2

Topological and Analytic Preliminaries

The neighborhood of a young child consists of the people very close on the left and right. As we get older we think in terms of two-dimensional neighborhoods (the people around the corner) or even three-dimensional neighborhoods (the people in the world). In this chapter we do likewise. We develop numerous methods for accurately describing sets in the real line (one-dimensional) and the plane (two-dimensional). In order to track down the elusive point at infinity, it becomes necessary to introduce the sphere (three-dimensional).

When a set is described in a satisfactory manner, we become concerned about its image. We investigate conditions under which properties of a set are preserved when the set is transformed into a new set. A remarkable outcome of our investigation is that the removal of a single point from one set may entirely change its character, whereas the removal of infinitely many points from a different set may be insignificant. The removal of two points from a set on the line may give it more affinity to a set in the plane than to its former self. In this chapter we learn that in a sense all points are equal but some points are more equal than others.

2.1 Point Sets in the Plane

A neighborhood of a real number x_0 is an interval in the form $(x_0-\epsilon, x_0+\epsilon)$, where ϵ is any positive real number. Thus we may say that an ϵ neighborhood of x_0 is the set of points $x \in \mathbb{R}$ for which $|x-x_0| < \epsilon$. There are different ways to extend this one-dimensional neighborhood concept to include points in the plane. A *square ϵ neighborhood* of a point (x_0, y_0) is the set of all points (x, y) whose coordinates satisfy the two inequalities

$$|x-x_0| < \epsilon, \quad |y-y_0| < \epsilon.$$

It consists of all points inside a square centered at (x_0, y_0). The sides of the square are parallel to the coordinate axes and have length 2ϵ. A *circular ϵ*

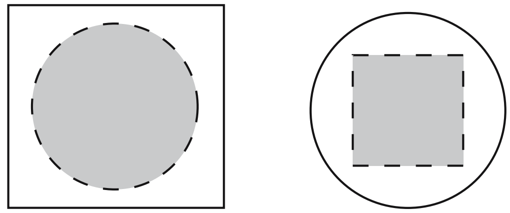

Figure 2.1. Illustration for open sets in the plane

neighborhood of (x_0, y_0) is the set of all points (x, y) whose distance from (x_0, y_0) is less than ϵ. It consists of all points (x, y) such that

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} < \epsilon,$$

i.e., points inside a circle centered at (x_0, y_0) whose radius is ϵ. Observe that every square neighborhood of a point contains a circular neighborhood of the point, and every circular neighborhood of a point contains a square neighborhood of the point (for a smaller ϵ, of course). This is illustrated in Figure 2.1. From our point of view (that a point in the plane represents a complex number), it will be more convenient to deal with circular neighborhoods, for then an ϵ neighborhood of the complex number z_0 consists of all points $z \in \mathbb{C}$ satisfying the inequality $|z - z_0| < \epsilon$. Such a neighborhood is denoted by $N(z_0; \epsilon)$.

Care must be taken to distinguish between a neighborhood on the real line and a neighborhood in the plane. For example, $\{x \in \mathbb{R} : -1 < x < 1\}$ is a neighborhood of 0, a point on the line; it is *not* a neighborhood of $(0, 0)$, a point in the plane. A point in the plane is not permitted to have a one-dimensional neighborhood.

The definitions and theorems in this section are valid simultaneously for points on the line and points in the plane, when the concepts of ϵ neighborhood are suitably interpreted. A set is said to be *bounded* if it is contained in some disk centered at the origin. A point is said to be an *interior point* of a set if there is some neighborhood of the point contained in the set. An important distinction between the bounded sets

$$A = \{z \in \mathbb{C} : |z - z_0| < \epsilon\} \quad \text{and} \quad B = \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}$$

is that every point in A is an *interior point*. To see this, let z_1 be any point in A. Then $|z - z_0| = \delta$ for some δ, $0 \leq \delta < \epsilon$. But for $\eta = (\epsilon - \delta)/2$, we have $N(z_1; \eta) \subset N(z_0; \delta)$ (see Figure 2.2). Of course, no point on the circle $|z - z_0| = \epsilon$ is an interior point of B. A set A is called an *open set* if every point in A is an interior point. We have shown that a neighborhood of a point in the plane is an open set. Other simple examples of open sets in the plane are

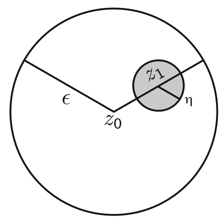

Figure 2.2. Description for an interior point

(a) the empty set,
(b) the set of all complex numbers,
(c) $\{z : |z| > r\}$, $r \geq 0$,
(d) $\{z : r_1 < |z| < r_2\}$, $0 \leq r_1 < r_2$,
(e) the intersection of any two open sets,
(f) the union of any collection of open sets.

Remark 2.1. An open interval on the real line is not an open set in the plane, since any neighborhood of a point will contain points in the plane that are not real. ●

A *deleted* ϵ neighborhood of z_0, denoted by $N'(z_0; \epsilon)$, is the set of all points z such that $0 < |z - z_0| < \epsilon$. That is, the point z_0 is "punched out" from the set. A point z_0 is called a *limit point* of a set A if every deleted neighborhood of z_0 contains a point of A. Note that a limit point z_0 may or may not be in the set A.

Examples 2.2. (i) The limit points of the open set $|z| < 1$ are $|z| \leq 1$; that is, all the points of the set and all the points on the unit circle $|z| = 1$. If $\partial\Delta = \{z : |z| = 1\}$ and $\overline{\Delta} = \{z : |z| \leq 1\}$, then all points of $\overline{\Delta}$ are its limit points and no other point is a limit point of $\overline{\Delta}$. The same is true for $\partial\Delta$. On the other hand, all points of $\Delta \setminus \{0\}$ together with 0 and the points of $\partial\Delta$ are limit points of $\Delta \setminus \{0\}$. Note that 0 and the points of $\partial\Delta$ are not in $\Delta \setminus \{0\}$.

(ii) The set $A = \{1/n : n \in \mathbb{N}\}$, where $\mathbb{N} = \{1, 2, 3, \ldots, n, \ldots\}$, has 0 as a limit point (regardless of whether the set is considered a subset of the line or the plane) and 0 is not in the set. Similarly, the set $A = \{e^{i\pi/n} : n \in \mathbb{N}\}$ has 1 as its only limit point, see Figure 2.3.

(iii) If A consists of the set of points that have both coordinates rational, then every point in the plane is a limit point of A. ●

A set is said to be *closed* if it contains all of its limit points. The union of a set A and its limit points is called the *closure* of A, and is denoted by $\overline{A}$. Some examples of closed sets in the plane are

(a) the empty set,
(b) the set of all complex numbers,

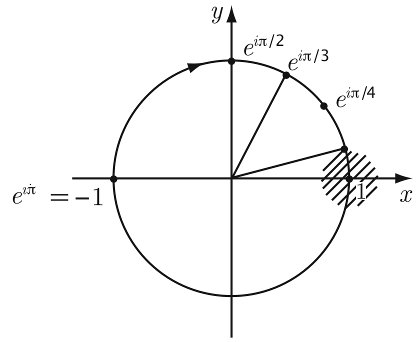

Figure 2.3. Description of limit point 1 of $\{e^{i\pi/n} : n \in \mathbb{N}\}$

(c) $\{z : |z| \geq r\}$, $r \geq 0$,
(d) $\{z : r_1 \leq |z| \leq r_2\}$, $0 \leq r_1 < r_2$,
(e) the union of any two closed sets,
(f) the intersections of any collection of closed sets,
(g) $\{z : |z| \leq 1\}$.

Some examples of sets that are not closed in the complex plane $\mathbb{C}$ are Δ, $\Delta \setminus \{0\}$, $\overline{\Delta} \setminus \{0\}$. Finally, we remark that the set $\overline{\Delta} \setminus \{0\}$ is neither closed nor open.

Theorem 2.3. *If z_0 is a limit point of A, then every neighborhood of z_0 contains infinitely many points of A.*

Proof. Assume that some deleted neighborhood of z_0 contains only a finite number of points of A. Let the points be $z_1, z_2, \dots, z_n$ and $\epsilon = \min_{i=1,2,\dots,n} |z_0 - z_i|$. Then $N'(z_0; \epsilon)$ contains no points of A, and z_0 can't be a limit point of A. ■

Corollary 2.4. *Every finite set is closed.*

Proof. The set contains all of its limit points—all "none" of them. ■

For the set $|z| \leq 1$, we would like to distinguish the interior points from the points on the unit circle. A point z_0 is called a *boundary point* of A if every neighborhood of z_0 contains points in A and points not in A (in the *complement* of A). The set of all boundary points of A is called *boundary* of A. For example, the circle $|z| = 1$ is the boundary for both the bounded set $|z| < 1$ and the unbounded set $|z| > 1$.

Remark 2.5. The boundary points determine the "openness" or "closedness" of a set. An open set cannot contain any of its boundary points, whereas a closed set must contain all of its boundary points (why?). Clearly, an interior point of a set A is a limit point of A but a limit point may or may not be an interior point of A. ●

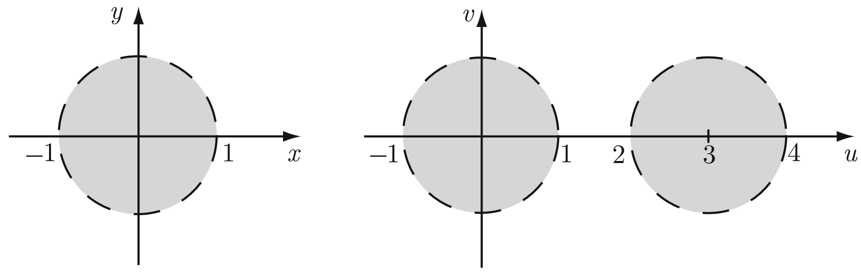

Figure 2.4. Description between open and connected sets

We would also like to distinguish between the two sets

$$A = \{z : |z| < 1\} \quad \text{and} \quad B = \{z : |z| < 1\} \cup \{z : |z - 3| < 1\}.$$

Set A is "all one piece", while set B consists of two pieces (Figure 2.4). A set S is said to be *connected* if there do not exist disjoint open sets U and V satisfying the following conditions:

$$\text{(i) } U \cup V \supset S, \qquad \text{(ii) } U \cap S \neq \phi, \quad V \cap S \neq \phi.$$

In particular, if an open connected set can be expressed as the disjoint union of two open sets U and V, then either $U = \phi$ or $V = \phi$. Set A above is connected and set B is not.

An open connected set is called a *domain*. A *region* is a domain together with some, none, or all of its boundary points.[1] We might think that the counterpart of a real-valued function of a real variable being defined on an open set is a complex-valued function of a complex variable being defined on an open set. But this is not the case. Actually, the counterpart of an open interval in $\mathbb{R}$ is a domain. Note that an open interval in $\mathbb{R}$ is a connected subset of $\mathbb{R}$. Likewise a domain is open as well as connected. The "negative" definition for connectedness is sometimes difficult to visualize. But when the connected set is a domain, we have the following useful property.

Theorem 2.6. *Any two points in a domain can be joined by a polygonal line that lies in the domain.*

Proof. Choose a point z_0 in the domain D. It suffices to show that every point in D can be joined to z_0 by a polygonal line that lies in D. Let A denote the set of all points in D that can be so joined to z_0 and let B denote all those points that cannot. Note that $A \cup B = D$ and $A \cap B = \phi$. We wish to show that B is empty.

[1] The reader is warned that some authors use the term "region" for what we call a domain (following the modern terminology), and others make no distinction between the terms.

If a point z_1 is in A, then z_1 is in D. Since D is open, there exists an $\epsilon_1 > 0$ such that $N(z_1;\epsilon_1) \subset D$. But all the points in $N(z_1;\epsilon_1)$ can be joined to z_1 by a straight line segment. Therefore, each point in $N(z_1;\epsilon_1)$ must be in A, which means that A is an open set.

Similarly, if a point z_2 is in B, then there exists an $\epsilon_2 > 0$ such that $N(z_2;\epsilon_2) \subset D$. All the points in this neighborhood must also lie in B, for if some point $b \in N(z_2;\epsilon_2)$ could be joined to z_0 by a polygonal line, then the straight line segment from z_2 to b could be connected to the polygonal line from z_0 to b in order to form a polygonal line from z_0 to z_2. Thus B is an open set. Consequently, neither A nor B can contain any boundary points. Since D is connected, either A or B must be empty. But $z_0 \in A$, so that B is empty. This completes the proof. ■

Note that a domain may contain two points that cannot be joined by a single *straight* line segment, as is illustrated in Figure 2.5.

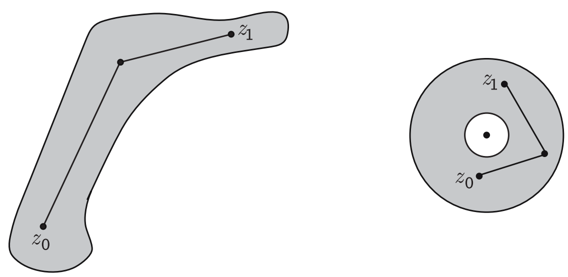

Figure 2.5. Connected domains

Remark 2.7. We could have required that the polygonal line of Theorem 2.6 be parallel to the coordinate axes. The only modification in the proof is the observation that any point in a disk can be joined to the center by combining a line segment parallel to the x axis with one parallel to the y axis. ●

The converse of Theorem 2.6 is also true: if any two points of an open set can be joined by a polygonal line, then the set is connected. The proof is left for the exercises. Also, in the exercises an example is given of a connected set, two of whose points cannot be joined by a polygonal line that lies in the set.

With the above definitions, we are furnished with a method for adequately characterizing most sets on either the line or the plane.

Examples 2.8. (i) Let $A = \{z \in \mathbb{C} : |z| \leq 1$, excluding the points $z_n = 1/n\,(n \in \mathbb{N})\}$. Then the set A is not open because the points on the unit circle have been included and is not closed because the limit points $z_n = 1/n\,(n \in \mathbb{N})$ have been excluded. The set is bounded, connected and has a boundary consisting of the unit circle, the points $z_n = 1/n$, and the origin.

(ii) Let $A = \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \cup \{z : \operatorname{Re} z < -2\}$. This set is open, not closed, not bounded, and not connected. Its boundary consists of all points on the lines $\operatorname{Re} z = 0$ and $\operatorname{Re} z = -2$.

(iii) Let $A = \{z \in \mathbb{C} : -\pi/4 \le \operatorname{Arg} z \le \pi/4\}$. This set is connected, closed, not open, and not bounded. Its boundary consists of the origin together with the rays $\operatorname{Arg} z = \pi/4$ and $\operatorname{Arg} z = -\pi/4$. ●

Questions 2.9.

1. What alternative definitions of "bounded" might we have used?
2. What can we say about unions and intersections of open and closed sets?
3. What can we say about the complements of open and closed sets?
4. What sets are open (closed) in both the plane and the line?
5. What sets are both open and closed?
6. Can a set have infinitely many points without having a limit point?
7. What is the relation between the boundary points and limit points?
8. How does the closure of the intersection of two sets compare with the intersection of their closures?
9. What can you say about intersections and unions of connected sets?
10. What can you say about a set in which every pair of points can be joined by a straight line segment lying in the set?
11. How does the set described in the previous question compare to a set in which there exists a point that can be joined to any other point by a straight line segment lying in the set? What is an example of such a set?
12. What are the boundary points of a deleted neighborhood of z_0?
13. What are the boundary points of the complex plane?

Exercises 2.10.

1. Prove that a neighborhood of a point on the real line (an open interval) is an open set in $\mathbb{R}$.
2. Show that a set A of complex numbers is bounded if and only if, given $z_0 \in \mathbb{C}$, there exists a real number M such that $z \in N(z_0; M)$ for every $z \in A$. Can M be chosen independent of z_0?
3. Show that a set of complex numbers is bounded if and only if both the sets of its real and imaginary parts are bounded.
4. Describe the following sets.
   (a) $\{z \in \mathbb{C} : 1 < |z| < 2, \text{ excluding points for which } z \in \mathbb{R}\}$
   (b) $\{z \in \mathbb{C} : z = (x, y), \ x \text{ and } y \text{ are rational}\}$
   (c) $\{x \in \mathbb{R} : x - \text{irrational}\}$
   (d) $\{x \in \mathbb{R} : x \in \mathbb{Z}\}$
   (e) $\{n \in \mathbb{N} : \bigcup_{n=1}^{\infty} [1/n, n]\}$
   (f) $\{z \in \mathbb{C} : |z| > 2, \ |\operatorname{Arg} z| < \pi/6\}$
   (g) $\{z \in \mathbb{C} : |z + 1| < |z - i|\}$

(h) $\{z \in \mathbb{C} : |z+1| = |z-i|\}$
(i) $\{z \in \mathbb{C} : |\operatorname{Re} z| + |\operatorname{Im} z| = 1\}$.

5. Which of the following subsets are connected?
   (a) $D = \{z \in \mathbb{C} : |z| < 1\} \cup \{z \in \mathbb{C} : |z+2| \leq 1\}$
   (b) $D = [0,2) \cup \{2 + 1/n : n \in \mathbb{N}\}$.
6. Prove that the union of an arbitrary collection of open sets is open and that the intersection of a *finite* number of open sets is open. Also, show that $\cap_{n=1}^{\infty}\{z : |z| < 1/n\}$ is not an open set.
7. Show that a set is open if and only if its complement is closed.
8. Show that the intersection of an arbitrary collection of closed sets is closed and the union of a *finite* number of closed sets is closed.
9. Show that the limit points of a set form a closed set.
10. Show that $\overline{A}$, the closure of A, is the smallest closed set containing A.
11. Show that a set is connected if any two of its points can be joined by a polygonal line.
12. Show that if a set A is connected, then $\overline{A}$ is connected. Is the converse true?
13. Show that the union of two domains is a domain if and only if they have a point in common.

2.2 Sequences

A *sequence* $\{z_n\}$ of complex numbers is formed by assigning to each positive integer n a complex number z_n. The point z_n is called the nth term of the sequence. Care must be taken to distinguish between the terms of the sequence and the set whose elements are the term of the sequence. For example, the sequence $\{2, 2, 2, \dots\}$ has infinitely many terms (as do all sequences), but the set $\{2, 2, 2, \dots\}$ contains only one point. In general, when we discuss set-theoretic properties of a sequence, we will mean the set associated with the terms of the sequence.

A sequence $\{z_n\}$ is said to have a *limit* z_0 (*converge* to z_0), written

$$\lim_{n\to\infty} z_n = z_0 \quad \text{or} \quad z_n \to z_0,$$

if for every $\epsilon > 0$, there exists an integer N (depending on ϵ) such that $|z - z_0| < \epsilon$ whenever $n > N$. Geometrically, this means that every neighborhood of z_0 contains all but a finite number of terms of sequence (see Figure 2.6). We must point out that $z_n \to z_0$ is equivalent to $z_n - z_0 \to 0$. To illustrate, the sequence $\{1/n\}$ converges to 0; but the sequence $\{(-1)^n\}$, which oscillates between 1 and -1, does not converge. Examples of convergent sequences that appear frequently are

(a) $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = 0 \quad (p > 0)$
(b) $\displaystyle\lim_{n\to\infty} |z|^n = 0 \quad (|z| < 1)$
(c) $\displaystyle\lim_{n\to\infty} n^{1/n} = 1$.

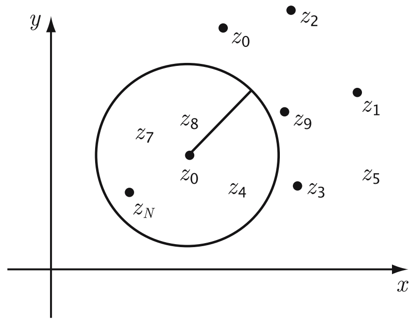

Figure 2.6. Geometric meaning of a convergence of a sequence

Example 2.11. We can easily see that $\{1+i^n\}_{n\geq 1}$ does not converge. Indeed, if $z_n = 1 + i^n$ then, for each fixed $k = 0, 1, 2, 3,$

$$z_{4n+k} = 1 + i^{4n+k} = 1 + i^k = \begin{cases} 2 & \text{if } k = 0 \\ 1+i & \text{if } k = 1 \\ 0 & \text{if } k = 2 \\ 1-i & \text{if } k = 3 \end{cases}$$

and so $\{1 + i^n\}$ diverges. Also we remark that $\{1 + i^n\}$ and $\{i^n\}$ diverge or converge together and so it suffices to deal with $\{i^n\}$ which is easier than the original sequence.

The convergence of the sequence $\{i^n/n\}_{n\geq 1}$ is easier to convince yourself of if you draw a figure representing these points. ●

There is a nice relationship between the convergence of a sequence of complex numbers and the convergence of its real and imaginary parts.

Theorem 2.12. *Let $z_n = x_n + iy_n$ be a sequence of complex numbers. Then $\{z_n\}$ converges to a complex number $z_0 = x_0+iy_0$ if and only if $\{x_n\}$ converges to x_0 and $\{y_n\}$ converges to y_0.*

Proof. The proof is simply a consequence of the inequalities

$$|\mathrm{Re}\, z|, \ |\mathrm{Im}\, z| \leq |z| \leq |\mathrm{Re}\, z| + |\mathrm{Im}\, z|.$$

To provide a detailed proof, we assume $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} y_n = y_0$. Then given $\epsilon > 0$, there exist an integer N such that $n > N$ implies

$$|x_n - x_0| < \epsilon/2, \quad |y_n - y_0| < \epsilon/2. \tag{2.1}$$

From (2.1) we obtain

$$|z_n - z_0| = |x_n - x_0 + i(y_n - y_0)| \leq |x_n - x_0| + |y_n - y_0| < \epsilon,$$

and $\{z_n\}$ converges to z_0. Conversely, if we assume that $\lim_{n\to\infty} z_n = z_0$, the inequalities

$$|x_n - x_0| \le |z_n - z_0|, \quad |y_n - y_0| \le |z_n - z_0|$$

show that $\{x_n\}$ and $\{y_n\}$ converge to x_0 and y_0, respectively. ∎

Theorem 2.12 essentially says that many properties of the complex sequences may be deduced from corresponding properties of real sequences. For example, the uniqueness of the limit of a complex sequence can be derived either directly or from the uniqueness property of real sequences.

A sequence of complex numbers $\{z_n\}$ is said to be *bounded* if there exists an $R > 0$ such that $|z_n| < R$ for all n. In other words, a sequence is said to be bounded if it is contained in some disk.

Since a convergent sequence eventually clusters about its limit, the next theorem is not too surprising.

Theorem 2.13. *A convergent sequence is bounded.*

Proof. If $\lim_{n\to\infty} z_n = z_0$, then $z_n \in N(z_0; 1)$ for $n > N$. Let

$$M = \max\{|z_1|, |z_2|, \dots, |z_N|\}.$$

Then, $|z_n| < M + |z_0| + 1$ for every n. ∎

The converse of Theorem 2.13 is not true. The sequence $\{1, 2, 1, 2, \dots\}$ is bounded and not convergent, although the odd terms and even terms both form convergent sequences.

A *subsequence* of a sequence $\{z_n\}$ is a sequence $\{z_{n_k}\}$ whose terms are selected from the terms of the original sequence and arranged in the same order. For the sequence $z_n = (-1)^n$, we have subsequence $\{z_{2k}\}$ converging to 1 and subsequence $\{z_{2k-1}\}$ converging to -1.

The next theorem shows that a subsequence must be at least as "well behaved" as the original sequence.

Theorem 2.14. *If a sequence $\{z_n\}$ converges to z_0, then every subsequence $\{z_{n_k}\}$ also converges to z_0.*

Proof. Given $\epsilon > 0$, we have $z_n \in N(z_0; \epsilon)$ for $n > N$. Hence $z_{n_k} \in N(z_0; \epsilon)$ for $n_k > N$. Since $n_k \ge n$ (why?), and there can be at most N terms of the subsequence for which $|z_{n_k} - z_0| \ge \epsilon$. ∎

We know that not all sets are bounded. However, if a set of real numbers is bounded, it has a "smallest" bound. A real number M is said to be the *least upper bound* (lub) of a nonempty set A of real numbers if

(i) $x \le M$ for every $x \in A$. That is A is bounded above by M and M is an upper bound for A.
(ii) For any $\epsilon > 0$, there exists a $y \in A$ such that $y > M - \epsilon$. That is, M is the smallest among all the upper bounds of A.

Similarly, the real number m is said to be the *greatest lower bound* (glb) of a nonempty set A if:

(i) $x \geq m$ for every $x \in A$; That is A is bounded below by m and m is a lower bound of A.

(ii) For any $\epsilon > 0$, there exists a $y \in A$ such that $y < m + \epsilon$. That is, m is the largest among all the lower bounds of A.

The *Dedekind property* states that *every nonempty bounded set of real numbers has a least upper bound and a greatest lower bound*. This is an amplified version of the result that $\mathbb{R}$ is complete. For a proof of this, see [R1].

As we have seen, the *converse* of Theorem 2.13 (even for real sequences) is not true. Bounded oscillating sequences need not converge. Eliminating the oscillation, however, will produce convergence. A real sequence $\{x_n\}$ is said to be *monotonically increasing* (*decreasing*) if $x_{n+1} \geq x_n (x_{n+1} \leq x_n)$ for every n. A sequence will be called *monotonic* if it is either monotonically increasing or monotonically decreasing.

Theorem 2.15. *Every bounded monotonic sequence of real numbers converges.*

Proof. Let the bounded sequence $\{x_n\}$ be monotonically increasing. According to the Dedekind property, there exists a least upper bound of $\{x_n\}$, call it x. By the definition of lub, given $\epsilon > 0$ there exists an integer N such that $x_N > x - \epsilon$. Since $\{x_n\}$ is monotonically increasing,

$$x - \epsilon < x_n \leq x \quad \text{for } n > N.$$

Hence $|x_n - x| < \epsilon$ for $n > N$, and $\{x_n\}$ converges to its least upper bound. The proof for monotonically decreasing sequences is identical, using the greatest lower bound instead of the least upper bound. ∎

The examples we have seen of bounded sequences that did not converge did have convergent subsequences. To show that this is true in general, we need the following

Lemma 2.16. *Every sequence of real numbers contains a monotonic subsequence.*

Proof. Assume that the real sequence $\{x_n\}$ has the property that there are infinitely many n such that $x_k \leq x_n$ for every $k \geq n$. Let n_1 be the first such n with this property, n_2 the second, etc. Then $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ is a monotonically decreasing subsequence of $\{x_n\}$.

On the other hand, if there are only finitely many n such that $x_k \leq x_n$ for every $k \geq n$, choose an integer m_1 such that no terms of the sequence $x_{m_1}, x_{m_1+1}, x_{m_1+2}, \ldots$ have this property. Let m_2 be the first integer greater than m_1 for which $x_{m_2} > x_{m_1}$. Continuing the process, we obtain a sequence $x_{m_1}, x_{m_2}, x_{m_3}, \ldots$ which is a monotonically increasing subsequence of $\{x_n\}$. This completes the proof. ∎

Although the converse of Theorem 2.13 is not true, here is slightly a weaker version of it.

Theorem 2.17. *Every bounded sequence of complex numbers contains a convergent subsequence.*

Proof. Let $z_n = x_n + iy_n$, with $|z_n| \leq M$. Then $|x_n| \leq M$ and $|y_n| \leq M$. By Lemma 2.16, $\{x_n\}$ contains a monotonic subsequence $\{x_{n_k}\}$. By Theorem 2.15, $\{x_{n_k}\}$ converges.

Now consider the corresponding subsequence $\{y_{n_k}\}$ of $\{y_n\}$. This may not converge, but by Theorem 2.15, it does contain a convergent subsequence $\{y_{n_{k(l)}}\}$. By Theorem 2.14, $\{x_{n_{k(l)}}\}$ also converges. Applying Theorem 2.12, the sequence

$$z_{n_{k(l)}} = x_{n_{k(l)}} + iy_{n_{k(l)}}$$

is a convergent subsequence of $\{z_n\}$, and this completes the proof. ∎

What are the relationships between the limit of a sequence, the limit points of a sequence, and lub or glb of a sequence? The lub and glb are meaningless in the complex number system, although (as we have just seen) these notions for real numbers may be used to prove theorems about complex numbers. For the sequence $\{n/(n+1)\}$, 1 is the lub, the limit, and the unique limit point. If a convergent sequence has only finitely many distinct elements, it will have no limit points; however, we do have the following theorem.

Theorem 2.18. *A point z_0 is a limit point of a set A if and only if there is a sequence of distinct points in A converging to z_0.*

Proof. If a sequence $\{z_n\}$ of distinct points in A converges to z_0, then every neighborhood of z_0 contains all but a finite number (hence infinitely many) of points of $\{z_n\}$. Therefore, z_0 is a limit point of A.

To prove the converse, let z_0 be a limit point of A. For every integer n, choose a point $z_n \in N(z_0; 1/n) \cap A$. Since every neighborhood of A contains infinitely many distinct points, we may assume the points of the sequence $\{z_n\}$ to be distinct.

Given $\epsilon > 0$, choose N such that $1/N < \epsilon$. Then $z_n \in N(z_0; \epsilon)$ for $n > N$, and the sequence $\{z_n\}$ converges to z_0. ∎

Combining the previous two theorems, we obtain

Theorem 2.19. (Bolzano–Weierstrass) *Every bounded infinite set in the complex plane has a limit point.*

Proof. Choose any sequence of distinct points in the set. By Theorem 2.17, this sequence contains a convergent subsequence; and by Theorem 2.18, the limit of this convergent subsequence is a limit point of the set. This completes the proof. ∎

A sequence $\{z_n\}$ of complex numbers is said to be a *Cauchy sequence* if for every $\epsilon > 0$, there exists an integer N (depending on ϵ) such that

$$|z_m - z_n| < \epsilon \quad \text{whenever} \quad m, n > N.$$

What is the difference between a Cauchy sequence and a convergent sequence? Geometrically, for a convergent sequence, all but a finite number of points are close to a fixed point (the limit of the sequence), while for a Cauchy sequence all but a finite number of points are close to each other. We will show, for complex sequences, that these concepts are equivalent. Moreover, from our exercises, we see that the algebra of complex sequences is essentially the same as that for the real sequences studied in real-variable theory.

Theorem 2.20. (Cauchy Criterion) *The sequence $\{z_n\}$ converges if and only if $\{z_n\}$ is a Cauchy sequence.*

Proof. Assume $\{z_n\}$ converges to z_0. By the triangle inequality,

$$|z_m - z_n| = |z_m - z_0 + z_0 - z_n| \le |z_m - z_0| + |z_n - z_0|. \tag{2.2}$$

Given $\epsilon > 0$, both terms on the right side of (2.2) can be made less than $\epsilon/2$ for $m, n > N$. Hence $\{z_n\}$ is a Cauchy sequence.

Conversely, assume $\{z_n\}$ is a Cauchy sequence. Then for $n > N$, we have $|z_n - z_N| < 1$. That is,

$$|z_n| < |z_N| + 1 \quad \text{for } n > N.$$

Thus $\{z_n\}$ is a bounded sequence. By Theorem 2.17, $\{z_n\}$ contains a subsequence $\{z_{n_k}\}$ that converges to a point (say z_0).

We will show that $\{z_n\}$ also converges to z_0. Once again using the triangle inequality, we obtain

$$|z_n - z_0| = |z_n - z_{n_k} + z_{n_k} - z_0| \le |z_n - z_{n_k}| + |z_{n_k} - z_0|. \tag{2.3}$$

Given $\epsilon > 0$, there exists an integer N such that, for $n > N$,

$$\begin{cases} |z_n - z_{n_k}| < \epsilon/2 & \text{(because } \{z_n\} \text{ is Cauchy)}, \\ |z_{n_k} - z_0| < \epsilon/2 & \text{(because } \{z_{n_k}\} \text{ converges to } z_0). \end{cases} \tag{2.4}$$

Combining (2.3) and (2.4), we see that $|z_n - z_0| < \epsilon$ for $n > N$. Hence, $\{z_n\}$ converges to z_0, and the proof is complete. ∎

Theorem 2.20 furnishes us with a general method for determining the convergence of a sequence of complex numbers even though we may not know in advance what its limit is. There are some systems in which not every Cauchy sequence converges. For instance, in the field of rational numbers, the Cauchy sequence $1, 1.41, 1.414, \ldots$ does not converge (because $\sqrt{2}$ is not rational). A system in which every Cauchy sequence converges is said to be *complete*. In Sprecher [S], it is shown that the real number system forms the only complete ordered field.

Example 2.21. Suppose that $z \neq 1$, but $|z| = 1$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} z^k = 0.$$

Indeed, as $(1-z)\sum_{k=1}^{n} z^k = z(1-z^n)$, we have

$$\left|\sum_{k=1}^{n} z^k\right| = \left|\frac{z(1-z^n)}{1-z}\right| \leq \frac{|z|(1+|z|^n)}{|1-z|} \leq \frac{2|z|}{|1-z|}$$

so that

$$\frac{1}{n}\left|\sum_{k=1}^{n} z^k\right| \leq \frac{2}{n}\left\{\frac{|z|}{|1-z|}\right\} \to 0 \quad \text{as} \quad n\to\infty. \qquad \bullet$$

Questions 2.22.

1. When a sequence $\{z_n\}$ converges to z_0, is the limit z_0 unique?
2. Let $\{x_n\}$ and $\{y_n\}$ be real sequences. If $\{(x_n + y_n)\}$ converges, does this mean that both $\{x_n\}$ and $\{y_n\}$ converge? How does this question compare with Theorem 2.12?
3. How many subsequences are there for a given sequence?
4. Can unbounded sequences have limit points? What about monotonic unbounded sequences?
5. When will the least upper bound of a set be an element of the set?
6. Can a real sequence converge to a value other than lub or glb of the sequence?
7. Can a sequence have infinitely many limit points?
8. Can you think of a sequence that converges without knowing what its limit is?
9. How could Theorem 2.18 have been proved without appealing to Theorem 2.3?
10. What can be said of the sequence $b_n = \text{glb}\,\{a_n, a_{n+1}, a_{n+2}, \dots\}$, where $\{a_n\}$ is a real sequence? What if $\{a_n\}$ is bounded?
11. Suppose that $\{z_n\}$ converges. Does $\{|z_n|\}$ converge? Does $\{\arg z_n\}$ converge? Does $\{\text{Arg}\, z_n\}$ converge?
12. Suppose that both $\{\text{Arg}\, z_n\}$ and $\{|z_n|\}$ converge. Does $\{z_n\}$ converge?

Exercises 2.23.

1. Let $\{z_n\}$ converge to z_0 and w_n converge to w_0. Show that
   (a) $\lim_{n\to\infty} (z_n + w_n) = z_0 + w_0$,
   (b) $\lim_{n\to\infty} z_n w_n = z_0 w_0$,
   (c) $\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{z_0}{w_0}$ provided $w_0 \neq 0$.

In particular, if

$$z_n = \frac{1+n+2i(n-1)}{n} \quad \text{and} \quad w_n = \frac{n^{1/2}+2i(3+4n^3)}{n^3},$$

find z_0, w_0 and z_0/w_0.

2. Show that no sequence having more than one limit point can converge.
3. If $\{z_n\}$ converges, show that $\{|z_n|\}$ converges. Is the converse true?
4. Which of the following sequences are convergent?
   (a) $\{i^n\}$
   (b) $\{z_0^n\}$, where $|z_0| < 1$
   (c) $\left\{\dfrac{\cos n + i \sin n}{n}\right\}$
   (d) $\left\{\dfrac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{n}\right\}$
   (f) $\left\{e^{n\pi i/3} + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^n\right\}$
   (g) $\left\{e^{n\pi i/6} + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^n\right\}$
   (f) $\left\{\frac{n\cos(n\pi)}{2n+1}\right\}$
   (i) $\left\{\sin\left(\frac{n\pi}{8}\right)\right\}$.
5. If $\{z_n\}_{n\geq 1}$ converges to 0, prove that $\{\frac{1}{n}\sum_{k=1}^{n} z_k\}$ converges to 0. Then show that $\{z_n\}$ converging to z_0 implies that $\{\frac{1}{n}\sum_{k=1}^{n} z_k\}$ converges to z_0.
6. Give an example of a sequence that
   (a) does not converge, but has exactly one limit point;
   (b) has n limit points, for any given integer n;
   (c) has infinitely many limit points.
7. Prove that the subsequential limits (the limits of all possible subsequences) of a sequence $\{z_n\}$ form a closed set.
8. Let $\{z_n\}$ be a sequence having the following property: Given $\epsilon > 0$, there exists an integer N such that for $n > N$, $|z_{n+1} - z_n| < \epsilon$. Give an example to show that $\{z_n\}$ need not be a Cauchy sequence.
9. Let $s_n = \sum_{k=1}^{n} 1/k!$. Use the Cauchy criterion to show that $\{s_n\}$ converges.

2.3 Compactness

The union of the open intervals $(n - \frac{1}{2}, n + \frac{1}{2})$ for $n = 1, 2, 3, \ldots$ contains the set of positive integers. Each interval is important in that the removal of any one of them will leave a positive integer uncovered. For the bounded set $S = \{x \in \mathbb{R} : 0 < x < 1\}$, the union of open intervals $(1/n, 1)$ for $n = 2, 3, 4, \ldots$ contains S. While the removal of any one of these intervals will prevent the union of the remaining intervals from covering S, the set S is not contained in any finite subcollection of the intervals.

A set is said to be *countable* if its elements can be put in a one-to-one correspondence with a subset of positive integers. A collection $\{O_\alpha\}$ of open sets is called an *open cover* of a set S if $S \subset \bigcup_\alpha O_\alpha$. Note that the collection $\{O_\alpha\}$ may contain uncountably many sets. A set S is *compact* if every open cover of S contains a finite subcover.

We have seen that neither the set of positive integers nor the open interval $(0,1)$ is compact. However, any finite set is compact because for any open cover we have a finite subcover formed by associating with each point one of the open sets containing the point.

The definition of compactness is not always easy to apply. We would like to work with a more geometrically intuitive method for determining compactness. To this end we will need the following.

Lemma 2.24. *Let $\{I_n\}$ be a sequence of closed and bounded intervals on the real line. If $I_{n+1} \subset I_n$ for every n and the length of I_n approaches 0 as $n \to \infty$, then there is exactly one point in common to all I_n.*

Proof. Let $I_n = \{x : a_n \le x \le b_n\}$. By hypothesis,

$$a_n \le a_{n+1}, \quad b_{n+1} \le b_n \quad (n = 1, 2, 3, \dots)$$

and

$$\lim_{n\to\infty} (b_n - a_n) = 0. \tag{2.5}$$

The sequences $\{a_n\}$ and $\{b_n\}$ are both monotonic and bounded $(a_n, b_n) \in [a_1, b_1]$ for every n. By Theorem 2.15 both sequences must converge; and by (2.5), they must converge to the same point, call it x_0. Since $x_0 = \operatorname{lub}\{a_n\} = \operatorname{glb}\{b_n\}$,

$$x_0 \in [a_n, b_n] \quad \text{for every } n$$

(see Figure 2.7). There cannot be another point x_1 in all the I_n. For, if $x_1\ (\neq x_0)$ were less than (resp. greater than) x_0, then x_0 would not be the lub $\{a_n\}$ (resp. glb $\{b_n\}$). ∎

Note that Lemma 2.24 is not true if *closed* is replaced by *open*. The collection of intervals $\{(0, 1/n) : n \in \mathbb{N}\}$ satisfy the hypotheses, although $\bigcap_{n=1}^{\infty} (0, 1/n) = \phi$.

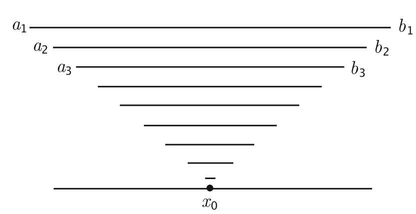

Figure 2.7.

Lemma 2.25. *Let $\{S_n\}$ be a sequence of closed and bounded rectangles in the plane. If $S_{n+1} \subset S_n$ for every n and the length of the sides of S_n approaches 0 as $n \to \infty$, then there is exactly one point in common to all the S_n.*

Proof. Let $\{I_n\}$ and $\{J_n\}$ be the projections of $\{S_n\}$ into real and imaginary axes respectively. Then $\{I_n\}$ and $\{J_n\}$ satisfy the conditions of Lemma 2.24. If

$$\{x_0\} = \cap_{n=1}^{\infty} I_n \quad \text{and} \quad \{y_0\} = \cap_{n=1}^{\infty} J_n,$$

then (see Figure 2.8) $\{z_0\} = \{(x_0, y_0)\} = \cap_{n=1}^{\infty} S_n$. ∎

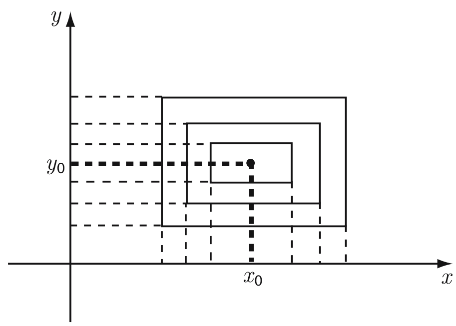

Figure 2.8.

Theorem 2.26. (Heine–Borel) *Every closed and bounded set is compact.*

Proof. Let S be a closed and bounded set. Assume $\{O_\alpha\}$ is an open cover of S that has no finite subcover. Since S is bounded, it is contained in some square S_0 whose vertices are $z = \pm a \pm ai$. The coordinate axes divide S_0 into four equal subsquares. At least one of these squares (call it S_1) has the property that $S \cap S_1$ cannot be covered by a finite subfamily of $\{O_\alpha\}$ (why?). We now divide S_1 into four *more* equal closed subsquares (see Figure 2.9). Again, for at least one of these squares, denoted by S_2, there is no finite subfamily of $\{O_\alpha\}$ that covers $S \cap S_2$.

We can continue the process indefinitely, forming a sequence $\{S_n\}$ of closed squares for which there is no finite subfamily of $\{O_\alpha\}$ that covers $S \cap S_n$. Note that the length of any side of S_n is $a/(2^{n-1})$. By Lemma 2.25, there is exactly one point, denoted by z_0, common to all squares S_n. This point z_0 must be a limit point of S, and hence an element of S.

Let O_{α_0} be an element of the cover $\{O_\alpha\}$ that contains z_0. Since O_{α_0} is an open set, $N(z_0; \epsilon) \subset O_{\alpha_0}$ for some $\epsilon > 0$. But $S_n \subset N(z_0; \epsilon)$ for n sufficiently large. Thus $S_n \cap S$ has a finite subcover of $\{O_\alpha\}$, namely, one element: O_{α_0}. This contradiction, concludes the proof. ∎

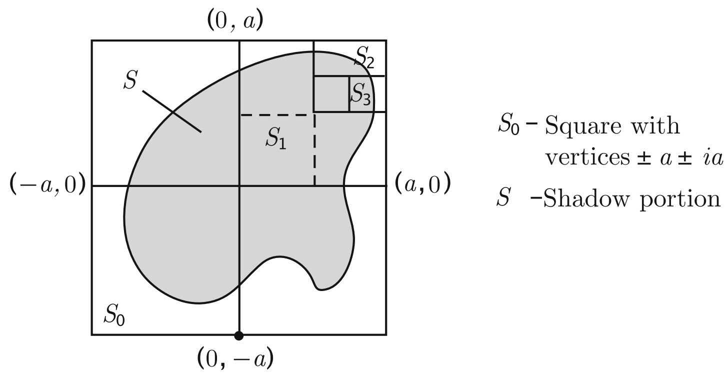

Figure 2.9. Illustration for Heine–Borel theorem

The gist of the above argument is that if no finite subcollection of $\{O_\alpha\}$ covers S, then no finite subcollection covers a carefully chosen sequence of subsets of S. On the other hand, this sequence of subsets can be made small enough to be contained in one of the open sets of the cover.

We are now ready to collect some of the important results of the last two sections to obtain the following major theorem.

Theorem 2.27. *Let S be a subset of the complex plane $\mathbb{C}$. The following statements are equivalent:*

(i) *S is closed and bounded.*
(ii) *S is compact.*
(iii) *Every infinite subset of S has a limit point in S.*
(iv) *Every sequence in S has a subsequence that converges to a point in S.*

Proof. The Heine–Borel theorem states that (i) implies (ii). We will show that (ii) implies (iii), (iii) implies (iv), and (iv) implies (i). Since each statement is clearly correct if S is a finite set, we may suppose that S is infinite.

Assume that (ii) holds. If A is an infinite subset of S having no limit point in S, then for every point in $S\setminus A$ we can find a neighborhood containing no points of A. Furthermore, for every point in A we can find a neighborhood containing no other points of A. The collection of all such neighborhoods is an open cover of S for which there is no finite subcover, contradicting the compactness of S.

Assume (iii) holds. Let $\{z_n\}$ be a sequence of distinct points in S. (Why is it sufficient to consider only such sequences?) By hypothesis, there exists a limit point z_0 of $\{z_n\}$ with $z_0 \in S$. By Theorem 2.18, there is some subsequence of $\{z_n\}$ converging to z_0.

Assume (iv) holds. If S is unbounded, then there exists a sequence of points $\{z_n\}$ in S such that $|z_n| > n$ for every n. Let $\{z_{n_k}\}$ be an arbitrary subsequence of $\{z_n\}$. For any point $z_0 \in S$, $N(z_0; 1)$ can contain no points of $\{z_{n_k}\}$ for which $n_k > |z_0| + 1$. Hence $\{z_{n_k}\}$ cannot converge to any point in

S, contradicting our assumption: To show that S is closed, let z_0 be a limit point of S. By Theorem 2.18, there is sequence of distinct points $\{z_n\}$ of S converges to z_0. By Theorem 2.14, every subsequence of $\{z_n\}$ converges to z_0. According to (iv), z_0 must therefore be in S. This completes the proof. ∎

Compactness is a nice property because reducing an open cover to a finite subcover often means that only a finite number of points need be considered in proving that a set has a certain property. For this reason, when we have compactness, many local properties (properties that hold in a neighborhood of each point in a set) can be shown to be global or uniform (a property of the set as a whole).

For example, from the fact that each point may be covered by a bounded neighborhood, we deduced that a compact set is bounded. Also, if each point in a compact set is a positive distance from a fixed point, the set itself is a positive distance from the point (see Exercise 2.29(3)). This, of course, is not true in general. Each point of the open interval $(0, 1)$ is a positive distance from 0, but we can not find a positive real number between 0 and the set.

What makes the addition of one or two points so important? Let us compare the open interval $(0, 1)$ with the closed interval $[0, 1]$. As we saw earlier, $\bigcup_{n=2}^{\infty}(1/n, 1)$ is an open cover of $(0, 1)$, that has no finite subcover. This cover does not contain the points $\{0\}$ and $\{1\}$. If these points were added to the set, intervals like $(-\epsilon, \epsilon)$ and $(1-\epsilon, 1+\epsilon)$ would also have to be added to obtain a cover. But then $(-\epsilon, \epsilon)$, $(1-\epsilon, 1+\epsilon)$ and $\bigcup_{n=2}^{N}(1/n, 1)$ for $N > 1/\epsilon$ would be a finite subcover.

Questions 2.28.

1. What can we say about the finite union (intersection) of compact sets?
2. What can we say about the infinite union (intersection) of compact sets?
3. What can we say about the complement of a compact set?
4. What can we say about Cauchy sequences in compact sets?
5. When can we say that every subset of a compact set is compact?
6. We have seen that the removal of one point from a set may destroy the compactness. How many points may be *added* to a set to destroy compactness?
7. What kind of generalizations to Lemma 2.25 might we have for compact sets?
8. Can we talk about "infinity" being a limit point?

Exercises 2.29.

1. Show that the union of any bounded set and its limit points is a compact set.
2. Show that a compact set of real numbers contains its greatest lower bound and its least upper bound. Can this occur for a set of real numbers that is not compact?

3. If S is compact and $z_0 \notin S$, prove that $\operatorname{glb}_{z \in S}|z - z_0| > 0$.
4. If $\{S_n\}$ is a sequence of nonempty compact sets with $S_{n+1} \subset S_n$ for every n, show that $\bigcap_{n=1}^{\infty} S_n \neq \phi$.
5. In Theorem 2.27, prove as many different implications as you can.
6. Show that the set of rational numbers are countable.
7. Show that any open cover of a subset of the plane has a *countable* subcover.

2.4 Stereographic Projection

Thus far, infinite limits have been carefully avoided. Consider the three real sequences:

$$a_n = n, \quad b_n = \begin{cases} n & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}, \quad c_n = (-1)^n n.$$

Even though all three sequences grow arbitrarily large, we do not want to say they all approach infinity. From our knowledge of finite limits, it seems appropriate that $\{a_n\}$ should approach infinity and that $\{b_n\}$ should not, since a subsequence of $\{b_n\}$ converges to 1. A case for $\{c_n\}$ can be made either way. The standard approach is to introduce the symbols $\pm\infty$, and adjoin them to the real numbers. The set $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty, -\infty\}$ is known as the *extended real number system*. In the extended real number system, we use the following conventions:

$$\begin{cases} \pm\infty + a = \pm\infty = a \pm \infty & \text{for } a \in \mathbb{R} \\ \infty \cdot a = a \cdot \infty = \infty & \text{for } a \in \mathbb{R}_\infty \setminus \{0\} \\ \dfrac{a}{\infty} = 0 & \text{for } a \in \mathbb{R} \setminus \{0\} \\ \dfrac{a}{0} = \infty & \text{for } a \in \mathbb{R}_\infty \setminus \{0\}. \end{cases}$$

The expressions $\infty + \infty = \infty$, $-\infty - \infty = -\infty$ hold while $\infty - \infty$ is not defined. In the extended real number system, $\{c_n\}$ does not converge because $\{c_{2n}\}$ approaches $+\infty$ and $\{c_{2n+1}\}$ approaches to $-\infty$.

A perfectly logical, if somewhat unusual, approach is to adjoin only one point, ∞, to $\mathbb{R}$. We then say that a sequence $\{a_n\}$ approaches ∞, written $\lim_{n\to\infty} a_n = \infty$, if, for any preassigned real number M, all but a finite number of terms lie outside the interval $(-M, M)$. According to this definition, the sequence $\{(-1)^n n\}$ does approach ∞.

This latter approach can be thought to arise from the former by grabbing the two points $-\infty$ and $+\infty$ (with two very long arms) and bringing them together. The real number line is then transformed into a circle. We now make this geometric notion more precise. Consider the unit circle $x^2 + y^2 = 1$. For any real number a, draw the straight line joining the points $(a, 0)$ and $(0, 1)$. This line intersects the unit circle at $(0, 1)$ and one other point (x_1, y_1), which we identify with the real number a. For example, points in the open interval

$(-1, 1)$ are identified with points in the lower half of the circle, the points -1 and 1 are identified with themselves, and the points outside the interval $[-1, 1]$ are identified with points in the upper half of the circle (see Figure 2.10).

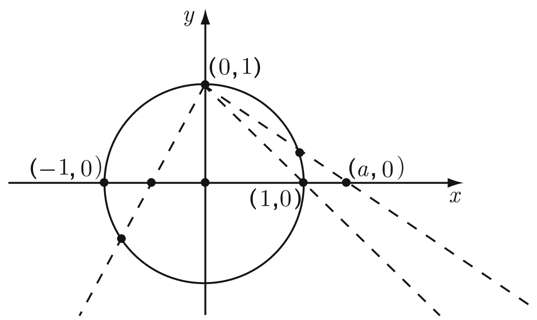

Figure 2.10. Illustration for the existence of $+\infty$ and $-\infty$ in $\mathbb{R}$

Observe that points close to one another on the real line are always identified with points close to one another on the circle. The converse is not true. Points "far out" in the positive and negative directions are identified with points close to one another on the unit circle. In fact, the greater the absolute value of a real number the closer is its identification with a point near $(0, 1)$, the only point on the unit circle not identified with a real number. For this reason, we identify the point $(0, 1)$ with the point ∞. This provide us with a one-to-one correspondence between points in the set $\mathbb{R} \cup \{\infty\}$ and the points on the unit circle. Since the set of real numbers is not compact, the identification of $\mathbb{R} \cup \{\infty\}$ with (compact) circle is called a *one-point compactification* of the real numbers.

Was the elimination of $-\infty$ worth all this effort? Not really. In fact, it is actually useful for $-\infty$ to mean "less than any real number". The set $\mathbb{R} \cup \{\infty\}$ was introduced in order to properly motivate our study of the extended complex plane. Consider the complex sequence $\{z_n\}$ defined by $z_n = n(\cos\theta + i \sin\theta)$, where $0 \leq \theta \leq 2\pi$. For each different value of θ, $\{z_n\}$ approaches ∞ along a different ray. Furthermore, since the complex numbers are not ordered, the symbol $-\infty$ would have no more meaning than the symbol $i\infty$.

In the case of complex numbers, by an *M neighborhood of* ∞, denoted by $N(\infty; M)$, we mean the set of all points whose absolute value is greater than M. That is the exterior of the disk with radius M and center at the origin. The sequence $\{z_n\}$ is said to approach ∞ if for any $M > 0$, $z_n \in N(\infty; M)$ for all but a finite number of n.

If we adjoin the point at ∞ to the set of complex numbers, we obtain the *extended complex number system*. Sometimes $\mathbb{C}$ is referred to as the finite complex plane and is designated also by $|z| < \infty$. Then $\mathbb{C} \cup \{\infty\} := \mathbb{C}_\infty$ is called the *extended complex plane*. Note that the extended complex number system is conceptually different from the extended real number system, in which two points ($+\infty$ and $-\infty$) are added. We first make the following algebraic rules as definitions:

$$\begin{cases} \infty \pm z = \infty = z \pm \infty & \text{for } z \in \mathbb{C} \\ \infty \cdot z = z \cdot \infty = \infty & \text{for } z \in \mathbb{C}_\infty \setminus \{0\} \\ \dfrac{z}{\infty} = 0 & \text{for } z \in \mathbb{C} \setminus \{0\} \\ \dfrac{z}{0} = \infty & \text{for } z \in \mathbb{C}_\infty \setminus \{0\}. \end{cases}$$

There is a difficulty in assigning meaning to the expressions $\infty + \infty$, $\infty - \infty$ ∞/∞, $\infty \cdot 0$ and $0/0$ and so none of these expressions has meaning in $\mathbb{C}_\infty$. The one-point compactification, $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$, of the plane has geometric model similar to that of the one-point compactification of the line, with the unit circle being replaced by the unit sphere

$$S = \{(x, y, z) : x^2 + y^2 + u^2 = 1\}$$

in the 3-dimensional Euclidean sphere in $\mathbb{R}^3$.

Identify the complex number $a+ib$ with the point $(a, b, 0)$ in $\mathbb{R}^3$. By doing so, we are free to imagine $\mathbb{C}$ as an object sitting inside $\mathbb{R}^3$ as xy plane. Having made this identification, for every number $a+ib$ in the complex plane, draw the straight line in $\mathbb{R}^3$ connecting the points $(a, b, 0)$ and $(0, 0, 1)$. This line intersects the sphere $x^2+y^2+u^2 = 1$ at $(0, 0, 1)$ and at one other point (x_1, y_1, u_1). The projection from the point $(0, 0, 1)$ on the sphere to the point $(a, b, 0)$ in the complex plane is called a *stereographic projection* (see Figure 2.11). The sphere S is called the Riemann sphere.

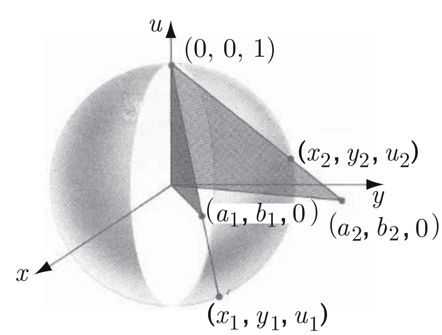

Figure 2.11. Stereographic projection

This one-to-one correspondence covers all points in the finite complex plane and all points in the sphere except $(0, 0, 1)$. The point at ∞ in the extended complex- number system is identified with the point $(0, 0, 1)$, sometimes called the *north pole*. Note that a neighborhood of ∞ in the complex plane corresponds to the interior of an arctic circle whose center is the north pole.

To find specifically the point (x_1, y_1, u_1) on the sphere identified with the point $(a, b, 0)$, observe that the three points

$$(0, 0, 1), \quad (x_1, y_1, u_1), \quad \text{and} \quad (a, b, 0)$$

are collinear. Hence,

$$\frac{x_1 - 0}{a} = \frac{y_1 - 0}{b} = \frac{u_1 - 1}{-1} = t \tag{2.6}$$

for some real scaler t. But

$$x_1^2 + y_1^2 + u_1^2 = (at)^2 + (bt)^2 + (1 - t)^2 = 1, \quad \text{i.e.,} \quad (a^2 + b^2 + 1)t^2 = 2t.$$

Solving for t, we obtain

$$t = \frac{2}{a^2 + b^2 + 1} = 1 - u_1$$

as $t = 0$ corresponds to $(0, 0, 1)$, the north pole. In view of (2.6), the complex number $a + ib$ is then identified with the point

$$(x_1, y_1, u_1) = \left(\frac{2a}{a^2 + b^2 + 1}, \frac{2b}{a^2 + b^2 + 1}, \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1} \right). \tag{2.7}$$

Rewriting (2.7), we identify the complex number $z = x + iy$ with the point on the sphere

$$\left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right).$$

From the second formula for t and (2.6), we conclude that

$$a = \frac{x_1}{1 - u_1} \quad \text{and} \quad b = \frac{y_1}{1 - u_1}.$$

Consequently, we identify the point (x, y, u) in $S \setminus \{(0, 0, 1)\}$ with the complex number in the plane

$$\left(\frac{x}{1 - u} \right) + i \left(\frac{y}{1 - u} \right).$$

For instance the points, $z = 0$ and $z = 1 - i$ correspond to the points $(0, 0, -1)$ and $(2/3, -2/3, 1/3)$, respectively.

Questions 2.30.

1. Which theorems for finite limit remain true for infinite limits?
2. What is the relationship between unbounded sets and neighborhoods of ∞?

3. How might we define ∞ to be a limit point of a sequence?
4. What might the symbol "$i\infty$" mean?
5. What might the symbol "$-i\infty$" mean?
6. What happens to the points on the unit circle in the complex plane under stereographic projection?
7. Could we have identified the complex plane with a different sphere?
8. What would be a one-point compactification of $\mathbb{R}^n$?
9. How are the images on the Riemann sphere of z and $\overline{z}$ related?
10. How are the images on the Riemann sphere of z and $-z$ related? How about for z and $-\overline{z}$?
11. What is the image of the line $x + y = 1$ in the complex plane, on the Riemann sphere?

Exercises 2.31.

1. Show that a sequence having a finite limit point cannot approach ∞.
2. If $\{z_n\}$ approaches ∞ and $\{w_n\}$ is bounded, show that $\{(z_n + w_n)\}$ approaches ∞.
3. Show that $\{z_n\}$ approaches ∞ if and only if $\{|z_n|\}$ approaches ∞.
4. Given a point (x_1, y_1, u_1) on the unit sphere, find its corresponding point in the complex plane.
5. Show that a circle on the sphere that does not pass through the north pole corresponds to a circle in the complex plane.
6. Show that a circle on the sphere passing through the north pole corresponds to a straight line in the complex plane.
7. Show that we may identify, by stereographic projection, the complex plane with the sphere $x^2 + y^2 + (u - \frac{1}{2})^2 = (\frac{1}{2})^2$.
8. Consider two antipodal points (x, y, u) and $(-x, -y, -u)$ on the Riemann sphere. Show that their stereographic projections z and z' are related by $zz' = -1$. Give a geometric interpretation.
9. Show that the image of the circle $|z| = \sqrt{3}$ under the stereographic projection is the set of all points (x_1, y_1, u_1) in the sphere described by $x_1^2 + y_1^2 = 3/4$ and $u_1 = 1/2$.

2.5 Continuity

A (single-valued) *function* or mapping f from a set A into a set B, written $f : A \to B$, is a rule that associates with each element x of A a unique element $f(x)$, the value of f at x, of B. The set A is called the *preimage* (or the domain set) of f and the subset of B associated with the element of A is called the *image* of f and is denoted by $f(A)$, i.e. $f(A) = \{f(x) : x \in A\}$. If the set B, called the *range* of the function, is equal to $f(A)$, the function is said to be *onto*. If no two elements of A are mapped onto the same element in B, the function is said to be *one-to-one* on A. By $f(a) = b$, we will mean that the element $a \in A$ is mapped onto the element of $b \in B$.

For each $b \in B$, we define $f^{-1}(b)$ to be the set of elements in A whose image is b. Note that $f^{-1}(b)$ may be empty if f is not onto. However, if f is one-to-one and onto, $f^{-1} : B \to A$ is also a one-to-one and onto function, called the inverse function of f.

Example 2.32. The function $w = f(z) = az + b,\ a \neq 0$, is one-to-one in $\mathbb{C}$ and the inverse function is defined by $z = (w - b)/a$. Note that both are defined in the whole plane $\mathbb{C}$.

On the other hand, the function f defined by $f(z) = z + 3z^2$ is not one-to-one in $|z| < 1$. For,

$$z_1 + 3z_1^2 = z_2 + 3z_2^2 \implies (z_1 - z_2) = 3(z_2 - z_1)(z_1 + z_2)$$

which implies $(z_1 - z_2)[1 + 3(z_1 + z_2)] = 0$ and we see that the last equality is true when $z_1 + z_2 = -1/3$. But there are many points $z_1,\ z_2 \in \Delta$ such that $z_1 + z_2 = -1/3$. However, this function is one-to-one in $|z| < 1/6$. ●

We have tacitly been dealing with functions. For example, a sequence of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$ and a sequence of complex numbers is a function $f : \mathbb{N} \to \mathbb{C}$, where $\mathbb{N}$ is the set of positive integers. In stereographic projection, a one-to-one function was found that mapped the extended complex plane onto the unit sphere. The reader (hopefully) is familiar with some of the properties of real-valued functions of a real variable, i.e., functions mapping sets of real numbers onto sets of real numbers. For example, the function $y = f(x) = x^2$, mapping the real variable x onto the real variable x^2, takes the set of real numbers onto the set of nonnegative real numbers, the closed interval $[0, 1]$ onto itself, and so on.

Remark 2.33. Strictly speaking, f stands for the function and $f(x)$ for the value of the function at the point x. However, when there is no ambiguity, we will sometimes use the time-honored notational abuse of referring to $f(x)$ as a function.

For $z = x + iy$, the complex-valued function $f(z)$ can be viewed as a function of the complex variable z or as a function of two real variables x and y. ●

For example, the function $f(z) = z^2$ may be expressed as

$$w = f(z) = f(x, y) = (x + iy)^2 = x^2 - y^2 + i(2xy),$$

where

$$\operatorname{Re} f(z) = u(x, y) = x^2 - y^2 \text{ and } \operatorname{Im} f(z) = v(x, y) = 2xy.$$

For this function, the points $(2, 1)$, $(1, 2)$, and $(3, -1)$ are mapped onto the points $(3, 4)$, $(-3, 4)$, and $(8, -6)$ respectively.

Just as a real-valued function of a real variable may be viewed as a mapping from the x axis to the y axis, so may a complex-valued function of a complex

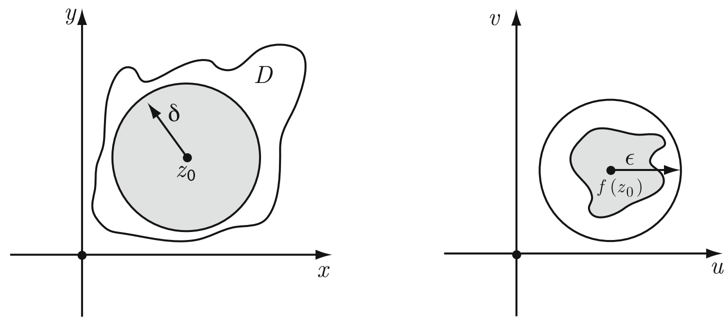

Figure 2.12. Concept of continuity at z_0

variable be viewed as a mapping from the xy plane (z plane) to the uv plane (w plane). While the y axis may be placed vertically on the x axis to obtain a complete two-dimensional picture of the real-valued function $y = f(x)$, the z plane and w plane must stay apart, at least in this three-dimensional world. In this book, we mostly deal with functions $f : A \to \mathbb{C}$ where A is a subset of $\mathbb{C}$.

In Chapter 3, we will be concerned with functions that map certain regions in the z plane onto certain regions in the w plane. Right now we have the more modest task of determining a class of functions that map points near one another in the z plane onto points near one another in the w plane.

A function $f(z)$, defined in a domain D, is said to be *continuous* at a point $z_0 \in D$ if for every $\epsilon > 0$, there exists a $\delta > 0$ (δ depending on ϵ and z_0) such that

$$|f(z) - f(z_0)| < \epsilon, \quad \text{whenever} \quad |z - z_0| < \delta. \tag{2.8}$$

Geometrically, this means that, for every neighborhood of $f(z_0)$ in the w plane, there corresponds a neighborhood of z_0 in the z plane whose image is contained in the neighborhood of $f(z_0)$. More formally, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$f(N(z_0; \delta)) \subset N(f(z_0); \epsilon) \tag{2.9}$$

(see Figure 2.12). If a function is continuous at every point of D, the function is said to be continuous in the domain D. A function $f : A \to \mathbb{C}$ is discontinuous (or has a discontinuity) at z_0 if $z_0 \in A$, yet f is not continuous at $z = z_0$.

Remark 2.34. We will use (2.8) and (2.9) interchangeably. The reader should convince himself of their equivalence and strive to be equally proficient with both.

Also, we will have occasion to discuss the continuity of a function in a region R that includes boundary points. By an ϵ neighborhood of a boundary point $z_0 \in R$, we will mean $N(z_0; \epsilon) \cap R$, and will call this an open set relative

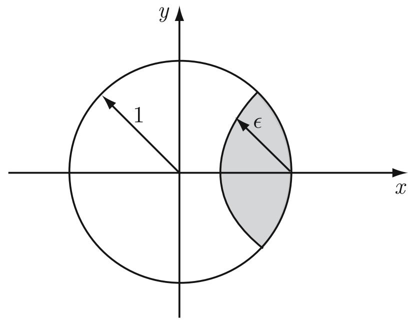

Figure 2.13. ϵ-neighborhood of a boundary point

to the region R. See Figure 2.13 for an ϵ neighborhood of a boundary point of the closed unit disk $|z| \leq 1$. ●

If $f(z)$ is continuous at z_0, we write $\lim_{z\to z_0} f(z) = f(z_0)$. A function may have a limit at a point without being continuous at the point. We say that $\lim_{z\to z_0} f(z) = L$ if for every neighborhood of L, there is a *deleted* neighborhood of z_0 whose image is contained in the neighborhood of L. If $L = f(z_0)$, the function is continuous at z_0 and the word "deleted" may be deleted from this definition.

Examples 2.35. (i) Let

$$f(z) = \begin{cases} z^2 & \text{if } z \neq 2, \\ 5 & \text{if } z = 2. \end{cases}$$

For this function, $\lim_{z\to 2} f(z) = 4$ although the function is not continuous at $z = 2$.

(ii) Let

$$f(z) = \begin{cases} \dfrac{z-2}{z^2-4} & \text{if } z \neq 2, \\ 4 & \text{if } z = 2. \end{cases}$$

Then $\lim_{z\to 2} f(z) = \lim_{z\to 2} 1/(z+2) = 1/4 = L$. Here $L \neq f(2)$. Hence f has a limit as $z \to 2$ but is not continuous at $z = 2$.

(iii) If $\lim_{z\to a} f(z) = L$, then for a given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\,|f(z)| - |L|\,| \leq |f(z) - L| < \epsilon \quad \text{whenever} \quad 0 < |z - a| < \delta$$

and therefore,

$$\lim_{z\to a} |f(z)| = |L|.$$

Clearly, if $L = 0$, $\lim_{z\to a} |f(z)| = |L|$ iff $\lim_{z\to a} f(z) = L$. What happens if $L \neq 0$? More precisely, if $\lim_{z\to a} |f(z)| = L'$, then is it always the case that $\lim_{z\to a} f(z)$ exists? Remember that if $\lim_{z\to a} f(z) = L$, then $|L| = L'$ and therefore, we have to examine when equality holds in

$$||f(z)| - L'| = ||f(z)| - |L|| \le |f(z) - L|.$$

Equality would imply that

$$\mathrm{Re}\,(f(z)\overline{L}) = |f(z)|\,|L| \quad \text{or} \quad |f(z)| = \mathrm{Re}\left(f(z)\frac{\overline{L}}{|L|}\right) = \mathrm{Re}\,(e^{i\theta}f(z))$$

where $\theta = \mathrm{Arg}\,(\overline{L}/|L|)$, or equivalently,

$$|e^{i\theta}f(z)| = \mathrm{Re}\,(e^{i\theta}f(z))$$

so that $e^{i\theta}f(z)$ is real and nonnegative which is impossible for a general complex-valued function $f(z)$. However, this is possible when $f(z) = L'$ or $f(z)$ is a real-valued function with constant sign.

(iv) The signum function *sgn* on $\mathbb{C}$ is defined by

$$sgn\,(z) := \begin{cases} \dfrac{|z|}{z} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases} = \begin{cases} \dfrac{\overline{z}}{|z|} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0. \end{cases}$$

This function is clearly continuous on $\mathbb{C}\setminus\{0\}$ and

$$|sgn\,(z)| = \begin{cases} 1 & \text{for } z \neq 0 \\ 0 & \text{for } z = 0. \end{cases}$$

●

A point z_0 in a set $D \subseteq \mathbb{C}$ that is not a limit point of D is called an *isolated point* of D. Clearly, at an isolated point z_0, there exists a $\delta > 0$ such that $N(z_0;\delta) \cap D = \{z_0\}$. A function $f : D \to \mathbb{C}$ is obviously continuous at all isolated points of D. For example, consider

$$f(z) = \begin{cases} z & \text{for } z \in \{1 - 1/n : n = 1, 2, \ldots\} \\ 1 & \text{for } z = 1 \end{cases}$$

and let $D = \{1 - \frac{1}{n} : n = 1, 2, \ldots\} \cup \{1\}$. The only limit point of D is 1 and so all other points of D are isolated. Since

$$\lim_{z\to 1} f(z) = f(1) = 1,$$

f is continuous at $z = 1$. By definition, f is obviously continuous at the isolated points $z = 1 - 1/n$, $n = 1, 2, \ldots$. Thus, f is continuous on D.

What is the relationship between limits of sequences and limits of more general functions? A complex sequences $\{z_n\}_{n\ge 1}$, which defines a mapping $f : \mathbb{N} \to \mathbb{C}$, converges to z_0 if for every $\epsilon > 0$, there exists an $M > 0$ such that

$$f(N(\infty; M) \cap \mathbb{N}) \subset N(z_0;\epsilon).$$

Recall that a real M neighborhood of ∞ is the set of points outside the interval $(-M, M)$.

If the preimage of f is an unbounded region instead of the set of positive integers, we have the following analog: Let $f : \mathbb{C} \to \mathbb{C}$. Then $\lim_{z\to\infty} f(z) = L$ if for $\epsilon > 0$, there exists an $M > 0$ with $f(N(\infty; M)) \subset N(L; \epsilon)$.

Even if our region is bounded, there are important similarities between limits of the sequences and limits of more general functions. A sequence has a limit if eventually its points are "close" to one another, while a function of a complex variable has a limit if closeness of points in different planes is preserved. Our next theorem shows that continuity may be viewed as an operation that preserves convergence of sequences.

Theorem 2.36. *The function $f(z)$, defined in a region R, is continuous at a point $z_0 \in R$ if and only if, for every sequence $\{z_n\}$ in R converging to z_0, the sequence $\{f(z_n)\}$ converges to $f(z_0)$.*

Proof. Let $f(z)$ be continuous at z_0. Then, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$ $(z \in R)$. If $\{z_n\}$ converges to z_0, then $|z_n - z_0| < \delta$ for $n > N$. By continuity, $|f(z_n) - f(z_0)| < \epsilon$ for $n > N$. Since ϵ was arbitrary, the sequence $\{f(z_n)\}$ converges to $f(z_0)$.

Conversely, suppose that $f(z)$ is not continuous at z_0. Now discontinuity of f at z_0 means that (see (2.9)) for some $\epsilon > 0$, $N(f(z_0); \epsilon)$ does not contain the image of any neighborhood of z_0. This means that we can find a sequence of points $\{z_n\}$ such that $z_n \in N(z_0; 1/n) \cap R$ and $f(z_n) \notin N(f(z_0); \epsilon)$. As $|z_n - z_0| < 1/n$ for all n, the sequence $\{z_n\}$ converges to z_0 although the sequence $\{f(z_n)\}$ does not converge to $f(z_0)$. This contradiction completes the proof. ■

Remark 2.37. Theorem 2.36 is equally valid for real-valued functions of a real variable. ●

Let f be a continuous function defined in a region A. What properties of A are inherited by its image $f(A)$? Theorem 2.36 states that convergent sequences in A give rise to convergent sequences in $f(A)$. But many properties, even for real-valued functions of a real variable, are not preserved under a continuous map.

Examples 2.38. (i) The function $f(z) = |z|$ maps the plane onto the real interval $[0, \infty)$. This shows that the continuous image of an open set need not be open. We then say that f is not an open map.

(ii) The function $f(x) = \tan^{-1} x$ maps the real line onto $(-\pi/2, \pi/2)$. This shows that the continuous image of a closed set need not be closed.

(iii) The function $f(z) = 1/z$ maps the punctured disk $0 < |z| < 1$ onto the exterior of the unit disk. This shows that the continuous image of a bounded set need not be bounded. ●

But all is not lost. If we combine the "nice" properties of the last two examples, the image must also be "nice".

Theorem 2.39. *The continuous image of a compact set is compact.*

Proof. Let $f : A \to f(A)$ be continuous on the compact set A. For any sequence $\{w_n\}$ in $f(A)$, we can find a corresponding sequence $\{z_n\}$ in A such that $f(z_n) = w_n$. By Theorem 2.27, there exists a subsequence $\{z_{n_k}\}$ that converges to a point $z_0 \in A$. By Theorem 2.36, $f(z_{n_k}) = w_{n_k}$ converges to a point $f(z_0) \in f(A)$. Since $\{w_n\}$ was arbitrary, every sequence in $f(A)$ has a subsequence that converges in $f(A)$. Hence $f(A)$ must be a compact set. ■

A function f is said to be *locally constant* if for each $a \in D$ there exists a neighborhood $N(a;\delta)$ of a on which $f(z) = f(a)$ for all z.

Theorem 2.40. *If a continuous function on a connected set D is locally constant, then f is constant throughout.*

Proof. Let a be such that $f(a) = b$. Define

$$S = \{z : f(z) = b\} = f^{-1}(b).$$

Now S is open because f is locally continuous. But S is closed because the singleton set $\{b\}$ is closed. Since S is not empty, we must have $D = S$. This completes the proof. ■

Because the complex field is not ordered, it makes no sense to talk about maximum and minimum values for a complex-valued function $f(z)$. However, the next best thing is a discussion of maxima and minima for the related real-valued function $|f(z)|$. It will be helpful to observe that $|f(z)|$ is continuous in any region where $f(z)$ is continuous. This follows from the inequality

$$\big|\,|f(z_2)| - |f(z_1)|\,\big| \le |f(z_2) - f(z_1)| \quad (z_1, z_2 \in \mathbb{C}).$$

Theorem 2.41. *If $f(z)$ is continuous on a compact set E, then $|f(z)|$ attains a maximum and minimum on E.*

Proof. According to Theorem 2.39, the image of E under $|f(z)|$, which we shall denote by E', is a compact set. Since E' is a bounded set of real numbers, it has a least upper bound b. As a consequence of Exercise 2.29(2), the point b is in the set E'. But this means that $|f(z_0)| = b$ for some $z_0 \in E$.

The proof that $|f(z)|$ attains its minimum is similar, with greatest lower bound being substituted for least upper bound. ■

A function $f(z)$ is said to *uniformly continuous* in a region R if for every $\epsilon > 0$, there exists a $\delta > 0$ (δ depending only on ϵ) such that if $z_1, z_2 \in R$ and $|z_1 - z_2| < \delta$, then $|f(z_1) - f(z_2)| < \epsilon$. This differs from continuity in a region in that the same δ may be used for every point in the region.

For example, the function $f(z) = z$ is uniformly continuous in every region, since we may always choose $\delta = \epsilon$.

Examples 2.42. The function $f(z) = 1/z$, although continuous, is not uniformly continuous in the region $0 < |z| < 1$.

To see this, assume that $f(z)$ is uniformly continuous. Then for $\epsilon > 0$ there exists a δ, $0 < \delta < 1$, to satisfy the conditions of the definition. We exploit the sensitivity of this function near the origin. Let $z_1 = \delta$ and $z_2 = \delta/(1+\epsilon)$. Then $|z_1 - z_2| = \delta\epsilon/(1+\epsilon) < \delta$, but

$$|f(z_1) - f(z_2)| = \left|\frac{1}{\delta} - \frac{1+\epsilon}{\delta}\right| = \frac{\epsilon}{\delta} > \epsilon,$$

showing that f is not uniformly continuous on the punctured unit disk.

Here is another example. The function $f(z) = z^2$ is not uniformly continuous in the complex plane $\mathbb{C}$.

Again, assume the contrary and let $\epsilon > 0$ be given. Then for any $\delta > 0$, choose

$$z_1 = 1/\delta \quad \text{and} \quad z_2 = 1/\delta + \delta/(1+\epsilon).$$

Then, we have $|z_1 - z_2| = \delta/(1+\epsilon) < \delta$ and

$$|z_1^2 - z_2^2| = 2/(1+\epsilon) + \delta^2/(1+\epsilon)^2 > 2/(1+\epsilon).$$

Note that this function is uniformly continuous in any bounded region. ●

Example 2.43. Consider $f(z) = x^2 - iy^2$. Clearly f is continuous on $\mathbb{C}$. But f is not uniformly continuous on $\mathbb{C}$, whereas it is uniformly continuous for $|z| < R$. To verify the second part we first note that, for $z = x + iy$ and $z_0 = x_0 + iy_0$,

$$\begin{aligned}|f(z) - f(z_0)| &= |(x+x_0)(x-x_0) - i(y+y_0)(y-y_0)| \\ &\le |x+x_0|\,|x-x_0| + |y+y_0|\,|y-y_0|.\end{aligned}$$

If z, z_0 are in the disk $|z| < R$, then $|x + x_0| < 2R$ and $|y + y_0| < 2R$. This implies that

$$|f(z) - f(z_0)| \le 2R[|x-x_0| + |y-y_0|] \le 2\sqrt{2}R|z-z_0|$$

(since $|x| + |y| \le \sqrt{2}|z|$). Now, given any $\epsilon > 0$, there exists a $\delta = \epsilon/(2\sqrt{2}R)$ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta = \frac{\epsilon}{2\sqrt{2}R}.$$

So, f is uniformly continuous on Δ_R.

The first part may now be verified as in the previous two examples, and so we leave this part as a simple exercise. ●

Theorem 2.44. *If $f(z)$ is continuous on a compact set A, then $f(z)$ is uniformly continuous on A.*

Proof. Let $\epsilon > 0$ be given. Then, for each point $z_\alpha \in A$, there is a neighborhood (depending on ϵ and z_α) such that

$$|f(z) - f(z_\alpha)| < \frac{\epsilon}{2} \tag{2.10}$$

whenever $|z - z_\alpha| < \delta_\alpha$, $z \in A$. The collection of all neighborhoods of the form $N(z_\alpha; \delta_\alpha/2)$ is a cover of A. By the compactness of A, there exists a finite subcover, say

$$A \subset \bigcup_{k=1}^{n} N\left(z_k; \frac{\delta_k}{2}\right). \tag{2.11}$$

Choose

$$\delta = \min\left\{\frac{\delta_1}{2}, \frac{\delta_2}{2}, \ldots, \frac{\delta_n}{2}\right\}.$$

We wish to show that this δ will work for the whole set A.

Let w_1 and w_2 be any two points in A such that $|w_1 - w_2| < \delta$. By (2.11), $w_1 \in N(z_k; \delta_k/2)$ for some k. According to (2.10), it follows that

$$|f(w_1) - f(z_k)| < \frac{\epsilon}{2}. \tag{2.12}$$

But we also have

$$|w_2 - z_k| \leq |w_2 - w_1| + |w_1 - z_k| < \delta + \frac{\delta_k}{2} < \frac{\delta_k}{2} + \frac{\delta_k}{2} = \delta_k.$$

Hence $w_2 \in N(z_k; \delta_k) \cap A$ and

$$|f(w_2) - f(z_k)| < \frac{\epsilon}{2}. \tag{2.13}$$

Combining (2.12)and (2.13) we obtain

$$|f(w_1) - f(w_2)| \leq |f(w_1) - f(z_k)| + |f(z_k) - f(w_2)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and this completes the proof. ∎

We end the section with a remark on stereographic projection discussed in the previous section. If $\pi : S \setminus \{(0,0,1)\} \to \mathbb{C}$ is a function, then, according to the rule of correspondence,

$$\pi(x, y, u) = \left(\frac{x}{1-u}, \frac{y}{1-u}, 0\right) = \left(\frac{x}{1-u}\right) + i\left(\frac{y}{1-u}\right)$$

and π has an inverse function $\pi^{-1} : \mathbb{C} \to S \setminus \{(0,0,1)\}$ with the rule of correspondence

$$\pi^{-1}(z) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right).$$

Thus, we have established the one-to-one correspondence between the Riemann sphere minus the north pole, namely $S \setminus \{(0,0,1)\}$, and $\mathbb{C}$. From these two formulas, it is evident that π and π^{-1} are continuous functions. In other words, the mapping π defined above is a homeomorphism, i.e., π is one-to-one onto, with both π and π^{-1} continuous. By allowing $(0,0,1)$ to map onto the point at infinity, it is evident that π maps S one-to-one onto $\mathbb{C}_\infty$. Moreover, if $s_1 = (x_1, y_1, u_1)$ and $s_2 = (x_2, y_2, u_2)$ are two points in S, then we define the distance function $d : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by the Euclidean distance

$$\begin{aligned} d(s_1, s_2) &= |(x_1, y_1, u_1) - (x_2, y_2, u_2)| \\ &= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (u_1-u_2)^2}\,. \end{aligned}$$

Suppose now that s_1, s_2 are the images under the stereographic projection of $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ and define $\chi : \mathbb{C}_\infty \times \mathbb{C}_\infty \to \mathbb{R}$ by

$$\chi(z_1, z_2) = d(s_1, s_2).$$

Then it is easy to verify that χ is a metric on $\mathbb{C} \cup \{\infty\}$. We call χ the *chordal metric* on $\mathbb{C} \cup \{\infty\}$ and $(\mathbb{C}_\infty, \chi)$ the extended complex plane which is indeed isometric (i.e., distance preserveness) with (S, d). A straightforward exercise shows that the chordal distance is

$$\chi(z_1, z_2) = \begin{cases} \dfrac{2|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}} & \text{if } z_1, z_2 \in \mathbb{C} \\ \dfrac{2}{\sqrt{1+|z_1|^2}} & \text{if } z_1 \in \mathbb{C}, z_2 = \infty \\ 0 & \text{if } z_1 = \infty,\ z_2 = \infty. \end{cases}$$

Let us see what the open disks look like that are centered at the point at infinity. A deleted ϵ-neighborhood of ∞ in $(\mathbb{C}_\infty, \chi)$ has the form

$$N_\chi(\infty; \epsilon) = \{z : \chi(z, \infty) < \epsilon\}.$$

According to the above formula

$$\chi(z, \infty) < \epsilon \iff (1+|z|^2)^{-1/2} < \epsilon/2 \iff 1 + |z|^2 > (2/\epsilon)^2.$$

Assuming $\epsilon < 2$, this means $|z| > \sqrt{(2/\epsilon)^2 - 1}$. This shows that a deleted neighborhood of ∞ in $\mathbb{C}_\infty$ is of the form

$$N'_\chi(\infty; R) = \{z \in \mathbb{C} : |z| > R\}, \quad R > 0.$$

Note that if $\epsilon \geq 2$, $N_\chi(\infty; \epsilon) = \mathbb{C}_\infty$.

Finally, we now briefly indicate certain concepts associated with $(\mathbb{C}_\infty, \chi)$. A sequence $\{z_n\}$ in $\mathbb{C}$ converges to ∞ in $(\mathbb{C}_\infty, \chi)$ if and only if given $R > 0$

there exists an index $N = N(R)$ such that $|z| > R$ for all $n \geq N$. Similarly, if $f : A \to \mathbb{C}$ and $z_0 \in \mathbb{C}$ is a limit point of A, then $\lim_{z\to z_0} f(z) = \infty$ iff given $R > 0$ there exists a $\delta > 0$ such that $|f(z)| > R$ whenever $0 < |z - z_0| < \delta$ and $z \in A$. If ∞ is a limit point of A, then $\lim_{z\to\infty} f(z) = L$ iff given $\epsilon > 0$ there exists an $R > 0$ such that $|f(z) - L| < \epsilon$ whenever $|z| > R$ and $z \in A$.

Questions 2.45.

1. What ambiguities might there be if we called the preimage the domain of the function?
2. What is the geometric significance of a complex-valued function of a real variable? A real-valued function of a complex variable?
3. What properties do functions and their inverses have in common?
4. For what kinds of functions will we have points closer (more distant) in the w plane than in the z plane?
5. What can we say about the continuity of sequences?
6. Can we talk about a function being continuous at ∞?
7. What can we say about the continuous image of a limit point of a set?
8. How do the proofs of Theorem 2.36 and Theorem 2.18 compare?
9. What is the largest region on which $f(z) = 1/z$ is uniformly continuous?
10. Can discontinuous functions map compact sets onto compact sets?
11. If a function is uniformly continuous on a set A, is it also uniformly continuous on every subset of A?
12. How can you define a piecewise continuous real-valued function of a real variable defined on an interval $[a, b]$?
13. How can you define a piecewise continuous complex function of a real variable defined on an interval $[a, b]$?
14. Are piecewise continuous real-valued functions of a real variable defined on an interval $[a, b]$ integrable and bounded?
15. Does $f(z) = \arg z$ define a complex function? How about
$$f(z) = \cos(\arg z) + i \sin(\arg z)?$$

Exercises 2.46.

1. Find the following limits when they exist:

(a) $\displaystyle\lim_{z\to 3i} \frac{z^2+9}{z-3i}$ (b) $\displaystyle\lim_{z\to 2i} \frac{\overline{z}+z^2}{1-\overline{z}}$

(c) $\displaystyle\lim_{z\to \infty} \frac{z+1}{z^2}$ (d) $\displaystyle\lim_{z\to \infty} \frac{z^2+10z+2}{2z^2-11z-6}$

(e) $\displaystyle\lim_{z\to 3i} \frac{z^3+27i}{z^2+9}$ (f) $\displaystyle\lim_{z\to 1} \frac{1-z^n}{z^2+5z-6} \quad (n \geq 1)$

2. Discuss continuity and uniform continuity for the following functions.

(a) $f(z) = \dfrac{1}{1-z} \quad (|z| < 1)$ (b) $f(z) = \dfrac{1}{z} \quad (|z| \geq 1)$

(c) $f(z) = \begin{cases} \dfrac{|z|}{z} & \text{if } 0 < |z| \leq 1 \\ 0 & \text{if } z = 0 \end{cases}$ (d) $f(z) = \begin{cases} \dfrac{\operatorname{Re} z}{z} & \text{if } 0 < |z| < 1 \\ 1 & \text{if } z = 0. \end{cases}$

3. Prove that $f(z) = 1/(1-z)$ is not uniformly continuous for $|z| < 1$.
4. Show that the function $f(z) = 1/z^2$ is not uniformly continuous for $0 < \operatorname{Re} z < 1/2$ but is uniformly continuous for $1/2 < \operatorname{Re} z < 1$.
5. Let $f(z)$ be one of the following functions each being defined in the punctured plane $\mathbb{C} \setminus \{0\}$:

$$\frac{\operatorname{Re} z}{z},\ \frac{\operatorname{Im} z}{z},\ \frac{z}{|z|},\ \frac{z}{\overline{z}},\ \frac{|z|}{z},\ \frac{\overline{z}}{z}\ .$$

Is it possible to suitably define any one of the these functions at $z = 0$ so that the resulting function will become continuous at $z = 0$. Answer the same question for the functions

$$\frac{z\operatorname{Re} z}{|z|} \text{ and } \frac{z\operatorname{Im} z}{|z|}.$$

How about for the functions

$$\frac{z}{\operatorname{Re} z} \text{ and } \frac{z}{\operatorname{Im} z}$$

when it is defined for $\mathbb{C} \setminus \{x + iy : x \neq 0\}$ and $\mathbb{C} \setminus \{x + iy : y \neq 0\}$, respectively?
6. Discuss continuity of

$$f(z) = \begin{cases} \dfrac{(\operatorname{Re} z)^2 (\operatorname{Im} z)}{|z|^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

at the all points of $\mathbb{C}$.
7. Find the following limits:
(a) $\lim_{z\to 0} f(z)$, where $f(z) = \dfrac{xy}{x^2+y^2} + 2xi$,
(b) $\lim_{z\to 0} f(z)$, where $f(z) = \dfrac{xy}{x^2+y} + 2\dfrac{x}{y}i$,
(c) $\lim_{z\to 0} f(z)$, where $f(z) = \dfrac{xy^3}{x^3+y^3} + \dfrac{x^8}{y^2+1}i$.
8. If $\lim_{z\to\infty} f(z) = a$, and $f(z)$ is defined for every positive integer n, prove that $\lim_{n\to\infty} f(n) = a$. Give an example to show that the converse is false.

9. Show that a monotonic real-valued function of a real variable cannot have uncountably many discontinuities.
10. Show that $f : A \to B$ is continuous if and only if for every open set O relative to B, $f^{-1}(O)$ is an open set relative to A.
11. Using Exercise 2.46(10), prove that the continuous image of a compact set is compact.
12. Show that $f : A \to B$ is continuous if and only if for every closed set F relative to B, $f^{-1}(F)$ is a closed set relative to A.
13. Prove that continuous image of a connected set is connected.
14. If a function, defined on a compact set, is continuous, one-to-one, and onto, show that the inverse function also has these properties. Can compactness be omitted?
15. Let f and g be continuous on a set A. Show that $f + g$, $f \cdot g$, and $f/g\,(g \neq 0)$ are also continuous on A. What can we say if f and g are uniformly continuous on A?
16. Show that $f(z)$ is continuous in a region R if and only if both $\operatorname{Re} f(z)$ and $\operatorname{Im} f(z)$ are continuous in R.
17. Show that every polynomial is continuous in the complex plane.
18. Let $f(z)$ be continuous in the complex plane. Let $A = \{z \in \mathbb{C} : f(z) = 0\}$. Show that A is a closed set.
19. Show that $\displaystyle\lim_{z\to 4} \frac{1}{z-4} = \infty$ and $\displaystyle\lim_{z\to\infty} \frac{1}{z^2+2} = 0$.
20. Suppose that $J : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is defined by $J(z) = 1/z$, $z \in \mathbb{C}_\infty$. Do our conventions imply $J(0) = \infty$ and $J(\infty) = \infty$? Does

$$\chi(J(z), J(w)) = \chi(z, w)$$

hold in $\mathbb{C}_\infty$?