PROBLEM SET 11.1

1–5 PERIOD, FUNDAMENTAL PERIOD

The *fundamental period* is the smallest positive period. Find it for

1. $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$

2. $\cos nx$, $\sin nx$, $\cos \dfrac{2\pi x}{k}$, $\sin \dfrac{2\pi x}{k}$, $\cos \dfrac{2\pi n x}{k}$, $\sin \dfrac{2\pi n x}{k}$

3. If $f(x)$ and $g(x)$ have period p, show that $h(x) = af(x) + bg(x)$ (a, b, constant) has the period p. Thus all functions of period p form a **vector space**.

4. Change of scale. If $f(x)$ has period p, show that $f(ax)$, $a \neq 0$, and $f(x/b)$, $b \neq 0$, are periodic functions of x of periods p/a and bp, respectively. Give examples.

5. Show that $f = \text{const}$ is periodic with any period but has no fundamental period.

6–10 GRAPHS OF 2π–PERIODIC FUNCTIONS

Sketch or graph $f(x)$ which for $-\pi < x < \pi$ is given as follows.

6. $f(x) = |x|$

7. $f(x) = |\sin x|$, $f(x) = \sin |x|$

8. $f(x) = e^{-|x|}$, $f(x) = |e^{-x}|$

9. $f(x) = \begin{cases} x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi \end{cases}$

10. $f(x) = \begin{cases} -\cos^2 x & \text{if } -\pi < x < 0 \\ \cos^2 x & \text{if } 0 < x < \pi \end{cases}$

11. Calculus review. Review integration techniques for integrals as they are likely to arise from the Euler formulas, for instance, definite integrals of $x \cos nx$, $x^2 \sin nx$, $e^{-2x} \cos nx$, etc.

12–21 FOURIER SERIES

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π. Show the details of your work. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$.

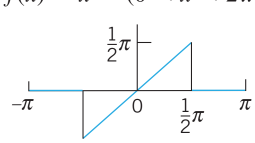
12. $f(x)$ in Prob. 6

13. $f(x)$ in Prob. 9

14. $f(x) = x^2 \quad (-\pi < x < \pi)$

15. $f(x) = x^2 \quad (0 < x < 2\pi)$

16.



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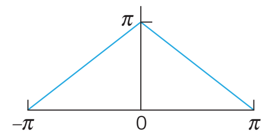

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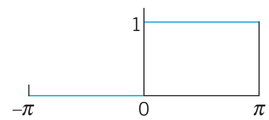

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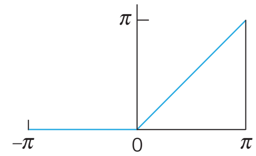

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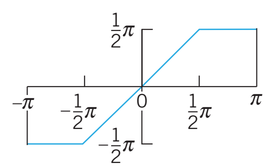

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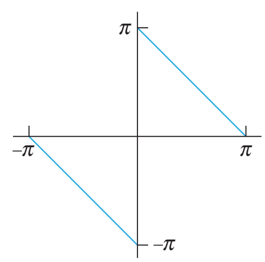


22. CAS EXPERIMENT. Graphing. Write a program for graphing partial sums of the following series. Guess from the graph what $f(x)$ the series may represent. Confirm or disprove your guess by using the Euler formulas.

(a) $2(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots)$
$\qquad - 2(\frac{1}{2}\sin 2x + \frac{1}{4}\sin 4x + \frac{1}{6}\sin 6x \cdots)$

(b) $\dfrac{1}{2} + \dfrac{4}{\pi^2}\left(\cos x + \dfrac{1}{9}\cos 3x + \dfrac{1}{25}\cos 5x + \cdots\right)$

(c) $\frac{2}{3}\pi^2 + 4(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + - \cdots)$

23. Discontinuities. Verify the last statement in Theorem 2 for the discontinuities of $f(x)$ in Prob. 21.

24. CAS EXPERIMENT. Orthogonality. Integrate and graph the integral of the product $\cos mx \cos nx$ (with various integer m and n of your choice) from $-a$ to a as a function of a and conclude orthogonality of $\cos mx$

We insert these two results into the formula for a_n. The sine terms cancel and so does a factor L^2. This gives

$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right).$$

Thus,

$$a_2 = -16k/(2^2\pi^2), \qquad a_6 = -16k/(6^2\pi^2), \qquad a_{10} = -16k/(10^2\pi^2), \cdots$$

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \cdots$. Hence the first half-range expansion of $f(x)$ is (Fig. 272a)

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right).$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$.

(b) ***Odd periodic extension.*** Similarly, from (6**) we obtain

$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}. \tag{5}$$

Hence the other half-range expansion of $f(x)$ is (Fig. 272b)

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - + \cdots\right).$$

The series represents the odd periodic extension of $f(x)$, of period $2L$.

Basic applications of these results will be shown in Secs. 12.3 and 12.5. ■

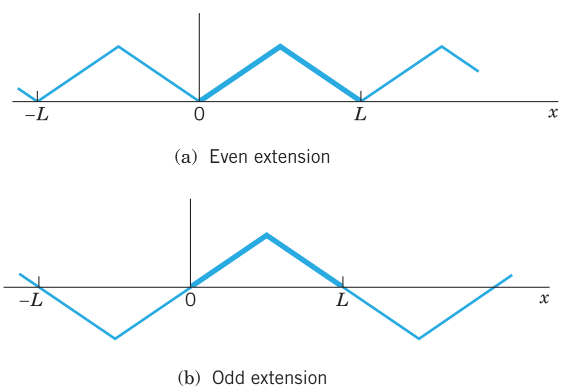


Fig. 272. Periodic extensions of $f(x)$ in Example 6

PROBLEM SET 11.2

1–7 EVEN AND ODD FUNCTIONS

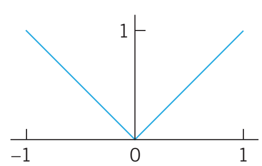
Are the following functions even or odd or neither even nor odd?

1. e^x, $e^{-|x|}$, $x^3 \cos nx$, $x^2 \tan \pi x$, $\sinh x - \cosh x$
2. $\sin^2 x$, $\sin (x^2)$, $\ln x$, $x/(x^2 + 1)$, $x \cot x$
3. Sums and products of even functions
4. Sums and products of odd functions
5. Absolute values of odd functions
6. Product of an odd times an even function
7. Find all functions that are both even and odd.

8–17 FOURIER SERIES FOR PERIOD $p = 2L$

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.

8.



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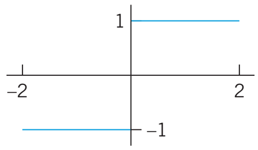

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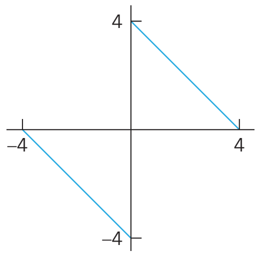
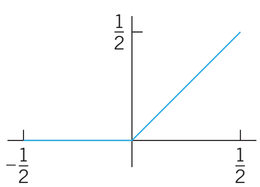


11. $f(x) = x^2 \quad (-1 < x < 1), \quad p = 2$

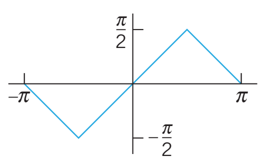
12. $f(x) = 1 - x^2/4 \quad (-2 < x < 2), \quad p = 4$

13.



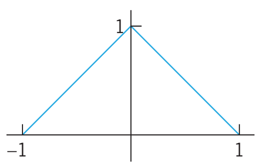
14. $f(x) = \cos \pi x \quad (-\frac{1}{2} < x < \frac{1}{2}), \quad p = 1$

15.



16. $f(x) = x|x| \quad (-1 < x < 1), \quad p = 2$

17.



18. Rectifier. Find the Fourier series of the function obtained by passing the voltage $v(t) = V_0 \cos 100\pi t$ through a half-wave rectifier that clips the negative half-waves.

19. Trigonometric Identities. Show that the familiar identities $\cos^3 x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x$ and $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$ can be interpreted as Fourier series expansions. Develop $\cos^4 x$.

20. Numeric Values. Using Prob. 11, show that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{1}{6}\pi^2$.

21. CAS PROJECT. Fourier Series of 2*L*-Periodic Functions. (a) Write a program for obtaining partial sums of a Fourier series (5).

(b) Apply the program to Probs. 8–11, graphing the first few partial sums of each of the four series on common axes. Choose the first five or more partial sums until they approximate the given function reasonably well. Compare and comment.

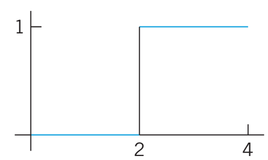
22. Obtain the Fourier series in Prob. 8 from that in Prob. 17.

23–29 HALF-RANGE EXPANSIONS

Find **(a)** the Fourier cosine series, **(b)** the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. Show the details.

23.

24.



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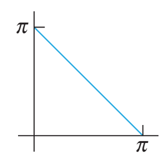

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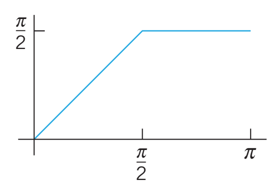

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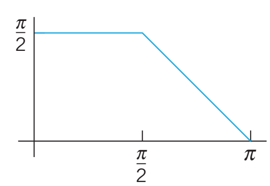

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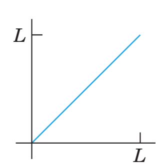

29. $f(x) = \sin x \ (0 < x < \pi)$

30. Obtain the solution to Prob. 26 from that of Prob. 27.

Problem Set 10.8, page 462

1. $x = 0, y = 0, z = 0$, no contributions. $x = a$: $\partial f/\partial n = \partial f/\partial x = -2x = -2a$, etc. Integrals $x = a$: $(-2a)bc$, $y = b$: $(-2b)ac$, $z = c$: $(4c)\,ab$. Sum 0

3. The volume integral of $8y^2 + [0, 8y] \cdot [2x, 0] = 8y^2$ is $8y^3/3 = \frac{8}{3}$. The surface integral of $f\partial g/\partial n = f \cdot 2x = 2f = 8y^2$ over $x = 1$ is $8y^3/3 = \frac{8}{3}$. Others 0.

5. The volume integral of $6y^2 \cdot 4 - 2x^2 \cdot 12$ is 0; $8(x = 1)$, $-8(y = 1)$, others 0.

7. $\mathbf{F} = [x, \ 0, \ 0]$, $\operatorname{div} \mathbf{F} = 1$, use (2*), Sec. 10.7, etc.

9. $z = 0$ and $z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2}$, $dx\,dy = r\,dr\,d\theta$, $-2\pi \cdot \frac{1}{2}(a^2 - r^2)^{3/2} \cdot \frac{2}{3}\,\Big|_0^a = \frac{2}{3}\pi a^3$

11. $r = a$, $\phi = 0$, $\cos\phi = 1$, $v = \frac{1}{3}a \cdot (4\pi a^2)$

Problem Set 10.9, page 468

1. S: $z = y \ (0 \leqq x \leqq 1, 0 \leqq y \leqq 4)$, $[0, 2z, -2z] \bullet [0, -1, 1]$, ± 20

3. $[2e^{-z}\cos y, \ -e^{-z}, \ 0] \bullet [0, \ -y, \ 1] = ye^{-z}$, $\pm(2 - 2/\sqrt{e})$

5. $[0, 2z, \frac{3}{2}] \bullet [0, 0, 1] = \frac{3}{2}$, $\pm\frac{3}{2}a^2$

7. $[-e^z, \ -e^x, \ -e^y] \bullet [-2x, \ 0, \ 1]$, $\pm(e^4 - 2e + 1)$

9. The sides contribute a, $3a^2/2$, $-a$, 0.

11. -2π; curl $\mathbf{F} = \mathbf{0}$

13. $5\mathbf{k}$, 80π

15. $[0, -1, 2x - 2y] \bullet [0, 0, 1]$, $\frac{1}{3}$

17. $\mathbf{r} = [\cos u, \ \sin u, \ v]$, $[-3v^2, \ 0, \ 0] \bullet [\cos u, \ \sin u, \ 0]$, -1

19. $\mathbf{r} = [u\cos v, \ u\sin v, \ u]$, $0 \leqq u \leqq 1$, $0 \leqq v \leqq \pi/2$, $[-e^z, \ 1, \ 0] \bullet [-u\cos v, \ -u\sin v, \ u]$. *Answer*: 1/2

Chapter 10 Review Questions and Problems, page 469

11. $\mathbf{r} = [4 - 10t, \ 2 + 8t]$, $\mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = [2(4 - 10t)^2, \ -4(2t + 8t)^2] \bullet [-10, \ 8]\,dt$; $-4528/3$. Or using exactness.

13. Not exact, curl $\mathbf{F} = (5\cos x)\mathbf{k}$, ± 10

15. 0 since curl $\mathbf{F} = \mathbf{0}$

17. By Stokes, $\pm 18\pi$

19. $\mathbf{F} = \operatorname{grad}(y^2 + xz)$, 2π

21. $M = 8$, $\bar{x} = \frac{8}{5}$, $\bar{y} = \frac{16}{5}$

23. $M = \frac{63}{20}$, $\bar{x} = \frac{8}{7} = 1.14$, $\bar{y} = \frac{118}{49} = 2.41$

25. $M = 4k/15$, $\bar{x} = \frac{5}{16}$, $\bar{y} = \frac{4}{7}$

27. $288(a + b + c)\pi$

29. $\operatorname{div} \mathbf{F} = 20 + 6z^2$. *Answer*: 21

31. $24 \sinh 1 = 28.205$

33. Direct integration, $\frac{224}{3}$

35. 72π

Problem Set 11.1, page 482

1. $2\pi, 2\pi, \pi, \pi, 1, 1, \frac{1}{2}, \frac{1}{2}$

5. There is no *smallest* $p > 0$.

13. $\dfrac{4}{\pi}\left(\cos x + \dfrac{1}{9}\cos 3x + \dfrac{1}{25}\cos 5x + \cdots\right) + 2\left(\sin x + \dfrac{1}{3}\sin 3x + \dfrac{1}{5}\sin 5x + \cdots\right)$

15. $\frac{4}{3}\pi^2 + 4\left(\cos x + \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x + \cdots\right) - 4\pi\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$

17. $\dfrac{\pi}{2} + \dfrac{4}{\pi}\left(\cos x + \dfrac{1}{9}\cos 3x + \dfrac{1}{25}\cos 5x + \cdots\right)$

19. $\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right) + \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots$

21. $2(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x + \cdots)$

Problem Set 11.2, page 490

1. Neither, even, odd, odd, neither **3.** Even **5.** Even

9. Odd, $L = 2$, $\frac{4}{\pi}\left(\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \cdots\right)$

11. Even, $L = 1$, $\frac{1}{3} - \frac{4}{\pi^2}\left(\cos \pi x - \frac{1}{4}\cos 2\pi x + \frac{1}{9}\cos 3\pi x - + \cdots\right)$

13. Rectifier, $L = \frac{1}{2}$, $\frac{1}{8} - \frac{1}{\pi^2}\left(\cos 2\pi x + \frac{1}{9}\cos 6\pi x + \frac{1}{25}\cos 10\pi x + \cdots\right) + \frac{1}{\pi}\left(\frac{1}{2}\sin 2\pi x - \frac{1}{4}\sin 4\pi x + \frac{1}{6}\sin 6\pi x - \frac{1}{8}\sin 8\pi x + - \cdots\right)$

15. Odd, $L = \pi$, $\frac{4}{\pi}\left(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - + \cdots\right)$

17. Even, $L = 1$, $\frac{1}{2} + \frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{9}\cos 3\pi x + \frac{1}{25}\cos 5\pi x + \cdots\right)$

19. $\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$

23. $L = 4$, **(a)** 1, **(b)** $\frac{4}{\pi}\left(\sin\frac{\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} + \frac{1}{5}\sin\frac{5\pi x}{4} + \cdots\right)$

25. $L = \pi$, **(a)** $\frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$,
(b) $2(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots)$

27. $L = \pi$, **(a)** $\frac{3\pi}{8} + \frac{2}{\pi}\left(\cos x - \frac{1}{2}\cos 2x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x - \frac{1}{18}\cos 6x + \frac{1}{49}\cos 7x + \frac{1}{81}\cos 9x - \frac{1}{50}\cos 10x + \frac{1}{121}\cos 11x + \cdots\right)$
(b) $\left(1 + \frac{2}{\pi}\right)\sin x + \frac{1}{2}\sin 2x + \left(\frac{1}{3} - \frac{2}{9\pi}\right)\sin 3x + \frac{1}{4}\sin 4x + \left(\frac{1}{5} + \frac{2}{25\pi}\right)\sin 5x + \frac{1}{6}\sin 6x + \cdots$

29. Rectifier, $L = \pi$,
(a) $\frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{1\cdot 3}\cos x + \frac{1}{3\cdot 5}\cos 3x + \frac{1}{5\cdot 7}\cos 5x + \cdots\right)$, **(b)** $\sin x$

Problem Set 11.3, page 494

3. The output becomes a pure cosine series.

5. For A_n this is similar to Fig. 54 in Sec. 2.8, whereas for the phase shift B_n the sense is the same for all n.