Calculus of Variations
solved problems

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1 Solving the Euler equation

Theorem.(Euler) Suppose $f(x, y, y')$ has continuous partial derivatives of the second order on the interval $[a, b]$. If a functional $F(y) = \int_a^b f(x, y, y')\mathrm{d}x$ attains a weak relative extrema at y_0, then y_0 is a solution of the following equation

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0\,.$$

It is called the *Euler equation*.

1.1 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b 12\,x\,\mathrm{y}(x) + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,dx$$

Hint:

$$elementary$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\ \mathrm{y}(x),\ \frac{\partial}{\partial x}\,\mathrm{y}(x)) = 12\,x\,\mathrm{y}(x) + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2$$

in the form

$$\mathrm{f}(x,\ y,\ z) = 12\,x\,y + z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\ \mathrm{y}(x),\ \frac{\partial}{\partial x}\,\mathrm{y}(x)) = 12\,x$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\ \mathrm{y}(x),\ \frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\ \mathrm{y}(x),\ \frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$12\,x - 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = x^3 + \_C1\,x + \_C2$$

Info.

$$cubic\_polynomial$$

Comment.

$$no\_comment$$

1.2 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b 3\,x + \sqrt{\frac{\partial}{\partial x}\,\mathrm{y}(x)}\,dx$$

Hint:

$$\textit{elementary}$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 3\,x + \sqrt{\frac{\partial}{\partial x}\,\mathrm{y}(x)}$$

in the form

$$\mathrm{f}(x,\,y,\,z) = 3\,x + \sqrt{z}$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \frac{1}{2}\,\frac{1}{\sqrt{\frac{\partial}{\partial x}\,y}}$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = -\frac{1}{4}\,\frac{\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)}{(\frac{\partial}{\partial x}\,\mathrm{y}(x))^{(3/2)}}$$

and finally obtain the Euler equation for our functional

$$\frac{1}{4}\,\frac{\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)}{(\frac{\partial}{\partial x}\,\mathrm{y}(x))^{(3/2)}} = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \_C1\,x + \_C2$$

Info.

$$\textit{linear\_function}$$

Comment.

$$\textit{no\_comment}$$

1.3 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b x + y(x)^2 + 3\,(\frac{\partial}{\partial x}\,y(x))\,dx$$

Hint:

$$elementary$$

Solution.
We denote auxiliary function f

$$f(x,\,y(x),\,\frac{\partial}{\partial x}\,y(x)) = x + y(x)^2 + 3\,(\frac{\partial}{\partial x}\,y(x))$$

in the form

$$f(x,\,y,\,z) = x + y^2 + 3\,z$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,f(x,\,y(x),\,\frac{\partial}{\partial x}\,y(x)) = 2\,y$$

and

$$\frac{\partial}{\partial z}\,f(x,\,y(x),\,\frac{\partial}{\partial x}\,y(x)) = 3$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,f(x,\,y(x),\,\frac{\partial}{\partial x}\,y(x)) = 0$$

and finally obtain the Euler equation for our functional

$$2\,y(x) = 0$$

We solve it using various tools and obtain the solution

$$y(x) = 0$$

Info.

$$constant\_solution$$

Comment.

$$no\_comment$$

1.4 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \mathrm{y}(x)^2\,dx$$

Hint:

$$elementary$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \mathrm{y}(x)^2$$

in the form

$$\mathrm{f}(x,\,y,\,z) = y^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,y$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

and finally obtain the Euler equation for our functional

$$2\,\mathrm{y}(x) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = 0$$

Info.

$$zero$$

Comment.

no_comment

1.5 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \mathrm{y}(x)^2 + x^2 \left(\frac{\partial}{\partial x}\, \mathrm{y}(x)\right) dx$$

Hint:

elementary

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\, \mathrm{y}(x),\, \frac{\partial}{\partial x}\, \mathrm{y}(x)) = \mathrm{y}(x)^2 + x^2 \left(\frac{\partial}{\partial x}\, \mathrm{y}(x)\right)$$

in the form

$$\mathrm{f}(x,\, y,\, z) = y^2 + x^2\, z$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\, \mathrm{f}(x,\, \mathrm{y}(x),\, \frac{\partial}{\partial x}\, \mathrm{y}(x)) = 2\, y$$

and

$$\frac{\partial}{\partial z}\, \mathrm{f}(x,\, \mathrm{y}(x),\, \frac{\partial}{\partial x}\, \mathrm{y}(x)) = x^2$$

and

$$\frac{\partial^2}{\partial x\, \partial z}\, \mathrm{f}(x,\, \mathrm{y}(x),\, \frac{\partial}{\partial x}\, \mathrm{y}(x)) = 2\, x$$

and finally obtain the Euler equation for our functional

$$2\, \mathrm{y}(x) - 2\, x = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = x$$

Info.

linear

Comment.

no_comment

1.6 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \mathrm{y}(x) + x\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))\,dx$$

Hint:

$$\textit{elementary}$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \mathrm{y}(x) + x\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))$$

in the form

$$\mathrm{f}(x,\,y,\,z) = y + x\,z$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 1$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = x$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 1$$

and finally obtain the Euler equation for our functional

$$0 = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \mathrm{Y}(x)$$

Info.

$$\textit{all\_functions}$$

Comment.

$$\textit{no\_comment}$$

1.7 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \frac{\partial}{\partial x}\,\mathrm{y}(x)\,dx$$

Hint:

$$elementary$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \frac{\partial}{\partial x}\,\mathrm{y}(x)$$

in the form

$$\mathrm{f}(x,\,y,\,z) = z$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 1$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

and finally obtain the Euler equation for our functional

$$0 = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \mathrm{Y}(x)$$

Info.

$$all\_functions$$

Comment.

$$no\_comment$$

1.8 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b y(x)^2 - (\frac{\partial}{\partial x} y(x))^2 \, dx$$

Hint:

$$missing\_x$$

Solution.
We denote auxiliary function f

$$f(x, y(x), \frac{\partial}{\partial x} y(x)) = y(x)^2 - (\frac{\partial}{\partial x} y(x))^2$$

in the form

$$f(x, y, z) = y^2 - z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y} f(x, y(x), \frac{\partial}{\partial x} y(x)) = 2\, y$$

and

$$\frac{\partial}{\partial z} f(x, y(x), \frac{\partial}{\partial x} y(x)) = -2\, (\frac{\partial}{\partial x} y)$$

and

$$\frac{\partial^2}{\partial x\, \partial z} f(x, y(x), \frac{\partial}{\partial x} y(x)) = -2\, (\frac{\partial^2}{\partial x^2} y(x))$$

and finally obtain the Euler equation for our functional

$$2\, y(x) + 2\, (\frac{\partial^2}{\partial x^2} y(x)) = 0$$

We solve it using various tools and obtain the solution

$$y(x) = \_C1 \cos(x) + \_C2 \sin(x)$$

Info.

$$trig\_functions$$

Comment.

$$no\_comment$$

1.9 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \sqrt{1 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}\,dx$$

Hint:

$$missing\_x\_y$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \sqrt{1 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}$$

in the form

$$\mathrm{f}(x,\,y,\,z) = \sqrt{1 + z^2}$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \frac{\frac{\partial}{\partial x}\,y}{\sqrt{1 + (\frac{\partial}{\partial x}\,y)^2}}$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = -\frac{(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))}{(1 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2)^{(3/2)}} + \frac{\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)}{\sqrt{1 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}}$$

and finally obtain the Euler equation for our functional

$$\frac{(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))}{(1 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2)^{(3/2)}} - \frac{\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)}{\sqrt{1 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}} = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \_C1\,x + \_C2$$

Info.

$$linear\_function$$

Comment.

$$no\_comment$$

1.10 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b x\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,dx$$

Hint:

$$missing\_y$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = x\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2$$

in the form

$$\mathrm{f}(x,\,y,\,z) = x\,z + z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = x + 2\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 1 + 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$-1 - 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = -\frac{1}{4}\,x^2 + \_C1\,x + \_C2$$

Info.

$$quadratic\_function$$

Comment.

$$no\_comment$$

1.11 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b (1+x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,dx$$

Hint:

no_hint

Solution.

We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = (1+x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2$$

in the form

$$\mathrm{f}(x,\,y,\,z) = (1+x)\,z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(1+x)\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + 2\,(1+x)\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$-2\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) - 2\,(1+x)\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \_C1 + \_C2\,\ln(1+x)$$

Info.

not_supplied

Comment.

no_comment

1.12 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b 2\,\mathrm{y}(x)\,e^x + \mathrm{y}(x)^2 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,dx$$

Hint:

no_hint

Solution.

We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,\mathrm{y}(x)\,e^x + \mathrm{y}(x)^2 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2$$

in the form

$$\mathrm{f}(x,\,y,\,z) = 2\,y\,e^x + y^2 + z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,e^x + 2\,y$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$2\,e^x + 2\,\mathrm{y}(x) - 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = (\frac{1}{2}\cosh(x)^2 + \frac{1}{2}\cosh(x)\sinh(x) + \frac{1}{2}\,x)\sinh(x)$$
$$+ (-\frac{1}{2}\cosh(x)\sinh(x) + \frac{1}{2}\,x - \frac{1}{2}\cosh(x)^2)\cosh(x) + \_C1\,\sinh(x) + \_C2\,\cosh(x)$$

Info.

not_supplied

Comment.

no_comment

1.13 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b 2\,\mathrm{y}(x) + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,dx$$

Hint:

$$\textit{no\_hint}$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,\mathrm{y}(x) + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2$$

in the form

$$\mathrm{f}(x,\,y,\,z) = 2\,y + z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$2 - 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \frac{1}{2}\,x^2 + \_C1\,x + \_C2$$

Info.

$$\textit{not\_supplied}$$

Comment.

$$\textit{no\_comment}$$

1.14 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b 2\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^3\,dx$$

Hint:

no_hint

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^3$$

in the form

$$\mathrm{f}(x,\,y,\,z) = 2\,z^3$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 6\,(\frac{\partial}{\partial x}\,y)^2$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 12\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$-12\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \_C1$$

Info.

not_supplied

Comment.

no_comment

1.15 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b 4\,x\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,dx$$

Hint:

no_hint

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 4\,x\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2$$

in the form

$$\mathrm{f}(x,\,y,\,z) = 4\,x\,z + z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 4\,x + 2\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 4 + 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$-4 - 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = -x^2 + \_C1\,x + \_C2$$

Info.

not_supplied

Comment.

no_comment

1.16 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b 7\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^3\,dx$$

Hint:

no_hint

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 7\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^3$$

in the form

$$\mathrm{f}(x,\,y,\,z) = 7\,z^3$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 21\,(\frac{\partial}{\partial x}\,y)^2$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 42\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$-42\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \_C1$$

Info.

not_supplied

Comment.

no_comment

1.17 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \sqrt{\mathrm{y}(x)\,(1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2)}\,dx$$

Hint:

$$no\_hint$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \sqrt{\mathrm{y}(x)\,(1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2)}$$

in the form

$$\mathrm{f}(x,\,y,\,z) = \sqrt{y\,(1+z^2)}$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \frac{1}{2}\,\frac{1+(\frac{\partial}{\partial x}\,y)^2}{\sqrt{y\,(1+(\frac{\partial}{\partial x}\,y)^2)}}$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \frac{y\,(\frac{\partial}{\partial x}\,y)}{\sqrt{y\,(1+(\frac{\partial}{\partial x}\,y)^2)}}$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) =$$
$$-\frac{1}{2}\,\frac{\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))\,((\frac{\partial}{\partial x}\,\mathrm{y}(x))\,(1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2)+2\,\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)))}{(\mathrm{y}(x)\,(1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2))^{(3/2)}}$$
$$+\frac{(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}{\sqrt{\mathrm{y}(x)\,(1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2)}}+\frac{\mathrm{y}(x)\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))}{\sqrt{\mathrm{y}(x)\,(1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2)}}$$

and finally obtain the Euler equation for our functional

$$\frac{1}{2}\,\frac{1}{\sqrt{\mathrm{y}(x)}}+\frac{1}{2}\,\frac{\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))\,((\frac{\partial}{\partial x}\,\mathrm{y}(x))\,(1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2)+2\,\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)))}{(\mathrm{y}(x)\,(1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2))^{(3/2)}}$$
$$-\frac{(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}{\sqrt{\mathrm{y}(x)\,(1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2)}}-\frac{\mathrm{y}(x)\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))}{\sqrt{\mathrm{y}(x)\,(1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2)}} = 0$$

We solve it using various tools and obtain the solution

$$y(x) = \frac{1}{4} \frac{4 + \_C1^2 x^2 + 2 \_C1^2 x \_C2 + \_C2^2 \_C1^2}{\_C1}$$

Info.

$$quadratic\_function$$

Comment.

$$no\_comment$$

1.18 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b x\,\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))\,dx$$

Hint:

$$no\_hint$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = x\,\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))$$

in the form

$$\mathrm{f}(x,\,y,\,z) = x\,y\,z$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = x\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = x\,y$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \mathrm{y}(x) + x\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$-\mathrm{y}(x) - x\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = 0$$

Info.

$$not\_supplied$$

Comment.

$$no\_comment$$

1.19 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b x\,\mathrm{y}(x) + 2\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,dx$$

Hint:

$$\textit{no\_hint}$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = x\,\mathrm{y}(x) + 2\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2$$

in the form

$$\mathrm{f}(x,\,y,\,z) = x\,y + 2\,z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = x$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 4\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 4\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$x - 4\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \frac{1}{24}\,x^3 + \_C1\,x + \_C2$$

Info.

$$\textit{not\_supplied}$$

Comment.

$$\textit{no\_comment}$$

1.20 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b x\,\mathrm{y}(x) + \mathrm{y}(x)^2 - 2\,\mathrm{y}(x)^2\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))\,dx$$

Hint:

$$no\_hint$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = x\,\mathrm{y}(x) + \mathrm{y}(x)^2 - 2\,\mathrm{y}(x)^2\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))$$

in the form

$$\mathrm{f}(x,\,y,\,z) = x\,y + y^2 - 2\,y^2\,z$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = x + 2\,y - 4\,y\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = -2\,y^2$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = -4\,\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$x + 2\,\mathrm{y}(x) + 4\,\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = -\frac{1}{2}\,x$$

Info.

$$no\_extremal$$

Comment.

$$no\_comment$$

1.21 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \mathrm{y}(x)\,\sqrt{1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}\,dx$$

Hint:

$$no\_hint$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \mathrm{y}(x)\,\sqrt{1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}$$

in the form

$$\mathrm{f}(x,\,y,\,z) = y\,\sqrt{1+z^2}$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \sqrt{1+(\frac{\partial}{\partial x}\,y)^2}$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \frac{y\,(\frac{\partial}{\partial x}\,y)}{\sqrt{1+(\frac{\partial}{\partial x}\,y)^2}}$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) =$$
$$\frac{(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}{\sqrt{1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}} - \frac{\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))}{(1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2)^{(3/2)}} + \frac{\mathrm{y}(x)\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))}{\sqrt{1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}}$$

and finally obtain the Euler equation for our functional

$$1 - \frac{(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}{\sqrt{1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}} + \frac{\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))}{(1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2)^{(3/2)}} - \frac{\mathrm{y}(x)\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))}{\sqrt{1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}} = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \frac{1}{2}\,\frac{1+(e^{(\sqrt{\_C1}\,(x+\_C2))})^2}{e^{(\sqrt{\_C1}\,(x+\_C2))}\,\sqrt{\_C1}}$$

Info.

$$not\_supplied$$

Comment.

$$no\_comment$$

1.22 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))\,dx$$

Hint:

$$\textit{no\_hint}$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))$$

in the form

$$\mathrm{f}(x,\,y,\,z) = y\,z$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \frac{\partial}{\partial x}\,y$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = y$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \frac{\partial}{\partial x}\,\mathrm{y}(x)$$

and finally obtain the Euler equation for our functional

$$-(\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \mathrm{Y}(x)$$

Info.

$$\textit{not\_supplied}$$

Comment.

$$\textit{no\_comment}$$

1.23 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b y(x)\,(\frac{\partial}{\partial x}\,y(x))^2\,dx$$

Hint:

no_hint

Solution.

We denote auxiliary function f

$$f(x,\,y(x),\,\frac{\partial}{\partial x}\,y(x)) = y(x)\,(\frac{\partial}{\partial x}\,y(x))^2$$

in the form

$$f(x,\,y,\,z) = y\,z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,f(x,\,y(x),\,\frac{\partial}{\partial x}\,y(x)) = (\frac{\partial}{\partial x}\,y)^2$$

and

$$\frac{\partial}{\partial z}\,f(x,\,y(x),\,\frac{\partial}{\partial x}\,y(x)) = 2\,y\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,f(x,\,y(x),\,\frac{\partial}{\partial x}\,y(x)) = 2\,(\frac{\partial}{\partial x}\,y(x))^2 + 2\,y(x)\,(\frac{\partial^2}{\partial x^2}\,y(x))$$

and finally obtain the Euler equation for our functional

$$-2\,(\frac{\partial}{\partial x}\,y(x))^2 - 2\,y(x)\,(\frac{\partial^2}{\partial x^2}\,y(x)) = 0$$

We solve it using various tools and obtain the solution

$$y(x) = 0$$

Info.

not_supplied

Comment.

no_comment

1.24 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \mathrm{y}(x)^2 + 2\,x\,\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))\,dx$$

Hint:

no_hint

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \mathrm{y}(x)^2 + 2\,x\,\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))$$

in the form

$$\mathrm{f}(x,\,y,\,z) = y^2 + 2\,x\,y\,z$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,y + 2\,x\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,x\,y$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,\mathrm{y}(x) + 2\,x\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$-2\,x\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \mathrm{Y}(x)$$

Info.

all_functions

Comment.

no_comment

1.25 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \mathrm{y}(x)^2 + 4\,\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + 4\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,dx$$

Hint:

no_hint

Solution.

We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \mathrm{y}(x)^2 + 4\,\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + 4\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2$$

in the form

$$\mathrm{f}(x,\,y,\,z) = y^2 + 4\,y\,z + 4\,z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,y + 4\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 4\,y + 8\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 4\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + 8\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$2\,\mathrm{y}(x) - 4\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) - 8\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \_C1\,\cosh(\frac{1}{2}\,x) + \_C2\,\sinh(\frac{1}{2}\,x)$$

Info.

two_exponentials

Comment.

no_comment

1.26 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \mathrm{y}(x)^2 + \mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,dx$$

Hint:

no_hint

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \mathrm{y}(x)^2 + \mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2$$

in the form

$$\mathrm{f}(x,\,y,\,z) = y^2 + y\,z + z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,y + (\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = y + 2\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = (\frac{\partial}{\partial x}\,\mathrm{y}(x)) + 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$2\,\mathrm{y}(x) - (\frac{\partial}{\partial x}\,\mathrm{y}(x)) - 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \_C1\,\sinh(x) + \_C2\,\cosh(x)$$

Info.

not_supplied

Comment.

no_comment

1.27 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b 2\,\mathrm{y}(x)\,e^x + \mathrm{y}(x)^2 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,dx$$

Hint:

no_hint

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,\mathrm{y}(x)\,e^x + \mathrm{y}(x)^2 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2$$

in the form

$$\mathrm{f}(x,\,y,\,z) = 2\,y\,e^x + y^2 + z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,e^x + 2\,y$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$2\,e^x + 2\,\mathrm{y}(x) - 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = (\frac{1}{2}\cosh(x)^2 + \frac{1}{2}\cosh(x)\sinh(x) + \frac{1}{2}\,x)\sinh(x)$$
$$+ (-\frac{1}{2}\cosh(x)\sinh(x) + \frac{1}{2}\,x - \frac{1}{2}\cosh(x)^2)\cosh(x) + \_C1\sinh(x) + \_C2\cosh(x)$$

Info.

not_supplied

Comment.

no_comment

1.28 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \mathrm{y}(x)^2 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2 - 2\,\mathrm{y}(x)\sin(x)\,dx$$

Hint:

no_hint

Solution.

We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \mathrm{y}(x)^2 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2 - 2\,\mathrm{y}(x)\sin(x)$$

in the form

$$\mathrm{f}(x,\,y,\,z) = y^2 + z^2 - 2\,y\sin(x)$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,y - 2\sin(x)$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$2\,\mathrm{y}(x) - 2\sin(x) - 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\begin{aligned}\mathrm{y}(x) = {} & (\frac{1}{4}\,e^x\cos(x) - \frac{1}{4}\sin(x)\,e^x + \frac{1}{4}\,e^{(-x)}\cos(x) + \frac{1}{4}\,e^{(-x)}\sin(x))\sinh(x) \\ & + (-\frac{1}{4}\,e^x\cos(x) + \frac{1}{4}\sin(x)\,e^x + \frac{1}{4}\,e^{(-x)}\cos(x) + \frac{1}{4}\,e^{(-x)}\sin(x))\cosh(x) \\ & + \_C1\sinh(x) + \_C2\cosh(x)\end{aligned}$$

Info.

exponentials_and_sin

Comment.

no_comment

1.29 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \mathrm{y}(x)^2 - 4\,\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + 4\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,dx$$

Hint:

no_hint

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \mathrm{y}(x)^2 - 4\,\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + 4\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2$$

in the form

$$\mathrm{f}(x,\,y,\,z) = y^2 - 4\,y\,z + 4\,z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,y - 4\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = -4\,y + 8\,(\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = -4\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + 8\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$2\,\mathrm{y}(x) + 4\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) - 8\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \_C1\,\cosh(\frac{1}{2}\,x) + \_C2\,\sinh(\frac{1}{2}\,x)$$

Info.

not_supplied

Comment.

no_comment

1.30 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \mathrm{y}(x) + \mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + (\frac{\partial}{\partial x}\,\mathrm{y}(x)) + \frac{1}{2}\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,dx$$

Hint:

no_hint

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \mathrm{y}(x) + \mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + (\frac{\partial}{\partial x}\,\mathrm{y}(x)) + \frac{1}{2}\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2$$

in the form

$$\mathrm{f}(x,\,y,\,z) = y + y\,z + z + \frac{1}{2}\,z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 1 + (\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = y + 1 + (\frac{\partial}{\partial x}\,y)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = (\frac{\partial}{\partial x}\,\mathrm{y}(x)) + (\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$1 - (\frac{\partial}{\partial x}\,\mathrm{y}(x)) - (\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \frac{1}{2}\,x^2 + \_C1\,x + \_C2$$

Info.

quadratic_function

Comment.

no_comment

1.31 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b y(x) - y(x) \left(\frac{\partial}{\partial x} y(x)\right) + x \left(\frac{\partial}{\partial x} y(x)\right)^2 dx$$

Hint:

$$no\_hint$$

Solution.
We denote auxiliary function f

$$f(x, y(x), \frac{\partial}{\partial x} y(x)) = y(x) - y(x) \left(\frac{\partial}{\partial x} y(x)\right) + x \left(\frac{\partial}{\partial x} y(x)\right)^2$$

in the form

$$f(x, y, z) = y - y z + x z^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y} f(x, y(x), \frac{\partial}{\partial x} y(x)) = 1 - \left(\frac{\partial}{\partial x} y\right)$$

and

$$\frac{\partial}{\partial z} f(x, y(x), \frac{\partial}{\partial x} y(x)) = -y + 2 x \left(\frac{\partial}{\partial x} y\right)$$

and

$$\frac{\partial^2}{\partial x \partial z} f(x, y(x), \frac{\partial}{\partial x} y(x)) = \left(\frac{\partial}{\partial x} y(x)\right) + 2 x \left(\frac{\partial^2}{\partial x^2} y(x)\right)$$

and finally obtain the Euler equation for our functional

$$1 - \left(\frac{\partial}{\partial x} y(x)\right) - 2 x \left(\frac{\partial^2}{\partial x^2} y(x)\right) = 0$$

We solve it using various tools and obtain the solution

$$y(x) = \frac{1}{2} x + \_C1 + \_C2 \ln(x)$$

Info.

$$not\_supplied$$

Comment.

$$no\_comment$$

1.32 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \left(\frac{\partial}{\partial x}\,\mathrm{y}(x)\right)\left(1 + x^2\left(\frac{\partial}{\partial x}\,\mathrm{y}(x)\right)\right) dx$$

Hint:

$$no\_hint$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \left(\frac{\partial}{\partial x}\,\mathrm{y}(x)\right)\left(1 + x^2\left(\frac{\partial}{\partial x}\,\mathrm{y}(x)\right)\right)$$

in the form

$$\mathrm{f}(x,\,y,\,z) = z\,(1 + x^2\,z)$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 1 + 2\,x^2\left(\frac{\partial}{\partial x}\,y\right)$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 4\,x\left(\frac{\partial}{\partial x}\,\mathrm{y}(x)\right) + 2\,x^2\left(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)\right)$$

and finally obtain the Euler equation for our functional

$$-4\,x\left(\frac{\partial}{\partial x}\,\mathrm{y}(x)\right) - 2\,x^2\left(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)\right) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \_C1 + \frac{\_C2}{x}$$

Info.

$$hyperbolic\_function$$

Comment.

$$no\_comment$$

1.33 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b \frac{(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}{x^3}\,dx$$

Hint:

$$\textit{no\_hint}$$

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = \frac{(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}{x^3}$$

in the form

$$\mathrm{f}(x,\,y,\,z) = \frac{z^2}{x^3}$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,\frac{\frac{\partial}{\partial x}\,y}{x^3}$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,\frac{\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)}{x^3} - 6\,\frac{\frac{\partial}{\partial x}\,\mathrm{y}(x)}{x^4}$$

and finally obtain the Euler equation for our functional

$$-2\,\frac{\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)}{x^3} + 6\,\frac{\frac{\partial}{\partial x}\,\mathrm{y}(x)}{x^4} = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \_C1 + \_C2\,x^4$$

Info.

$$\textit{not\_supplied}$$

Comment.

$$\textit{no\_comment}$$

1.34 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2 + 2\,\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) - 16\,\mathrm{y}(x)^2\,dx$$

Hint:

no_hint

Solution.

We denote auxiliary function f

$$\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2 + 2\,\mathrm{y}(x)\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) - 16\,\mathrm{y}(x)^2$$

in the form

$$\mathrm{f}(x,\,y,\,z) = z^2 + 2\,y\,z - 16\,y^2$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial}{\partial x}\,y) - 32\,y$$

and

$$\frac{\partial}{\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial}{\partial x}\,y) + 2\,y$$

and

$$\frac{\partial^2}{\partial x\,\partial z}\,\mathrm{f}(x,\,\mathrm{y}(x),\,\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) + 2\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$-32\,\mathrm{y}(x) - 2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) - 2\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = \_C1\,\cos(4\,x) + \_C2\,\sin(4\,x)$$

Info.

trig_functions

Comment.

no_comment

1.35 Problem.

Using the Euler equation find the extremals for the following functional

$$\int_a^b (\frac{\partial}{\partial x} \mathrm{y}(x))^2 + \cos(\mathrm{y}(x))\, dx$$

Hint:

no_hint

Solution.
We denote auxiliary function f

$$\mathrm{f}(x,\, \mathrm{y}(x),\, \frac{\partial}{\partial x} \mathrm{y}(x)) = (\frac{\partial}{\partial x} \mathrm{y}(x))^2 + \cos(\mathrm{y}(x))$$

in the form

$$\mathrm{f}(x,\, y,\, z) = z^2 + \cos(y)$$

Substituting we x, $y(x)$ and $y'(x)$ for x, y and z we obtain the integrand in the given functional. We compute partial derivatives

$$\frac{\partial}{\partial y} \mathrm{f}(x,\, \mathrm{y}(x),\, \frac{\partial}{\partial x} \mathrm{y}(x)) = -\sin(y)$$

and

$$\frac{\partial}{\partial z} \mathrm{f}(x,\, \mathrm{y}(x),\, \frac{\partial}{\partial x} \mathrm{y}(x)) = 2\,(\frac{\partial}{\partial x}\, y)$$

and

$$\frac{\partial^2}{\partial x\, \partial z} \mathrm{f}(x,\, \mathrm{y}(x),\, \frac{\partial}{\partial x} \mathrm{y}(x)) = 2\,(\frac{\partial^2}{\partial x^2} \mathrm{y}(x))$$

and finally obtain the Euler equation for our functional

$$-\sin(\mathrm{y}(x)) - 2\,(\frac{\partial^2}{\partial x^2} \mathrm{y}(x)) = 0$$

We solve it using various tools and obtain the solution

$$\mathrm{y}(x) = 0$$

Info.

not_supplied

Comment.

no_comment