LECTURE 12: STOCHASTIC DIFFERENTIAL EQUATIONS, DIFFUSION PROCESSES, AND THE FEYNMAN-KAC FORMULA

1. Existence and Uniqueness of Solutions to SDEs

It is frequently the case that economic or financial considerations will suggest that a stock price, exchange rate, interest rate, or other economic variable evolves in time according to a stochastic differential equation of the form

$$dX_t = \alpha(t, X_t)\, dt + \beta(t, X_t)\, dW_t \tag{1}$$

where W_t is a standard Brownian motion and α and β are given functions of time t and the current state x. More generally, when several related economic variables $X^1, X^2, \dots, X^N$ are considered, the *vector* $X_t = (X_t^1, X_t^2, \dots, X_t^N)^T$ may evolve in time according to a *system* of stochastic differential equations of the form

$$dX_t^i = \alpha_i(t, X_t)\, dt + \sum_{j=1}^{d} \beta_{ij}(t, X_t)\, dW_t^j, \tag{2}$$

where $W_t = (W_t^1, W_t^2, \dots, W_t^d)$ is a $d-$dimensional Brownian motion. Notice that this system of equations may be written in vector form as (1), where now X_t and $\alpha(t, x)$ are $N-$vectors with entries X_t^i and $\alpha_i(t, x)$, respectively; dW_t is the $d-$vector of increments dW_t^j of the component Brownian motions W_t^j; and $\beta(t, x)$ is the $N \times d-$matrix with entries $\beta_{ij}(t, x)$.

In certain cases, such as the Black-Scholes model for the behavior of a stock price, where $\alpha(t, x) = r_t x$ and $\beta(t, x) = \sigma_t x$ with r_t and σ_t deterministic (nonrandom) functions of t, it is possible to guess a solution, and then to verify, using Itô's formula, that the guess does indeed obey (1). If we knew that for each initial condition X_0 there is at most one solution to the stochastic differential equation (1), then we could conclude that our guess must be that solution. How do we know that solutions to (1) are unique? Unfortunately, it is not always the case that they are!

Example 1: (Courtesy of Itô and Watanabe, 1978) Consider the stochastic differential equation

$$dX_t = 3X_t^{1/3}\, dt + 3X_t^{2/3} dW_t \tag{3}$$

with the initial condition $X_0 = 0$. Clearly, the process $X_t \equiv 0$ is a solution. But so is

$$X_t = W_t^3. \tag{4}$$

□

The problem in this example is that the coefficients $\alpha(t, x) = 3x^{1/3}$ and $\beta(t, x) = 3x^{2/3}$, although continuous in x, are not smooth at $x = 0$. Fortunately, mild smoothness hypotheses on the coefficients $\alpha(t, x)$ and $\beta(t, x)$ ensure uniqueness of solutions.

Definition 1. *The function $f(t, x)$ is* locally Lipschitz *in the space variable(s) x if, for every integer n, there exists a constant C_n such that, for all x, y, and t no larger than n in absolute value*[1],

$$|f(t, x) - f(t, y)| \le C_n |x - y|. \tag{5}$$

[1] If $x = (x_1, x_2, \dots, x_n)$ is a vector, then its absolute value is defined to be $|x| = \sqrt{\sum_i x_i^2}$.

In practice, one rarely needs to verify this condition, because the following is true, by the mean value theorem of calculus: *If $f(t, x)$ is continuously differentiable, then it locally Lipschitz.* Occasionally one encounters situations where one of the coefficients in a stochastic differential equation has "corners" (for instance, the function $f(x) = |x|$); such functions are locally Lipschitz but not continuously differentiable.

Theorem 1. *Suppose that the functions $\alpha_i(t, x)$ and $\beta_{ij}(t, x)$ are all locally Lipschitz in the space variable x. Then for each initial condition $X_0 = x_0$, there is at most one solution to the system of stochastic differential equations (2).*

Are there always solutions to stochastic differential equations of the form (1)? No! In fact, existence of solutions for all time $t \geq 0$ is not guaranteed *even for ordinary differential equations* (that is, differential equations with no random terms). It is important to understand why this is so. A differential equation (or a system of differential equations) prescribes how the state vector X_t will evolve in any small time interval dt, for as long as the state vector remains finite. However, there is no reason why the state vector must remain finite for all times $t \geq 0$.

Example 2: Consider the ordinary differential equation

$$\frac{dx}{dt} = \frac{1}{1-t} \qquad \text{for } 0 \leq t < 1. \tag{6}$$

The solution for the initial condition x_0 is

$$x(t) = x_0 + \int_0^t (1-s)^{-1}\, ds = x_0 - \log(1-t) \qquad \text{for } 0 \leq t < 1. \tag{7}$$

As t approaches 1, $x(t)$ converges to ∞. □

Example 3: The previous example shows that if the coefficients in a differential equation depend explicitly on t then solutions may "explode" in finite time. This example shows that explosion of solutions may occur even if the differential equation is *autonomous*, that is, if the coefficients have no explicit time dependence. The differential equation is

$$\frac{dx}{dt} = x^2. \tag{8}$$

The general solution is

$$x(t) = -(t-C)^{-1}. \tag{9}$$

For the initial condition $x(0) = x_0 > 0$, the value of C must be $C = 1/x_0$. Consequently, the solution $x(t)$ explodes as t approaches $1/x_0$. □

Observe that in Example 2, the coefficient $\alpha(t, x) = x^2$ is not only time-independent, but continuously differentiable, and therefore locally Lipschitz. Hence, the hypotheses of Theorem 1 do not guarantee existence of solutions for all t. The next theorem gives useful sufficient conditions for the existence of solutions for all t.

Theorem 2. *Assume that the coefficients in the system (1) of stochastic differential equations satisfy the following* global *Lipschitz and growth conditions: for some $C < \infty$,*

$$
\begin{aligned}
(10)\qquad |\alpha_i(t,x) - \alpha_i(t,y)| &\le C|x-y| && \text{for all } t \in \mathbb{R} \text{ and } x, y \in \mathbb{R}^N \\
|\beta_{ij}(t,x) - \beta_{ij}(t,y)| &\le C|x-y| && \text{for all } t \in \mathbb{R} \text{ and } x, y \in \mathbb{R}^N \\
|\alpha_i(t,x)| &\le C|x| && \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^N, \text{ and} \\
|\beta_{ij}(t,x)| &\le C|x| && \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^N.
\end{aligned}
$$

Then for each $x_0 \in \mathbb{R}^N$ there is a (unique) solution to the system (1) such that $X_0 = x_0$.

The existence and uniqueness theorems 1 and 2 stated above were proved by Itô in 1951. The proofs follow the lines of the classical proofs for existence and uniqueness of solutions of ordinary differential equations, with appropriate modifications for the random terms. See, for instance, Karatzas & Shreve for details.

More properties of the Ornstein-Uhlenbeck process are given in the exercises.

2. The Ornstein-Uhlenbeck Process

In the parlance of professional probability, a *diffusion process* is a continuous-time stochastic process that satisfies an autonomous (meaning that the coefficients α and β do not depend explicitly on the time variable t) stochastic differential equation of the form (1). Such processes are necessarily (strong) Markov processes.[2] Apart from Brownian motion, perhaps the most important diffusion process is the *Ornstein-Uhlenbeck* process, known also in finance circles as the *Vasicek model*. The Ornstein-Uhlenbeck process is the prototypical *mean-reverting* process: although random, the process exhibits a pronounced tendency toward an equilibrium value, just as an oscillating pendulum or spring is always pulled toward its rest position. In financial applications, the Ornstein-Uhlenbeck process is often used to model quantities that tend to fluctuate about equilibrium values, such as *interest rates* or *volatilities* of stock prices. The stochastic differential equation for the Ornstein-Uhlenbeck process is

$$
dY_t = -\alpha(Y_t - \mu)\,dt + \sigma\,dW_t, \tag{11}
$$

where $\alpha, \mu \in \mathbb{R}$ and $\sigma > 0$ are parameters. Observe that, if $\sigma = 0$ then this becomes an ordinary differential equation with an attractive rest point at μ. (Exercise: Find the general solution when $\sigma = 0$, and verify that as $t \to \infty$ every solution curve converges to μ.) The term $\sigma\,dW_t$ allows for the possibility of random fluctuations about the rest position μ; however, if Y_t begins to randomly wander very far from μ then the "mean-reversion" term $-\alpha(Y_t - \mu)\,dt$ becomes larger, forcing Y_t back toward μ.

The coefficients of the stochastic differential equation (11) satisfy the hypotheses of Theorem 2, and so for every possible initial state $y_0 \in \mathbb{R}$ there is a unique solution Y_t. In fact, it is possible to give an explicit representation of the solution. Let's try the simplest case, where $\mu = 0$. To guess such a representation, try a combination of ordinary and stochastic (Itô) integrals; more generally, try a combination of nonrandom functions and Itô integrals:

$$
Y_t = A(t)\left(y_0 + \int_0^t B(s)\,dW_s\right),
$$

[2] A thorough discussion of such issues is given in the XXX-rated book *Multidimensional Diffusion Processes* be Stroock and Varadhan. For a friendlier introduction, try Steele's new book *Stochastic Calculus with Financial Applications*.

where $A(0) = 1$. If $A(t)$ is differentiable and $B(t)$ is continuous, then

$$
\begin{aligned}
dY_t &= A'(t)\left(y_0 + \int_0^t B(s)\,dW_s\right)\,dt + A(t)B(t)\,dW_t \\
&= \frac{A'(t)}{A(t)} Y_t\,dt + A(t)B(t)\,dW_t.
\end{aligned}
$$

Matching coefficients with (11) shows that $A'(t)/A(t) = -\alpha$ and $A(t)B(t) = \sigma$. Since $A(0) = 1$, this implies that $A(t) = \exp\{-\alpha t\}$ and $B(t) = \sigma \exp\{\alpha t\}$. Thus, for any initial condition $Y_0 = y_0$, the solution of (11) is given by

$$
\boxed{Y_t = \exp\{-\alpha t\}\,y_0 + \sigma \exp\{-\alpha t\} \int_0^t \exp\{\alpha s\}\,dW_s} \tag{12}
$$

The explicit formula (12) allows us to read off a large amount of important information about the Ornstein-Uhlenbeck process. First, recall that it is always the case that the integral of a *nonrandom* function $f(s)$ against dW_s is a normal (Gaussian) random variable, with mean zero and variance $\int f(s)^2\,ds$. Thus, for each t,

$$
Y_t \sim \text{Normal}\left(y_0 e^{-\alpha t}, \sigma^2 \frac{1 - e^{-2\alpha t}}{2\alpha}\right). \tag{13}
$$

As $t \to \infty$, the mean and variance converge (rapidly!) to 0 and $\sigma^2/2\alpha$, respectively, and so

$$
Y_t \xrightarrow{\mathcal{D}} \text{Normal}\left(0, \frac{\sigma^2}{2\alpha}\right). \tag{14}
$$

This shows that the Ornstein-Uhlenbeck process has a *stationary* (or *equilibrium*, or *steady-state*) distribution, and that it is the Gaussian distribution with the paramaters shown above. In fact, formula (12) implies even more. Consider two different initial states y_0 and y_0', and let Y_t and Y_t' be the solutions (12) to the stochastic differential equation (11) with these initial conditions, respectively. Then

$$
Y_t - Y_t' = \exp\{-\alpha t\}\,(y_0 - y_0'). \tag{15}
$$

Thus, the difference between the two solutions Y_t and Y_t' decays *exponentially* in time, at rate α. For this reason α is sometimes called the *relaxation parameter*.

3. Diffusion Equations and the Feynman-Kac Formula

Diffusion processes (specifically, Brownian motion) originated in physics as mathematical models of the motions of individual molecules undergoing random collisions with other molecules in a gas or fluid. Long before the mathematical foundations of the subject were laid[3], Albert Einstein realized that the microscopic random motion of molecules was ultimately responsible for the macroscopic physical phenomenon of *diffusion*, and made the connection between the volatility parameter σ of the random process and the diffusion constant in the partial differential equation governing diffusion.[4] The connection between the differential equations of diffusion and heat flow and the

[3]around 1920, by Norbert Wiener, who proved that there is a probability distribution (measure) P on the space of continuous paths such that, if one chooses a path B_t at random from this distribution then the resulting stochastic process is a Brownian motion, as defined in Lecture 5.

[4]This observation led to the first accurate determination of Avagadro's number, and later, in 1921, to a Nobel prize in physics for Mr Einstein.

random process of Brownian motion has been a recurring theme in mathematical research ever since.

In the 1940s, Richard Feynman discovered that the Schrodinger equation (the differential equation governing the time evolution of quantum states in quantum mechanics) could be solved by (a kind of) averaging over paths, an observation which led him to a far-reaching reformulation of the quantum theory in terms of "path integrals".[5] Upon learning of Feynman's ideas, Mark Kac (a mathematician at Cornell University, where Feynman was, at the time, an Assistant Professor of Physics) realized that a similar representation could be given for solutions of the heat equation (and other related diffusion equations) with external cooling terms. This representation is now known as the *Feynman-Kac formula*. Later it became evident that the expectation occurring in this representation is of the same type that occurs in derivative security pricing.

The simplest heat equation with a cooling term is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - K(x)u, \tag{16}$$

where $K(x)$ is a function of the space variable x representing the amount of external cooling at location x.

Theorem 3. *(Feynman-Kac Formula) Let $K(x)$ be a nonnegative, continuous function, and let $f(x)$ be bounded and continuous. Suppose that $u(t, x)$ is a bounded function that satisfies the partial differential equation (16) and the initial condition*

$$u(0, x) = \lim_{(t,y)\to(0,x)} u(t, y) = f(x). \tag{17}$$

Then

$$u(t, x) = E^x \exp\left\{-\int_0^t K(W_s)\, ds\right\} f(W_t), \tag{18}$$

where, under the probability measure P^x, the process $\{W_t\}_{t\geq 0}$ is Brownian motion started at x.

The hypotheses given are not the most general under which the theorem remains valid, but suffice for many important applications. Occasionally one encounters functions $K(x)$ and $f(x)$ that are not continuous everywhere, but have only isolated discontinuities; the Feynman-Kac formula remains valid for such functions, but the initial condition (17) holds only at points x where f is continuous.

An obvious consequence of the formula is uniqueness of solutions to the *Cauchy problem* (the partial differential equation (16) together with the initial condition (17).

Corollary 1. *Under the hypotheses of Theorem 3, there is at most one solution of the heat equation (16) with initial condition (17), specifically, the function u defined by the expectation (18).*

□

The Feynman-Kac formula may also be used as the basis for an existence theorem, but this is not so simple, and since it is somewhat tangential to our purposes, we shall omit it.

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[5]The theory is spelled out in considerable detail in the book *Quantum Mechanics and Path Integrals* by Feynman and Hibbs. For a nontechnical explanation, read Feynman's later book *QED*, surely one of the finest popular expositions of a scientific theory ever written.

Proof of the Feynman-Kac Formula. Fix $t > 0$, and consider the stochastic process

$$Y_s = e^{-R(s)} u(t-s, W_s), \qquad \text{where}$$
$$R(s) = \exp\left\{ -\int_0^s K(W_r)\, dr \right\}$$

with s now serving as the time parameter. Because $u(t,x)$ is, by hypothesis, a solution of the heat equation (16), it is continuously differentiable once in t and twice in x. Moreover, since u is bounded, so is the process Y_t. By Itô's theorem,

$$\begin{aligned} dY_s = &- K(W_s) e^{-R(s)} u(t-s, W_s)\, ds \\ &- u_t(t-s, W_s) e^{-R(s)}\, ds \\ &+ u_x(t-s, W_s) e^{-R(s)}\, dW_s \\ &+ (1/2) u_{xx}(t-s, W_s) e^{-R(s)}\, ds. \end{aligned}$$

Since u satisfies the partial differential equation (16), the ds terms in the last expression sum to zero, leaving

$$dY_s = u_x(t-s, W_s) e^{-R(s)}\, dW_s. \tag{19}$$

Thus, Y_s is a martingale up to time t.[6] By the "Conservation of Expectation" law for martingales, it follows that

$$Y_0 = u(t,x) = E^x Y_t = E^x e^{-R(t)} u(0, W_t) = E^x e^{-R(t)} f(W_t). \tag{20}$$

As we have remarked, the hypotheses of Theorem 3 may be relaxed considerably, but this is a technically demanding task. The primary difficulty has to do with convergence issues: when f is an unbounded function the expectation in the Feynman-Kac formula (18) need not even be well-defined. Nonuniqueness of solutions to the Cauchy problem is also an obstacle. Consider, for instance, the simple case where $f \equiv 0$ and $K \equiv 0$; then the function

$$v(t,x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \frac{d^n}{dt^n} e^{-1/t^2} \tag{21}$$

is a solution of the heat equation (16) that satisfies the initial condition (17). This example should suffice to instill, if not fear, at least caution in anyone using the Feynman-Kac formula, because it implies nonuniqueness of solutions to the Cauchy problem for *every* initial condition. To see this, observe that if $u(t,x)$ is a solution to the heat equation (16) with initial condition $u(0,x) = f(x)$, then so is $u(t,x) + v(t,x)$. Notice, however, that the function $v(t,x)$ grows exponentially as $x \to \infty$. In many applications, the solution u of interest grows subexponentially in the space variable x. The following result states that, under mild hypotheses on the functions f and K, there is only one solution to the Cauchy problem that grows subexponentially in x.

Proposition 1. *Let f and K be piecewise continuous functions such that $K \geq 0$ and f is of subexponential growth. Then the function $u(t,x)$ defined by the Feynman-Kac formula (18) satisfies*

[6] To make this argument airtight, one must verify that the process $u_x(t-s, W_s)$ is of class $\mathcal{H}^2$ up to time $t - \varepsilon$, for any $\varepsilon > 0$. This may be accomplished by showing that the partial derivative $u_x(s,x)$ remains bounded for $x \in \mathbb{R}$ and $s \geq \varepsilon$, for any $\varepsilon > 0$. Details are omitted. One then applies the Conservation of Expectation law at $t - \varepsilon$, and uses the boundedness of u and the dominated convergence theorem to complete the proof.

the heat equation (16) and the initial condition

$$\lim_{(t,y)\to(0,x)} u(t,y) = f(x) \tag{22}$$

at every x where f is continuous. Moreover, the function u defined by (18) is the unique solution of the Cauchy problem that is of subexponential growth in the space variable x, specifically, such that for each $T < \infty$ and $\varepsilon > 0$,

$$\lim_{x\to\infty} \sup_{0\le t\le T} \sup_{|y|\le x} \frac{|u(t,y)|}{\exp\{\varepsilon x\}} = 0. \tag{23}$$

See Karatzas & Shreve for a proof of the first statement, and consult your local applied mathematician for the second.

The Feynman-Kac formula and the argument given above both generalize in a completely straightforward way to $d-$dimensional Brownian motion.

Theorem 4. *Let $K : \mathbb{R}^d \to [0,\infty)$ and $f : \mathbb{R}^d \to \mathbb{R}$ be continuous functions, with f bounded. Suppose that $u(t,x)$ is a bounded function that satisfies the partial differential equation*

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} - Ku \\ &= \frac{1}{2}\Delta u - Ku \end{aligned} \tag{24}$$

and the initial condition

$$)u(0,x) = f(x). \tag{25}$$

Assume that, under the probability measure P^x the process W_t is a $d-$dimensional Brownian motion started at x. Then

$$u(t,x) = E^x \exp\left\{-\int_0^t K(W_s)\,ds\right\} f(W_t). \tag{26}$$

Moreover, the function u defined by (26) is the only solution to the heat equation (24) satisfying the initial condition (25) that grows subexponentially in the space variable x.

4. Generalizations of the Feynman-Kac Formula

The Feynman-Kac formula is now over 50 years old; thus, it should come as no surprise that the mathematical literature is rich in generalizations and variations. Two types of generalizations are of particular usefulness in financial applications: (1) those in which the Brownian motion W_t is replaced by another diffusion process, and (2) those where the Brownian motion (or more generally diffusion process) is restricted to stay within a certain region of space.

4.1. **Feynman-Kac for other diffusion processes.** Let P^x be a family of probability measures on some probability space, one for each possible initial point x, under which the stochastic process X_t is a diffusion process started at x with local drift $\mu(x)$ and local volatility $\sigma(x)$. That is, suppose that under each P^x the process X_t obeys the stochastic differential equation and initial condition

$$\begin{aligned} dX_t &= \mu(X_t)\,dt + \sigma(X_t)\,dW_t \\ X_0 &= x. \end{aligned} \tag{27}$$

Define the *infinitesimal generator* of the process X_t to be the differential operator

$$\mathcal{G} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}. \tag{28}$$

Theorem 5. *Assume that $\alpha(t,x) = \mu(x)$ and $\beta(t,x) = \sigma(x)$ satisfy the global Lipschitz and growth hypotheses of Theorem 2. Let $f(x)$ and $K(x)$ be continuous functions such that $K \geq 0$ and $f(x) = O(|x|)$ as $|x| \to \infty$. Then the function $u(t,x)$ defined by*

$$u(t,x) = E^x \exp\left\{-\int_0^t K(X_s)\,ds\right\} f(X_t) \tag{29}$$

satisfies the diffusion equation

$$\frac{\partial u}{\partial t} = \mathcal{G}u - Ku \tag{30}$$

and the initial condition $u(0,x) = f(x)$. Moreover, u is the only solution to the Cauchy problem that is of at most polynomial growth in x.

Exercise: Mimic the argument used to prove Theorem 3 to prove this in the special case where f, K, μ, and σ are all bounded functions.

Example: Consider once again the Black-Scholes problem of pricing a European call option with strike price C on a stock whose share price obeys the stochastic differential equation

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t \tag{31}$$

where the short rate r and the stock volatility σ are constant. The risk-neutral price of the call at time $t = 0$ is given by the expectation

$$V_0 = E^x e^{-rT} f(S_T) \tag{32}$$

where T is the expiration time, $S_0 = x$ is the initial share price of the stock, and $f(x) = (x - C)_+$. This expectation is of the form that occurs in the Feynman-Kac formula (37), with the identification $K(x) \equiv r$. Therefore, if $v(t,x)$ is defined by

$$v(t,x) = E^x e^{-rt} f(S_t) \tag{33}$$

then v must satisfy the diffusion equation

$$v_t = rxv_x + (1/2)\sigma^2 x^2 v_{xx} - rv \tag{34}$$

with the initial condition $v(0,x) = f(x)$. Observe that equation (34) is the backward (time-reversed) form of the Black-Scholes equation. It is possible to solve this initial value problem by making the substitution $y = e^x$ and then solving the resulting constant-coefficient PDE by intelligent guesswork. (Exercise: Try it! Rocket scientists should be comfortable with this kind of calculation. And, of course, financial engineers should be capable of intelligent guesswork.) One then arrives at the Black-Scholes formula.

4.2. **Feynman-Kac for Multidimensional Diffusion Processes.** Just as the Feynman-Kac theorem for one-dimensional Brownian motion extends naturally to multidimensional Brownian motion, so does the Feynman-Kac theorem for one-dimensional diffusion processes extend to multidimensional diffusions. A $d-$dimensional diffusion process X_t follows a stochastic differential equation of the form (27), but where X_t and $\mu(x)$ are $d-$vectors, W_t is an $N-$dimensional Brownian motion, and $\sigma(x)$ is a $d \times N$ matrix-valued function of x. The *generator* of the diffusion process X_t is the differential operator

$$\mathcal{G} = \frac{1}{2}\sum_{i=1}^{d}\sum_{i'=1}^{d}\left(\sum_{j=1}^{N}\sigma^{ij}(x)\sigma^{i'j}(x)\right)\frac{\partial^2}{\partial x_i \partial x_{i'}} + \sum_{i=1}^{d}\mu^i(x)\frac{\partial}{\partial x_i}. \tag{35}$$

Note that the terms in this expression correspond to terms in the multidimensional Itô formula. In fact, if X_t obeys the stochastic differential equation (27) and $u(t,x)$ is sufficiently differentiable, then the Itô formula reads

$$du(t,X_t) = u_t(t,X_t)\,dt + \mathcal{G}u(t,X_t)\,dt + \text{terms involving } dW_t^j \tag{36}$$

Theorem 6. *Assume that the coefficient $\alpha(t,x) = \mu(x)$ and $\beta(t,x) = \sigma(x)$ in the stochastic differential equation (27) satisfy the global Lipschitz and growth hypotheses of Theorem 2. Let $f(x)$ and $K(x)$ be continuous functions such that $K \geq 0$ and $f(x) = O(|x|)$ as $|x| \to \infty$. Assume that under P^x the process X_t has initial state $x \in \mathbb{R}^d$. Then the function $u(t,x)$ defined by*

$$u(t,x) = E^x \exp\left\{-\int_0^t K(X_s)\,ds\right\} f(X_t) \tag{37}$$

satisfies the diffusion equation

$$\frac{\partial u}{\partial t} = \mathcal{G}u - Ku \tag{38}$$

and the initial condition $u(0,x) = f(x)$. Moreover, u is the only solution to the Cauchy problem that is of at most polynomial growth in x.

4.3. **Feynman-Kac Localized.** Certain exotic options pay off only when the price process of the underlying asset reaches (or fails to reach) an agreed "knockin" (or "knockout") value before the expiration of the option. The expectations that occur as arbitrage prices of such contracts are then evaluated only on the event that knockin has occurred (or knockout has not occurred). There are Feynman-Kac theorems for such expectations. The function $u(t,x)$ defined by the expectation (see below) satisfies a heat equation of the same type as (16), but only in the domain where payoff remains a possibility; in addition, there is a boundary condition at the knockin point(s). Following is the simplest instance of such a theorem.

Assume that, under the probability measure P^x the process W_t is a $1-$dimensional Brownian motion started at $W_0 = x$. Let $J = (a,b)$ be an open interval of $\mathbb{R}$, and define

$$\tau = \tau_{\mathcal{J}} = \min\{t \geq 0 \,:\, W_t \notin J\} \tag{39}$$

to be the time of first exit from J.

Theorem 7. *Let $K : [a,b] \to [0,\infty)$ and $f : (a,b) \to \mathbb{R}$ be continuous functions such that f has compact support (that is, there is a* closed *interval I contained in (a,b) such that $f(x) = 0$ for all $x \notin I$). Then*

$$u(t,x) = E^x \exp\left\{-\int_0^t K(W_s)\,ds\right\} f(W_t)\mathbf{1}\{t < \tau\} \tag{40}$$

is the unique bounded solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{d^2 u}{dx^2} - Ku \qquad \forall\, x \in J \text{ and } t > 0 \tag{41}$$

with boundary and initial conditions

$$u(t,a) = 0 \qquad (BC)_a$$
$$u(t,b) = 0 \qquad (BC)_b \tag{42}$$
$$u(0,x) = f(x) \qquad (IC).$$

Proof. **Exercise.** □

5. Application of the Feynman-Kac Theorems

In financial applications, the Feynman-Kac theorems are most useful in problems where the expectation giving the arbitrage price of a contract cannot be evaluated in closed form. One must then resort to numerical approximations. Usually, there are two avenues of approach: (1) simulation; or (2) numerical solution of a PDE (or system of PDEs). The Feynman-Kac theorems provide the PDEs.

It is not our business in this course to discuss methods for the solution of PDEs. Nevertheless, we cannot leave the subject of the Feynman-Kac formula without doing at least one substantial example. This example will show how the method of *eigenfunction expansion* works, in one of the simplest cases. The payoff will be an explicit formula for the transition probabilities of Brownian motion restricted to an interval.

5.1. **Transition probabilities for Brownian motion in an interval.** As in section 4.3 above, consider one-dimensional Brownian motion with "killing" (or "absorption" at the endpoints of an interval J. For simplicity, take $J = (0, 2\pi)$. Recall that $\tau = \tau_J$ is the time of first exit from J by the process W_t. We are interested in the expectation

$$u(t,x) = E^x f(W_t)\mathbf{1}\{t < \tau\}, \tag{43}$$

where $f : J \to \mathbb{R}$ is a continuous function with compact support in J. This expectation is an instance of the expectation in equation (40), with $K(x) \equiv 0$. By Theorem 7, the function u satisfies the heat equation (41), with $K = 0$. Our objective is to find a solution to this differential equation that also satisfies the initial and boundary conditions (42).

Our strategy is based on the *superposition principle.* Without the constraints of the initial and boundary conditions, there are *infinitely many* solutions to the heat equation, as we have already seen (look again at equation (21)). Because the the heat equation is linear, any linear combination of solutions is also a solution. Thus, one may attempt to find a solution that satisfies the infinitely and boundary conditions by looking at superpositions of simpler solutions.

What are the simplest bounded solutions of the heat equation? Other than the constants, probably the simplest are the exponentials

$$u(t,x:\theta) := \exp\{i\theta x\}\exp\left\{-\theta^2 t/2\right\}. \tag{44}$$

By themselves, these solutions are of no use, as they are complex-valued, and the function $u(t,x)$ defined by (43) is real-valued. However, the functions $u(t,x;\theta)$ come naturally in pairs, indexed by $\pm\theta$. Adding and subtracting the functions in these pairs leads to another large simple class of solutions:

$$v(t,x;\theta) = (\sin\theta x)e^{-\theta^2 t/2} \qquad \text{and} \tag{45}$$

$$w((t,x;\theta) = (\cos\theta x)e^{-\theta^2 t/2} \tag{46}$$

These are real-valued. Moreover, if $\theta = n/2$ for integer n then the functions $v(t, x; \theta)$ satisfy the boundary conditions in (42). This suggest that we look for the desired solution among the (infinite) linear combinations

$$u(t,x) = \sum_{n=0}^{\infty} a_n v(t,x;n/2) = \sum_{n=0}^{\infty} a_n e^{-n^2 t/2} \sin(nx/2) \tag{47}$$

Since each term satisfies both the heat equation and the boundary conditions, so will the sum (provided the interchange of derivatives and infinite sum can be justified – we won't worry about such details here). Thus, we need only find coefficients a_n so that the initial condition $u(0, x) = f(x)$ is satisfied. But

$$u(0,x) = \sum_{n=-\infty}^{\infty} a_n \sin(nx/2) \tag{48}$$

is a Fourier series! Thus, if we match the coefficients a_n with the corresponding Fourier coefficients of f, the initial condition will be met. The Fourier coefficients of f are defined by

$$a_n := \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin nx \, dx \tag{49}$$

Therefore, the solution is given by

$$u(t,x) = \sum_{n=0}^{\infty} e^{-n^2 t/2} \sin(nx/2) \left(\frac{1}{2\pi} \int_0^{2\pi} f(x) \sin ny \, dy \right), \tag{50}$$

or alternatively,

$$u(t,x) = \int_0^{2\pi} \left(\sum_{n=0}^{\infty} e^{-n^2 t/2} \sin(nx/2) \sin(ny/2) \right) f(y) \, dy \tag{51}$$

Finally, compare the integral formula (51) with the expectation equation (43). In both formulas, the function f is integrated against a kernel: in (43), against a probability density, and in (51), against an infinite series of sinewaves. Since these formulas apply for all continuous functions f with support in $(0, 2\pi)$, it follows that the kernels must be identical. Thus, we have proved the following interesting formula:

$$\boxed{P^x \{W_t \in dy \text{ and } t < \tau\} = \sum_{n=0}^{\infty} e^{-n^2 t/2} \sin(nx/2) \sin(ny/2).} \tag{52}$$

6. Exercises

1. **Linear systems of stochastic differential equations.** Let W_t be a standard $d-$dimensional Brownian motion, and let A be a (nonrandom) $d \times d$ matrix. Consider the stochastic differential equation

$$dX_t = AX_t \, dt + dW_t, \tag{53}$$

where X_t is a $d-$vector-valued process.

(A) Find an explicit formula for the solution when the initial condition is $X_0 = x$, and show that the process X_t is Gaussian.

(B) Under what conditions on the matrix A will the process X_t have a stationary distribution?

HINT: You will need to know something about the matrix-valued exponential function e^{At}. This is defined by

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} \tag{54}$$

What is $(d/dt)e^{At}$?

2. **Transition probabilities of the Ornstein-Uhlenbeck process.** Consider the Ornstein-Uhlenbeck process Y_t defined by equation (12). Let $\mathcal{F}_t$ be the $\sigma-$algebra consisting of all events observable by time t. Show that

$$P(Y_{t+s} \in dy \,|\, \mathcal{F}_s) = f_t(Y_s, y)\, dy \tag{55}$$

for some probability density $f_t(x, y)$, and identify $f_t(x, y)$.

3. **Brownian representation of the Ornstein-Uhlenbeck process.** Let W_t be a standard one-dimensional Brownian motion. Define

$$Y_t = e^{-t} W_{e^{2t}}. \tag{56}$$

Show that the process Y_t is a Markov process with exactly the same transition probabilities as the Ornstein-Uhlenbeck process.

4. **Ornstein-Uhlenbeck process and Brownian motion with quadratic cooling.** Let Y_t be the Ornstein-Uhlenbeck process, with relaxation parameter $\alpha = 1$ and diffusion parameter $\sigma = 1$, and let W_t be one-dimensional Brownian motion. Let $f : \mathbb{R} \to \mathbb{R}$ be any bounded continuous function. Show that for any $t > 0$,

$$E^x f(Y_t) = E^x f(W_t) \exp\left\{ -\frac{1}{2}(W_t - x)^2 - \frac{1}{2}\int_0^t (W_s - x)^2\, ds \right\} \tag{57}$$

NOTE: The superscript x on the expectation operators indicates that $Y_0 = x$ and $W_0 = x$. (Beware that the formula above may have incorrect factors of 2, wrong minus signs, and other similar mistakes.)