THE FEYNMAN–KAC FORMULA AND SOME APPLICATIONS

SAMANTHA XU

In this expository note we explore the Feynman–Kac formula, along with some of its applications. This formula gives a connection between measures on the space of continuous functions and (parabolic) partial differential equations. While extremely useful as a black-box, we will actually utilize the proof of the Feynman–Kac formula to obtain much stronger results.

1. Brownian motion and Wiener Measure

A *Gaussian random variable* X of mean μ and variance σ^2 obeys the law

$$\mathbb{P}\{X \in [a,b]\} = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\, dx.$$

There are many equivalent formulations of Brownian motion. We present two here (though we will mostly work the latter):

Lemma 1.1. *For a collection $\{B(t)\}_{t\in\mathbb{Q}^+}$ of Gaussian random variables, the following conditions are equivalent*

(1) *If $0 < t_1 < \ldots < t_n$, then $B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ are independent and have mean 0 and variance $t_1, t_2 - t_1, \ldots, t_n - t_{n-1}$, respectively.*

(2) *For $t > s$, define $p(t, x; s, y) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}}$. If $0 < t_1 < \ldots < t_n$ and $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are Borel subsets of $\dot{\mathbb{R}} = \mathbb{R} \cup \infty$, then*

$$\mathbb{P}(B(t_j) \in \mathcal{A}_j, j = 1, \ldots, n) = \int_{\mathcal{A}_1}\int_{\mathcal{A}_2}\cdots\int_{\mathcal{A}_n} p(t_n, x_n; t_{n-1}, x_{n-1})\cdots p(t_2, x_2; t_1, x_1)p(t_1, x_1; 0, 0)dx_n \ldots dx_2dx_1.$$

Proof. To show (1) $\implies$ (2), the hypothesis imply that

$$\mathbb{P}(B(t_2) \in [a,b] \mid B(t_1) = x_1) = \int_a^b p(t_2, x_2; t_1, x_1)\, dx_2.$$

For the rest, it is best to draw pictures here.

To show (2) $\implies$ (1), observe that $B(0) = 0$ and, for $0 \le t_1 < t_2 \le t_3 < t_4$,

$$\begin{aligned}\mathbb{P}(B(t_2) - B(t_1) \in (a,b)) &= \int_{-\infty}^{\infty}\int_{x_2 - x_1 \in (a,b)} p(t_2, x_2; t_1, x_1)p(t_1, x_1; 0, 0)\, dx_2dx_1 \\ &= \int_a^b \frac{1}{\sqrt{2\pi(t_2 - t_1)}} e^{-\frac{x^2}{2(t_2-t_1)}}\, dx\end{aligned}$$

and

$$\mathbb{P}(B(t_2) - B(t_1) \in (a,b) \text{ and } B(t_4) - B(t_3) \in (c,d))$$

$$= \int_{-\infty}^{\infty} \int_{x_2 - x_1 \in (a,b)} \int_{-\infty}^{\infty} \int_{x_4 - x_3 \in (c,d)} p(t_4, x_4; t_3, x_3) \times$$
$$\times\, p(t_3, x_3; t_2, x_2) p(t_2, x_2; t_1, x_1) p(t_1, x_1; 0, 0)\, dx_4 dx_3 dx_2 dx_1$$
$$= \int_a^b \frac{1}{\sqrt{2\pi(t_2 - t_1)}} e^{-\frac{x^2}{2(t_2-t_1)}}\, dx \cdot \int_c^d \frac{1}{\sqrt{2\pi(t_4 - t_3)}} e^{-\frac{y^2}{2(t_4-t_3)}}\, dy$$
$$= \mathbb{P}(B(t_2) - B(t_1) \in (a,b)) \cdot \mathbb{P}(B(t_4) - B(t_3) \in (c,d))$$

□

Theorem 1.2 (Existence of Wiener Measure)**.** *There exists a measure W on $\{f : \mathbb{Q}^+ \to \mathbb{R}\}$ such that*

$$W(f(t_j) \in \mathcal{A}_j, j = 1, \ldots, n) = \int_{\mathcal{A}_1} \int_{\mathcal{A}_2} \cdots \int_{\mathcal{A}_n} p(t_n, x_n; t_{n-1}, x_{n-1}) \cdots$$
$$p(t_2, x_2; t_1, x_1) p(t_1, x_1; 0, 0) dx_n \ldots dx_2 dx_1.$$

when $0 < t_1 < \ldots < t_n$ and $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are Borel subsets of $\dot{\mathbb{R}}$.

The construction of Wiener measure relies on the fact that probability measures on infinite dimensional spaces can be characterized by their finite dimensional distributions. This is the content of the Kolmogorov Consistency theorem. To begin, we define the *cylinder σ-algebra* on $(\dot{\mathbb{R}})^{\mathbb{N}}$ as the smallest σ-algebra containing the sets $A_1 \times A_2 \times \cdots \times A_n \times \dot{\mathbb{R}} \times \dot{\mathbb{R}} \times \cdots$.

Theorem 1.3 (Kolmogorov Consistency Theorem)**.** *Let μ_d be (Borel) probability measures on $\dot{\mathbb{R}}^d$ such that for every Borel set $A \subseteq \dot{\mathbb{R}}^d$ $\mu_{d+1}(A \times \dot{\mathbb{R}}) = \mu_d(A)$. Then there exists a (Borel) probability measure μ on $\dot{\mathbb{R}}^{\mathbb{N}}$ such that $\mu\big(\{(x_1, x_2, \ldots) : (x_1, \ldots, x_d) \in A\}\big) = \mu_d(A)$.*

Proof. Letting $C_{fin}((\dot{\mathbb{R}})^{\mathbb{N}})$ denote the space of all continuous functions on $(\dot{\mathbb{R}})^{\mathbb{N}}$ that depend on only finitely many coordinates, we define the map[1] $\ell : C_{fin}((\dot{\mathbb{R}})^{\mathbb{N}}) \to \mathbb{R}$ by

$$\ell(g) = \int g(x_\alpha) d\mu_d(x_\alpha)$$

if g is a function of $(x_1, \ldots, x_d)$. Since the μ_d's are consistent, therefore $\ell(\cdot)$ is a well-defined linear functional. It is in fact continuous on $C_{fin}((\dot{\mathbb{R}})^{\mathbb{N}})$ because $|\ell(g)| \leq \|g\|_\infty$. Note that the identity map is in $C_{fin}((\dot{\mathbb{R}})^{\mathbb{N}})$ and that it clearly separates points. Thus, the Stone-Weierstrass theorem asserts that $C_{fin}((\dot{\mathbb{R}})^{\mathbb{N}})$ is dense in $C((\dot{\mathbb{R}})^{\mathbb{N}})$ and so ℓ extends uniquely to a positive linear function on $C((\dot{\mathbb{R}})^{\mathbb{N}})$. It follows by the Riesz theorem that there is a unique Borel measure μ on $(\dot{\mathbb{R}})^{\mathbb{N}}$ such that

$$\ell(g) = \int g(\vec{x}) d\mu(\vec{x}).$$

□

Proof of Existence of Wiener Measure. We need to check

$$W(f(t_j) \in \mathcal{A}_j, j = 1, \ldots, n \text{ and } f(T) \in \dot{\mathbb{R}}) = W(f(t_j) \in \mathcal{A}_j, j = 1, \ldots, n).$$

[1] Here we extend μ_d to $(\dot{\mathbb{R}})^d$ via $\mu_d(A) := \mu_d(A \cap \mathbb{R}^d)$ for all Borel $A \subseteq (\dot{\mathbb{R}})^d$

This is an immediate consequence of the identities

$$\int_{-\infty}^{\infty} p(T, w; t, x)\, dw = 1$$

$$\int_{-\infty}^{\infty} p(t_2, x_2; T, w) p(T, w; t_1, x_1)\, dw = p(t_2, x_2; t_1, x_1),$$

which, in turn, are easy computations. □

Theorem 1.4. *Brownian motion is almost surely locally Hölder continuous of exponent almost* $\frac{1}{2}$.

This regularity result is essentially the content of a result known as Kolmogorov's Continuity Criterion and the fact that $B(t) - B(s)$, $t > s$ is gaussian with mean 0 and variance $t - s$.

Proposition 1.5.

$$\mathbb{E}[|B(t) - B(s)|^p] \lesssim_p |t - s|^{\frac{p}{2}}.$$

Proof.

$$\mathbb{P}(|B(t) - B(s)| > \lambda) = 2\int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{2(t-s)}}\, dx = 2\int_{\lambda\sqrt{t-s}}^{\infty} e^{-\frac{x^2}{2}}\, dx \lesssim e^{-\frac{\lambda^2}{2(t-s)}}.$$

Thus,

$$\mathbb{E}[|B(t) - B(s)|^p] = \int_0^{\infty} \lambda^{p-1} \mathbb{P}(|B(t) - B(s)| > \lambda)\, d\lambda \lesssim |t - s|^{\frac{p}{2}}.$$

□

Proposition 1.6 (Kolmogorov Continuity Criterion)**.** *Let* $X(t)$, $t \in [0, 1) \cap Q_{dyad}$, *be a family of (real-valued) random variables such that there exists* $\gamma > 1$, $\varepsilon > 0$ *such that*

$$\mathbb{E}[|X(t) - X(s)|^{\gamma}] \lesssim |t - s|^{1+\varepsilon},$$

then, for $\alpha < \varepsilon/\gamma$.

$$\mathbb{E}\left[\left(\sup_{\substack{s \neq t \\ \text{dyadic}}} \frac{|X(t) - X(s)|}{|t - s|^{\alpha}}\right)^{\gamma}\right] < \infty.$$

Remark. This follows the treatment of Revuz, Yor, "Continuous Martingales and Brownian Motion."

Proof. For each integer $n \geq 0$, define $D_n = \{\frac{k}{2^n} : 0 \leq k \leq 2^n\}$ and

$$K_n = \sup_{\substack{|s-t| = 2^{-n} \\ s,t \in D_n}} |X(t) - X(s)|.$$

Because there are at most 2^{n+1} number of such pairs (s, t), therefore the supremum is actually achieved. Furthermore,

$$\mathbb{E}[K_n^{\gamma}] \leq \sum_{\substack{|s-t| = 2^{-n} \\ s,t \in D_n}} \mathbb{E}[|X(t) - X(s)|^{\gamma}] \lesssim 2^{N+n} 2^{-n(1+\varepsilon)} \lesssim 2^{-n\varepsilon}.$$

Let s, t be dyadic numbers such that $|s-t| \leq 2^{-n}$, then we claim that

$$|X(t) - X(s)| \leq 2 \sum_{j=n}^{\infty} K_j.$$

Indeed, say $s \in D_{N_1}$ and $t \in D_{N_2}$, which we may assume[2] to be greater than n. Write $s = \sum_{k=1}^{N_1} \frac{a_k}{2^k}, t = \sum_{k=1}^{N_2} \frac{b_k}{2^k}$ with $a_k, b_k \in \{0, 1\}$, and define $\{s_j\}_{j=1}^{N_1}, \{t_j\}_{j=1}^{N_2}$ by

$$s_j = \sum_{k=1}^{j} \frac{a_k}{2^k} \qquad \text{and} \qquad t_j = \sum_{k=1}^{j} \frac{b_k}{2^k}.$$

Then

$$X(t) - X(s) = X(s_n) - X(t_n) + \sum_{j=n}^{N_1 - 1} [X(s_{j+1}) - X(s_j)] + \sum_{j=n}^{N_2 - 1} [X(t_{j+1}) - X(t_j)]$$

and $|s - t| \leq 2^{-n}$ implies $s_{n-1} = t_{n-1}$ [otherwise the difference becomes irreconcilable] and thus $|s_n - t_n| = 0$ or 2^{-n}. The claim follows.

Let $M_\alpha = \sup_{\substack{t \neq s \\ \text{dyadic}}} \frac{|X(t)-X(s)|}{|t-s|^\alpha}$. Then, by the above argument

$$\begin{aligned}
M_\alpha &\leq \sup_{n \geq 1} 2^{(n+1)\alpha} \sup_{\substack{2^{-(n+1)} \leq |t-s| \leq 2^{-n} \\ \text{dyadic}}} |X(t) - X(s)| \\
&\leq \sup_{n \geq 1} 2^{(n+1)\alpha} \sup_{\substack{0 \leq |t-s| \leq 2^{-n} \\ \text{dyadic}}} |X(t) - X(s)| \\
&\leq \sup_{n \geq 1} 2^{(n+1)\alpha} \sum_{j=n}^{\infty} K_j \\
&\lesssim_\alpha \sum_{j=1}^{\infty} 2^{j\alpha} K_j.
\end{aligned}$$

Then, with $\alpha < \varepsilon/\gamma$, we have

$$[\mathbb{E}[M_\alpha^\gamma]]^{1/\gamma} \lesssim_\alpha \sum_{j=1}^{\infty} 2^{j\alpha} [\mathbb{E}[K_j^\gamma]]^{1/\gamma} = \sum_{j=1}^{\infty} 2^{j(\alpha - \frac{\varepsilon}{\gamma})} < \infty.$$

□

Proof of Hölder regularity of Brownian Motion. We can take $\alpha < \frac{p-2}{2p}$, which in turn can be made arbitrarily close to $\frac{1}{2}$ as $p \to \infty$. □

Remark. (1) We may construct measures $W_{t,x}$ where the random walk begins at time t and at point x. This is the same as conditioning $B(t)$ to be equal to x. Rigorously speaking, we define a measure $W_{t,x}$ on $f : \mathbb{Q}^{\geq t} \to \dot{\mathbb{R}}$ by

$$W_{t,x}(f(t_j) \in \mathcal{A}_j, j = 1, \dots, n) = \int_{\mathcal{A}_1} \int_{\mathcal{A}_2} \cdots \int_{\mathcal{A}_n} p(t_n, x_n; t_{n-1}, x_{n-1}) \cdots \\ p(t_2, x_2; t_1, x_1) p(t_1, x_1; t, x) dx_n \dots dx_2 dx_1.$$

[2]This follows from the inclusion $D_1 \subset D_2 \subset \cdots$.

with $t < t_1 < t_2 < \cdots < t_n$ and $\mathcal{A}_j \subset \dot{\mathbb{R}}$ being Borel. The same consistency check remains valid here.

(2) We may also define the *Brownian bridge measure* on $[0,1]$ by

$$BB(f(t_j) \in \mathcal{A}_j, j = 1, \ldots, n) = \int_{\mathcal{A}_1} \int_{\mathcal{A}_2} \cdots \int_{\mathcal{A}_n} \frac{p(1,0;t_n,x_n)}{p(1,0;0,0)} p(t_n,x_n;t_{n-1},x_{n-1}) \cdots \\ p(t_2,x_2;t_1,x_1)p(t_1,x_1;t,x)dx_n \ldots dx_2 dx_1$$

with $0 < t_1 < t_2 < \cdots < t_n < 1$ and $\mathcal{A}_j \subset \dot{\mathbb{R}}$ being Borel. The consistency check follows from $\int_{-\infty}^{\infty} p(t_2,x_2;T,w)p(T,w;t_1,x_1)\, dw = p(t_2,x_2;t_1,x_1)$.

2. Feynman–Kac

The Feynman–Kac formula gives a connection between measures on path space to parabolic partial differential equations. The underlying idea of this connection is the fact that $p(t,x;s,y)$, the transitional probability for Brownian motion, obeys

$$\partial_t p(t,x;s,y) = \frac{1}{2}\partial_x^2 p(t,x;s,y) \qquad \text{and} \qquad -\partial_s p(t,x;s,y) = \frac{1}{2}\partial_y^2 p(t,x;s,y).$$

As a basic example, we illustrate the connection in the case of Brownian motion.

(1) For bounded, continuous $g : \mathbb{R} \to \mathbb{R}$, the function

$$\begin{aligned} \phi(t,x) &:= \mathbb{E}^{W_{0,x}}[g(B(t)] \\ &= \int_{-\infty}^{\infty} p(t,y;0,x)g(y)\, dy \\ &= \int_{-\infty}^{\infty} p(t,x;0,y)g(y)\, dy \end{aligned}$$

obeys the PDE

$$\partial_t \phi = \frac{1}{2}\partial_x^2 \phi$$

with initial condition $\phi(0,\cdot) = g(\cdot)$. It is best to draw an interpretation of $\phi(t,x)$ here.

(2) Fix $T > 0$. For $0 < t < T$ and for bounded, continuous $f : \mathbb{R} \to \mathbb{R}$, the function

$$\begin{aligned} \psi(t,x) &:= \mathbb{E}^{W_{t,x}}[f(B(T))] \\ &= \int_{-\infty}^{\infty} p(T,w;t,x)f(w)\, dw \end{aligned}$$

obeys the PDE

$$-\partial_t \psi = \frac{1}{2}\partial_x^2 \psi$$

with terminal condition $\phi(T,\cdot) = f(\cdot)$. It is best to draw an interpretation of $\psi(t,x)$ here.

Theorem 2.1 (Feynman–Kac)**.** *Fix $T > 0$. Let $V(t,x)$ be continuous and bounded and let $f : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Then , for $0 < t < T$ and $x \in \mathbb{R}$, the function*

$$u(t,x) := \mathbb{E}^{W_{t,x}}\left[f(B(T)) \exp\left(\int_t^T V(\tau, B(\tau))\, d\tau \right) \right]$$

obeys the PDE

$$-\partial_t u = \frac{1}{2}\partial_x^2 u + V(t,x)u$$

with terminal condition $u(T,\cdot) = f(\cdot)$.

Remark. A similar result for the initial condition version of the Feynman–Kac formula also holds. See Varadhan, "Stochastic Processes" for further details.

Proof. The terminal condition is clear. Expanding the exponential, we have

$$u(t,x) = \mathbb{E}^{W_{t,x}}[f(B(T))] + \sum_{n=1}^{\infty} P_n(t,x)$$

where

$$\begin{aligned} P_n(t,x) &= \frac{1}{n!}\mathbb{E}^{W_{t,x}}\left[f(B(T))\int_t^T \cdots \int_t^T V(\tau_1, B(\tau_1)) \times \cdots \right. \\ &\qquad\qquad \left. \cdots \times V(\tau_n, B(\tau_n))\, d\tau_1 \cdots d\tau_n\right] \\ &= \mathbb{E}^{W_{t,x}}\left[f(B(T))\int \cdots \int_{t\le\tau_1\le\cdots\le\tau_n\le T} V(\tau_1, B(\tau_1)) \times \cdots \right. \\ &\qquad\qquad \left. \cdots \times V(\tau_n, B(\tau_n))\, d\tau_1 \cdots d\tau_n\right] \\ &= \int \cdots \int_{t\le\tau_1\le\cdots\le\tau_n\le T} \mathbb{E}^{W_{t,x}}\left[f(B(T))V(\tau_1, B(\tau_1)) \times \cdots \right. \\ &\qquad\qquad \left. \cdots \times V(\tau_n, B(\tau_n))\right]\, d\tau_1 \cdots d\tau_n. \end{aligned}$$

The integrand in the last expression can be evaluated as

$$\begin{aligned} &\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(y)p(T,y;\tau_n,x_n)V(\tau_n,x_n)p(\tau_n,x_n;\tau_{n-1}x_{n-1})\cdots \\ &\qquad\qquad V(\tau_1,x_1)p(\tau_1,x_1;t,x)\, dx_1\cdots dx_n dy. \end{aligned}$$

The time derivative of P_n will then produce two terms, one for differentiating the integral bounds and one for differentiating inside the integral. Because

$$\lim_{\tau_1 \downarrow t} p(\tau_1, x_1; t, x) = \delta_{x_1 - x}$$

it follows that, for integer $n \ge 1$,

$$-\partial_t P_n(t,x) = \frac{1}{2}\partial_x^2 P_n(t,x) + V(t,x)P_{n-1}(t,x)$$

with $P_0 := \mathbb{E}^{W_{t,x}}[f(B(T))]$. Because $-\partial_t \mathbb{E}^{W_{t,x}}[f(B(T))] = \frac{1}{2}\partial_x^2 \mathbb{E}^{W_{t,x}}[f(B(T))]$ as above, therefore

$$-\partial_t u = \frac{1}{2}\partial_x^2 \mathbb{E}^{W_{t,x}}[f(B(T))] + \sum_{n=1}^{\infty} \frac{1}{2}\partial_x^2 P_n + V(t,x)P_{n-1} = \frac{1}{2}\partial_x^2 u + V(t,x)u,$$

as desired. □

The result gives an immediate corollary in the case that the fundamental solution to $-\partial_t u = \frac{1}{2}\partial_x^2 u + V(t,x)u$.

Corollary 2.2. *Fix $T > 0$. Let $V(t,x)$ be continuous and bounded and let $f : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Suppose that $\Phi(T,w;t,x)$ is the fundamental solution to $-\partial_t u = \frac{1}{2}\partial_x^2 u + V(t,x)u$. Then , for $0 < t < T$ and $x \in \mathbb{R}$, the function*

$$\mathbb{E}^{W_{t,x}}\left[f(B(T))\exp\left(\int_t^T V(\tau,B(\tau))\,d\tau\right)\right] = \int_{\mathbb{R}} f(w)\Phi(T,w;t,x)\,dw.$$

3. Multi-time Feynman–Kac

The Feynman–Kac formula is a very powerful result which has many applications as a black box. In this section, we explore an alternate route, which is to strengthen the formula itself by utilizing the proof of the Feynman–Kac itself.

Theorem 3.1 (Multi-time Feynman–Kac)**.** *Let $0 < T_1 < \cdots < T_N < 1$ be fixed and let $f_1,\ldots,f_N, g : \mathbb{R} \to \mathbb{R}$ be bounded, measurable functions. Let $V(t,x)$ be bounded and continuous and suppose that $\Phi(T,w;t,x)$ is the fundamental solution to $-\partial_t\Phi = \frac{1}{2}\partial_x^2\Phi + V(t,x)\Phi$. Then*

$$\mathbb{E}^W\left[e^{\int_0^1 V(\tau,B(\tau))\,d\tau} f_1(B(T_1))\cdots f_N(B(T_N))g(B(1))\right]$$

$$= \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} g(x)\Phi(1,x;T_N,x_N)f_N(x_N)\Phi(T_N,x_N;T_{N-1},x_{N-1})\times\cdots$$
$$\cdots\times f_1(x_1)\Phi(T_1,x_1;0,0)\,dx dx_N\cdots dx_1.$$

Proof. We only treat the case $N = 1$; the induction step works in much the same way. Then

$$\mathbb{E}^W\left[e^{\int_0^{T_1} V(\tau,B(\tau))\,d\tau} f_1(B(T_1)) e^{\int_{T_1}^1 V(\tau,B(\tau))\,d\tau} g(B(1))\right]$$
$$= \mathbb{E}^W\left[\left(\sum_{n=1}^{\infty}\right)(f_1(B(T_1)))\left(\sum_{m=1}^{\infty} B_m\right)g(B(1))\right]$$

where

$$A_n = \frac{1}{n!}\int_0^{T_1}\cdots\int_0^{T_1} V(t_1,B(t_1))\cdots V(t_n,B(t_n))\,dt_n\cdots dt_1$$
$$B_n = \frac{1}{n!}\int_{T_1}^1\cdots\int_{T_1}^1 V(s_1,B(s_1))\cdots V(s_m,B(s_m))\,ds_m\cdots ds_1.$$

Fix an $n,m \in \mathbb{N}$ and let $d\vec{t} = dt_n\cdots dt_1$, $d\vec{s} = ds_m\cdots ds_1$, and

$$\mathcal{T}_n := \{(t_1,\ldots,t_n) \in \mathbb{R}^n : 0 < t_1 < \cdots < t_n < T_1\}$$
$$\mathcal{S}_m := \{(s_1,\ldots,s_m) \in \mathbb{R}^m : T_1 < s_1 < \cdots < s_m < 1\}.$$

Then, denoting $s_0 = T_1$ and $t_0 = 0$,

$$\mathbb{E}^W[A_n f(B(T_1))B_m g(B(1))] = \int_{\mathcal{T}_n}\int_{\mathcal{S}_m}\int_{\mathbb{R}^n}\int_{\mathbb{R}}\int_{\mathbb{R}^m}\int_{\mathbb{R}} g(\beta)p(1,\beta;s_m,x_m)$$
$$\prod_{j=2}^m V(s_j,y_j)p(s_j,y_j;s_{j-1},y_{j-1})\cdot V(s_1,y_1)p(t_1,x_1;T_1,\alpha)f(\alpha)p(T_1,\alpha;t_n,x_n)$$
$$\prod_{j=2}^n V(t_j,y_j)p(t_j,y_j;t_{j-1},y_{j-1})\cdot V(t_1,x_1)p(t_1,x_1;0,0)\,d\beta\,d\vec{y}\,d\alpha\,dx\,d\vec{s}\,d\vec{t}$$

Recall from the proof of the Feynman–Kac formula,

$$P_m(T_1,\alpha) = \mathbb{E}^{W_{T_1,\alpha}}\left[\frac{1}{m!}\left(\int_{T_1}^1 V(\tau,B(\tau))\,d\tau\right)^m f(B(1))\right]$$

and it's corollary, $\sum_{m=0}^\infty P_m(T_1,\alpha) = \mathbb{E}^{W_{T_1,\alpha}}\left[g(B(1))\exp\left(\int_{T_1}^1 V(\tau,B(\tau))\,d\tau\right)\right] = \int_{\mathbb{R}} g(\beta)\Phi(1,\beta;T_1,\alpha)\,d\beta$, then

$$\sum_{m=0}^\infty \mathbb{E}^W\left[A_n f(B(T_1))B_m g(B(1))\right] = \int_{\mathbb{R}}\int_{\mathcal{T}_n}\int_{\mathbb{R}_n}\left[f(\alpha)\int_{\mathbb{R}} g(\beta)\Phi(1,\beta;T_1,\alpha)\,d\beta\right]$$

$$p(T_1,\alpha;t_n,x_n)\prod_{k=2}^n V(t_k,x_k)p(t_k,x_k;t_{k-1},x_{k-1})\cdot V(t_1,x_1)p(t_1,x_1;0,0)\,d\vec{x}\,d\vec{t}\,d\alpha.$$

Note that, if $h(\alpha) := f(\alpha)\int_{\mathbb{R}} g(\beta)\Phi(1,\beta;T_1,\alpha)\,d\beta$, then it follows that

$$\begin{aligned}
&\mathbb{E}^W\left[e^{\int_0^1 V(\tau,B(\tau))\,d\tau} f_1(B(T_1))g(B(1))\right]\\
&\quad= \sum_{n=0}^\infty\sum_{m=0}^\infty \mathbb{E}^W\left[A_n f(B(T_1))B_m g(B(1))\right]\\
&\quad= \mathbb{E}^W\left[e^{\int_0^{T_1} V(\tau,B(\tau))\,d\tau} h(B(T_1))\right]\\
&\quad= \int_{\mathbb{R}}\left[f(\alpha)\int_{\mathbb{R}} g(\beta)\Phi(1,\beta;T_1,\alpha)\,d\beta\right]\Phi(T_1,\alpha;0,0)\,d\alpha.
\end{aligned}$$

□

In a similar manner, we also have the following result.

Theorem 3.2 (Multi-time Feynman–Kac, Brownian Bridge Version)**.** *Let* $0 < T_1 < \cdots < T_N < 1$ *be fixed and let* $f_1,\ldots,f_N : \mathbb{R}\to\mathbb{R}$ *be bounded, measurable functions. Let* $V(t,x)$ *be bounded and continuous and suppose that* $\Phi(T,w;t,x)$ *is the fundamental solution to* $-\partial_t\Phi = \frac{1}{2}\partial_x^2\Phi + V(t,x)\Phi$*. Then*

$$\begin{aligned}
&p(1,0;0,0)\mathbb{E}^{BB}\left[e^{\int_0^1 V(\tau,B(\tau))\,d\tau} f_1(B(T_1))\cdots f_N(B(T_N))\right]\\
&\quad= \int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\Phi(1,0;T_N,x_N)f_N(x_N)\Phi(T_N,x_N;T_{N-1},x_{N-1})\times\cdots\\
&\qquad\qquad\cdots\times f_1(x_1)\Phi(T_1,x_1;0,0)\,dx_N\cdots dx_1.
\end{aligned}$$

4. An application to a non-linear wave equation

As an application, we wish to analyze the statistical mechanics of the nonlinear wave equation on $B(0,1)\subset\mathbb{R}^3$

$$\begin{cases} -\partial_t^2 u + \Delta u = u^3\\ u(t,x):\mathbb{R}^+\times B(0,1)\to\mathbb{R},\\ u \text{ is radial in } x,\ \text{and } u|_{R^+\times\partial B(0,1)} = 0\end{cases}$$

To do this, we wish to make sense of the "Gibbs measure"

$$\frac{1}{Z_1}e^{-\int_{B(0,1)}\frac{1}{2}|\nabla u|^2+\frac{1}{4}|u|^4}\text{“}du\text{”}\otimes\frac{1}{Z_2}e^{-\int_{B(0,1)}\frac{1}{2}|u_t|^2}\text{“}du_t\text{”}.$$

We will focus primarily on the former measure $\frac{1}{Z_1} e^{-\int_{B(0,1)} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4}$ "du" (the latter is known as "white noise").

First, we make sense of the "free measure" $\mu = \frac{1}{Z} e^{-\int_{B(0,1)} \frac{1}{2}|\nabla u|^2}$ "du". It can be shown that it has the same law as that of the stochastic series $\sum_{n=1}^{\infty} \frac{a_n}{n\pi} \frac{\sqrt{2}\sin(n\pi r)}{r}$, $a_n \sim N(0,1)$. It follows, say by Karhunen–Loeve theory, that $r_*\mu$ has the same law as that of Brownian bridge. More later.

Lemma 4.1. *The map $C([0,1]) \to \mathbb{R}$ given by $f(r) \mapsto e^{-\frac{\pi}{2}\int_0^1 \frac{(f(r))^4}{r^2}\,dr}$ is measurable.*

Proof. Indeed, $f(r) \mapsto \int_0^1 \frac{(f(r))^4}{\max(r^2, \frac{1}{n})}\,dr$ is a continuous map from $C([0,1])$ to $\mathbb{R}$ for each $n \in \mathbb{N}$. It follows that $f(r) \mapsto \exp\left(-\frac{\pi}{2}\int_0^1 \frac{(f(r))^4}{\max(r^2,\frac{1}{n})}\,dr\right)$ is also continuous. Taking the limit $n \to \infty$ shows that $f(r) \mapsto \exp\left(-\frac{\pi}{2}\int_0^1 \frac{(f(r))^4}{r^2}\,dr\right)$ is measurable. □

Lemma 4.2. $\mathbb{E}^{BB}\left[\exp\left(-\frac{\pi}{2}\int_0^1 \frac{(BB(r))^4}{r^2}\,dr\right)\right] > 0.$

Proof. It suffices to show that $\int_0^1 \frac{(BB(r))^4}{r^2}\,dr$ is finite almost surely, but this follows from the fact that Brownian bridge is almost surely Hölder-$\frac{1}{3}$ continuous. □

More things here.