t) = F(L)G(t) = 0.$$

Since $G \equiv 0$ would give $u \equiv 0$, we require $F(0) = 0$, $F(L) = 0$ and get $F(0) = A = 0$ by (7) and then $F(L) = B \sin pL = 0$, with $B \neq 0$ (to avoid $F \equiv 0$); thus,

$$\sin pL = 0, \qquad \text{hence} \qquad p = \frac{n\pi}{L}, \qquad n = 1, 2, \cdots.$$

Setting $B = 1$, we thus obtain the following solutions of (5) satisfying (2):

$$F_n(x) = \sin \frac{n\pi x}{L}, \qquad n = 1, 2, \cdots.$$

(As in Sec. 12.3, we need not consider *negative* integral values of n.)

All this was literally the same as in Sec. 12.3. From now on it differs since (6) differs from (6) in Sec. 12.3. We now solve (6). For $p = n\pi/L$, as just obtained, (6) becomes

$$\dot{G} + \lambda_n{}^2 G = 0 \qquad \text{where} \qquad \lambda_n = \frac{cn\pi}{L}.$$

It has the general solution

$$G_n(t) = B_n e^{-\lambda_n{}^2 t}, \qquad n = 1, 2, \cdots$$

where B_n is a constant. Hence the functions

$$u_n(x, t) = F_n(x)G_n(t) = B_n \sin \frac{n\pi x}{L} e^{-\lambda_n{}^2 t} \qquad (n = 1, 2, \cdots) \tag{8}$$

are solutions of the heat equation (1), satisfying (2). These are the **eigenfunctions** of the problem, corresponding to the **eigenvalues** $\lambda_n = cn\pi/L$.

Step 3. **Solution of the entire problem. Fourier series.** So far we have solutions (8) satisfying the boundary conditions (2). To obtain a solution that also satisfies the initial condition (3), we consider a series of these eigenfunctions,

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\lambda_n{}^2 t} \qquad \left(\lambda_n = \frac{cn\pi}{L}\right). \tag{9}$$

From this and (3) we have

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x).$$

Hence for (9) to satisfy (3), the B_n's must be the coefficients of the **Fourier sine series,** as given by (4) in Sec. 11.3; thus

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \qquad (n = 1, 2, \cdots). \tag{10}$$

The solution of our problem can be established, assuming that $f(x)$ is piecewise continuous (see Sec. 6.1) on the interval $0 \leqq x \leqq L$ and has one-sided derivatives (see Sec. 11.1) at all interior points of that interval; that is, under these assumptions the series (9) with coefficients (10) is the solution of our physical problem. A proof requires knowledge of uniform convergence and will be given at a later occasion (Probs. 19, 20 in Problem Set 15.5).

Because of the exponential factor, all the terms in (9) approach zero as t approaches infinity. The rate of decay increases with n.

EXAMPLE 1 Sinusoidal Initial Temperature

Find the temperature $u(x, t)$ in a laterally insulated copper bar 80 cm long if the initial temperature is $100 \sin (\pi x/80)$ °C and the ends are kept at 0°C. How long will it take for the maximum temperature in the bar to drop to 50°C? First guess, then calculate. *Physical data for copper:* density 8.92 gm/cm^3, specific heat 0.092 cal/(gm °C), thermal conductivity 0.95 cal/(cm sec °C).

Solution. The initial condition gives

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{80} = f(x) = 100 \sin \frac{\pi x}{80}.$$

Hence, by inspection or from (9) we get $B_1 = 100$, $B_2 = B_3 = \cdots = 0$. In (9) we need $\lambda_1^2 = c^2\pi^2/L^2$, where $c^2 = K/(\sigma\rho) = 0.95/(0.092 \cdot 8.92) = 1.158$ [cm^2/sec]. Hence we obtain

$$\lambda_1^2 = 1.158 \cdot 9.870/80^2 = 0.001785 \text{ [sec}^{-1}\text{]}.$$

The solution (9) is

$$u(x, t) = 100 \sin \frac{\pi x}{80}\, e^{-0.001785t}.$$

Also, $100e^{-0.001785t} = 50$ when $t = (\ln 0.5)/(-0.001785) = 388$ [sec] ≈ 6.5 [min]. Does your guess, or at least its order of magnitude, agree with this result? ▩

EXAMPLE 2 Speed of Decay

Solve the problem in Example 1 when the initial temperature is $100 \sin (3\pi x/80)$ °C and the other data are as before.

Solution. In (9), instead of $n = 1$ we now have $n = 3$, and $\lambda_3^2 = 3^2\lambda_1^2 = 9 \cdot 0.001785 = 0.01607$, so that the solution now is

$$u(x, t) = 100 \sin \frac{3\pi x}{80}\, e^{-0.01607t}.$$

Hence the maximum temperature drops to 50°C in $t = (\ln 0.5)/(-0.01607) \approx 43$ [seconds], which is much faster (9 times as fast as in Example 1; why?).

Had we chosen a bigger n, the decay would have been still faster, and in a sum or series of such terms, each term has its own rate of decay, and terms with large n are practically 0 after a very short time. Our next example is of this type, and the curve in Fig. 292 corresponding to $t = 0.5$ looks almost like a sine curve; that is, it is practically the graph of the first term of the solution. ∎

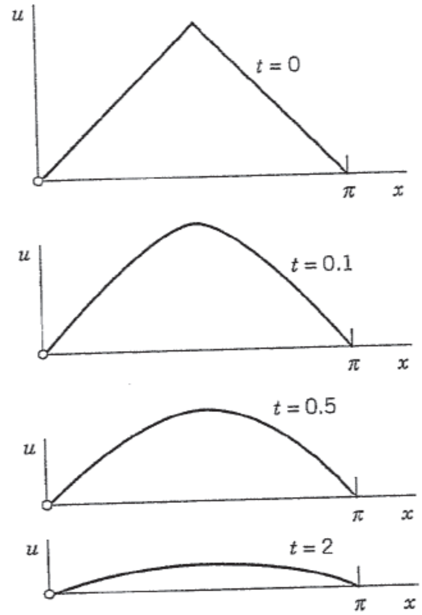


Fig. 292. Example 3. Decrease of temperature with time t for $L = \pi$ and $c = 1$

EXAMPLE 3 **"Triangular" Initial Temperature in a Bar**

Find the temperature in a laterally insulated bar of length L whose ends are kept at temperature 0, assuming that the initial temperature is

$$f(x) = \begin{cases} x & \text{if} \quad 0 < x < L/2, \\ L - x & \text{if} \quad L/2 < x < L. \end{cases}$$

(The uppermost part of Fig. 292 shows this function for the special $L = \pi$.)

Solution. From (10) we get

$$B_n = \frac{2}{L}\left(\int_0^{L/2} x \sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} (L - x)\sin\frac{n\pi x}{L}\,dx\right). \tag{10*}$$

Integration gives $B_n = 0$ if n is even,

$$B_n = \frac{4L}{n^2\pi^2} \quad (n = 1, 5, 9, \cdots) \qquad \text{and} \qquad B_n = -\frac{4L}{n^2\pi^2} \quad (n = 3, 7, 11, \cdots).$$

(see also Example 4 in Sec. 11.3 with $k = L/2$). Hence the solution is

$$u(x, t) = \frac{4L}{\pi^2}\left[\sin\frac{\pi x}{L}\exp\left[-\left(\frac{c\pi}{L}\right)^2 t\right] - \frac{1}{9}\sin\frac{3\pi x}{L}\exp\left[-\left(\frac{3c\pi}{L}\right)^2 t\right] + - \cdots\right].$$

Figure 292 shows that the temperature decreases with increasing t, because of the heat loss due to the cooling of the ends.

Compare Fig. 292 and Fig. 288 in Sec. 12.3 and comment. ∎

EXAMPLE 4 **Bar with Insulated Ends. Eigenvalue 0**

Find a solution formula of (1), (3) with (2) replaced by the condition that both ends of the bar are insulated.

Solution. Physical experiments show that the rate of heat flow is proportional to the gradient of the temperature. Hence if the ends $x = 0$ and $x = L$ of the bar are insulated, so that no heat can flow through the ends, we have grad $u = u_x = \partial u/\partial x$ and the boundary conditions

$$(2^*) \qquad u_x(0, t) = 0, \qquad u_x(L, t) = 0 \qquad \text{for all } t.$$

Since $u(x, t) = F(x)G(t)$, this gives $u_x(0, t) = F'(0)G(t) = 0$ and $u_x(L, t) = F'(L)G(t) = 0$. Differentiating (7), we have $F'(x) = -Ap \sin px + Bp \cos px$, so that

$$F'(0) = Bp = 0 \qquad \text{and then} \qquad F'(L) = -Ap \sin pL = 0.$$

The second of these conditions gives $p = p_n = n\pi/L$, $(n = 0, 1, 2, \cdots)$. From this and (7) with $A = 1$ and $B = 0$ we get $F_n(x) = \cos(n\pi x/L)$, $(n = 0, 1, 2, \cdots)$. With G_n as before, this yields the eigenfunctions

$$(11) \qquad u_n(x, t) = F_n(x)G_n(t) = A_n \cos \frac{n\pi x}{L} e^{-\lambda_n^2 t} \qquad (n = 0, 1, \cdots)$$

corresponding to the eigenvalues $\lambda_n = cn\pi/L$. The latter are as before, but we now have the additional eigenvalue $\lambda_0 = 0$ and eigenfunction $u_0 = const$, which is the solution of the problem if the initial temperature $f(x)$ is constant. This shows the remarkable fact that ***a separation constant can very well be zero, and zero can be an eigenvalue.***

Furthermore, whereas (8) gave a Fourier sine series, we now get from (11) a Fourier cosine series

$$(12) \qquad u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-\lambda_n^2 t} \qquad \left(\lambda_n = \frac{cn\pi}{L}\right).$$

Its coefficients result from the initial condition (3),

$$u(x, 0) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} = f(x),$$

in the form (2), Sec. 11.3, that is,

$$(13) \qquad A_0 = \frac{1}{L}\int_0^L f(x)\, dx, \qquad A_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, \cdots.$$

EXAMPLE 5 **"Triangular" Initial Temperature in a Bar with Insulated Ends**

Find the temperature in the bar in Example 3, assuming that the ends are insulated (instead of being kept at temperature 0).

Solution. For the triangular initial temperature, (13) gives $A_0 = L/4$ and (see also Example 4 in Sec. 11.3 with $k = L/2$)

$$A_n = \frac{2}{L}\left[\int_0^{L/2} x \cos \frac{n\pi x}{L}\, dx + \int_{L/2}^{L} (L - x) \cos \frac{n\pi x}{L}\, dx\right] = \frac{2L}{n^2\pi^2}\left(2 \cos \frac{n\pi}{2} - \cos n\pi - 1\right).$$

Hence the solution (12) is

$$u(x, t) = \frac{L}{4} - \frac{8L}{\pi^2}\left\{\frac{1}{2^2} \cos \frac{2\pi x}{L} \exp\left[-\left(\frac{2c\pi}{L}\right)^2 t\right] + \frac{1}{6^2} \cos \frac{6\pi x}{L} \exp\left[-\left(\frac{6c\pi}{L}\right)^2 t\right] + \cdots\right\}.$$

We see that the terms decrease with increasing t, and $u \to L/4$ as $t \to \infty$; this is the mean value of the initial temperature. This is plausible because no heat can escape from this totally insulated bar. In contrast, the cooling of the ends in Example 3 led to heat loss and $u \to 0$, the temperature at which the ends were kept.

Steady Two-Dimensional Heat Problems. Laplace's Equation

We shall now extend our discussion from one to two space dimensions and consider the two-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2\nabla^2 u = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

for **steady** (that is, *time-independent*) problems. Then $\partial u/\partial t = 0$ and the heat equation reduces to **Laplace's equation**

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{14}$$

(which has already occurred in Sec. 10.8 and will be considered further in Secs. 12.7–12.10). A heat problem then consists of this PDE to be considered in some region R of the xy-plane and a given boundary condition on the boundary curve C of R. This is a **boundary value problem (BVP).** One calls it:

First BVP or **Dirichlet Problem** if u is prescribed on C (**"Dirichlet boundary condition"**)

Second BVP or **Neumann Problem** if the normal derivative $u_n = \partial u/\partial n$ is prescribed on C (**"Neumann boundary condition"**)

Third BVP, Mixed BVP, or **Robin Problem** if u is prescribed on a portion of C and u_n on the rest of C (**"Mixed boundary condition").**

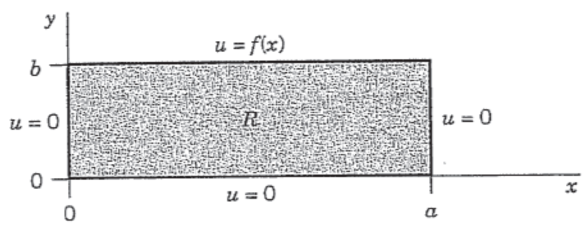


Fig. 293. Rectangle R and given boundary values

Dirichlet Problem in a Rectangle R (Fig. 293). We consider a Dirichlet problem for Laplace's equation (14) in a rectangle R, assuming that the temperature $u(x, y)$ equals a given function $f(x)$ on the upper side and 0 on the other three sides of the rectangle.

We solve this problem by separating variables. Substituting $u(x, y) = F(x)G(y)$ into (14) written as $u_{xx} = -u_{yy}$, dividing by FG, and equating both sides to a negative constant, we obtain

$$\frac{1}{F} \cdot \frac{d^2F}{dx^2} = -\frac{1}{G} \cdot \frac{d^2G}{dy^2} = -k.$$

From this we get

$$\frac{d^2F}{dx^2} + kF = 0,$$

and the left and right boundary conditions imply

$$F(0) = 0, \qquad \text{and} \qquad F(a) = 0.$$

This gives $k = (n\pi/a)^2$ and corresponding nonzero solutions

$$F(x) = F_n(x) = \sin \frac{n\pi}{a} x, \qquad n = 1, 2, \cdots. \tag{15}$$

The ODE for G with $k = (n\pi/a)^2$ then becomes

$$\frac{d^2G}{dy^2} - \left(\frac{n\pi}{a}\right)^2 G = 0.$$

Solutions are

$$G(y) = G_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}.$$

Now the boundary condition $u = 0$ on the lower side of R implies that $G_n(0) = 0$; that is, $G_n(0) = A_n + B_n = 0$ or $B_n = -A_n$. This gives

$$G_n(y) = A_n(e^{n\pi y/a} - e^{-n\pi y/a}) = 2A_n \sinh \frac{n\pi y}{a}.$$

From this and (15), writing $2A_n = A_n^*$, we obtain as the **eigenfunctions** of our problem

$$u_n(x, y) = F_n(x)G_n(y) = A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}. \tag{16}$$

These solutions satisfy the boundary condition $u = 0$ on the left, right, and lower sides.

To get a solution also satisfying the boundary condition $u(x, b) = f(x)$ on the upper side, we consider the infinite series

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y).$$

From this and (16) with $y = b$ we obtain

$$u(x, b) = f(x) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a}.$$

We can write this in the form

$$u(x, b) = \sum_{n=1}^{\infty} \left(A_n^* \sinh \frac{n\pi b}{a}\right) \sin \frac{n\pi x}{a}.$$

This shows that the expressions in the parentheses must be the Fourier coefficients b_n of $f(x)$; that is, by (4) in Sec. 11.3,

$$b_n = A_n^* \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx.$$

From this and (16) we see that the solution of our problem is

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \tag{17}$$

where

$$A_n^* = \frac{2}{a \sinh (n\pi b/a)} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx. \tag{18}$$

We have obtained this solution formally, neither considering convergence nor showing that the series for u, u_{xx}, and u_{yy} have the right sums. This can be proved if one assumes that f and f' are continuous and f'' is piecewise continuous on the interval $0 \leqq x \leqq a$. The proof is somewhat involved and relies on uniform convergence. It can be found in [C4] listed in App. 1.

Unifying Power of Methods. Electrostatics, Elasticity

The Laplace equation (14) also governs the electrostatic potential of electrical charges in any region that is free of these charges. Thus our steady-state heat problem can also be interpreted as an electrostatic potential problem. Then (17), (18) is the potential in the rectangle R when the upper side of R is at potential $f(x)$ and the other three sides are grounded.

Actually, in the steady-state case, the two-dimensional wave equation (to be considered in Secs. 12.7, 12.8) also reduces to (14). Then (17), (18) is the displacement of a rectangular elastic membrane (rubber sheet, drumhead) that is fixed along its boundary, with three sides lying in the xy-plane and the fourth side given the displacement $f(x)$.

This is another impressive demonstration of the ***unifying power*** of mathematics. It illustrates that ***entirely different physical systems may have the same mathematical model*** and can thus be treated by the same mathematical methods.

PROBLEM SET 12.5

1. **WRITING PROJECT. Wave and Heat Equations.** Compare the two PDEs with respect to type, general behavior of eigenfunctions, and kind of boundary and initial conditions and resulting practical problems. Also discuss the difference between Figs. 288 in Sec. 12.3 and 292.

2. **(Eigenfunctions)** Sketch (or graph) and compare the first three eigenfunctions (8) with $B_n = 1$, $c = 1$, $L = \pi$ for $t = 0, 0.2, 0.4, 0.6, 0.8, 1.0$.

3. **(Decay)** How does the rate of decay of (8) with fixed n depend on the specific heat, the density, and the thermal conductivity of the material?

4. If the first eigenfunction (8) of the bar decreases to half its value within 10 sec, what is the value of the diffusivity?

5–9 LATERALLY INSULATED BAR

A laterally insulated bar of length 10 cm and constant cross-sectional area 1 cm^2, of density 10.6 gm/cm^3, thermal conductivity 1.04 cal/(cm sec °C), and specific heat 0.056 cal/(gm °C) (this corresponds to silver, a good heat conductor) has initial temperature $f(x)$ and is kept at 0°C at the ends $x = 0$ and $x = 10$. Find the temperature $u(x, t)$ at later times. Here, $f(x)$ equals:

5. $f(x) = \sin 0.4\pi x$
6. $f(x) = \sin 0.1\pi x + \frac{1}{2}\sin 0.2\pi x$
7. $f(x) = 0.2x$ if $0 < x < 5$ and 0 otherwise
8. $f(x) = 1 - 0.2|x - 5|$
9. $f(x) = x$ if $0 < x < 2.5$, $f(x) = 2.5$ if $2.5 < x < 7.5$, $f(x) = 10 - x$ if $7.5 < x < 10$

10. **(Arbitrary temperatures at ends)** If the ends $x = 0$ and $x = L$ of the bar in the text are kept at constant temperatures U_1 and U_2, respectively, what is the temperature $u_I(x)$ in the bar after a long time (theoretically, as $t \to \infty$)? First guess, then calculate.
11. In Prob. 10 find the temperature at any time.
12. **(Changing end temperatures)** Assume that the ends of the bar in Probs. 5–9 have been kept at 100°C for a long time. Then at some instant, call it $t = 0$, the temperature at $x = L$ is suddenly changed to 0°C and kept at 0°C, whereas the temperature at $x = 0$ is kept at 100°C. Find the temperature in the middle of the bar at $t = 1, 2, 3, 10, 50$ sec. First guess, then calculate.

BAR UNDER ADIABATIC CONDITIONS

"Adiabatic" means no heat exchange with the neighborhood, because the bar is completely insulated, also at the ends. *Physical Information:* The heat flux at the ends is proportional to the value of $\partial u/\partial x$ there.

13. Show that for the completely insulated bar, $u_x(0, t) = 0$, $u_x(L, t) = 0$, $u(x, t) = f(x)$ and separation of variables gives the following solution, with A_n given by (2) in Sec. 11.3.

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}\, e^{-(cn\pi/L)^2 t}$$

14–19 Find the temperature in Prob. 13 with $L = \pi$, $c = 1$, and

14. $f(x) = x$
15. $f(x) = 1$
16. $f(x) = 0.5 \cos 4x$
17. $f(x) = \pi^2 - x^2$
18. $f(x) = \frac{1}{2}\pi - |x - \frac{1}{2}\pi|$
19. $f(x) = (x - \frac{1}{2}\pi)^2$

20. Find the temperature of the bar in Prob. 13 if the left end is kept at 0°C, the right end is insulated, and the initial temperature is $U_0 = const.$

21. **The boundary condition of heat transfer**

$$(19) \qquad -u_x(\pi, t) = k[u(\pi, t) - u_0]$$

applies when a bar of length π with $c = 1$ is laterally insulated, the left end $x = 0$ is kept at 0°C, and at the right end heat is flowing into air of constant temperature u_0. Let $k = 1$ for simplicity, and $u_0 = 0$. Show that a solution is $u(x, t) = \sin px\, e^{-p^2 t}$, where p is a solution of $\tan p\pi = -p$. Show graphically that this equation has infinitely many positive solutions $p_1, p_2, p_3, \cdots$, where $p_n > n - \frac{1}{2}$ and $\lim_{n\to\infty} (p_n - n + \frac{1}{2}) = 0$. (Formula (19) is also known as **radiation boundary condition,** but this is misleading; see Ref. [C3], p. 19.)

22. **(Discontinuous f)** Solve (1), (2), (3) with $L = \pi$ and $f(x) = U_0 = const$ ($\neq 0$) if $0 < x < \pi/2$, $f(x) = 0$ if $\pi/2 < x < \pi$.
23. **(Heat flux)** The *heat flux* of a solution $u(x, t)$ across $x = 0$ is defined by $\phi(t) = -Ku_x(0, t)$. Find $\phi(t)$ for the solution (9). Explain the name. Is it physically understandable that ϕ goes to 0 as $t \to \infty$?

OTHER HEAT EQUATIONS

24. **(Bar with heat generation)** If heat is generated at a constant rate throughout a bar of length $L = \pi$ with initial temperature $f(x)$ and the ends at $x = 0$ and π are kept at temperature 0, the heat equation is $u_t = c^2 u_{xx} + H$ with constant $H > 0$. Solve this problem. *Hint.* Set $u = v - Hx(x - \pi)/(2c^2)$.
25. **(Convection)** If heat in the bar in the text is free to flow through an end into the surrounding medium kept at 0°C, the PDE becomes $v_t = c^2 v_{xx} - \beta v$. Show that it can be reduced to the form (1) by setting $v(x, t) = u(x, t)w(t)$.
26. Consider $v_t = c^2 v_{xx} - v \quad (0 < x < L,\ t > 0)$, $v(0, t) = 0$, $v(L, t) = 0$, $v(x, 0) = f(x)$, where the term $-v$ models heat transfer to the surrounding medium kept at temperature 0. Reduce this PDE by setting $v(x, t) = u(x, t)w(t)$ with w such that u is given by (9), (10).
27. **(Nonhomogeneous heat equation)** Show that the problem modeled by

$$u_t - c^2 u_{xx} = Ne^{-\alpha x}$$

and (2), (3) can be reduced to a problem for the homogeneous heat equation by setting

$$u(x, t) = v(x, t) + w(x)$$

and determining w so that v satisfies the homogeneous PDE and the conditions $v(0, t) = v(L, t) = 0$, $v(x, 0) = f(x) - w(x)$. (The term $Ne^{-\alpha x}$ may represent heat loss due to radioactive decay in the bar.)

28–35 TWO-DIMENSIONAL PROBLEMS

28. **(Laplace equation)** Find the potential in the rectangle $0 \leqq x \leqq 20$, $0 \leqq y \leqq 40$ whose upper side is kept at potential 220 V and whose other sides are grounded.

29. Find the potential in the square $0 \leqq x \leqq 2$, $0 \leqq y \leqq 2$ if the upper side is kept at the potential $\sin \frac{1}{2}\pi x$ and the other sides are grounded.

30. **CAS PROJECT. Isotherms.** Find the steady-state solutions (temperatures) in the square plate in Fig. 294 with $a = 2$ satisfying the following boundary conditions. Graph isotherms.

(a) $u = \sin \pi x$ on the upper side, 0 on the others.

(b) $u = 0$ on the vertical sides, assuming that the other sides are perfectly insulated.

(c) Boundary conditions of your choice (such that the solution is not identically zero).

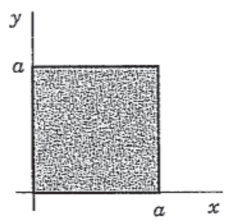


Fig. 294. Square plate

31. **(Heat flow in a plate)** The faces of the thin square plate in Fig. 294 with side $a = 24$ are perfectly insulated. The upper side is kept at 20°C and the other sides are kept at 0°C. Find the steady-state temperature $u(x, y)$ in the plate.

32. Find the steady-state temperature in the plate in Prob. 31 if the lower side is kept at U_0°C, the upper side at U_1°C, and the other sides are kept at 0°C. *Hint:* Split into two problems in which the boundary temperature is 0 on three sides for each problem.

33. **(Mixed boundary value problem)** Find the steady-state temperature in the plate in Prob. 31 with the upper and lower sides perfectly insulated, the left side kept at 0°C, and the right side kept at $f(y)$°C.

34. **(Radiation)** Find steady-state temperatures in the rectangle in Fig. 293 with the upper and left sides perfectly insulated and the right side radiating into a medium at 0°C according to $u_x(a, y) + hu(a, y) = 0$, $h > 0$ constant. (You will get many solutions since no condition on the lower side is given.)

35. Find formulas similar to (17), (18) for the temperature in the rectangle R of the text when the lower side of R is kept at temperature $f(x)$ and the other sides are kept at 0°C.

12.6 Heat Equation: Solution by Fourier Integrals and Transforms

Our discussion of the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

in the last section extends to bars of infinite length, which are good models of very long bars or wires (such as a wire of length, say, 300 ft). Then the role of Fourier series in the solution process will be taken by **Fourier integrals** (Sec. 11.7).

Let us illustrate the method by solving (1) for a bar that extends to infinity on both sides (and is laterally insulated as before). Then we do not have boundary conditions, but only the **initial condition**

$$u(x, 0) = f(x) \qquad (-\infty < x < \infty) \tag{2}$$

where $f(x)$ is the given initial temperature of the bar.

To solve this problem, we start as in the last section, substituting $u(x, t) = F(x)G(t)$ into (1). This gives the two ODEs

$$F'' + p^2 F = 0 \qquad \text{[see (5), Sec. 12.5]} \tag{3}$$

and

$$\dot{G} + c^2p^2G = 0 \qquad \text{[see (6), Sec. 12.5].} \tag{4}$$

Solutions are

$$F(x) = A \cos px + B \sin px \qquad \text{and} \qquad G(t) = e^{-c^2p^2t},$$

respectively, where A and B are any constants. Hence a solution of (1) is

$$u(x, t; p) = FG = (A \cos px + B \sin px)e^{-c^2p^2t}. \tag{5}$$

Here we had to choose the separation constant k negative, $k = -p^2$, because positive values of k would lead to an increasing exponential function in (5), which has no physical meaning.

Use of Fourier Integrals

Any series of functions (5), found in the usual manner by taking p as multiples of a fixed number, would lead to a function that is periodic in x when $t = 0$. However, since $f(x)$ in (2) is not assumed to be periodic, it is natural to use **Fourier integrals** instead of Fourier series. Also, A and B in (5) are arbitrary and we may regard them as functions of p, writing $A = A(p)$ and $B = B(p)$. Now, since the heat equation (1) is linear and homogeneous, the function

$$u(x, t) = \int_0^\infty u(x, t; p)\, dp = \int_0^\infty [A(p) \cos px + B(p) \sin px]\, e^{-c^2p^2t}\, dp \tag{6}$$

is then a solution of (1), provided this integral exists and can be differentiated twice with respect to x and once with respect to t.

Determination of $A(p)$ and $B(p)$ from the Initial Condition. From (6) and (2) we get

$$u(x, 0) = \int_0^\infty [A(p) \cos px + B(p) \sin px]\, dp = f(x). \tag{7}$$

This gives $A(p)$ and $B(p)$ in terms of $f(x)$; indeed, from (4) in Sec. 11.7 we have

$$A(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos pv\, dv, \qquad B(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin pv\, dv. \tag{8}$$

According to (1*), Sec. 11.9, our Fourier integral (7) with these $A(p)$ and $B(p)$ can be written

$$u(x, 0) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^{\infty} f(v) \cos (px - pv)\, dv \right] dp.$$

Similarly, (6) in this section becomes

$$u(x, t) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^{\infty} f(v) \cos (px - pv)\, e^{-c^2p^2t}\, dv \right] dp.$$

Assuming that we may reverse the order of integration, we obtain

$$\textbf{(9)} \qquad u(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \left[\int_0^{\infty} e^{-c^2p^2t} \cos(px - pv)\, dp \right] dv.$$

Then we can evaluate the inner integral by using the formula

$$(10) \qquad \int_0^{\infty} e^{-s^2} \cos 2bs\, ds = \frac{\sqrt{\pi}}{2} e^{-b^2}.$$

[A derivation of (10) is given in Problem Set 16.4 (Team Project 28).] This takes the form of our inner integral if we choose $p = s/(c\sqrt{t})$ as a new variable of integration and set

$$b = \frac{x - v}{2c\sqrt{t}}.$$

Then $2bs = (x - v)p$ and $ds = c\sqrt{t}\, dp$, so that (10) becomes

$$\int_0^{\infty} e^{-c^2p^2t} \cos(px - pv)\, dp = \frac{\sqrt{\pi}}{2c\sqrt{t}} \exp\left\{ -\frac{(x - v)^2}{4c^2t} \right\}.$$

By inserting this result into (9) we obtain the representation

$$\textbf{(11)} \qquad u(x, t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(v) \exp\left\{ -\frac{(x - v)^2}{4c^2t} \right\} dv.$$

Taking $z = (v - x)/(2c\sqrt{t})$ as a variable of integration, we get the alternative form

$$\textbf{(12)} \qquad u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2cz\sqrt{t})\, e^{-z^2}\, dz.$$

If $f(x)$ is bounded for all values of x and integrable in every finite interval, it can be shown (see Ref. [C10]) that the function (11) or (12) satisfies (1) and (2). Hence this function is the required solution in the present case.

EXAMPLE 1 Temperature in an Infinite Bar

Find the temperature in the infinite bar if the initial temperature is (Fig. 295)

$$f(x) = \begin{cases} U_0 = \textit{const} & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

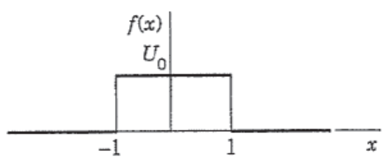


Fig. 295. Initial temperature in Example 1

Solution. From (11) we have

$$u(x, t) = \frac{U_0}{2c\sqrt{\pi t}} \int_{-1}^{1} \exp\left\{-\frac{(x-v)^2}{4c^2 t}\right\} dv.$$

If we introduce the above variable of integration z, then the integration over v from -1 to 1 corresponds to the integration over z from $(-1 - x)/(2c\sqrt{t})$ to $(1 - x)/(2c\sqrt{t})$, and

$$u(x, t) = \frac{U_0}{\sqrt{\pi}} \int_{-(1+x)/(2c\sqrt{t})}^{(1-x)/(2c\sqrt{t})} e^{-z^2}\, dz \qquad (t > 0). \tag{13}$$

We mention that this integral is not an elementary function, but can be expressed in terms of the error function, whose values have been tabulated. (Table A4 in App. 5 contains a few values; larger tables are listed in Ref. [GR1] in App. 1. See also CAS Project 10, p. 568.) Figure 296 shows $u(x, t)$ for $U_0 = 100°C$, $c^2 = 1\ \text{cm}^2/\text{sec}$, and several values of t. ■

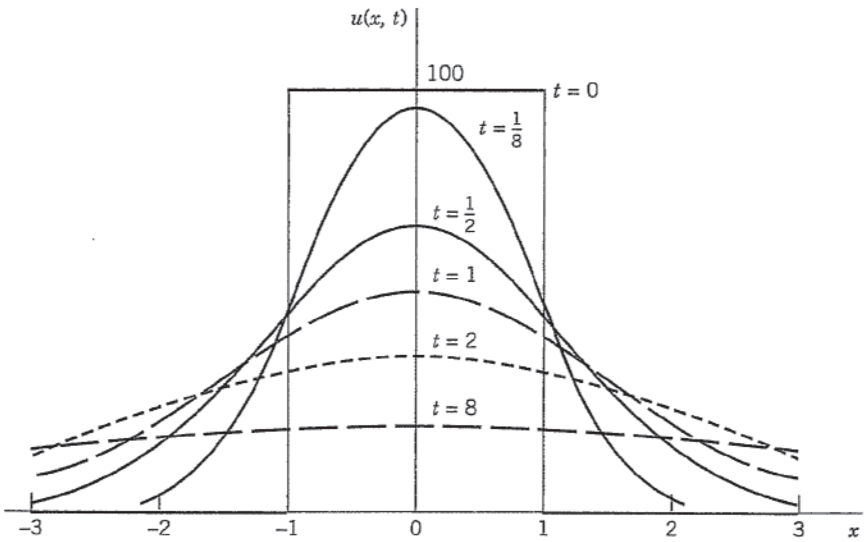


Fig. 296. Solution $u(x, t)$ in Example 1 for $U_0 = 100°C$, $c^2 = 1\ \text{cm}^2/\text{sec}$, and several values of t

Use of Fourier Transforms

The Fourier transform is closely related to the Fourier integral, from which we obtained the transform in Sec. 11.9. And the transition to the Fourier cosine and sine transform in Sec. 11.8 was even simpler. (You may perhaps wish to review this before going on.) Hence it should not surprise you that we can use these transforms for solving our present or similar problems. The Fourier transform applies to problems concerning the entire axis, and the Fourier cosine and sine transforms to problems involving the positive half-axis. Let us explain these transform methods by typical applications that fit our present discussion.

EXAMPLE 2 Temperature in the Infinite Bar in Example 1

Solve Example 1 using the Fourier transform.

Solution. The problem consists of the heat equation (1) and the initial condition (2), which in this example is

$$f(x) = U_0 = const \quad \text{if } |x| < 1 \qquad \text{and 0 otherwise.}$$

Our strategy is to take the Fourier transform with respect to x and then to solve the resulting ***ordinary*** DE in t. The details are as follows.

Let $\hat{u} = \mathcal{F}(u)$ denote the Fourier transform of u, ***regarded as a function of x.*** From (10) in Sec. 11.9 we see that the heat equation (1) gives

$$\mathcal{F}(u_t) = c^2\mathcal{F}(u_{xx}) = c^2(-w^2)\mathcal{F}(u) = -c^2w^2\hat{u}.$$

On the left, assuming that we may interchange the order of differentiation and integration, we have

$$\mathcal{F}(u_t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t e^{-iwx}\,dx = \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} u e^{-iwx}\,dx = \frac{\partial\hat{u}}{\partial t}.$$

Thus

$$\frac{\partial\hat{u}}{\partial t} = -c^2w^2\hat{u}.$$

Since this equation involves only a derivative with respect to t but none with respect to w, this is a first-order ***ordinary DE,*** with t as the independent variable and w as a parameter. By separating variables (Sec. 1.3) we get the general solution

$$\hat{u}(w, t) = C(w)e^{-c^2w^2t}$$

with the arbitrary "constant" $C(w)$ depending on the parameter w. The initial condition (2) yields the relationship $\hat{u}(w, 0) = C(w) = \hat{f}(w) = \mathcal{F}(f)$. Our intermediate result is

$$\hat{u}(w, t) = \hat{f}(w)e^{-c^2w^2t}.$$

The inversion formula (7), Sec. 11.9, now gives the solution

$$u(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{-c^2w^2t}e^{iwx}\,dw. \tag{14}$$

In this solution we may insert the Fourier transform

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-ivw}\,dv.$$

Assuming that we may invert the order of integration, we then obtain

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\left[\int_{-\infty}^{\infty} e^{-c^2w^2t}e^{i(wx-wv)}\,dw\right]dv.$$

By the Euler formula (3). Sec. 11.9, the integrand of the inner integral equals

$$e^{-c^2w^2t}\cos(wx - wv) + ie^{-c^2w^2t}\sin(wx - wv).$$

We see that its imaginary part is an odd function of w, so that its integral is 0. (More precisely, this is the principal part of the integral; see Sec. 16.4.) The real part is an even function of w, so that its integral from $-\infty$ to ∞ equals twice the integral from 0 to ∞:

$$u(x, t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_{0}^{\infty} e^{-c^2w^2t}\cos(wx - wv)\,dw\right]dv.$$

This agrees with (9) (with $p = w$) and leads to the further formulas (11) and (13). ▨

EXAMPLE 3 **Solution in Example 1 by the Method of Convolution**

Solve the heat problem in Example 1 by the method of convolution.

Solution. The beginning is as in Example 2 and leads to (14), that is,

$$u(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{-c^2w^2t}e^{iwx}\,dw. \tag{15}$$

Now comes the crucial idea. We recognize that this is of the form (13) in Sec. 11.9, that is,

$$u(x, t) = (f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(w)\hat{g}(w)e^{iwx}\, dw \tag{16}$$

where

$$\hat{g}(w) = \frac{1}{\sqrt{2\pi}}\, e^{-c^2w^2t}. \tag{17}$$

Since, by the definition of convolution [(11), Sec. 11.9],

$$(f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)\, dp, \tag{18}$$

as our next and last step we must determine the inverse Fourier transform g of $\hat{g}$. For this we can use formula 9 in Table III of Sec. 11.10,

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}}\, e^{-w^2/(4a)}$$

with a suitable a. With $c^2t = 1/(4a)$ or $a = 1/(4c^2t)$, using (17) we obtain

$$\mathcal{F}(e^{-x^2/(4c^2t)}) = \sqrt{2c^2t}\, e^{-c^2w^2t} = \sqrt{2c^2t}\sqrt{2\pi}\,\hat{g}(w).$$

Hence $\hat{g}$ has the inverse

$$\frac{1}{\sqrt{2c^2t}\sqrt{2\pi}}\, e^{-x^2/(4c^2t)}.$$

Replacing x with $x - p$ and substituting this into (18) we finally have

$$u(x, t) = (f * g)(x) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(p) \exp\left\{-\frac{(x-p)^2}{4c^2t}\right\} dp. \tag{19}$$

This solution formula of our problem agrees with (11). We wrote $(f * g)(x)$, without indicating the parameter t with respect to which we did not integrate. ■

EXAMPLE 4 Fourier Sine Transform Applied to the Heat Equation

If a laterally insulated bar extends from $x = 0$ to infinity, we can use the Fourier sine transform. We let the initial temperature be $u(x, 0) = f(x)$ and impose the boundary condition $u(0, t) = 0$. Then from the heat equation and (9b) in Sec. 11.8, since $f(0) = u(0, 0) = 0$, we obtain

$$\mathcal{F}_s(u_t) = \frac{\partial \hat{u}_s}{\partial t} = c^2\mathcal{F}_s(u_{xx}) = -c^2w^2\mathcal{F}_s(u) = -c^2w^2\hat{u}_s(w, t).$$

This is a first-order ODE $\partial\hat{u}_s/\partial t + c^2w^2\hat{u}_s = 0$. Its solution is

$$\hat{u}_s(w, t) = C(w)e^{-c^2w^2t}.$$

From the initial condition $u(x, 0) = f(x)$ we have $\hat{u}_s(w, 0) = \hat{f}_s(w) = C(w)$. Hence

$$\hat{u}_s(w, t) = \hat{f}_s(w)e^{-c^2w^2t}.$$

Taking the inverse Fourier sine transform and substituting

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(p) \sin wp\, dp$$

on the right, we obtain the solution formula

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(p) \sin wp \, e^{-c^2w^2t} \sin wx \, dp \, dw. \tag{20}$$

Figure 297 shows (20) with $c = 1$ for $f(x) = 1$ if $0 \leqq x \leqq 1$ and 0 otherwise, graphed over the xt-plane for $0 \leqq x \leqq 2$, $0.01 \leqq t \leqq 1.5$. Note that the curves of $u(x, t)$ for constant t resemble those in Fig. 296 on p. 565. ■

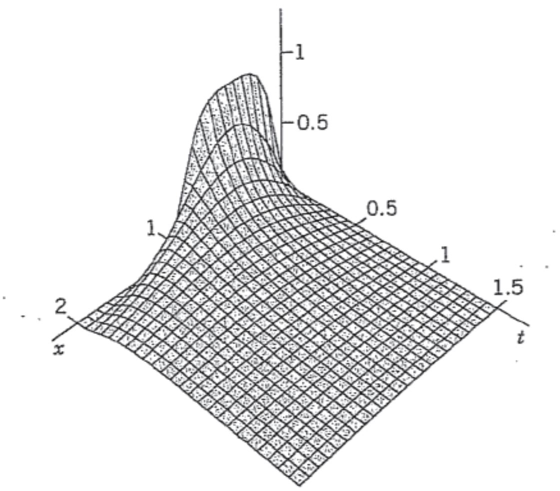


Fig. 297. Solution (20) in Example 4

PROBLEM SET 12.6

1–7 SOLUTION IN INTEGRAL FORM

Using (6), obtain the solution of (1) in integral form satisfying the initial condition $u(x, 0) = f(x)$, where

1. $f(x) = 1$ if $|x| < a$ and 0 otherwise
2. $f(x) = e^{-k|x|}$ $(k > 0)$
3. $f(x) = 1/(1 + x^2)$. [Use (15) in Sec. 11.7.]
4. $f(x) = (\sin x)/x$. [Use Prob. 4 in Sec. 11.7.]
5. $f(x) = (\sin \pi x)/x$. [Use Prob. 4 in Sec. 11.7.]
6. $f(x) = x$ if $|x| < 1$ and 0 otherwise
7. $f(x) = |x|$ if $|x| < 1$ and 0 otherwise.

8. Verify that u in Prob. 5 satisfies the initial condition.
9. **CAS PROJECT. Heat Flow. (a)** Graph the basic Fig. 296.

   (b) In (a) apply animation to "see" the heat flow in terms of the decrease of temperature.

   (c) Graph $u(x, t)$ with $c = 1$ as a surface over the upper xt-half-plane.
10. **CAS PROJECT. Error Function**

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-w^2} \, dw \tag{21}$$

This function is important in applied mathematics and physics (probability theory and statistics, thermodynamics, etc.) and fits our present discussion. Regarding it as a typical case of a special function defined by an integral that cannot be evaluated as in elementary calculus, do the following.

(a) Sketch or graph the **bell-shaped curve** [the curve of the integrand in (21)]. Show that erf x is odd. Show that

$$\int_a^b e^{-w^2} \, dw = \frac{\sqrt{\pi}}{2} (\operatorname{erf} b - \operatorname{erf} a),$$

$$\int_{-b}^b e^{-w^2} \, dw = \sqrt{\pi} \operatorname{erf} b.$$

(b) Obtain the Maclaurin series of erf x from that of the integrand. Use that series to compute a table of erf x for $x = 0(0.01)3$ (meaning $x = 0, 0.01, 0.02, \cdots, 3$).

(c) Obtain the values required in (b) by an integration command of your CAS. Compare accuracy.

(d) It can be shown that erf $(\infty) = 1$. Confirm this experimentally by computing erf x for large x.

(e) Let $f(x) = 1$ when $x > 0$ and 0 when $x < 0$. Using $\operatorname{erf}(\infty) = 1$, show that (12) then gives

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-x/(2c\sqrt{t})}^{\infty} e^{-z^2}\, dz$$

$$= \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(-\frac{x}{2c\sqrt{t}}\right) \qquad (t > 0).$$

(f) Express the temperature (13) in terms of the error function.

(g) Show that $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2}\, ds$

$$= \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right).$$

Here, the integral is the definition of the "distribution function of the normal distribution" to be discussed in Sec. 24.8.

12.7 Modeling: Membrane, Two-Dimensional Wave Equation

The vibrating string in Sec. 12.2 is a basic one-dimensional vibrational problem. Equally important is its two-dimensional analog, namely, the motion of an elastic membrane, such as a drumhead, that is stretched and then fixed along its edge. Indeed, setting up the model will proceed almost as in Sec. 12.2.

Physical Assumptions

1. The mass of the membrane per unit area is constant ("homogeneous membrane"). The membrane is perfectly flexible and offers no resistance to bending.
2. The membrane is stretched and then fixed along its entire boundary in the xy-plane. The tension per unit length T caused by stretching the membrane is the same at all points and in all directions and does not change during the motion.
3. The deflection $u(x, y, t)$ of the membrane during the motion is small compared to the size of the membrane, and all angles of inclination are small.

Although these assumptions cannot be realized exactly, they hold relatively accurately for small transverse vibrations of a thin elastic membrane, so that we shall obtain a good model, for instance, of a drumhead.

Derivation of the PDE of the Model ("Two-Dimensional Wave Equation") from Forces. As in Sec. 12.2 the model will consist of a PDE and additional conditions. The PDE will be obtained by the same method as in Sec. 12.2, namely, by considering the forces acting on a small portion of the physical system, the membrane in Fig. 298 on the next page, as it is moving up and down.

Since the deflections of the membrane and the angles of inclination are small, the sides of the portion are approximately equal to Δx and Δy. The tension T is the force per unit length. Hence the forces acting on the sides of the portion are approximately $T\,\Delta x$ and $T\,\Delta y$. Since the membrane is perfectly flexible, these forces are tangent to the moving membrane at every instant.

Horizontal Components of the Forces. We first consider the horizontal components of the forces. These components are obtained by multiplying the forces by the cosines of the angles of inclination. Since these angles are small, their cosines are close to 1. Hence

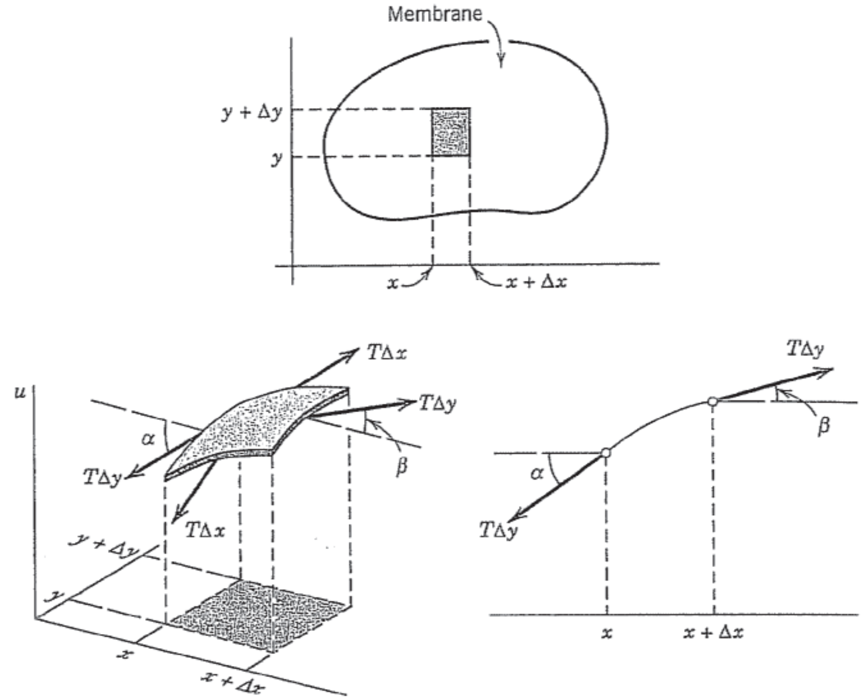


Fig. 298. Vibrating membrane

the horizontal components of the forces at opposite sides are approximately equal. Therefore, the motion of the particles of the membrane in a horizontal direction will be negligibly small. From this we conclude that we may regard the motion of the membrane as transversal; that is, each particle moves vertically.

Vertical Components of the Forces. These components along the right side and the left side are (Fig. 298), respectively,

$$T\,\Delta y \sin\beta \qquad \text{and} \qquad -T\,\Delta y \sin\alpha.$$

Here α and β are the values of the angle of inclination (which varies slightly along the edges) in the middle of the edges, and the minus sign appears because the force on the left side is directed downward. Since the angles are small, we may replace their sines by their tangents. Hence the resultant of those two vertical components is

$$\begin{aligned} T\,\Delta y\,(\sin\beta - \sin\alpha) &\approx T\,\Delta y\,(\tan\beta - \tan\alpha) \\ &= T\,\Delta y\,[u_x(x + \Delta x, y_1) - u_x(x, y_2)] \end{aligned} \tag{1}$$

where subscripts x denote partial derivatives and y_1 and y_2 are values between y and $y + \Delta y$. Similarly, the resultant of the vertical components of the forces acting on the other two sides of the portion is

$$T\,\Delta x\,[u_y(x_1, y + \Delta y) - u_y(x_2, y)] \tag{2}$$

where x_1 and x_2 are values between x and $x + \Delta x$.

Newton's Second Law Gives the PDE of the Model. By Newton's second law (see Sec. 2.4) the sum of the forces given by (1) and (2) is equal to the mass $\rho\Delta A$ of that small portion times the acceleration $\partial^2 u/\partial t^2$; here ρ is the mass of the undeflected membrane per unit area, and $\Delta A = \Delta x\, \Delta y$ is the area of that portion when it is undeflected. Thus

$$\rho\, \Delta x\, \Delta y\, \frac{\partial^2 u}{\partial t^2} = T\, \Delta y\, [u_x(x + \Delta x, y_1) - u_x(x, y_2)] + T\, \Delta x\, [u_y(x_1, y + \Delta y) - u_y(x_2, y)]$$

where the derivative on the left is evaluated at some suitable point $(\tilde{x}, \tilde{y})$ corresponding to that portion. Division by $\rho\, \Delta x\, \Delta y$ gives

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left[\frac{u_x(x + \Delta x, y_1) - u_x(x, y_2)}{\Delta x} + \frac{u_y(x_1, y + \Delta y) - u_y(x_2, y)}{\Delta y}\right].$$

If we let Δx and Δy approach zero, we obtain the PDE of the model

$$\text{(3)} \qquad \frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad c^2 = \frac{T}{\rho}.$$

This PDE is called the **two-dimensional wave equation.** The expression in parentheses is the Laplacian $\nabla^2 u$ of u (Sec. 10.8). Hence (3) can be written

$$\text{(3')} \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.$$

Solutions of the wave equation (3) will be obtained and discussed in the next section.

12.8 Rectangular Membrane. Double Fourier Series

The model of the vibrating membrane for obtaining the displacement $u(x, y, t)$ of a point (x, y) of the membrane from rest ($u = 0$) at time t is

$$\text{(1)} \qquad \frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\text{(2)} \qquad u = 0 \text{ on the boundary}$$

$$\text{(3a)} \qquad u(x, y, 0) = f(x, y)$$

$$\text{(3b)} \qquad u_t(x, y, 0) = g(x, y).$$

Here (1) is the **two-dimensional wave equation** with $c^2 = T/\rho$ just derived, (2) is the **boundary condition** (membrane fixed along the boundary in the xy-plane for all times $t \geqq 0$), and (3) are the **initial conditions** at $t = 0$, consisting of the given *initial displacement* (initial shape) $f(x, y)$ and the given *initial velocity* $g(x, y)$, where $u_t = \partial u/\partial t$. We see that these conditions are quite similar to those for the string in Sec. 12.2.

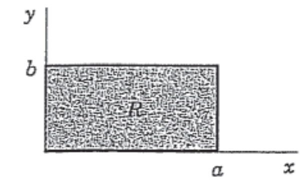


Fig. 299. Rectangular membrane

As a first important model, let us consider the **rectangular membrane** R in Fig. 299, which is simpler than the circular drumhead to follow. Then the boundary in (2) is the rectangle in Fig. 299. We shall solve this problem in three steps:

Step 1. By separating variables, setting $u(x, y, t) = F(x, y)G(t)$ and later $F(x, y) = H(x)Q(y)$ we obtain from (1) an ODE (4) for G and later from a PDE (5) for F two ODEs (6) and (7) for H and Q.

Step 2. From the solutions of those ODEs we determine solutions (13) of (1) ("**eigenfunctions**" u_{mn}) that satisfy the boundary condition (2).

Step 3. We compose the u_{mn} into a double series (14) solving the whole model (1), (2), (3).

Step 1. Three ODEs From the Wave Equation (1)

To obtain ODEs from (1), we apply two successive separations of variables. In the first separation we set $u(x, y, t) = F(x, y)G(t)$. Substitution into (1) gives

$$F\ddot{G} = c^2(F_{xx}G + F_{yy}G)$$

where subscripts denote partial derivatives and dots denote derivatives with respect to t. To separate the variables, we divide both sides by c^2FG:

$$\frac{\ddot{G}}{c^2G} = \frac{1}{F}(F_{xx} + F_{yy}).$$

Since the left side depends only on t, whereas the right side is independent of t, both sides must equal a constant. By a simple investigation we see that only negative values of that constant will lead to solutions that satisfy (2) without being identically zero; this is similar to Sec. 12.3. Denoting that negative constant by $-\nu^2$, we have

$$\frac{\ddot{G}}{c^2G} = \frac{1}{F}(F_{xx} + F_{yy}) = -\nu^2.$$

This gives two equations: for the **"time function"** $G(t)$ we have the ODE

$$\ddot{G} + \lambda^2 G = 0 \qquad \text{where } \lambda = c\nu, \tag{4}$$

and for the **"amplitude function"** $F(x, y)$ a PDE, called the *two-dimensional* **Helmholtz**[3] **equation**

$$F_{xx} + F_{yy} + \nu^2 F = 0. \tag{5}$$

[3]HERMANN VON HELMHOLTZ (1821–1894), German physicist, known for his basic work in thermodynamics, fluid flow, and acoustics.

Separation of the Helmholtz equation is achieved if we set $F(x, y) = H(x)Q(y)$. By substitution of this into (5) we obtain

$$\frac{d^2H}{dx^2}\, Q = -\left(H\,\frac{d^2Q}{dy^2} + \nu^2 HQ\right).$$

To separate the variables, we divide both sides by HQ, finding

$$\frac{1}{H}\,\frac{d^2H}{dx^2} = -\frac{1}{Q}\left(\frac{d^2Q}{dy^2} + \nu^2 Q\right).$$

Both sides must equal a constant, by the usual argument. This constant must be negative, say, $-k^2$, because only negative values will lead to solutions that satisfy (2) without being identically zero. Thus

$$\frac{1}{H}\,\frac{d^2H}{dx^2} = -\frac{1}{Q}\left(\frac{d^2Q}{dy^2} + \nu^2 Q\right) = -k^2.$$

This yields two ODEs for H and Q, namely,

$$\text{(6)}\qquad \frac{d^2H}{dx^2} + k^2 H = 0$$

and

$$\text{(7)}\qquad \frac{d^2Q}{dy^2} + p^2 Q = 0 \qquad \text{where } p^2 = \nu^2 - k^2.$$

Step 2. Satisfying the Boundary Condition

General solutions of (6) and (7) are

$$H(x) = A\cos kx + B\sin kx \qquad \text{and} \qquad Q(y) = C\cos py + D\sin py$$

with constant A, B, C, D. From $u = FG$ and (2) it follows that $F = HQ$ must be zero on the boundary, that is, on the edges $x = 0$, $x = a$, $y = 0$, $y = b$; see Fig. 299. This gives the conditions

$$H(0) = 0, \qquad H(a) = 0, \qquad Q(0) = 0, \qquad Q(b) = 0.$$

Hence $H(0) = A = 0$ and then $H(a) = B\sin ka = 0$. Here we must take $B \neq 0$ since otherwise $H(x) \equiv 0$ and $F(x, y) \equiv 0$. Hence $\sin ka = 0$ or $ka = m\pi$, that is,

$$k = \frac{m\pi}{a} \quad (m \text{ integer}).$$

In precisely the same fashion we conclude that $C = 0$ and p must be restricted to the values $p = n\pi/b$ where n is an integer. We thus obtain the solutions $H = H_m$, $Q = Q_n$, where

$$H_m(x) = \sin\frac{m\pi x}{a} \quad \text{and} \quad Q_n(y) = \sin\frac{n\pi y}{b}, \qquad \begin{array}{l} m = 1, 2, \cdots, \\ n = 1, 2, \cdots. \end{array}$$

As in the case of the vibrating string, it is not necessary to consider $m, n = -1, -2, \cdots$ since the corresponding solutions are essentially the same as for positive m and n, except for a factor -1. Hence the functions

$$(8) \qquad F_{mn}(x, y) = H_m(x)Q_n(y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad \begin{array}{l} m = 1, 2, \cdots, \\ n = 1, 2, \cdots, \end{array}$$

are solutions of the Helmholtz equation (5) that are zero on the boundary of our membrane.

Eigenfunctions and Eigenvalues. Having taken care of (5), we turn to (4). Since $p^2 = \nu^2 - k^2$ in (7) and $\lambda = c\nu$ in (4), we have

$$\lambda = c\sqrt{k^2 + p^2}.$$

Hence to $k = m\pi/a$ and $p = n\pi/b$ there corresponds the value

$$(9) \qquad \lambda = \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \qquad \begin{array}{l} m = 1, 2, \cdots, \\ n = 1, 2, \cdots, \end{array}$$

in the ODE (4). A corresponding general solution of (4) is

$$G_{mn}(t) = B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t.$$

It follows that the functions $u_{mn}(x, y, t) = F_{mn}(x, y)G_{mn}(t)$, written out

$$(10) \qquad u_{mn}(x, y, t) = (B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

with λ_{mn} according to (9), are solutions of the wave equation (1) that are zero on the boundary of the rectangular membrane in Fig. 299. These functions are called the **eigenfunctions** or *characteristic functions,* and the numbers λ_{mn} are called the **eigenvalues** or *characteristic values* of the vibrating membrane. The frequency of u_{mn} is $\lambda_{mn}/2\pi$.

Discussion of Eigenfunctions. It is very interesting that, depending on a and b, several functions F_{mn} may correspond to the same eigenvalue. Physically this means that there may exist vibrations having the same frequency but entirely different **nodal lines** (curves of points on the membrane that do not move). Let us illustrate this with the following example.

EXAMPLE 1 **Eigenvalues and Eigenfunctions of the Square Membrane**

Consider the square membrane with $a = b = 1$. From (9) we obtain its eigenvalues

$$\lambda_{mn} = c\pi\sqrt{m^2 + n^2}. \tag{11}$$

Hence $\lambda_{mn} = \lambda_{nm}$, but for $m \neq n$ the corresponding functions

$$F_{mn} = \sin m\pi x \sin n\pi y \qquad \text{and} \qquad F_{nm} = \sin n\pi x \sin m\pi y$$

are certainly different. For example, to $\lambda_{12} = \lambda_{21} = c\pi\sqrt{5}$ there correspond the two functions

$$F_{12} = \sin \pi x \sin 2\pi y \qquad \text{and} \qquad F_{21} = \sin 2\pi x \sin \pi y.$$

Hence the corresponding solutions

$$u_{12} = (B_{12} \cos c\pi\sqrt{5}t + B^*_{12} \sin c\pi\sqrt{5}t)F_{12} \qquad \text{and} \qquad u_{21} = (B_{21} \cos c\pi\sqrt{5}t + B^*_{21} \sin c\pi\sqrt{5}t)F_{21}$$

have the nodal lines $y = \frac{1}{2}$ and $x = \frac{1}{2}$, respectively (see Fig. 300). Taking $B_{12} = 1$ and $B^*_{12} = B^*_{21} = 0$, we obtain

$$u_{12} + u_{21} = \cos c\pi\sqrt{5}t\,(F_{12} + B_{21}F_{21}) \tag{12}$$

which represents another vibration corresponding to the eigenvalue $c\pi\sqrt{5}$. The nodal line of this function is the solution of the equation

$$F_{12} + B_{21}F_{21} = \sin \pi x \sin 2\pi y + B_{21} \sin 2\pi x \sin \pi y = 0$$

or, since $\sin 2\alpha = 2 \sin \alpha \cos \alpha$,

$$\sin \pi x \sin \pi y\,(\cos \pi y + B_{21} \cos \pi x) = 0. \tag{13}$$

This solution depends on the value of B_{21} (see Fig. 301).

From (11) we see that even more than two functions may correspond to the same numerical value of λ_{mn}. For example, the four functions F_{18}, F_{81}, F_{47}, and F_{74} correspond to the value

$$\lambda_{18} = \lambda_{81} = \lambda_{47} = \lambda_{74} = c\pi\sqrt{65}, \qquad \text{because} \qquad 1^1 + 8^2 = 4^2 + 7^2 = 65.$$

This happens because 65 can be expressed as the sum of two squares of positive integers in several ways. According to a theorem by Gauss, this is the case for every sum of two squares among whose prime factors there are at least two different ones of the form $4n + 1$ where n is a positive integer. In our case we have $65 = 5 \cdot 13 = (4 + 1)(12 + 1)$. ▩

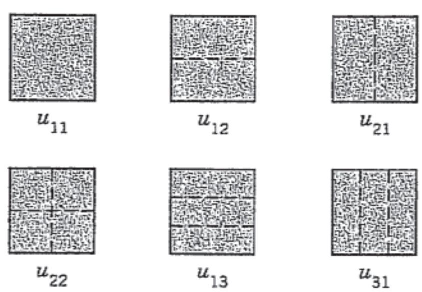


Fig. 300. Nodal lines of the solutions $u_{11}, u_{12}, u_{21}, u_{22}, u_{13}, u_{31}$ in the case of the square membrane

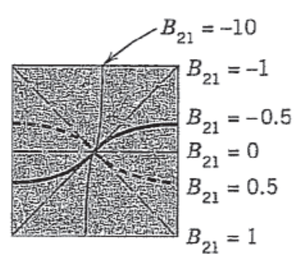


Fig. 301. Nodal lines of the solution (12) for some values of B_{21}

Step 3. Solution of the Model (1), (2), (3). Double Fourier Series

So far we have solutions (10) satisfying (1) and (2) only. To obtain the solution that also satisfies (3), we proceed as in Sec. 12.3. We consider the double series

$$\begin{aligned} u(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}(x, y, t) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (B_{mn} \cos \lambda_{mn} t + B^*_{mn} \sin \lambda_{mn} t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \end{aligned} \tag{14}$$

(without discussing convergence and uniqueness). From (14) and (3a), setting $t = 0$, we have

$$u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = f(x, y). \tag{15}$$

Suppose that $f(x, y)$ can be represented by (15). (Sufficient for this is the continuity of f, $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x \partial y$ in R.) Then (15) is called the **double Fourier series** of $f(x, y)$. Its coefficients can be determined as follows. Setting

$$K_m(y) = \sum_{n=1}^{\infty} B_{mn} \sin \frac{n\pi y}{b} \tag{16}$$

we can write (15) in the form

$$f(x, y) = \sum_{m=1}^{\infty} K_m(y) \sin \frac{m\pi x}{a} .$$

For fixed y this is the Fourier sine series of $f(x, y)$, considered as a function of x. From (4) in Sec. 11.3 we see that the coefficients of this expansion are

$$K_m(y) = \frac{2}{a} \int_0^a f(x, y) \sin \frac{m\pi x}{a} \, dx. \tag{17}$$

Furthermore, (16) is the Fourier sine series of $K_m(y)$, and from (4) in Sec. 11.3 it follows that the coefficients are

$$B_{mn} = \frac{2}{b} \int_0^b K_m(y) \sin \frac{n\pi y}{b} \, dy.$$

From this and (17) we obtain the **generalized Euler formula**

$$B_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy \qquad \begin{matrix} m = 1, 2, \cdots \\ n = 1, 2, \cdots \end{matrix} \tag{18}$$

for the **Fourier coefficients** of $f(x, y)$ in the double Fourier series (15).

The B_{mn} in (14) are now determined in terms of $f(x, y)$. To determine the B^*_{mn}, we differentiate (14) termwise with respect to t; using (3b), we obtain

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B^*_{mn}\lambda_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = g(x, y).$$

Suppose that $g(x, y)$ can be developed in this double Fourier series. Then, proceeding as before, we find that the coefficients are

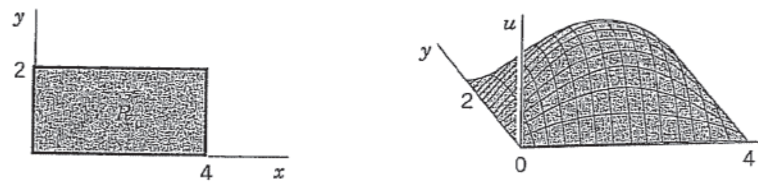
$$\text{(19)} \qquad B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy \qquad \begin{matrix} m = 1, 2, \cdots \\ n = 1, 2, \cdots. \end{matrix}$$

Result. *If f and g in* (3) *are such that u can be represented by* (14), *then* (14) *with coefficients* (18) *and* (19) *is the solution of the model* (1), (2), (3).

EXAMPLE 2 **Vibration of a Rectangular Membrane**

Find the vibrations of a rectangular membrane of sides $a = 4$ ft and $b = 2$ ft (Fig. 302) if the tension is 12.5 lb/ft, the density is 2.5 slugs/ft^2 (as for light rubber), the initial velocity is 0, and the initial displacement is

$$\text{(20)} \qquad f(x, y) = 0.1(4x - x^2)(2y - y^2) \text{ ft.}$$



Membrane Initial displacement

Fig. 302. Example 2

Solution. $c^2 = T/\rho = 12.5/2.5 = 5$ [ft^2/sec^2]. Also, $B^*_{mn} = 0$ from (19). From (18) and (20),

$$B_{mn} = \frac{4}{4\cdot 2}\int_0^2\int_0^4 0.1(4x - x^2)(2y - y^2)\sin\frac{m\pi x}{4}\sin\frac{n\pi y}{2}\,dx\,dy$$

$$= \frac{1}{20}\int_0^4 (4x - x^2)\sin\frac{m\pi x}{4}\,dx\int_0^2 (2y - y^2)\sin\frac{n\pi y}{2}\,dy.$$

Two integrations by parts give for the first integral on the right

$$\frac{128}{m^3\pi^3}[1 - (-1)^m] = \frac{256}{m^3\pi^3} \qquad (m \text{ odd})$$

and for the second integral

$$\frac{16}{n^3\pi^3}[1 - (-1)^n] = \frac{32}{n^3\pi^3} \qquad (n \text{ odd}).$$

For even m or n we get 0. Together with the factor 1/20 we thus have $B_{mn} = 0$ if m or n is even and

$$B_{mn} = \frac{256\cdot 32}{20m^3n^3\pi^6} \approx \frac{0.426\,050}{m^3n^3} \qquad (m \text{ and } n \text{ both odd}).$$

From this, (9), and (14) we obtain the answer

$$u(x, y, t) = 0.426\,050 \sum_{m,n \text{ odd}} \sum \frac{1}{m^3 n^3} \cos\left(\frac{\sqrt{5}\pi}{4} \sqrt{m^2 + 4n^2}\right) t \sin\frac{m\pi x}{4} \sin\frac{n\pi y}{2}$$

$$(21) \qquad = 0.426\,050 \left(\cos\frac{\sqrt{5}\pi\sqrt{5}}{4} t \sin\frac{\pi x}{4} \sin\frac{\pi y}{2} + \frac{1}{27}\cos\frac{\sqrt{5}\pi\sqrt{37}}{4} t \sin\frac{\pi x}{4} \sin\frac{3\pi y}{2}\right.$$

$$\left. + \frac{1}{27}\cos\frac{\sqrt{5}\pi\sqrt{13}}{4} t \sin\frac{3\pi x}{4} \sin\frac{\pi y}{2} + \frac{1}{729}\cos\frac{\sqrt{5}\pi\sqrt{45}}{4} t \sin\frac{3\pi x}{4} \sin\frac{3\pi y}{2} + \cdots\right).$$

To discuss this solution, we note that the first term is very similar to the initial shape of the membrane, has no nodal lines, and is by far the dominating term because the coefficients of the next terms are much smaller. The second term has two horizontal nodal lines ($y = 2/3, 4/3$), the third term two vertical ones ($x = 4/3, 8/3$), the fourth term two horizontal and two vertical ones, and so on.

PROBLEM SET 12.8

1. **(Frequency)** How does the frequency of the eigenfunctions of the rectangular membrane change if (a) we double the tension, (b) we take a membrane of half the mass of the original one, (c) we double the sides of the membrane? (Give reason.)

SQUARE MEMBRANE

2. Determine and sketch the nodal lines of the eigenfunctions of the square membrane for $m = 1, 2, 3, 4$ and $n = 1, 2, 3, 4$.

3–8 **Double Fourier Series.** Represent $f(x, y)$ by a series (15), where $0 < x < 1$, $0 < y < 1$.

3. $f(x, y) = 1$
4. $f(x, y) = x$
5. $f(x, y) = y$
6. $f(x, y) = x + y$
7. $f(x, y) = xy$
8. $f(x, y) = xy(1 - x)(1 - y)$

9. **CAS PROJECT. Double Fourier Series. (a)** Write a program that gives and graphs partial sums of (15). Apply it to Probs. 4 and 5. Do the graphs show that those partial sums satisfy the boundary condition (3a)? Explain why. Why is the convergence rapid?

   (b) Do the tasks in (a) for Prob. 3. Graph a portion, say, $0 < x < \frac{1}{2}$, $0 < y < \frac{1}{2}$, of several partial sums on common axes, so that you can see how they differ. (See Fig. 303.)

   (c) Do the tasks in (b) for functions of your choice.

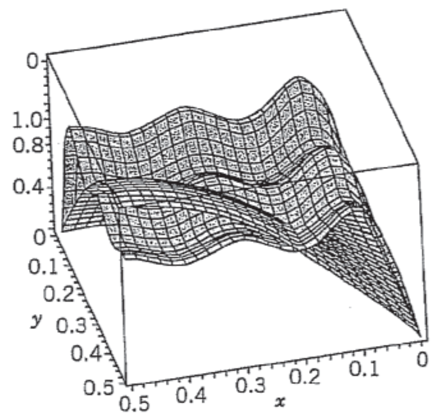


Fig. 303. Partial sums $S_{2,2}$ and $S_{10,10}$ in CAS Project 9b

10. **CAS EXPERIMENT. Quadruples of F_{mn}.** Write a program that gives you four numerically equal λ_{mn} in Example 1, so that four different F_{mn} correspond to it. Sketch the nodal lines of F_{18}, F_{81}, F_{47}, F_{74} in Example 1 and similarly for further F_{mn} that you will find.

11–13 **Deflection.** Find the deflection $u(x, y, t)$ of the square membrane of side π and $c^2 = 1$ if the initial velocity is 0 and the initial deflection is

11. $k \sin 2x \sin 5y$
12. $0.1 \sin x \sin y$
13. $0.1xy(\pi - x)(\pi - y)$

RECTANGULAR MEMBRANE

14. Verify the discussion of the terms of (21) in Example 2.
15. Repeat the task of Prob. 2 when $a = 4$ and $b = 1$.

16. Verify the calculation of B_{mn} in Example 2 by integration by parts.

17. Find eigenvalues of the rectangular membrane of sides $a = 2$ and $b = 1$ to which there correspond two or more different (independent) eigenfunctions.

18. (Minimum property) Show that among all rectangular membranes of the same area $A = ab$ and the same c the square membrane is that for which u_{11} [see (10)] has the lowest frequency.

19–22 **Double Fourier Series.** Represent $f(x, y)$ $(0 < x < a, 0 < y < b)$ by a double Fourier series (15).

19. $f(x, y) = k$

20. $f(x, y) = 0.25xy$

21. $f(x, y) = xy(a^2 - x^2)(b^2 - y^2)$

22. $f(x, y) = xy(a - x)(b - y)$

23. (Deflection) Find the deflection of the membrane of sides a and b with $c^2 = 1$ for the initial deflection

$$f(x, y) = \sin \frac{3\pi x}{a} \sin \frac{4\pi y}{b} \text{ and initial velocity 0.}$$

24. Repeat the task in Prob. 23 with $c^2 = 1$, for $f(x, y)$ as in Prob. 22 and initial velocity 0.

25. (Forced vibrations) Show that forced vibrations of a membrane are modeled by the PDE $u_{tt} = c^2\nabla^2 u + P/\rho$, where $P(x, y, t)$ is the external force per unit area acting perpendicular to the xy-plane.

12.9 Laplacian in Polar Coordinates. Circular Membrane. Fourier–Bessel Series

In boundary value problems for PDEs it is a ***general principle*** to use coordinates in which the formula for the boundary is as simple as possible. Since we want to discuss circular membranes (drumheads), we first transform the Laplacian in the wave equation (1), Sec. 12.8,

$$u_{tt} = c^2\nabla^2 u = c^2(u_{xx} + u_{yy}) \tag{1}$$

(subscripts denoting partial derivatives) into **polar coordinates**

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan \frac{y}{x}.$$

Hence $x = r \cos \theta$, $y = r \sin \theta$. By the chain rule (Sec. 9.6) we obtain

$$u_x = u_r r_x + u_\theta \theta_x.$$

Differentiating once more with respect to x and using the product rule and then again the chain rule gives

$$\begin{aligned} u_{xx} &= (u_r r_x)_x + (u_\theta \theta_x)_x \\ &= (u_r)_x r_x + u_r r_{xx} + (u_\theta)_x \theta_x + u_\theta \theta_{xx} \\ &= (u_{rr} r_x + u_{r\theta}\theta_x) r_x + u_r r_{xx} + (u_{\theta r} r_x + u_{\theta\theta}\theta_x)\theta_x + u_\theta \theta_{xx}. \end{aligned} \tag{2}$$

Also, by differentiation of r and θ we find

$$r_x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}, \qquad \theta_x = \frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2}.$$

Differentiating these two formulas again, we obtain

$$r_{xx} = \frac{r - xr_x}{r^2} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{y^2}{r^3}, \qquad \theta_{xx} = -y\left(-\frac{2}{r^3}\right) r_x = \frac{2xy}{r^4}.$$

We substitute all these expressions into (2). Assuming continuity of the first and second partial derivatives, we have $u_{r\theta} = u_{\theta r}$, and by simplifying,

$$u_{xx} = \frac{x^2}{r^2} u_{rr} - 2\frac{xy}{r^3} u_{r\theta} + \frac{y^2}{r^4} u_{\theta\theta} + \frac{y^2}{r^3} u_r + 2\frac{xy}{r^4} u_\theta. \tag{3}$$

In a similar fashion it follows that

$$u_{yy} = \frac{y^2}{r^2} u_{rr} + 2\frac{xy}{r^3} u_{r\theta} + \frac{x^2}{r^4} u_{\theta\theta} + \frac{x^2}{r^3} u_r - 2\frac{xy}{r^4} u_\theta. \tag{4}$$

By adding (3) and (4) we see that the **Laplacian of u in polar coordinates** is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{5}$$

Circular Membrane

Circular membranes occur in drums, pumps, microphones, telephones, and so on. This accounts for their great importance in engineering. Whenever a circular membrane is plane and its material is elastic, but offers no resistance to bending (this excludes thin metallic membranes!), its vibrations are modeled by the **two-dimensional wave equation in polar coordinates** obtained from (1) with $\nabla^2 u$ given by (5), that is,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) \qquad c^2 = \frac{T}{\rho}. \tag{6}$$

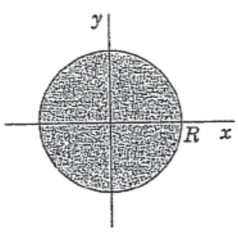


Fig. 304. Circular membrane

We shall consider a membrane of radius R (Fig. 304) and determine solutions $u(r, t)$ that are radially symmetric. (Solutions also depending on the angle θ will be discussed in the problem set.) Then $u_{\theta\theta} = 0$ in (6) and the model of the problem (the analog of (1), (2), (3) in Sec. 12.8) is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{7}$$

$$u(R, t) = 0 \text{ for all } t \geqq 0 \tag{8}$$

$$u(r, 0) = f(r) \tag{9a}$$

$$u_t(r, 0) = g(r). \tag{9b}$$

Here (8) means that the membrane is fixed along the boundary circle $r = R$. The initial deflection $f(r)$ and the initial velocity $g(r)$ depend only on r, not on θ, so that we can expect radially symmetric solutions $u(r, t)$.

Step 1. Two ODEs From the Wave Equation (7). Bessel's Equation

Using the **method of separation of variables,** we first determine solutions $u(r, t) = W(r)G(t)$. (We write W, not F because W depends on r, whereas F, used before, depended on x.) Substituting $u = WG$ and its derivatives into (7) and dividing the result by c^2WG, we get

$$\frac{\ddot{G}}{c^2G} = \frac{1}{W}\left(W'' + \frac{1}{r}W'\right)$$

where dots denote derivatives with respect to t and primes denote derivatives with respect to r. The expressions on both sides must equal a constant. This constant must be negative, say, $-k^2$, in order to obtain solutions that satisfy the boundary condition without being identically zero. Thus,

$$\frac{\ddot{G}}{c^2G} = \frac{1}{W}\left(W'' + \frac{1}{r}W'\right) = -k^2.$$

This gives the two linear ODEs

(10) $$\ddot{G} + \lambda^2 G = 0 \qquad \text{where } \lambda = ck$$

and

(11) $$W'' + \frac{1}{r}W' + k^2W = 0.$$

We can reduce (11) to Bessel's equation (Sec. 5.5) if we set $s = kr$. Then $1/r = k/s$ and, retaining the notation W for simplicity, we obtain by the chain rule

$$W' = \frac{dW}{dr} = \frac{dW}{ds}\frac{ds}{dr} = \frac{dW}{ds}k \qquad \text{and} \qquad W'' = \frac{d^2W}{ds^2}k^2.$$

By substituting this into (11) and omitting the common factor k^2 we have

(12) $$\frac{d^2W}{ds^2} + \frac{1}{s}\frac{dW}{ds} + W = 0.$$

This is **Bessel's equation** (1), Sec. 5.5, with parameter $\nu = 0$.

Step 2. Satisfying the Boundary Condition (8)

Solutions of (12) are the Bessel functions J_0 and Y_0 of the first and second kind (see Secs. 5.5, 5.6). But Y_0 becomes infinite at 0, so that we cannot use it because the deflection of the membrane must always remain finite. This leaves us with

(13) $$W(r) = J_0(s) = J_0(kr) \qquad (s = kr).$$

On the boundary $r = R$ we get $W(R) = J_0(kR) = 0$ from (8) (because $G \equiv 0$ would imply $u \equiv 0$). We can satisfy this condition because J_0 has (infinitely many) positive zeros, $s = \alpha_1, \alpha_2, \cdots$ (see Fig. 305), with numerical values

$$\alpha_1 = 2.4048, \quad \alpha_2 = 5.5201, \quad \alpha_3 = 8.6537, \quad \alpha_4 = 11.7915, \quad \alpha_5 = 14.9309$$

and so on. (For further values, consult your CAS or Ref. [GR1] in App. 1.) These zeros are slightly irregularly spaced, as we see. Equation (13) now implies

$$kR = \alpha_m \qquad \text{thus} \qquad k = k_m = \frac{\alpha_m}{R}, \qquad m = 1, 2, \cdots. \tag{14}$$

Hence the functions

$$W_m(r) = J_0(k_m r) = J_0\left(\frac{\alpha_m}{R}\, r\right), \qquad m = 1, 2, \cdots \tag{15}$$

are solutions of (11) that are zero on the boundary circle $r = R$.

Eigenfunctions and Eigenvalues. For W_m in (15), a corresponding general solution of (10) with $\lambda = \lambda_m = ck_m = c\alpha_m/R$ is

$$G_m(t) = A_m \cos \lambda_m t + B_m \sin \lambda_m t.$$

Hence the functions

$$u_m(r, t) = W_m(r)G_m(t) = (A_m \cos \lambda_m t + B_m \sin \lambda_m t)J_0(k_m r) \tag{16}$$

with $m = 1, 2, \cdots$ are solutions of the wave equation (7) satisfying the boundary condition (8). These are the **eigenfunctions** of our problem. The corresponding **eigenvalues** are λ_m.

The vibration of the membrane corresponding to u_m is called the mth **normal mode;** it has the frequency $\lambda_m/2\pi$ cycles per unit time. Since the zeros of the Bessel function J_0 are not regularly spaced on the axis (in contrast to the zeros of the sine functions appearing in the case of the vibrating string), the sound of a drum is entirely different from that of a violin. The forms of the normal modes can easily be obtained from Fig. 305 and are shown in Fig. 306. For $m = 1$, all the points of the membrane move up (or down) at the same time. For $m = 2$, the situation is as follows. The function $W_2(r) = J_0(\alpha_2 r/R)$ is zero for $\alpha_2 r/R = \alpha_1$, thus $r = \alpha_1 R/\alpha_2$. The circle $r = \alpha_1 R/\alpha_2$ is, therefore, **nodal line,** and when at some instant the central part of the membrane moves up, the outer part ($r > \alpha_1 R/\alpha_2$) moves down, and conversely. The solution $u_m(r, t)$ has $m - 1$ nodal lines, which are circles (Fig. 306).

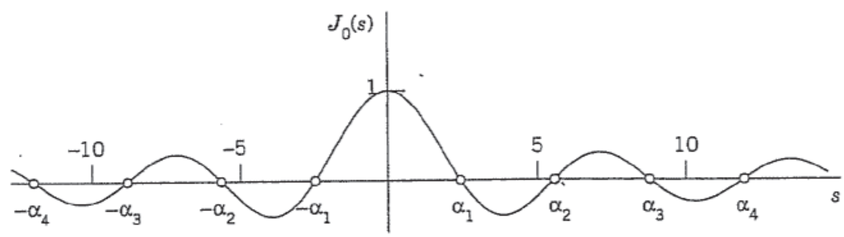


Fig. 305. Bessel function $J_0(s)$

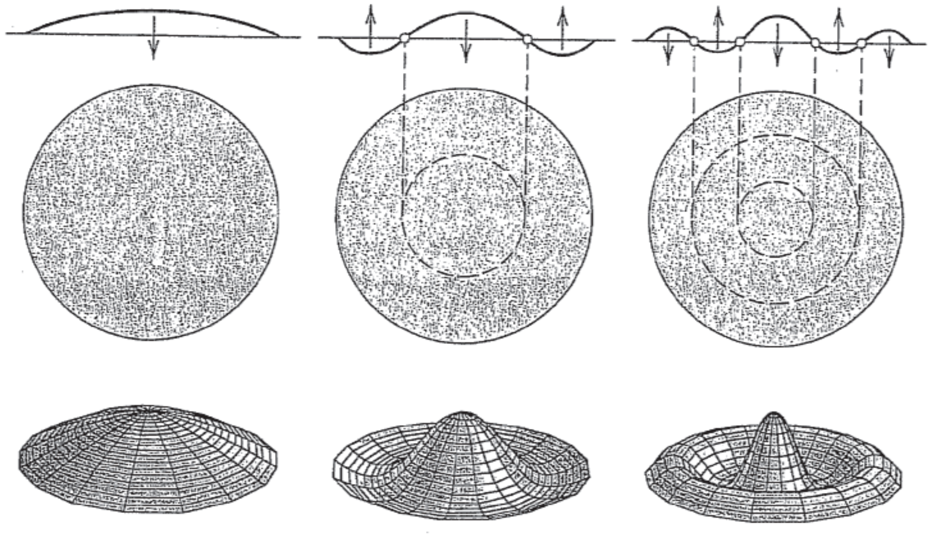


Fig. 306. Normal modes of the circular membrane in the case of vibrations independent of the angle

Step 3. Solution of the Entire Problem

To obtain a solution $u(r, t)$ that also satisfies the initial conditions (9), we may proceed as in the case of the string. That is, we consider the series

$$u(r, t) = \sum_{m=1}^{\infty} W_m(r)G_m(t) = \sum_{m=1}^{\infty} (A_m \cos \lambda_m t + B_m \sin \lambda_m t)\, J_0\left(\frac{\alpha_m}{R} r\right) \tag{17}$$

(leaving aside the problems of convergence and uniqueness). Setting $t = 0$ and using (9a), we obtain

$$u(r, 0) = \sum_{m=1}^{\infty} A_m J_0\left(\frac{\alpha_m}{R} r\right) = f(r). \tag{18}$$

Thus for the series (17) to satisfy the condition (9a), the constants A_m must be the coefficients of the **Fourier–Bessel series** (18) that represents $f(r)$ in terms of $J_0(\alpha_m r/R)$; that is [see (10) in Sec. 5.8 with $n = 0$, $\alpha_{0,m} = \alpha_m$, and $x = r$],

$$A_m = \frac{2}{R^2 J_1^{\,2}(\alpha_m)} \int_0^R r f(r) J_0\left(\frac{\alpha_m}{R} r\right) dr \qquad (m = 1, 2, \cdots). \tag{19}$$

Differentiability of $f(r)$ in the interval $0 \leqq r \leqq R$ is sufficient for the existence of the development (18); see Ref. [A13]. The coefficients B_m in (17) can be determined from (9b) in a similar fashion. Numeric values of A_m and B_m may be obtained from a CAS or by a numeric integration method, using tables of J_0 and J_1. However, numeric integration can sometimes be ***avoided,*** as the following example shows.

EXAMPLE 1 Vibrations of a Circular Membrane

Find the vibrations of a circular drumhead of radius 1 ft and density 2 slugs/ft^2 if the tension is 8 lb/ft, the initial velocity is 0, and the initial displacement is

$$f(r) = 1 - r^2 \text{ [ft]}.$$

Solution. $c^2 = T/\rho = 8/2 = 4$ [ft^2/sec^2]. Also $B_m = 0$, since the initial velocity is 0. From (19) and Example 3 in Sec. 5.8, since $R = 1$, we obtain

$$A_m = \frac{2}{J_1^{\,2}(\alpha_m)} \int_0^1 r(1 - r^2)\, J_0(\alpha_m r)\, dr = \frac{4J_2(\alpha_m)}{\alpha_m^{\,2} J_1^{\,2}(\alpha_m)} = \frac{8}{\alpha_m^{\,3} J_1(\alpha_m)}$$

where the last equality follows from (24c), Sec. 5.5, with $\nu = 1$, that is,

$$J_2(\alpha_m) = \frac{2}{\alpha_m} J_1(\alpha_m) - J_0(\alpha_m) = \frac{2}{\alpha_m} J_1(\alpha_m).$$

Table 9.5 on p. 409 of [GR1] gives α_m and $J_0'(\alpha_m)$. From this we get $J_1(\alpha_m) = -J_0'(\alpha_m)$ by (24b), Sec. 5.5, with $\nu = 0$, and compute the coefficients A_m:

m	α_m	$J_1(\alpha_m)$	$J_2(\alpha_m)$	A_m
1	2.40483	0.51915	0.43176	1.10801
2	5.52008	−0.34026	−0.12328	−0.13978
3	8.65373	0.27145	0.06274	0.04548
4	11.79153	−0.23246	−0.03943	−0.02099
5	14.93092	0.20655	0.02767	0.01164
6	18.07106	−0.18773	−0.02078	−0.00722
7	21.21164	0.17327	0.01634	0.00484
8	24.35247	−0.16170	−0.01328	−0.00343
9	27.49348	0.15218	0.01107	0.00253
10	30.63461	−0.14417	−0.00941	−0.00193

Thus

$$f(r) = 1.108 J_0(2.4048r) - 0.140 J_0(5.5201r) + 0.045 J_0(8.6537r) - \cdots .$$

We see that the coefficients decrease relatively slowly. The sum of the explicitly given coefficients in the table is 0.99915. The sum of *all* the coefficients should be 1. (Why?) Hence by the Leibniz test in App. A3.3 the partial sum of those terms gives about three correct decimals of the amplitude $f(r)$.

Since

$$\lambda_m = ck_m = c\alpha_m/R = 2\alpha_m,$$

from (17) we thus obtain the solution (with r measured in feet and t in seconds)

$$u(r, t) = 1.108 J_0(2.4048r) \cos 4.8097t - 0.140 J_0(5.5201r) \cos 11.0402t + 0.045 J_0(8.6537r) \cos 17.3075t - \cdots .$$

In Fig. 306, $m = 1$ gives an idea of the motion of the first term of our series, $m = 2$ of the second term, and $m = 3$ of the third term, so that we can "see" our result about as well as for a violin string in Sec. 12.3. ∎

PROBLEM SET 12.9

1. Why did we use polar coordinates in this section?

2. Work out the details of the calculation leading to the Laplacian in polar coordinates.

3. If u is independent of θ, then (5) reduces to $\nabla^2 u = u_{rr} + u_r/r$. Derive this directly from the Laplacian in Cartesian coordinates.

4. An **alternative form** of (5) is $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$. Derive this from (5).

5. **(Radial solution)** Show that the only solution of $\nabla^2 u = 0$ depending only on $r = \sqrt{x^2 + y^2}$ is $u = a \ln r + b$ with constant a and b.

6. **TEAM PROJECT. Series for Dirichlet and Neumann Problems**

   (a) Show that $u_n = r^n \cos n\theta$, $u_n = r^n \sin n\theta$, $n = 0, 1, \cdots$, are solutions of Laplace's equation $\nabla^2 u = 0$ with $\nabla^2 u$ given by (5). (What would u_n be in Cartesian coordinates? Experiment with small n.)

   (b) **Dirichlet problem** (See Sec. 12.5) Assuming that termwise differentiation is permissible, show that a solution of the Laplace equation in the disk $r < R$ satisfying the boundary condition $u(R, \theta) = f(\theta)$ (f given) is

$$
(20)\qquad u(r, \theta) = a_0 + \sum_{n=1}^{\infty}\left[a_n\left(\frac{r}{R}\right)^n \cos n\theta + b_n\left(\frac{r}{R}\right)^n \sin n\theta\right]
$$

   where a_n, b_n are the Fourier coefficients of f (see Sec. 11.1).

   (c) **Dirichlet problem** Solve the Dirichlet problem using (20) if $R = 1$ and the boundary values are $u(\theta) = -100$ volts if $-\pi < \theta < 0$, $u(\theta) = 100$ volts if $0 < \theta < \pi$. (Sketch this disk, indicate the boundary values.)

   (d) **Neumann problem** Show that the solution of the Neumann problem $\nabla^2 u = 0$ if $r < R$, $u_N(R, \theta) = f(\theta)$ (where $u_N = \partial u/\partial N$ is the directional derivative in the direction of the outer normal) is

$$
u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta)
$$

   with arbitrary A_0 and

$$
A_n = \frac{1}{\pi n R^{n-1}}\int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta,
$$

$$
B_n = \frac{1}{\pi n R^{n-1}}\int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta.
$$

   (e) **Compatibility condition** Show that (9), Sec. 10.4, imposes on $f(\theta)$ in (d) the *"compatibility condition"*

$$
\int_{-\pi}^{\pi} f(\theta)\, d\theta = 0.
$$

   (f) **Neumann problem** Solve $\nabla^2 u = 0$ in the annulus $1 < r < 3$ if $u_r(1, \theta) = \sin\theta$, $u_r(3, \theta) = 0$.

7–12 ELECTROSTATIC POTENTIAL. STEADY-STATE HEAT PROBLEMS

The electrostatic potential satisfies Laplace's equation $\nabla^2 u = 0$ in any region free of charges. Also the heat equation $u_t = c^2\nabla^2 u$ (Sec. 12.5) reduces to Laplace's equation if the temperature u is time-independent (**"steady-state case"**). Using (20), find the potential (equivalently: the steady-state temperature) in the disk $r < 1$ if the boundary values are (sketch them, to see what is going on).

7. $u(1, \theta) = 40 \cos^3 \theta$
8. $u(1, \theta) = 800 \sin^3 \theta$
9. $u(1, \theta) = 110$ if $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ and 0 otherwise
10. $u(1, \theta) = \theta$ if $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ and 0 otherwise
11. $u(1, \theta) = |\theta|$ if $-\pi < \theta < \pi$
12. $u(1, \theta) = \theta^2$ if $-\pi < \theta < \pi$

13. **CAS EXPERIMENT. Equipotential Lines.** Guess what the equipotential lines $u(r, \theta) = const$ in Probs. 9 and 11 may look like. Then graph some of them, using partial sums of the series.

14. **(Semidisk)** Find the electrostatic potential in the semidisk $r < 1$, $0 < \theta < \pi$ which equals $110\,\theta(\pi - \theta)$ on the semicircle $r = 1$ and 0 on the segment $-1 < x < 1$.

15. **(Semidisk)** Find the steady-state temperature in a semicircular thin plate $r < a$, $0 < \theta < \pi$ with the semicircle $r = a$ kept at constant temperature u_0 and the segment $-a < x < a$ at 0.

16. **(Invariance)** Show that $\nabla^2 u$ is invariant under translations $x^* = x + a$, $y^* = y + b$ and under rotations $x^* = x \cos\alpha - y \sin\alpha$, $y^* = x \sin\alpha + y \cos\alpha$.

CIRCULAR MEMBRANE

17. (Frequency) What happens to the frequency of an eigenfunction of a drum if you double the tension?

18. (Size of a drum) A small drum should have a higher fundamental frequency than a large one, tension and density being the same. How does this follow from our formulas?

19. (Tension) Find a formula for the tension required to produce a desired fundamental frequency f_1 of a drum.

20. CAS PROJECT. Normal Modes. **(a)** Graph the normal modes u_4, u_5, u_6 as in Fig. 306.

(b) Write a program for calculating the A_m's in Example 1 and extend the table to $m = 15$. Verify numerically that $\alpha_m \approx (m - \frac{1}{4})\pi$ and compute the error for $m = 1, \cdots, 10$.

(c) Graph the initial deflection $f(r)$ in Example 1 as well as the first three partial sums of the series. Comment on accuracy.

(d) Compute the radii of the nodal lines of u_2, u_3, u_4 when $R = 1$. How do these values compare to those of the nodes of the vibrating string of length 1? Can you establish any empirical laws by experimentation with further u_m?

21. (Nodal lines) Is it possible that for fixed c and R two or more u_m [see (16)] with different nodal lines correspond to the same eigenvalue? (Give a reason.)

22. Why is $A_1 + A_2 + \cdots = 1$ in Example 1? Compute the first few partial sums until you get 3-digit accuracy. What does this problem mean in the field of music?

23. (Nonzero initial velocity) Show that for (17) to satisfy (9b) we must have

$$(21) \qquad B_m = \frac{2}{c\alpha_m R J_1{}^2(\alpha_m)} \times \int_0^R r g(r) J_0(\alpha_m r/R)\, dr.$$

VIBRATIONS OF A CIRCULAR MEMBRANE DEPENDING ON BOTH r AND θ

24. (Separations) Show that substitution of $u = F(r, \theta)G(t)$ into the wave equation (6), that is,

$$(22) \qquad u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right)$$

gives an ODE and a PDE

$$(23) \qquad \ddot{G} + \lambda^2 G = 0, \qquad \text{where } \lambda = ck,$$

$$(24) \qquad F_{rr} + \frac{1}{r} F_r + \frac{1}{r^2} F_{\theta\theta} + k^2 F = 0.$$

Show that the PDE can now be separated by substituting $F = W(r)Q(\theta)$, giving

$$(25) \qquad Q'' + n^2 Q = 0,$$

$$(26) \qquad r^2 W'' + rW' + (k^2 r^2 - n^2) W = 0.$$

25. (Periodicity) Show that $Q(\theta)$ must be periodic with period 2π and, therefore, $n = 0, 1, 2, \cdots$ in (25) and (26). Show that this yields the solutions $Q_n = \cos n\theta$, $Q_n{}^* = \sin n\theta$, $W_n = J_n(kr)$, $n = 0, 1, \cdots$.

26. (Boundary condition) Show that the boundary condition

$$(27) \qquad u(R, \theta, t) = 0$$

leads to $k = k_{mn} = \alpha_{mn}/R$, where $s = \alpha_{mn}$ is the mth positive zero of $J_n(s)$.

27. (Solutions depending on both r and θ) Show that solutions of (22) satisfying (27) are (see Fig. 307)

$$(28) \qquad \begin{aligned} u_{mn} &= (A_{mn} \cos ck_{mn}t + B_{mn} \sin ck_{mn}t) \times \\ &\quad \times J_n(k_{mn}r) \cos n\theta \\ u^*_{mn} &= (A^*_{mn} \cos ck_{mn}t + B^*_{mn} \sin ck_{mn}t) \times \\ &\quad \times J_n(k_{mn}r) \sin n\theta \end{aligned}$$

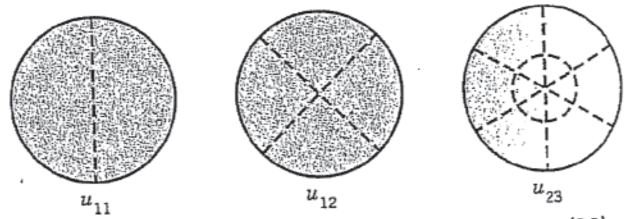


Fig. 307. Nodal lines of some of the solutions (28)

28. (Initial condition) Show that $u_t(r, \theta, 0) = 0$ gives $B_{mn} = 0$, $B^*_{mn} = 0$ in (28).

29. Show that $u^*_{m0} = 0$ and u_{m0} is identical with (16) in the current section.

30. (Semicircular membrane) Show that u_{11} represents the fundamental mode of a semicircular membrane and find the corresponding frequency when $c^2 = 1$ and $R = 1$.

12.10 Laplace's Equation in Cylindrical and Spherical Coordinates. Potential

Laplace's equation

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0 \tag{1}$$

is one of the most important PDEs in physics and its engineering applications. Here, x, y, z are Cartesian coordinates in space (Fig. 165 in Sec. 9.1), $u_{xx} = \partial^2 u/\partial x^2$, etc. The expression $\nabla^2 u$ is called the **Laplacian** of u. The theory of the solutions of (1) is called **potential theory.** Solutions of (1) that have ***continuous*** second partial derivatives are known as **harmonic functions.**

Laplace's equation occurs mainly in **gravitation, electrostatics** (see Theorem 3, Sec. 9.7). steady-state **heat flow** (Sec. 12.5), and **fluid flow** (to be discussed in Chap. 18.4).

Recall from Sec. 9.7 that the gravitational **potential** $u(x, y, z)$ at a point (x, y, z) resulting from a single mass located at a point (X, Y, Z) is

$$u(x, y, z) = \frac{c}{r} = \frac{c}{\sqrt{(x-X)^2 + (y-Y)^2 + (z-Z)^2}} \qquad (r > 0) \tag{2}$$

and u satisfies (1). Similarly, if mass is distributed in a region T in space with density $\rho(X, Y, Z)$, its potential at a point (x, y, z) not occupied by mass is

$$u(x, y, z) = k \iiint_T \frac{\rho(X, Y, Z)}{r}\, dX\, dY\, dZ. \tag{3}$$

It satisfies (1) because $\nabla^2(1/r) = 0$ (Sec. 9.7) and ρ is not a function of x, y, z.

Practical problems involving Laplace's equation are boundary value problems in a region T in space with boundary surface S. Such a problem is called (see also Sec. 12.5 for the two-dimensional case):

(I) First boundary value problem or **Dirichlet problem** if u is prescribed on S.

(II) Second boundary value problem or **Neumann problem** if the normal derivative $u_n = \partial u/\partial n$ is prescribed on S.

(III) Third or **mixed boundary value problem** or **Robin problem** if u is prescribed on a portion of S and u_n on the remaining portion of S.

Laplacian in Cylindrical Coordinates

The first step in solving a boundary value problem is generally the introduction of coordinates in which the boundary surface S has a simple representation. Cylindrical symmetry (a cylinder as a region T) calls for cylindrical coordinates r, θ, z related to x, y, z by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z \qquad \text{(Fig. 308, p. 588).} \tag{4}$$

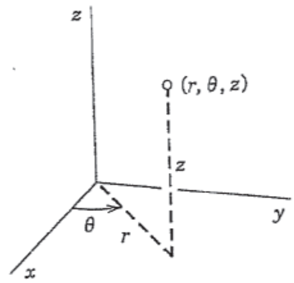


Fig. 308. Cylindrical coordinates

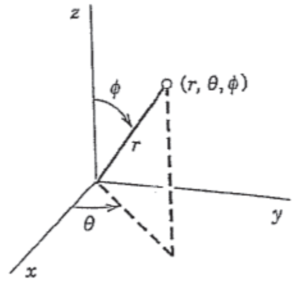


Fig. 309. Spherical coordinates

For these we get $\nabla^2 u$ immediately by adding u_{zz} to (5) in Sec. 12.9; thus,

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}. \tag{5}$$

Laplacian in Spherical Coordinates

Spherical symmetry (a ball as region T bounded by a sphere S) requires **spherical coordinates** r, θ, ϕ related to x, y, z by

$$x = r\cos\theta\sin\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\phi \qquad \text{(Fig. 309)}. \tag{6}$$

Using the chain rule (as in Sec. 12.9), we obtain $\nabla^2 u$ in spherical coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}. \tag{7}$$

We leave the details as an exercise. It is sometimes practical to write (7) in the form

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}\right]. \tag{7'}$$

Remark on Notation. Equation (6) is used in calculus and extends the familiar notation for polar coordinates. Unfortunately, some books use θ and ϕ interchanged, an extension of the notation $x = r\cos\phi$, $y = r\sin\phi$ for polar coordinates (used in some European countries).

Boundary Value Problem in Spherical Coordinates

We shall solve the following **Dirichlet problem** in spherical coordinates:

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right)\right] = 0. \tag{8}$$

$$u(R, \phi) = f(\phi) \tag{9}$$

$$\lim_{r\to\infty} u(r, \phi) = 0. \tag{10}$$

The PDE (8) follows from (7) by assuming that the solution u will not depend on θ because the Dirichlet condition (9) is independent of θ. This may be an electrostatic potential (or a temperature) $f(\phi)$ at which the sphere S: $r = R$ is kept. Condition (10) means that the potential at infinity will be zero.

Separating Variables by substituting $u(r, \phi) = G(r)H(\phi)$ into (8). Multiplying (8) by r^2, making the substitution and then dividing by GH, we obtain

$$\frac{1}{G}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = -\frac{1}{H\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH}{d\phi}\right).$$

By the usual argument both sides must be equal to a constant k. Thus we get the two ODEs

$$\text{(11)}\qquad \frac{1}{G}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = k \qquad \text{or} \qquad r^2\frac{d^2G}{dr^2} + 2r\frac{dG}{dr} = kG$$

and

$$\text{(12)}\qquad \frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH}{d\phi}\right) + kH = 0.$$

The solutions of (11) will take a simple form if we set $k = n(n+1)$. Then, writing $G' = dG/dr$, etc., we obtain

$$\text{(13)}\qquad r^2G'' + 2rG' - n(n+1)G = 0.$$

This is an **Euler–Cauchy equation.** From Sec. 2.5 we know that it has solutions $G = r^a$. Substituting this and dropping the common factor r^a gives

$$a(a-1) + 2a - n(n+1) = 0. \qquad \text{The roots are} \qquad a = n \quad \text{and} \quad -n-1.$$

Hence solutions are

$$\text{(14)}\qquad G_n(r) = r^n \qquad \text{and} \qquad G_n^*(r) = \frac{1}{r^{n+1}}.$$

We now solve (12). Setting $\cos\phi = w$, we have $\sin^2\phi = 1 - w^2$ and

$$\frac{d}{d\phi} = \frac{d}{dw}\frac{dw}{d\phi} = -\sin\phi\frac{d}{dw}.$$

Consequently, (12) with $k = n(n+1)$ takes the form

$$\text{(15)}\qquad \frac{d}{dw}\left[(1-w^2)\frac{dH}{dw}\right] + n(n+1)H = 0.$$

This is **Legendre's equation** (see Sec. 5.3), written out

$$(15') \qquad (1 - w^2)\frac{d^2H}{dw^2} - 2w\frac{dH}{dw} + n(n+1)H = 0.$$

For integer $n = 0, 1, \cdots$ the Legendre polynomials

$$H = P_n(w) = P_n(\cos\phi) \qquad n = 0, 1, \cdots,$$

are solutions of Legendre's equation (15). We thus obtain the following two sequences of solution $u = GH$ of Laplace's equation (8), with constant A_n and B_n, where $n = 0, 1, \cdots$,

$$(16) \qquad \text{(a)}\ u_n(r, \phi) = A_n r^n P_n(\cos\phi), \qquad \text{(b)}\ u_n^*(r, \phi) = \frac{B_n}{r^{n+1}} P_n(\cos\phi)$$

Use of Fourier–Legendre Series

Interior Problem: Potential Within the Sphere S. We consider a series of terms from (16a),

$$(17) \qquad u(r, \phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\phi) \qquad (r \leqq R).$$

Since S is given by $r = R$, for (17) to satisfy the Dirichlet condition (9) on the sphere S, we must have

$$(18) \qquad u(R, \phi) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\phi) = f(\phi);$$

that is, (18) must be the **Fourier–Legendre series** of $f(\phi)$. From (7) in Sec. 5.8 we get the coefficients

$$(19^*) \qquad A_n R^n = \frac{2n+1}{2}\int_{-1}^{1} \tilde{f}(w) P_n(w)\, dw$$

where $\tilde{f}(w)$ denotes $f(\phi)$ as a function of $w = \cos\phi$. Since $dw = -\sin\phi\, d\phi$, and the limits of integration -1 and 1 correspond to $\phi = \pi$ and $\phi = 0$, respectively, we also obtain

$$(19) \qquad A_n = \frac{2n+1}{2R^n}\int_0^{\pi} f(\phi) P_n(\cos\phi) \sin\phi\, d\phi, \qquad n = 0, 1, \cdots.$$

If $f(\phi)$ and $f'(\phi)$ are piecewise continuous on the interval $0 \leqq \phi \leqq \pi$, then the series (17) with coefficients (19) solves our problem for points inside the sphere because it can be shown that under these continuity assumptions the series (17) with coefficients (19) gives the derivatives occurring in (8) by termwise differentiation, thus justifying our derivation.

Exterior Problem: Potential Outside the Sphere S. Outside the sphere we cannot use the functions u_n in (16a) because they do not satisfy (10). But we can use the u_n^* in (16b), which do satisfy (10) (but could not be used inside S; why?). Proceeding as before leads to the solution of the exterior problem

$$u(r, \phi) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos \phi) \qquad (r \geqq R) \tag{20}$$

satisfying (8), (9), (10), with coefficients

$$B_n = \frac{2n+1}{2} R^{n+1} \int_0^{\pi} f(\phi) P_n(\cos \phi) \sin \phi \, d\phi. \tag{21}$$

The next example illustrates all this for a sphere of radius 1 consisting of two hemispheres that are separated by a small strip of insulating material along the equator, so that these hemispheres can be kept at different potentials (110 V and 0 V).

EXAMPLE 1 Spherical Capacitor

Find the potential inside and outside a spherical capacitor consisting of two metallic hemispheres of radius 1 ft separated by a small slit for reasons of insulation, if the upper hemisphere is kept at 110 V and the lower is grounded (Fig. 310).

Solution. The given boundary condition is (recall Fig. 309)

$$f(\phi) = \begin{cases} 110 & \text{if} \quad 0 \leqq \phi < \pi/2 \\ 0 & \text{if} \quad \pi/2 < \phi \leqq \pi. \end{cases}$$

Since $R = 1$, we thus obtain from (19)

$$\begin{aligned} A_n &= \frac{2n+1}{2} \cdot 110 \int_0^{\pi/2} P_n(\cos \phi) \sin \phi \, d\phi \\ &= \frac{2n+1}{2} \cdot 110 \int_0^1 P_n(w) \, dw \end{aligned}$$

where $w = \cos \phi$. Hence $P_n(\cos \phi) \sin \phi \, d\phi = -P_n(w) \, dw$, we integrate from 1 to 0, and we finally get rid of the minus by integrating from 0 to 1. You can evaluate this integral by your CAS or continue by using (11) in Sec. 5.3, obtaining

$$A_n = 55(2n+1) \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m! (n-m)! (n-2m)!} \int_0^1 w^{n-2m} \, dw$$

where $M = n/2$ for even n and $M = (n-1)/2$ for odd n. The integral equals $1/(n - 2m + 1)$. Thus

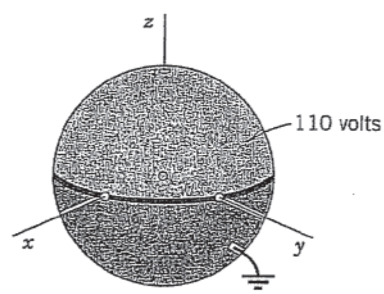


Fig. 310. Spherical capacitor in Example 1

$$A_n = \frac{55(2n+1)}{2^n} \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{m!(n-m)!(n-2m+1)!} \,. \tag{22}$$

Taking $n = 0$, we get $A_0 = 55$ (since $0! = 1$). For $n = 1, 2, 3, \cdots$ we get

$$A_1 = \frac{165}{2} \cdot \frac{2!}{0!1!2!} = \frac{165}{2},$$

$$A_2 = \frac{275}{4}\left(\frac{4!}{0!2!3!} - \frac{2!}{1!1!1!}\right) = 0,$$

$$A_3 = \frac{385}{8}\left(\frac{6!}{0!3!4!} - \frac{4!}{1!2!2!}\right) = -\frac{385}{8}, \qquad \text{etc.}$$

Hence the ***potential*** **(17)** ***inside the sphere*** is (since $P_0 = 1$)

$$u(r, \phi) = 55 + \frac{165}{2} r P_1(\cos\phi) - \frac{385}{8} r^3 P_3(\cos\phi) + \cdots \tag{23}$$ (Fig. 311)

with $P_1, P_3, \cdots$ given by (11′), Sec. 5.3. Since $R = 1$, we see from (19) and (21) in this section that $B_n = A_n$, and (20) thus gives the ***potential outside the sphere***

$$u(r, \phi) = \frac{55}{r} + \frac{165}{2r^2} P_1(\cos\phi) - \frac{385}{8r^4} P_3(\cos\phi) + \cdots. \tag{24}$$

Partial sums of these series can now be used for computing approximate values of the inner and outer potential. Also, it is interesting to see that far away from the sphere the potential is approximately that of a point charge, namely, $55/r$. (Compare with Theorem 3 in Sec. 9.7.) ■

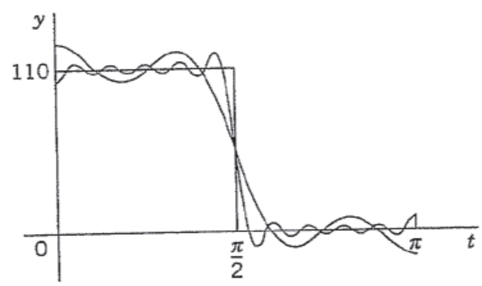


Fig. 311. Partial sums of the first 4, 6, and 11 nonzero terms of (23) for $r = R = 1$

EXAMPLE 2 **Simpler Cases. Help with Problems**

The technicalities occurring in cases like that of Example 1 can often be avoided. For instance, find the potential inside the sphere $S: r = R = 1$ when S is kept at the potential $f(\phi) = \cos 2\phi$. (Can you see the potential on S? What is it at the North Pole? The equator? The South Pole?)

Solution. $w = \cos\phi$, $\cos 2\phi = 2\cos^2\phi - 1 = 2w^2 - 1 = \frac{4}{3}P_2(w) - \frac{1}{3} = \frac{4}{3}(\frac{3}{2}w^2 - \frac{1}{2}) - \frac{1}{3}$. Hence the potential in the interior of the sphere is

$$u = \tfrac{4}{3}r^2 P_2(w) - \tfrac{1}{3} = \tfrac{4}{3}r^2 P_2(\cos\phi) - \tfrac{1}{3} = \tfrac{2}{3}r^2(3\cos^2\phi - 1) - \tfrac{1}{3}.$$ ■

PROBLEM SET 12.10

1. Derive (7) from $\nabla^2 u$ in Cartesian coordinates. (Show the details.)

2. Find the surfaces on which the functions u_1, u_2, u_3 are zero.

3. Sketch the functions $P_n(\cos \phi)$ for $n = 0, 1, 2$ (see (11′) in Sec. 5.3).

4. Sketch the functions $P_3(\cos \phi)$ and $P_4(\cos \phi)$.

5. Verify that u_n and $u_n{}^*$ in (16) are solutions of (8).

6–11 POTENTIALS DEPENDING ONLY ON r

6. **(Dimension 3)** Show that the only solution of the Laplace equation depending only on $r = \sqrt{x^2 + y^2 + z^2}$ is $u = c/r + k$ with constant c and k.

7. **(Dimension 3)** Verify that $u = c/r$, $r = \sqrt{x^2 + y^2 + z^2}$, satisfies Laplace's equation in spherical coordinates.

8. **(Dirichlet problem).** Find the electrostatic potential between two concentric spheres of radii $r_1 = 10$ cm and $r_2 = 20$ cm kept at potentials $U_1 = 260$ V and $U_2 = 110$ V, respectively.

9. **(Dimension 2, logarithmic potential)** Show that the only solution of the two-dimensional Laplace equation depending only on $r = \sqrt{x^2 + y^2}$ is $u = c \ln r + k$ with constant c and k.

10. **(Logarithmic potential)** Find the electrostatic potential between two coaxial cylinders of radii $r_1 = 10$ cm and $r_2 = 20$ cm kept at potentials $U_1 = 260$ V and $U_2 = 110$ V, respectively. Compare with Prob. 8. Comment.

11. **(Heat problem)** If the surface of the ball $r^2 = x^2 + y^2 + z^2 \leq R^2$ is kept at temperature zero and the initial temperature in the ball is $f(r)$, show that the temperature $u(r, t)$ in the ball is a solution of $u_t = c^2(u_{rr} + 2u_r/r)$ satisfying the conditions $u(R, t) = 0$, $u(r, 0) = f(r)$. Show that setting $v = ru$ gives $v_t = c^2 v_{rr}$, $v(R, t) = 0$, $v(r, 0) = rf(r)$. Include the condition $v(0, t) = 0$ (which holds because u must be bounded at $r = 0$), and solve the resulting problem by separating variables.

12. **(Two-dimensional potential problems)** Show that the functions $x^2 - y^2$, xy, $x/(x^2 + y^2)$, $e^x \cos y$, $e^x \sin y$, $\cos x \cosh y$, $\ln (x^2 + y^2)$, and $\arctan (y/x)$ satisfy Laplace's equation $u_{xx} + u_{yy} = 0$. (Two-dimensional potential problems are best solved by ***complex analysis,*** as we shall see in Chap. 18.)

13–17 BOUNDARY VALUE PROBLEMS IN SPHERICAL COORDINATES r, θ, ϕ

Find the potential in the interior of the sphere S: $r = R = 1$ if this interior is free of charges and the potential on S is:

13. $f(\phi) = 100$

14. $f(\phi) = \cos \phi$

15. $f(\phi) = \cos 3\phi$

16. $f(\phi) = \sin^2 \phi$

17. $f(\phi) = 35 \cos 4\phi + 20 \cos 2\phi + 9$

18. Show that in Prob. 13 the potential exterior to the sphere is the same as that of a point charge at the origin. Is this physically plausible?

19. Sketch the intersection of the equipotential surfaces in Prob. 14 with the xz-plane.

20. Find the potential exterior to the sphere in Example 2 of the text and in Prob. 15.

21. What is the temperature in a ball of radius 1 and of homogeneous material if its lower boundary hemisphere is kept at 0°C and its upper at 100°C?

22. **(Reflection in a sphere)** Let r, θ, ϕ be spherical coordinates. If $u(r, \theta, \phi)$ satisfies $\nabla^2 u = 0$, show that $v(r, \theta, \phi) = u(1/r, \theta, \phi)/r$ satisfies $\nabla^2 v = 0$. What does this give for (16)?

23. **(Reflection in a circle)** Let r, θ be polar coordinates. If $u(r, \theta)$ satisfies $\nabla^2 u = 0$, show that the function $v(r, \theta) = u(1/r, \theta)$ satisfies $\nabla^2 v = 0$. What are $u = r \cos \theta$ and v in terms of x and y? Answer the same question for $u = r^2 \cos \theta \sin \theta$ and v.

24. **TEAM PROJECT. Transmission Line and Related PDEs.** Consider a long cable or telephone wire (Fig. 312) that is imperfectly insulated, so that leaks occur along the entire length of the cable. The source S of the current $i(x, t)$ in the cable is at $x = 0$, the receiving end T at $x = l$. The current flows from S to T, through the load, and returns to the ground. Let the constants R, L, C, and G denote the resistance, inductance, capacitance to ground, and conductance to ground, respectively, of the cable per unit length.

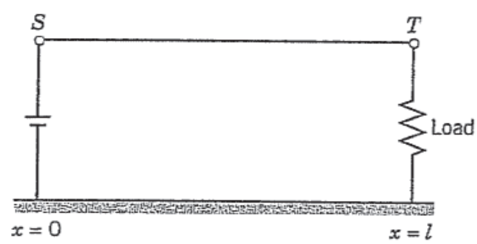


Fig. 312. Transmission line

(a) Show that (**"first transmission line equation"**)

$$-\frac{\partial u}{\partial x} = Ri + L\frac{\partial i}{\partial t}$$

where $u(x, t)$ is the potential in the cable. *Hint:* Apply Kirchhoff's voltage law to a small portion of the cable between x and $x + \Delta x$ (difference of the potentials at x and $x + \Delta x$ = resistive drop + inductive drop).

(b) Show that for the cable in (a) (**"second transmission line equation"**),

$$-\frac{\partial i}{\partial x} = Gu + C\frac{\partial u}{\partial t}$$

Hint: Use Kirchhoff's current law (difference of the currents at x and $x + \Delta x$ = loss due to leakage to ground + capacitive loss).

(c) **Second-order PDEs.** Show that elimination of i or u from the transmission line equations leads to

$$u_{xx} = LCu_{tt} + (RC + GL)u_t + RGu,$$
$$i_{xx} = LCi_{tt} + (RC + GL)i_t + RGi.$$

(d) **Telegraph equations.** For a submarine cable, G is negligible and the frequencies are low. Show that this leads to the so-called *submarine cable equations* or **telegraph equations**

$$u_{xx} = RCu_t, \qquad i_{xx} = RCi_t.$$

Find the potential in a submarine cable with ends ($x = 0$, $x = l$) grounded and initial voltage distribution $U_0 = const.$

(e) **High-frequency line equations.** Show that in the case of alternating currents of high frequencies the equations in (c) can be approximated by the so-called **high-frequency line equations**

$$u_{xx} = LCu_{tt}, \qquad i_{xx} = LCi_{tt}.$$

Solve the first of them, assuming that the initial potential is

$$U_0 \sin(\pi x/l),$$

and $u_t(x, 0) = 0$ and $u = 0$ at the ends $x = 0$ and $x = l$ for all t.

12.11 Solution of PDEs by Laplace Transforms

Readers familiar with Chap. 6 may wonder whether Laplace transforms can also be used for solving *partial* differential equations. The answer is yes, particularly if one of the independent variables ranges over the positive axis. The steps to obtain a solution are similar to those in Chap. 6. For a PDE in two variables they are as follows.

1. Take the Laplace transform with respect to one of the two variables, usually t. This gives an ***ODE for the transform*** of the unknown function. This is so since the derivatives of this function with respect to the other variable slip into the transformed equation. The latter also incorporates the given boundary and initial conditions.
2. Solving that ODE, obtain the transform of the unknown function.
3. Taking the inverse transform, obtain the solution of the given problem.

If the coefficients of the given equation do not depend on t, the use of Laplace transforms will simplify the problem.

We explain the method in terms of a typical example.

EXAMPLE 1 Semi-Infinite String

Find the displacement $w(x, t)$ of an elastic string subject to the following conditions. (We write w since we need u to denote the unit step function.)

(i) The string is initially at rest on the x-axis from $x = 0$ to ∞ (*"semi-infinite string"*).

(ii) For $t > 0$ the left end of the string ($x = 0$) is moved in a given fashion, namely, according to a single sine wave

$$w(0, t) = f(t) = \begin{cases} \sin t & \text{if } 0 \leqq t \leqq 2\pi \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Fig. 313).}$$

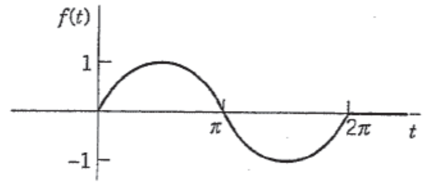


Fig. 313. Motion of the left end of the string in Example 1 as a function of time t

(iii) Furthermore, $\lim_{x\to\infty} w(x, t) = 0$ for $t \geqq 0$.

Of course there is no infinite string, but our model describes a long string or rope (of negligible weight) with its right end fixed far out on the x-axis.

Solution. We have to solve the wave equation (Sec. 12.2)

$$(1) \qquad \frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}, \qquad c^2 = \frac{T}{\rho}$$

for positive x and t, subject to the "boundary conditions"

$$(2) \qquad w(0, t) = f(t), \qquad \lim_{x\to\infty} w(x, t) = 0 \qquad (t \geqq 0)$$

with f as given above, and the initial conditions

$$(3) \qquad \text{(a)} \quad w(x, 0) = 0, \qquad \text{(b)} \quad w_t(x, 0) = 0.$$

We take the Laplace transform ***with respect to t***. By (2) in Sec. 6.2,

$$\mathcal{L}\left\{\frac{\partial^2 w}{\partial t^2}\right\} = s^2\mathcal{L}\{w\} - sw(x, 0) - w_t(x, 0) = c^2\mathcal{L}\left\{\frac{\partial^2 w}{\partial x^2}\right\}.$$

The expression $-sw(x, 0) - w_t(x, 0)$ drops out because of (3). On the right we assume that we may interchange integration and differentiation. Then

$$\mathcal{L}\left\{\frac{\partial^2 w}{\partial x^2}\right\} = \int_0^\infty e^{-st} \frac{\partial^2 w}{\partial x^2}\, dt = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st} w(x, t)\, dt = \frac{\partial^2}{\partial x^2}\mathcal{L}\{w(x, t)\}.$$

Writing $W(x, s) = \mathcal{L}\{w(x, t)\}$, we thus obtain

$$s^2 W = c^2 \frac{\partial^2 W}{\partial x^2}, \qquad \text{thus} \qquad \frac{\partial^2 W}{\partial x^2} - \frac{s^2}{c^2} W = 0.$$

Since this equation contains only a derivative with respect to x, it may be regarded as an ***ordinary differential equation*** for $W(x, s)$ considered as a function of x. A general solution is

$$(4) \qquad W(x, s) = A(s)e^{sx/c} + B(s)e^{-sx/c}.$$

From (2) we obtain, writing $F(s) = \mathcal{L}\{f(t)\}$,

$$W(0, s) = \mathcal{L}\{w(0, t)\} = \mathcal{L}\{f(t)\} = F(s).$$

Assuming that we can interchange integration and taking the limit, we have

$$\lim_{x\to\infty} W(x, s) = \lim_{x\to\infty}\int_0^\infty e^{-st} w(x, t)\, dt = \int_0^\infty e^{-st} \lim_{x\to\infty} w(x, t)\, dt = 0.$$

This implies $A(s) = 0$ in (4) because $c > 0$, so that for every fixed positive s the function $e^{sx/c}$ increases as x increases. Note that we may assume $s > 0$ since a Laplace transform generally exists for *all* s greater than some fixed k (Sec. 6.2). Hence we have

$$W(0, s) = B(s) = F(s),$$

so that (4) becomes

$$W(x, s) = F(s)e^{-sx/c}.$$

From the second shifting theorem (Sec. 6.3) with $a = x/c$ we obtain the inverse transform

$$w(x, t) = f\left(t - \frac{x}{c}\right) u\left(t - \frac{x}{c}\right) \tag{5}$$ (Fig. 314)

that is,

$$w(x, t) = \sin\left(t - \frac{x}{c}\right) \quad \text{if} \quad \frac{x}{c} < t < \frac{x}{c} + 2\pi \quad \text{or} \quad ct > x > (t - 2\pi)c$$

and zero otherwise. This is a single sine wave traveling to the right with speed c. Note that a point x remains at rest until $t = x/c$, the time needed to reach that x if one starts at $t = 0$ (start of the motion of the left end) and travels with speed c. The result agrees with our physical intuition. Since we proceeded formally, we must verify that (5) satisfies the given conditions. We leave this to the student. ∎

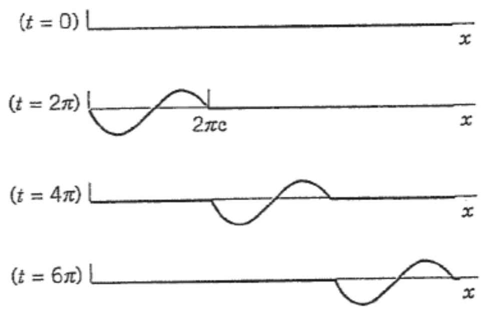


Fig. 314. Traveling wave in Example 1

This is the end of Chap. 12, in which we concentrated on the most important partial differential equations (PDEs) in physics and engineering. This is also the end of Part C on Fourier analysis and PDEs.

We have seen that PDEs have various basic engineering applications, which make them the subject of many ongoing research projects.

Numerics for PDEs follows in Secs. 21.4–21.7, which are independent of the other sections in Part E on numerics.

In the next part, Part D on **complex analysis,** we turn to an area of a different nature that is also highly important to the engineer, as our examples and problems will show. This will include another approach to the (two-dimensional) **Laplace equation** and its applications in Chap. 18.

PROBLEM SET 12.11

1. Sketch a figure similar to Fig. 314 if $c = 1$ and f is "triangular" as in Example 1, Sec. 12.3.
2. How does the speed of the wave in Example 1 depend on the tension and on the mass of the string?
3. Verify the solution in Example 1. What traveling wave do we obtain in Example 1 in the case of a (nonterminating) sinusoidal motion of the left end starting at $t = 0$?

4–6 SOLVE BY LAPLACE TRANSFORMS

4. $\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial t} = x,\ w(x, 0) = 1,\ w(0, t) = 1$

5. $x\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} = xt, \quad w(x, 0) = 0$ if $x \geqq 0$,
$w(0, t) = 0$ if $t \geqq 0$

6. $\frac{\partial^2 w}{\partial x^2} = 100\frac{\partial^2 w}{\partial t^2} + 100\frac{\partial w}{\partial t} + 25w$,
$w(x, 0) = 0$ if $x \geqq 0$, $\quad w_t(x, 0) = 0$ if $t \geqq 0$,
$w(0, t) = \sin t$ if $t \geqq 0$

7. Solve Prob. 5 by another method.

8–10 HEAT PROBLEM

Find the temperature $w(x, t)$ in a semi-infinite laterally insulated bar extending from $x = 0$ along the x-axis to infinity, assuming that the initial temperature is 0, $w(x, t) \to 0$ as $x \to \infty$ for every fixed $t \geqq 0$, and $w(0, t) = f(t)$. Proceed as follows.

8. Set up the model and show that the Laplace transform leads to

$$sW = c^2 \frac{\partial^2 W}{\partial x^2} \quad (W = \mathcal{L}\{w\})$$

and

$$W = F(s)e^{-\sqrt{s}x/c} \quad (F = \mathcal{L}\{f\}).$$

Applying the convolution theorem, show that

$$w(x, t) = \frac{x}{2c\sqrt{\pi}} \int_0^t f(t - \tau)\tau^{-3/2}e^{-x^2/(4c^2\tau)}\, d\tau.$$

9. Let $w(0, t) = f(t) = u(t)$ (Sec. 6.3). Denote the corresponding w, W, and F by w_0, W_0, and F_0. Show that then in Prob. 8,

$$w_0(x, t) = \frac{x}{2c\sqrt{\pi}} \int_0^t \tau^{-3/2}e^{-x^2/(4c^2\tau)}\, d\tau$$
$$= 1 - \operatorname{erf}\left(\frac{x}{2c\sqrt{t}}\right)$$

with the error function erf as defined in Problem Set 12.6.

10. **(Duhamel's formula[4])** Show that in Prob. 9,

$$W_0(x, s) = \frac{1}{s}\, e^{-\sqrt{s}x/c}$$

and the convolution theorem gives *Duhamel's formula*

$$w(x, t) = \int_0^t f(t - \tau)\, \frac{\partial w_0}{\partial \tau}\, d\tau.$$

CHAPTER 12 REVIEW QUESTIONS AND PROBLEMS

1. Write down the three probably most important PDEs from memory and state their main applications.
2. What is the method of separating variables for PDEs? Give an example from memory.
3. What is the superposition principle? Give a typical application.
4. What role did Fourier series play in this chapter? Fourier integrals?
5. What are the eigenfunctions and their frequencies of the vibrating string? Of the heat equation?
6. What additional conditions did we consider for the wave equation? For the heat equation?
7. Name and explain the three kinds of boundary conditions.
8. What do you know about types of PDEs? About transformation to normal forms?
9. What is d'Alembert's method? To what PDE does it apply?
10. When and why did we use polar coordinates? Spherical coordinates?
11. When and why did Legendre's equation occur in this chapter? Bessel's equation?
12. What are the eigenfunctions of the circular membrane? How do their frequencies differ in principle from those of the eigenfunctions of the vibrating string?
13. Explain mathematically (not physically) why we got exponential functions in separating the heat equation, but not for the wave equation.
14. What is the error function? Why did it occur and where?
15. Explain the idea of using Laplace transform methods for PDEs. Give an example from memory.
16. For what k and m are $x^4 + kx^2y^2 + y^4$ and $\sin mx \sinh y$ solutions of Laplace's equation?
17. Verify that $(x^2 - y^2)/(x^2 + y^2)^2$ satisfies Laplace's equation.

18–21 Solve for $u = u(x, y)$:

18. $u_{yy} + 16u = 0$
19. $u_{xx} + u_x - 2u = 0$
20. $u_{xy} + u_y + x + y + 1 = 0$
21. $u_{yy} + u_y = 0$, $u(x, 0) = f(x)$, $u_y(x, 0) = g(x)$
22. Find all solution $u(x, y) = F(x)G(y)$ of Laplace's equation in two variables.

[4] JEAN-MARIE CONSTANT DUHAMEL (1797–1872), French mathematician.

23–26 Find and sketch or graph (as in Fig. 285 in Sec. 12.3) the deflection $u(x, t)$ of a vibrating string of length π, extending from $x = 0$ to $x = \pi$, and $c^2 = T/\rho = 1$, starting with velocity 0 and deflection

23. $f(x) = \sin x - \frac{1}{2} \sin 2x$

24. $f(x) = \frac{1}{2}\pi - |x - \frac{1}{2}\pi|$

25. $f(x) = \sin^3 x$

26. $f(x) = x(\pi - x)$

27–30 Find the temperature distribution in a laterally insulated thin copper bar ($c^2 = K/\rho\sigma = 1.158$ cm²/sec), 50 cm long and of constant cross section with endpoints at $x = 0$ and 50 kept at 0°C and initial temperature

27. $f(x) = \sin(\pi x/50)$

28. $f(x) = x(50 - x)$

29. $f(x) = 25 - |25 - x|$

30. $f(x) = 4 \sin^3 (\pi x/10)$

31–33 Find the temperature $u(x, t)$ in a laterally insulated bar of length π, extending from $x = 0$ to $x = \pi$, with $c^2 = 1$ for adiabatic boundary condition (see Problem Set 12.5) and initial temperature

31. $100 \cos 4x$

32. $3x^2$

33. $\pi - 2|x - \frac{1}{2}\pi|$

34. Using partial sums, graph $u(x, t)$ in Prob. 33 for several constant t on common axes. Do these graphs agree with your physical intuition?

35. Let $f(x, y) = u(x, y, 0)$ be the initial temperature in a thin square plate of side π with edges kept at 0°C and faces perfectly insulated. Separating variables, obtain from $u_t = c^2\nabla^2 u$ the solution

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin mx \sin ny\, e^{-c^2(m^2+n^2)t}$$

where

$$B_{mn} = \frac{4}{\pi^2} \int_0^{\pi}\int_0^{\pi} f(x, y) \sin mx \sin ny \, dx\, dy.$$

36. Find the temperature in Prob. 35 if $f(x, y) = x(\pi - x)y(\pi - y)$.

37–42 Transform to normal form and solve (showing the details!)

37. $u_{xy} = u_{xx}$

38. $u_{xx} + 4u_{xy} + 4u_{yy} = 0$

39. $u_{xx} + 4u_{yy} = 0$

40. $2u_{xx} + 5u_{xy} + 2u_{yy} = 0$

41. $u_{xx} + 2u_{xy} + u_{yy} = 0$

42. $u_{yy} + u_{xy} - 2u_{xx} = 0$

43–47 Show that the following membranes of area 1 with $c^2 = 1$ have the frequencies of the fundamental mode as given (4-decimal values). Compare.

43. Circle: $\alpha_1/(2\sqrt{\pi}) = 0.6784$

44. Square: $1\sqrt{2} = 0.7071$

45. Rectangle (sides 1:2): $\sqrt{5/8} = 0.7906$

46. Semicircle: $3.832/\sqrt{8\pi} = 0.7644$

47. Quadrant of circle: $\alpha_{12}/(4\sqrt{\pi}) = 0.7244$ ($\alpha_{12} = 5.13562$ = first positive zero of J_2)

48–50 Find the electrostatic potential in the following (charge-free) regions:

48. Between two concentric spheres of radii r_0 and r_1 kept at the potentials u_0 and u_1, respectively.

49. Between two coaxial circular cylinders of radii r_0 and r_1 kept at the potential u_0 and u_1, respectively. (Compare with Prob. 48.)

50. In the interior of a sphere of radius 1 kept at the potential $f(\phi) = \cos 3\phi + 3 \cos \phi$ (referred to our usual spherical coordinates).

SUMMARY OF CHAPTER 12
Partial Differential Equations (PDEs)

Whereas ODEs (Chaps. 1–6) serve as models of problems involving only *one* independent variable, problems involving *two or more* independent variables (space variables or time t and one or several space variables) lead to PDEs. This accounts for the enormous importance of PDEs to the engineer and physicist. Most important are:

(1) $u_{tt} = c^2 u_{xx}$ One-dimensional wave equation (Secs. 12.2–12.4)

(2) $u_{tt} = c^2(u_{xx} + u_{yy})$ Two-dimensional wave equation (Secs. 12.7–12.9)

(3) $u_t = c^2 u_{xx}$ One-dimensional heat equation (Secs. 12.5, 12.6)

(4) $\nabla^2 u = u_{xx} + u_{yy} = 0$ Two-dimensional Laplace equation (Secs. 12.5, 12.9)

(5) $\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$ Three-dimensional Laplace equation (Sec. 12.10).

Equations (1) and (2) are hyperbolic, (3) is parabolic, (4) and (5) are elliptic.

In practice, one is interested in obtaining the solution of such an equation in a given region satisfying given additional conditions, such as **initial conditions** (conditions at time $t = 0$) or **boundary conditions** (prescribed values of the solution u or some of its derivatives on the boundary surface S, or boundary curve C, of the region) or both. For (1) and (2) one prescribes two initial conditions (initial displacement and initial velocity). For (3) one prescribes the initial temperature distribution. For (4) and (5) one prescribes a boundary condition and calls the resulting problem a (see Sec. 12.5)

Dirichlet problem if u is prescribed on S,
Neumann problem if $u_n = \partial u/\partial n$ is prescribed on S,
Mixed problem if u is prescribed on one part of S and u_n on the other.

A general method for solving such problems is the method of **separating variables** or **product method,** in which one assumes solutions in the form of products of functions each depending on one variable only. Thus equation (1) is solved by setting $u(x, t) = F(x)G(t)$; see Sec. 12.3; similarly for (3) (see Sec. 12.5). Substitution into the given equation yields ***ordinary*** differential equations for F and G, and from these one gets infinitely many solutions $F = F_n$ and $G = G_n$ such that the corresponding functions

$$u_n(x, t) = F_n(x)G_n(t)$$

are solutions of the PDE satisfying the given boundary conditions. These are the **eigenfunctions** of the problem, and the corresponding **eigenvalues** determine the frequency of the vibration (or the rapidity of the decrease of temperature in the case of the heat equation, etc.). To satisfy also the initial condition (or conditions), one must consider infinite series of the u_n, whose coefficients turn out to be the Fourier coefficients of the functions f and g representing the given initial conditions (Secs. 12.3, 12.5). Hence **Fourier series** (and *Fourier integrals*) are of basic importance here (Secs. 12.3, 12.5, 12.6, 12.8).

Steady-state problems are problems in which the solution does not depend on time t. For these, the heat equation $u_t = c^2\nabla^2 u$ becomes the Laplace equation.

Before solving an initial or boundary value problem, one often transforms the PDE into coordinates in which the boundary of the region considered is given by simple formulas. Thus in polar coordinates given by $x = r\cos\theta$, $y = r\sin\theta$, the **Laplacian** becomes (Sec. 12.9)

$$\nabla^2 u = u_{rr} + \frac{1}{r}\, u_r + \frac{1}{r^2}\, u_{\theta\theta}; \tag{6}$$

for spherical coordinates see Sec. 12.10. If one now separates the variables, one gets **Bessel's equation** from (2) and (6) (vibrating circular membrane, Sec. 12.9) and **Legendre's equation** from (5) transformed into spherical coordinates (Sec. 12.10).