Anti-Final

1. Users call help desk every 15 minutes, on the average. There is one help desk specialist on duty, and her average service time is 9 minutes. Modeling the help desk as an M/M/1 queuing system, compute:

   (a) the expected number of users in the system at any time, including the users talking to the specialist and those waiting to be served;
   (b) the probability that the next caller does not have to wait until the help desk specialist responds to him;
   (c) the proportion of time when exactly two users are waiting.

   <u>Solution</u>. *Given:* $\mu_A = 15$ *min*, $\mu_S = 9$ *min.*
   Then $r = \lambda_A/\lambda_S = \mu_S/\mu_A = 9/15 = 0.6$.

   (a) $E(X) = \dfrac{r}{1-r} = \dfrac{0.6}{0.4} = \boxed{1.5 \text{ users}}$
   (b) $P(W = 0) = \pi_0 = 1 - r = \boxed{0.4}$
   (c) $P(X_w = 2) = P(X = 3) = \pi_3 = (1-r)r^3 = (0.4)(0.6)^3 = \boxed{0.0864}$

2. Unauthorized attempts to log into a certain computer system occur according to a Poisson process with the rate of one attempts every five days.

(a) What is the probability of at least 3 unauthorized attempts during the next week?
(b) After the 5th unauthorized attempt, the computer manager will change the password. What is the probability that this will happen during the next 12 days?

Solution. *(a) We need $P(X \geq 3)$, where X is the number of unauthorized attempts during the next 7 days. This X has Poisson distribution with parameter $\lambda t = (1/5)(7) = 1.4$. From the given table of Poisson distribution,*

$$P(X \geq 3) = 1 - F(2) = 1 - 0.833 = \boxed{0.167}$$

(b) We need $P(T \leq 12)$, where T is the time of the 5th event. This T has Gamma distribution with parameters $r = 5$ and $\lambda = 1/5$. Using the Poisson-Gamma formula,

$$P(T \leq 12) = P(Y \geq 5) = 1 - F(4) = 1 - 0.904 = \boxed{0.096},$$

from the Poisson table, where Y has Poisson distribution with parameter $\lambda t = (1/5)(12) = 2.4$.

3. A dog can eat one piece of meat at a time. When he is busy eating, the other pieces of meat will be eaten by other pets. On the average, the dog's owner throws him a piece of meat every 20 minutes, and it takes the dog 10 minutes to eat it. Assume Bernoulli single-server queuing process with 5-minute frames and capacity limited by 1 piece of meat.

(a) Find the transition probability matrix for the amount of meat (the number of pieces) that a dog has at any time.
(b) Find the steady-state proportion of time when the dog has some meat to eat.

Solution. *(a) We are given $\lambda_A = 1/20$ min^{-1}, $\lambda_S = 1/10$ min^{-1}, $\Delta = 5$ min, and $C = 2$. Compute probabilities*

$$p_A = \lambda_A \Delta = (1/20)(5) = 1/4, \quad p_S = \lambda_S \Delta = (1/10)(5) = 1/2,$$

and the transition probability matrix (with $X \in \{0, 1, 2\}$ due to the limited capacity)

$$P = \begin{pmatrix} 1 - p_A & p_A \\ p_S(1 - p_A) & 1 - p_S(1 - p_A) \end{pmatrix} = \begin{pmatrix} 3/4 & 1/4 \\ 3/8 & 5/8 \end{pmatrix}.$$

(b) Solve the steady-state equations

$$\begin{cases} \pi P = \pi \\ \sum \pi_i = 1 \end{cases} \Rightarrow \begin{cases} (3/4)\pi_0 + (3/8)\pi_1 &= \pi_0 \\ (1/4)\pi_0 + (5/8)\pi_1 &= \pi_1 \\ \pi_0 + \pi_1 &= 1 \end{cases} \Rightarrow \begin{cases} (3/8)\pi_1 &= (1/4)\pi_0 \\ (1/4)\pi_0 &= (3/8)\pi_1 \\ \pi_0 + \pi_1 &= 1 \end{cases} \Rightarrow \begin{cases} \pi_1 = (2/3)\pi_0 \\ \\ (5/3)\pi_0 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \pi_0 &= 3/5 \\ \pi_1 &= (2/3)\pi_0 = 2/5 \end{cases}$$

The dog has some meat to eat 2/5 or 40% of the time.

4. In a recent survey of 500 U.S. workers, 18% say it is likely that they will lose their job in the next year. Construct a 95% confidence interval for the true proportion of U.S. workers who think they are likely to lose their job in the next year.

Solution. *Let p be the proportion of U.S. workers who think they are likely to lose their job in the next year. We want a 95% confidence interval for p. In this case, we would like to use a z-interval. Using the formula, the confidence interval is:*

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.18 \pm 1.96\sqrt{\frac{(0.18)(0.82)}{500}} = 0.18 \pm 0.034 = (0.146, 0.214).$$

5. Suppose a researcher claims that the mean age of founders of start-up companies in Silicon Valley is less than 30 years. To test his claim, he randomly selected 7 start-up companies and obtained the following data on the age of their founders:

$$27, 28, 32, 33, 24, 28, 31 \text{ years}$$

Is there significant evidence that the mean age of founders is less than 30 years? Answer this question by performing an appropriate hypothesis test at 5% level of significance. Assume normal distribution for the age random variable.

Solution. *Let μ denote the mean age (in years) of founders of start-up companies in Silicon Valley. We want to test the null hypothesis $H_0 : \mu = 30$ (or $\mu \geq 30$) against the alternative hypothesis $H_1 : \mu < 30$. Since the data are normally distributed with unknown population variance, we would like to use a left-tailed t-test for these hypotheses. From the given data, we can compute $\overline{x} = 29$, $s = 3.162$ and*

$$t = \frac{\overline{x} - 30}{s/\sqrt{7}} = -0.837.$$

Further, the critical point for this test is

$$t_{n-1,\alpha} = t_{6,0.05} = 1.943.$$

Since $t = -0.837 > -t_{6,0.05} = -1.943$, we accept H_0. Therefore, at 5% level, there is no significant evidence that μ is less than 30 years.

6. Let the random variable X denote the severity of a virus attack on a scale of zero to one. Suppose X follows a continuous distribution with probability density function

$$f_\theta(x) = \begin{cases} (\theta+1)x^\theta, & 0 < x < 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > -1$ is an unknown parameter. The severity of the last four virus attacks are as follows:

$$0.10, 0.30, 0.50, 0.20.$$

Use these data to estimate θ using either the method of moments or the method of maximum likelihood. Assume that the virus attacks are independent.

Solution. **Method of moments:** *First, we compute*

$$E(X) = \int_0^1 x f_\theta(x) dx = (\theta+1)\int_0^1 x^{\theta+1} dx = \frac{\theta+1}{\theta+2},$$

and $\overline{x} = 0.275$. *Next, we solve* $0.275 = (\theta+1)/(\theta+2)$ *for* θ *to get*

$$\hat{\theta}_{MOME} = -0.45/0.725 = -0.621.$$

Method of maximum likelihood: *The log-likelihood function is:*

$$\ln L(\theta) = \ln\left(\prod_{i=1}^{4}(\theta+1)x_i^\theta\right) = 4\ln(\theta+1) + \theta\sum_{i=1}^{4}\ln x_i.$$

Differentiating this function with respect to θ *and equating it to zero, we get the likelihood equation as:*

$$0 = \frac{\partial}{\partial\theta}L(\theta) = \frac{4}{\theta+1} + \sum_{i=1}^{4}\ln x_i.$$

Solving it for θ, *we get the MLE as*

$$\hat{\theta}_{MLE} = -1 - \frac{4}{\sum_{i=1}^{4}\ln x_i} = -0.311$$

Cheat sheet for the Final Exam

Discrete Distributions

Expected value	$\mu = E(X) = \sum_x xP(x)$
Expected value of a function	$Eg(X) = \sum_x g(x)P(x)$
Variance	$\sigma^2 = Var(X) = \sum_x (x-\mu)^2 P(x)$
Binomial probability mass function	$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$ for $x = 0, 1, ..., n,$
Geometric probability mass function	$P(x) = (1-p)^{x-1} p$ for $x = 1, 2, ...$
Poisson probability mass function	$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ for $x = 0, 1, ...$

Continuous Distributions

Expected value	$\mu = E(X) = \int x f(x) dx$
Expected value of a function	$Eg(X) = \int g(x) f(x) dx$
Variance	$\sigma^2 = Var(X) = \int (x-\mu)^2 f(x) dx$
Exponential density	$f(x) = \lambda e^{-\lambda x}$ for $0 < x < \infty$
Uniform density	$f(x) = \frac{1}{b-a}$ for $a < x < b$
Gamma density	$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$ for $0 < x < \infty$
Gamma-Poisson formula	$P(X < x) = P(Y \geq r)$; $P(X > x) = P(Y < r)$ for $X \sim$ Gamma(r, λ), $Y \sim$ Poisson(λx)
Normal density	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$ for $-\infty < x < \infty$
Normal approximation	Binomial(n, p) $\approx$ Normal$\left(\mu = np, \sigma = \sqrt{np(1-p)}\right)$ for $n \geq 30$, $0.05 \leq p \leq 0.95$
Central Limit Theorem	$\{(X_1 + \ldots + X_n) - n\mu\}/(\sigma\sqrt{n}) \to$ Normal(0,1) as $n \to \infty$

Expected values and variances of some distributions

Distribution	Bernoulli (p)	Binomial (n, p)	Geometric (p)	Poisson (λ)	Exponential (λ)	Gamma (r, λ)	Uniform (a, b)	Normal (μ, σ)
$E(X)$	p	np	$\frac{1}{p}$	λ	$\frac{1}{\lambda}$	$\frac{r}{\lambda}$	$\frac{a+b}{2}$	μ
Var(X)	$p(1-p)$	$np(1-p)$	$\frac{1-p}{p^2}$	λ	$\frac{1}{\lambda^2}$	$\frac{r}{\lambda^2}$	$\frac{(b-a)^2}{12}$	σ^2

Stochastic processes

Binomial process: $p = \lambda\Delta$, number of frames $n = \dfrac{t}{\Delta}$; number of events in time t is $X(t) \sim$Binomial(n, p); interarrival time is $T = Y\Delta$, where $Y \sim$Geometric(p)

Poisson process: number of events in time t is $X(t) \sim$Poisson(λt); interarrival time is $T \sim$Exponential(λ)

Markov chains

k-step transition probability matrix $P_k = P^k$

Steady state distribution is a solution of $\begin{cases} \pi P &= \pi \\ \sum \pi_i &= 1 \end{cases}$

Queuing Systems

Bernoulli single-server queuing system with capacity C

Arrival probability $p_A = \lambda_A \Delta$, departure probability $p_S = \lambda_S \Delta$.

Transition probabilities: $p_{00} = 1 - p_A$, $p_{01} = p_A$;
$p_{k,k-1} = p_S(1-p_A)$, $p_{k,k} = p_A p_S + (1-p_A)(1-p_S)$, $p_{k,k+1} = p_A(1-p_S)$ for $1 \le k \le C-1$;
$p_{C,C-1} = p_S(1-p_A)$, $p_{C,C} = p_A p_S + (1-p_A)(1-p_S) + p_A(1-p_S) = 1 - p_{C,C-1}$.

M/M/1 queueing system

Distribution of the number of jobs $\pi_x = P\{X = x\} = r^x(1-r)$ for $x = 0, 1, 2, \ldots$

$E(X) = \dfrac{r}{1-r}$, $\mathrm{Var}(X) = \dfrac{r}{(1-r)^2}$, where $r = \lambda_A/\lambda_S = \mu_S/\mu_A$

Performance characteristics $E(T) = \dfrac{\mu_S}{1-r} = \dfrac{1}{\lambda_S(1-r)}$, $E(W) = \dfrac{\mu_S\, r}{1-r} = \dfrac{r}{\lambda_S(1-r)}$, $E(X_w) = \dfrac{r^2}{1-r}$

Statistics

Sample mean $\bar{X} = \frac{1}{n}\sum X_i$, sample variance $s^2 = \frac{1}{n-1}\sum (X_i - \bar{X})^2$.

Confidence interval for the mean $\bar{X} \pm z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}$ (known σ), $\bar{X} \pm t_{\alpha/2}\dfrac{s}{\sqrt{n}}$ (unknown σ)

Confidence interval for the proportion $\hat{p} \pm z_{\alpha/2}\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

Hypothesis testing

H_0	H_A	Test statistic	Rejection region
$\mu = \mu_0$	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \ne \mu_0$	$Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$ (known σ)	$Z < -z_\alpha$ $Z > z_\alpha$ $\lvert Z\rvert > \lvert z_{\alpha/2}\rvert$

H_0	H_A	Test statistic	Rejection region
$\mu = \mu_0$	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \ne \mu_0$	$t = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}}$ (unknown σ)	$t < -t_\alpha$ $t > t_\alpha$ $\lvert t\rvert > \lvert t_{\alpha/2}\rvert$

H_0	H_A	Test statistic	Rejection region
$p = p_0$	$p < p_0$ $p > p_0$ $p \ne p_0$	$Z = \dfrac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$	$Z < -z_\alpha$ $Z > z_\alpha$ $\lvert Z\rvert > \lvert z_{\alpha/2}\rvert$

Binomial Cumulative Distribution Function

		p																		
n	*x*	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50	.55	.60	.65	.70	.75	.80	.85	.90	.95
5	0	.774	.590	.444	.328	.237	.168	.116	.078	.050	.031	.018	.010	.005	.002	.001	.000	.000	.000	.000
	1	.977	.919	.835	.737	.633	.528	.428	.337	.256	.188	.131	.087	.054	.031	.016	.007	.002	.000	.000
	2	.999	.991	.973	.942	.896	.837	.765	.683	.593	.500	.407	.317	.235	.163	.104	.058	.027	.009	.001
	3	1.0	1.0	.998	.993	.984	.969	.946	.913	.869	.813	.744	.663	.572	.472	.367	.263	.165	.081	.023
	4	1.0	1.0	1.0	1.0	.999	.998	.995	.990	.982	.969	.950	.922	.884	.832	.763	.672	.556	.410	.226
10	0	.599	.349	.197	.107	.056	.028	.013	.006	.003	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1	.914	.736	.544	.376	.244	.149	.086	.046	.023	.011	.005	.002	.001	.000	.000	.000	.000	.000	.000
	2	.988	.930	.820	.678	.526	.383	.262	.167	.100	.055	.027	.012	.005	.002	.000	.000	.000	.000	.000
	3	.999	.987	.950	.879	.776	.650	.514	.382	.266	.172	.102	.055	.026	.011	.004	.001	.000	.000	.000
	4	1.0	.998	.990	.967	.922	.850	.751	.633	.504	.377	.262	.166	.095	.047	.020	.006	.001	.000	.000
	5	1.0	1.0	.999	.994	.980	.953	.905	.834	.738	.623	.496	.367	.249	.150	.078	.033	.010	.002	.000
	6	1.0	1.0	1.0	.999	.996	.989	.974	.945	.898	.828	.734	.618	.486	.350	.224	.121	.050	.013	.001
	7	1.0	1.0	1.0	1.0	1.0	.998	.995	.988	.973	.945	.900	.833	.738	.617	.474	.322	.180	.070	.012
	8	1.0	1.0	1.0	1.0	1.0	1.0	.999	.998	.995	.989	.977	.954	.914	.851	.756	.624	.456	.264	.086
15	0	.463	.206	.087	.035	.013	.005	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1	.829	.549	.319	.167	.080	.035	.014	.005	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	2	.964	.816	.604	.398	.236	.127	.062	.027	.011	.004	.001	.000	.000	.000	.000	.000	.000	.000	.000
	3	.995	.944	.823	.648	.461	.297	.173	.091	.042	.018	.006	.002	.000	.000	.000	.000	.000	.000	.000
	4	.999	.987	.938	.836	.686	.515	.352	.217	.120	.059	.025	.009	.003	.001	.000	.000	.000	.000	.000
	5	1.0	.998	.983	.939	.852	.722	.564	.403	.261	.151	.077	.034	.012	.004	.001	.000	.000	.000	.000
	6	1.0	1.0	.996	.982	.943	.869	.755	.610	.452	.304	.182	.095	.042	.015	.004	.001	.000	.000	.000
	7	1.0	1.0	.999	.996	.983	.950	.887	.787	.654	.500	.346	.213	.113	.050	.017	.004	.001	.000	.000
	8	1.0	1.0	1.0	.999	.996	.985	.958	.905	.818	.696	.548	.390	.245	.131	.057	.018	.004	.000	.000
	9	1.0	1.0	1.0	1.0	.999	.996	.988	.966	.923	.849	.739	.597	.436	.278	.148	.061	.017	.002	.000
	10	1.0	1.0	1.0	1.0	1.0	.999	.997	.991	.975	.941	.880	.783	.648	.485	.314	.164	.062	.013	.001
	11	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.998	.994	.982	.958	.909	.827	.703	.539	.352	.177	.056	.005
20	1	.736	.392	.176	.069	.024	.008	.002	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	2	.925	.677	.405	.206	.091	.035	.012	.004	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	3	.984	.867	.648	.411	.225	.107	.044	.016	.005	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000
	4	.997	.957	.830	.630	.415	.238	.118	.051	.019	.006	.002	.000	.000	.000	.000	.000	.000	.000	.000
	5	1.0	.989	.933	.804	.617	.416	.245	.126	.055	.021	.006	.002	.000	.000	.000	.000	.000	.000	.000
	6	1.0	.998	.978	.913	.786	.608	.417	.250	.130	.058	.021	.006	.002	.000	.000	.000	.000	.000	.000
	7	1.0	1.0	.994	.968	.898	.772	.601	.416	.252	.132	.058	.021	.006	.001	.000	.000	.000	.000	.000
	8	1.0	1.0	.999	.990	.959	.887	.762	.596	.414	.252	.131	.057	.020	.005	.001	.000	.000	.000	.000
	9	1.0	1.0	1.0	.997	.986	.952	.878	.755	.591	.412	.249	.128	.053	.017	.004	.001	.000	.000	.000
	10	1.0	1.0	1.0	.999	.996	.983	.947	.872	.751	.588	.409	.245	.122	.048	.014	.003	.000	.000	.000
	11	1.0	1.0	1.0	1.0	.999	.995	.980	.943	.869	.748	.586	.404	.238	.113	.041	.010	.001	.000	.000
	12	1.0	1.0	1.0	1.0	1.0	.999	.994	.979	.942	.868	.748	.584	.399	.228	.102	.032	.006	.000	.000
	13	1.0	1.0	1.0	1.0	1.0	1.0	.998	.994	.979	.942	.870	.750	.583	.392	.214	.087	.022	.002	.000
	14	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.998	.994	.979	.945	.874	.755	.584	.383	.196	.067	.011	.000
	15	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.998	.994	.981	.949	.882	.762	.585	.370	.170	.043	.003

Poisson Cumulative Distribution Function

	λ														
x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
0	.905	.819	.741	.670	.607	.549	.497	.449	.407	.368	.333	.301	.273	.247	.223
1	.995	.982	.963	.938	.910	.878	.844	.809	.772	.736	.699	.663	.627	.592	.558
2	1.00	.999	.996	.992	.986	.977	.966	.953	.937	.920	.900	.879	.857	.833	.809
3	1.00	1.00	1.00	.999	.998	.997	.994	.991	.987	.981	.974	.966	.957	.946	.934
4	1.00	1.00	1.00	1.00	1.00	1.00	.999	.999	.998	.996	.995	.992	.989	.986	.981
5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.999	.998	.998	.997	.996

	λ														
x	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
0	.202	.183	.165	.150	.135	.122	.111	.100	.091	.082	.074	.067	.061	.055	.050
1	.525	.493	.463	.434	.406	.380	.355	.331	.308	.287	.267	.249	.231	.215	.199
2	.783	.757	.731	.704	.677	.650	.623	.596	.570	.544	.518	.494	.469	.446	.423
3	.921	.907	.891	.875	.857	.839	.819	.799	.779	.758	.736	.714	.692	.670	.647
4	.976	.970	.964	.956	.947	.938	.928	.916	.904	.891	.877	.863	.848	.832	.815
5	.994	.992	.990	.987	.983	.980	.975	.970	.964	.958	.951	.943	.935	.926	.916
6	.999	.998	.997	.997	.995	.994	.993	.991	.988	.986	.983	.979	.976	.971	.966
7	1.00	1.00	.999	.999	.999	.999	.998	.997	.997	.996	.995	.993	.992	.990	.988
8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.999	.999	.999	.998	.998	.997	.996
9	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.999	.999	.999

Standard Normal Cumulative Distribution Function

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990

Table of Student's T-distribution. Contains t_α; critical values, such that $P\{t > t_\alpha\} = \alpha$

	α, the right-tail probability									
ν (d.f.)	.10	.05	.025	.02	.01	.005	.0025	.001	.0005	.0001
1	3.078	6.314	12.706	15.89	31.82	63.66	127.3	318.3	636.6	3185
2	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60	70.71
3	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92	22.20
4	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610	13.04
5	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.894	6.869	9.676
6	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959	8.023
7	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408	7.064
8	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041	6.442
9	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781	6.009
10	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587	5.694
11	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437	5.453
12	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318	5.263
13	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221	5.111
14	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140	4.985
15	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073	4.880
16	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015	4.790
17	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965	4.715
18	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.610	3.922	4.648
19	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883	4.590
20	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850	4.539
22	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792	4.452
24	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745	4.382
26	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707	4.324
28	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674	4.276
30	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646	4.234
32	1.309	1.694	2.037	2.141	2.449	2.738	3.015	3.365	3.622	4.198
34	1.307	1.691	2.032	2.136	2.441	2.728	3.002	3.348	3.601	4.168
36	1.306	1.688	2.028	2.131	2.434	2.719	2.990	3.333	3.582	4.140
38	1.304	1.686	2.024	2.127	2.429	2.712	2.980	3.319	3.566	4.115
40	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551	4.094
45	1.301	1.679	2.014	2.115	2.412	2.690	2.952	3.281	3.520	4.049
50	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496	4.014
60	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460	3.962
70	1.294	1.667	1.994	2.093	2.381	2.648	2.899	3.211	3.435	3.926
80	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416	3.899
90	1.291	1.662	1.987	2.084	2.368	2.632	2.878	3.183	3.402	3.878
100	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390	3.861
200	1.286	1.653	1.972	2.067	2.345	2.601	2.838	3.131	3.340	3.789
∞	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.090	3.290	3.719