In the table, the functions $f\colon \mathbb{R} \to \mathbb{R}$ are understood to be 2π-periodic[1] and $a \in \mathbb{R}$ is a constant. Recall that the formal Fourier series of f is given by

$$f(\theta) \sim \sum_{n\in\mathbb{Z}} c_n e^{in\theta} = \frac{a_0}{2} + \sum_{n\in\mathbb{N}} \left[a_n \cos(n\,\theta) + b_n \sin(n\,\theta)\right] ,$$

where

$$\begin{aligned} c_n &= \widehat{f}(n) &&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\, d\theta && \text{for all } n \in \mathbb{Z} , \\ a_n &= c_n + c_{-n} &&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta && \text{for all } n \in \mathbb{N}_0 , \\ b_n &= i(c_n - c_{-n}) &&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta && \text{for all } n \in \mathbb{N} . \end{aligned}$$

Table of Fourier Series for periodic $f\colon \mathbb{R} \to \mathbb{R}$

	f	defining interval of f	Fourier series of f
1	$f(\theta) = \lvert\theta\rvert$	$[-\pi, \pi)$	$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos((2n-1)\theta)}{(2n-1)^2}$
2	$f(\theta) = \theta$	$[-\pi, \pi)$	$2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\theta)$
3	$f(\theta) = \pi - \theta$	$[0, 2\pi)$	$2\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n}$
4	$f(\theta) = \begin{cases} 0 & \text{if } -\pi \le \theta \le 0 \\ \theta & \text{if } 0 < \theta < \pi \end{cases}$	$[-\pi, \pi)$	$\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos((2n-1)\theta)}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\theta)$
5	$f(\theta) = \sin^2(\theta)$	$[-\pi, \pi)$	$\frac{1}{2} - \frac{1}{2}\cos(2\theta)$
6	$f(\theta) = \begin{cases} -1 & \text{if } -\pi \le \theta \le 0 \\ 1 & \text{if } 0 < \theta < \pi \end{cases}$	$[-\pi, \pi)$	$\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin((2n-1)\theta)}{2n-1}$
7	$f(\theta) = \begin{cases} 0 & \text{if } -\pi \le \theta \le 0 \\ 1 & \text{if } 0 < \theta < \pi \end{cases}$	$[-\pi, \pi)$	$\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin((2n-1)\theta)}{2n-1}$
8	$f(\theta) = \lvert\sin(\theta)\rvert$	$[-\pi, \pi)$	$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n\theta)}{4n^2-1}$
9	$f(\theta) = \lvert\cos(\theta)\rvert$	$[-\pi, \pi)$	$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n\cos(2n\theta)}{4n^2-1}$
10	$f(\theta) = \begin{cases} 0 & \text{if } -\pi \le \theta \le 0 \\ \sin(\theta) & \text{if } 0 < \theta < \pi \end{cases}$	$[-\pi, \pi)$	$\frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n\theta)}{4n^2-1} + \frac{1}{2}\sin(\theta)$
11	$a \in [0, \pi)$ $f(\theta) = \begin{cases} a\,\frac{\pi+\theta}{a-\pi} & \text{if } -\pi \le \theta \le -a \\ \theta & \text{if } -a < \theta < a \\ a\,\frac{\pi-\theta}{\pi-a} & \text{if } a \le \theta < \pi \end{cases}$	$[-\pi, \pi)$	$\frac{2}{\pi - a}\sum_{n=1}^{\infty} \frac{\sin(an)}{n^2} \sin(n\theta)$

[1] Specifically, the formula for f is given in a specified defining interval of length 2π and then it is understood that f is extended 2π-periodically to the whole of $\mathbb{R}$.

	f	defining interval of f	Fourier series of f
12	$a \in (0,\pi]$ $f(\theta) = \begin{cases} (2a)^{-1} & \text{if } \lvert\theta\rvert \le a \\ 0 & \text{if } \lvert\theta\rvert > a \end{cases}$	$[-\pi,\pi)$	$\frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin(an)}{an} \cos(n\theta)$
13	$a \in (0,\pi]$ $f(\theta) = \begin{cases} (2a)^{-1} & \text{if } \lvert\theta-\theta_0\rvert \le a \\ 0 & \text{if } \lvert\theta-\theta_0\rvert > a \end{cases}$	$[\theta_0-\pi, \theta_0+\pi)$	$\frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin(an)}{an} \cos(n\theta_0)\ \cos(n\theta)\ +$ $\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin(an)}{an} \sin(n\theta_0)\ \sin(n\theta)$
14	$a \in (0,\pi/4]$ $f(\theta) = \begin{cases} +1 & \text{if } -a < \theta < a \\ -1 & \text{if } 2a < \theta < 4a \\ 0 & \text{elsewhere in } [-\pi,\pi) \end{cases}$	$[-\pi,\pi)$	$\sum_{n=1}^{\infty} \frac{\sin(an)}{n} (1-\cos(3an))\ \cos(n\theta)\ -$ $\sum_{n=1}^{\infty} \frac{\sin(an)}{n} \sin(3an)\ \sin(n\theta)$
15	$a \in (0,\pi]$ $f(\theta) = \begin{cases} a^{-2}(a-\lvert\theta\rvert) & \text{if } \lvert\theta\rvert \le a \\ 0 & \text{if } \lvert\theta\rvert > a \end{cases}$	$[-\pi,\pi)$	$\frac{1}{2\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-\cos(an)}{a^2 n^2} \cos(n\theta)$
16	$f(\theta) = \theta^2$	$[-\pi,\pi)$	$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\theta)$
17	$f(\theta) = \theta(\pi - \lvert\theta\rvert)$	$[-\pi,\pi)$	$\frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin((2n-1)\theta)$
18	$a \in \mathbb{R}\setminus\{0\}$ $f(\theta) = e^{a\theta}$ Let $\gamma_a := \frac{\sinh(a\pi)}{\pi}$.	$[-\pi,\pi)$	$\gamma_a \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a-in} e^{in\theta} =$ $\frac{2\gamma_a}{a} + 2a\gamma_a \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} \cos(n\theta) - 2\gamma_a \sum_{n=1}^{\infty} \frac{(-1)^n n}{a^2+n^2} \sin(n\theta)$
19	$a \in \mathbb{R}\setminus\{0\}$ $f(\theta) = e^{a\theta}$ Let $\gamma_a := \frac{e^{2\pi a}-1}{2\pi}$.	$[0,2\pi)$	$\gamma_a \sum_{n=-\infty}^{\infty} \frac{1}{a-in} e^{in\theta} =$ $\frac{2\gamma_a}{a} + 2a\gamma_a \sum_{n=1}^{\infty} \frac{1}{a^2+n^2} \cos(n\theta) - 2\gamma_a \sum_{n=1}^{\infty} \frac{n}{a^2+n^2} \sin(n\theta)$
20	$f(\theta) = \sinh(\theta)$	$[-\pi,\pi)$	$\frac{2\sinh(\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1} \sin(n\theta)$

The hyperbolic cosine (denoted by cosh) and hyperbolic sine (denoted by sinh) are defined by:

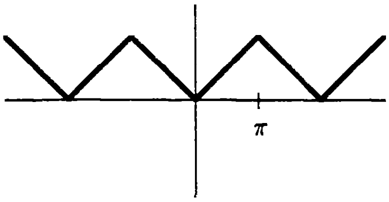
$$\cosh(\theta) := \frac{e^{\theta}+e^{-\theta}}{2} \qquad \text{and} \qquad \sinh(\theta) := \frac{e^{\theta}-e^{-\theta}}{2}\ .$$

References

[F] Gerald B. Folland, *Fourier analysis and its applications*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1992.

Below are sketches of the functions f from the table.

2.1 The Fourier series of a periodic function 27



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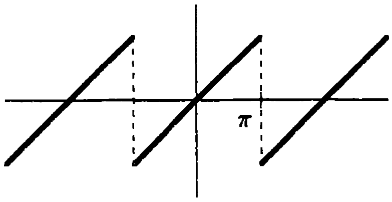

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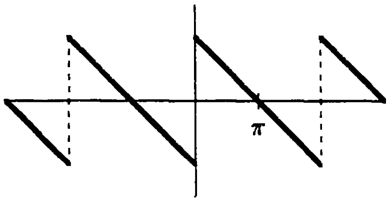

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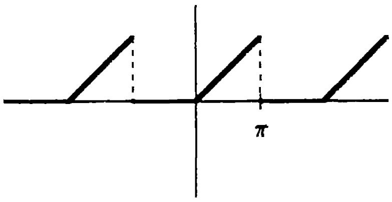

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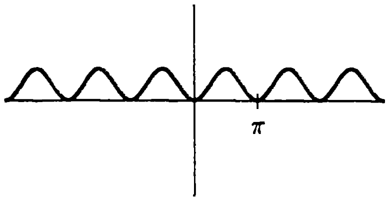

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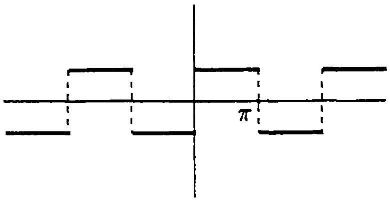

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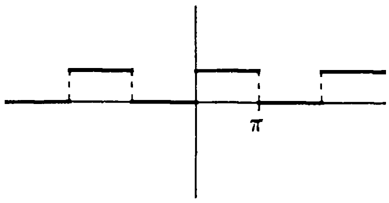

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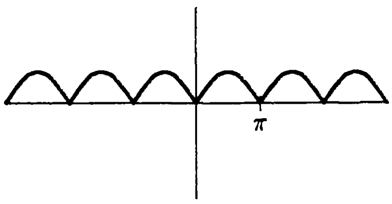

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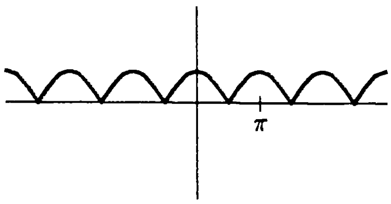

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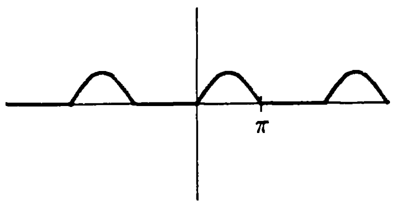

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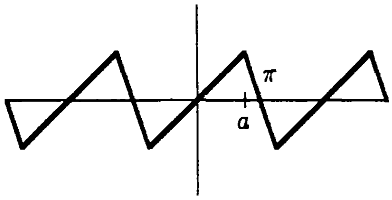

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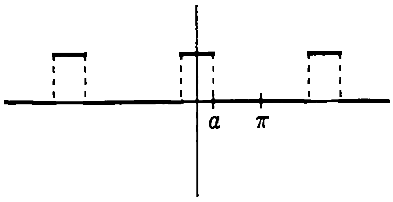

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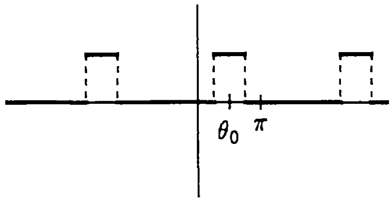

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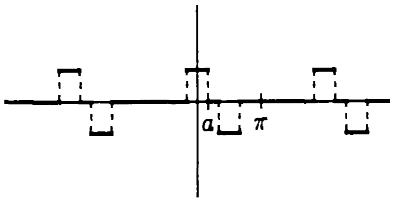

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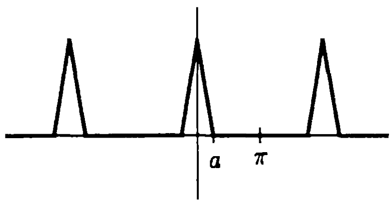

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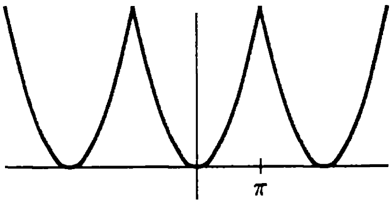

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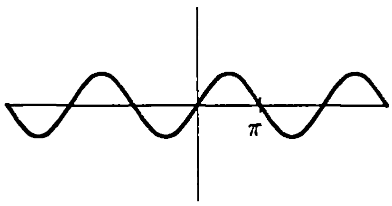

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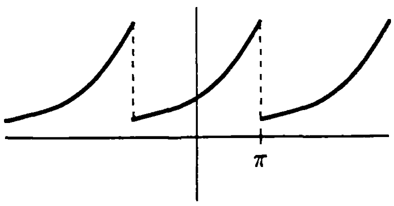

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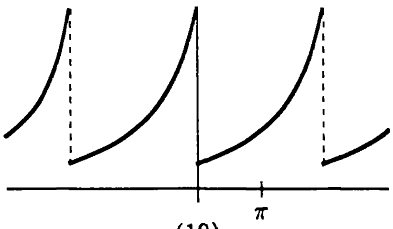

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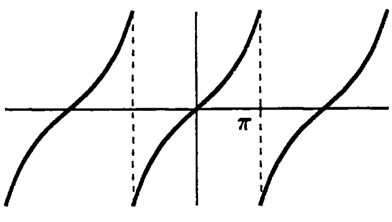

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