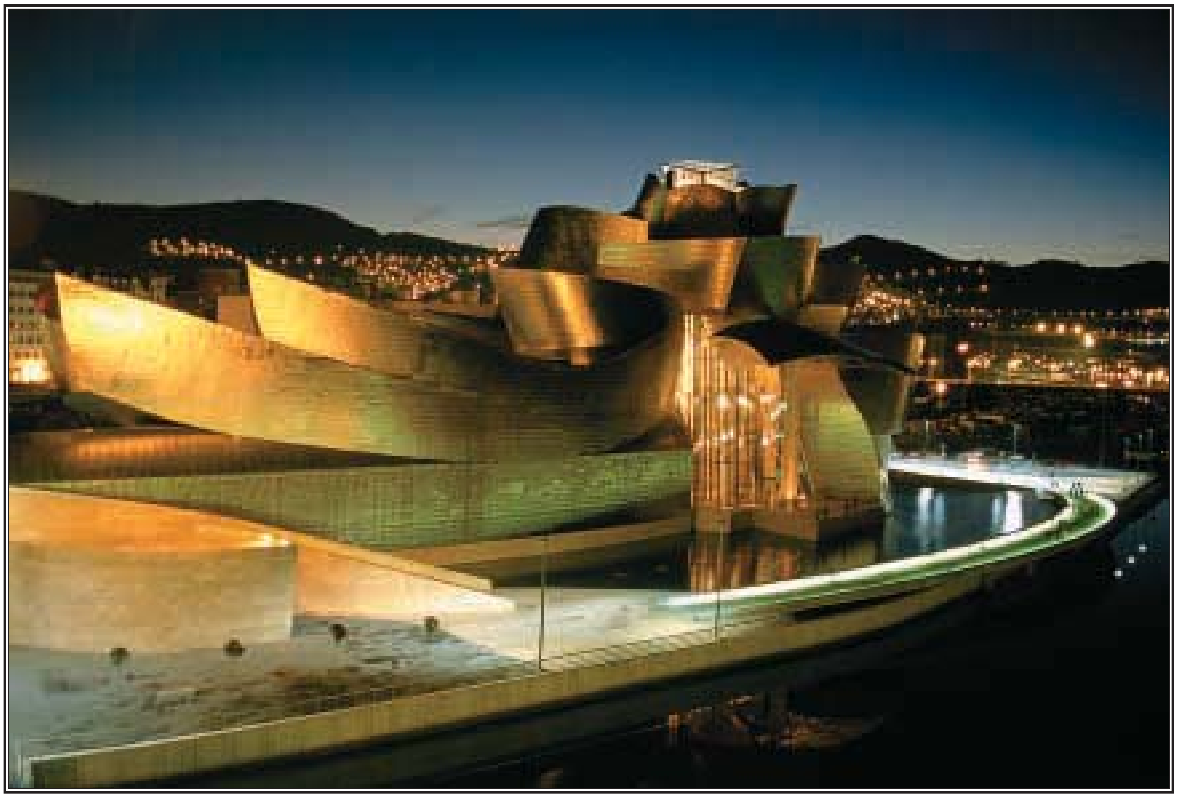


2 Limits and Derivatives

In *A Preview of Calculus* (page 2) we saw how the idea of a limit underlies the various branches of calculus. Thus, it is appropriate to begin our study of calculus by investigating limits and their properties. The special type of limit that is used to find tangents and velocities gives rise to the central idea in differential calculus, the derivative. We see how derivatives can be interpreted as rates of change in various situations and learn how the derivative of a function gives information about the original function.

2.1 The Tangent and Velocity Problems

In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.

The Tangent Problem

Locate tangents interactively and explore them numerically.
Resources / Module 1
/ Tangents
/ What Is a Tangent?

The word *tangent* is derived from the Latin word *tangens*, which means "touching." Thus, a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact. How can this idea be made precise?

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once as in Figure 1(a). For more complicated curves this definition is inadequate. Figure l(b) shows two lines l and t passing through a point P on a curve C. The line l intersects C only once, but it certainly does not look like what we think of as a tangent. The line t, on the other hand, looks like a tangent but it intersects C twice.

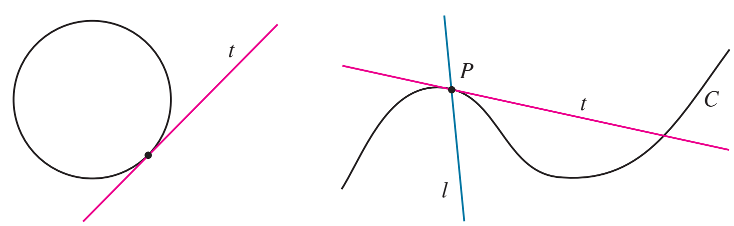


FIGURE 1 (a) (b)

To be specific, let's look at the problem of trying to find a tangent line t to the parabola $y = x^2$ in the following example.

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION We will be able to find an equation of the tangent line t as soon as we know its slope m. The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope. But observe that we can compute an

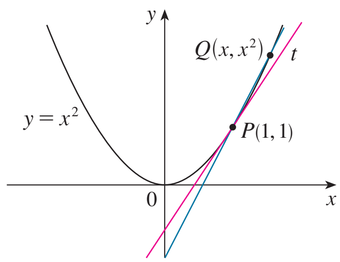


FIGURE 2

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ.

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

The tables in the margin show the values of m_{PQ} for several values of x close to 1. The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2. This suggests that the slope of the tangent line t should be $m = 2$.

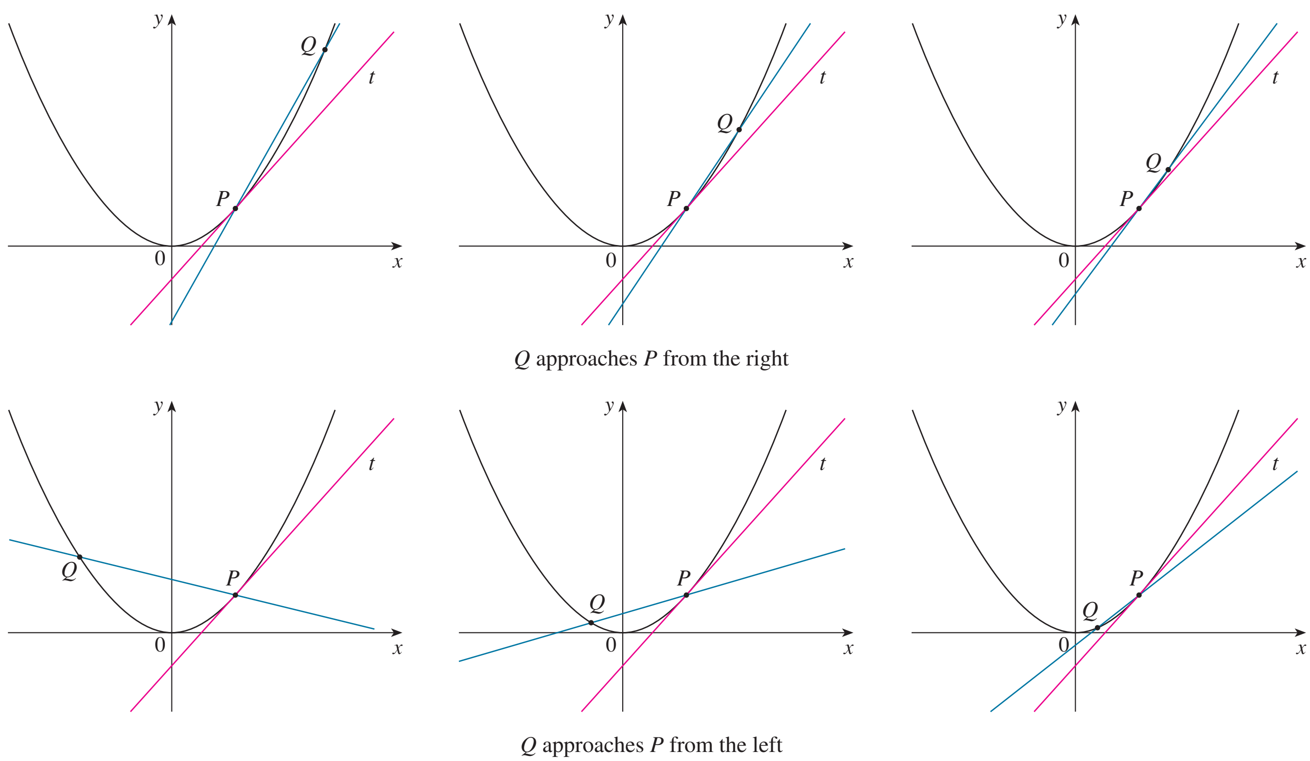
We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m \qquad \text{and} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line (see Appendix B) to write the equation of the tangent line through $(1, 1)$ as

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

Figure 3 illustrates the limiting process that occurs in this example. As Q



Q approaches *P* from the right

Q approaches *P* from the left

FIGURE 3

TEC In Module 2.1 you can see how the process in Figure 3 works for five additional functions.

approaches P along the parabola, the corresponding secant lines rotate about P and approach the tangent line t. ■

Many functions that occur in science are not described by explicit equations; they are defined by experimental data. The next example shows how to estimate the slope of the tangent line to the graph of such a function.

t	Q
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

EXAMPLE 2 The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data at the left describe the charge Q remaining on the capacitor (measured in microcoulombs) at time t (measured in seconds after the flash goes off). Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where $t = 0.04$. [*Note:* The slope of the tangent line represents the electric current flowing from the capacitor to the flash bulb (measured in microamperes).]

SOLUTION In Figure 4 we plot the given data and use them to sketch a curve that approximates the graph of the function.

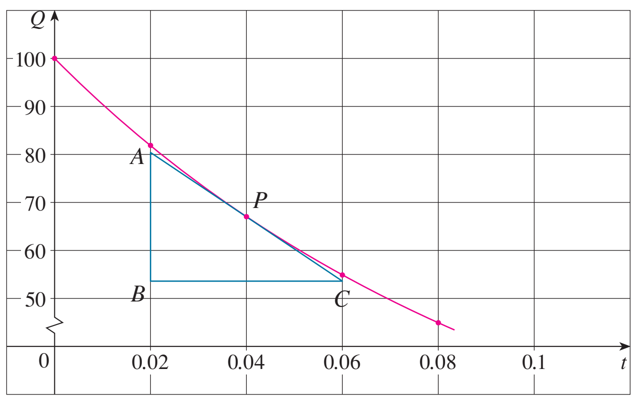


FIGURE 4

Given the points $P(0.04, 67.03)$ and $R(0.00, 100.00)$ on the graph, we find that the slope of the secant line PR is

$$m_{PR} = \frac{100.00 - 67.03}{0.00 - 0.04} = -824.25$$

R	m_{PR}
(0.00, 100.00)	−824.25
(0.02, 81.87)	−742.00
(0.06, 54.88)	−607.50
(0.08, 44.93)	−552.50
(0.10, 36.76)	−504.50

The table at the left shows the results of similar calculations for the slopes of other secant lines. From this table we would expect the slope of the tangent line at $t = 0.04$ to lie somewhere between -742 and -607.5. In fact, the average of the slopes of the two closest secant lines is

$$\tfrac{1}{2}(-742 - 607.5) = -674.75$$

So, by this method, we estimate the slope of the tangent line to be -675.

Another method is to draw an approximation to the tangent line at P and measure the sides of the triangle ABC, as in Figure 4. This gives an estimate of the slope of the tangent line as

$$-\frac{|AB|}{|BC|} \approx -\frac{80.4 - 53.6}{0.06 - 0.02} = -670$$ ■

▲ The physical meaning of the answer in Example 2 is that the electric current flowing from the capacitor to the flash bulb after 0.04 second is about −670 microamperes.

The Velocity Problem

If you watch the speedometer of a car as you travel in city traffic, you see that the needle doesn't stay still for very long; that is, the velocity of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined? Let's investigate the example of a falling ball.

The CN Tower in Toronto is currently the tallest freestanding building in the world.

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

SOLUTION Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time ($t = 5$) so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\begin{aligned} \text{average velocity} &= \frac{\text{distance traveled}}{\text{time elapsed}} \\ &= \frac{s(5.1) - s(5)}{0.1} \\ &= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s} \end{aligned}$$

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus, the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$

■

You may have the feeling that the calculations used in solving this problem are very similar to those used earlier in this section to find tangents. In fact, there is a close connection between the tangent problem and the problem of finding velocities. If we draw the graph of the distance function of the ball (as in Figure 5) and we consider the points $P(a, 4.9a^2)$ and $Q(a + h, 4.9(a + h)^2)$ on the graph, then the slope of the secant line PQ is

$$m_{PQ} = \frac{4.9(a + h)^2 - 4.9a^2}{(a + h) - a}$$

which is the same as the average velocity over the time interval $[a, a + h]$. Therefore, the velocity at time $t = a$ (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).

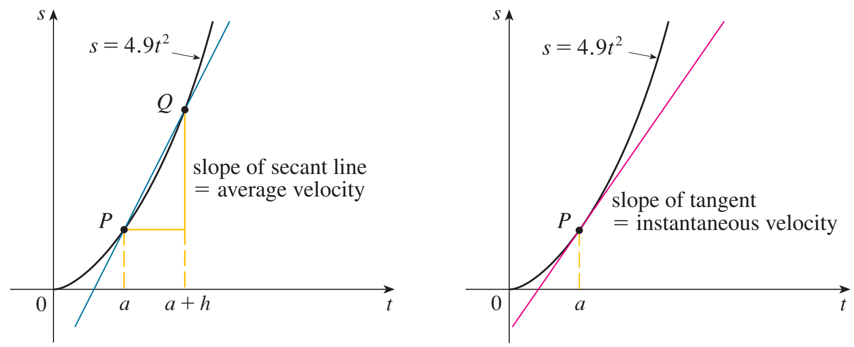


FIGURE 5

Examples 1 and 3 show that in order to solve tangent and velocity problems we must be able to find limits. After studying methods for computing limits in the next four sections, we will return to the problems of finding tangents and velocities in Section 2.6.

2.1 Exercises

1. A tank holds 1000 gallons of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in gallons) after t minutes.

t (min)	5	10	15	20	25	30
V (gal)	694	444	250	111	28	0

(a) If P is the point (15, 250) on the graph of V, find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5$, 10, 20, 25, and 30.

(b) Estimate the slope of the tangent line at P by averaging the slopes of two secant lines.

(c) Use a graph of the function to estimate the slope of the tangent line at P. (This slope represents the rate at which the water is flowing from the tank after 15 minutes.)

2. A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after t minutes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

t (min)	36	38	40	42	44
Heartbeats	2530	2661	2806	2948	3080

The monitor estimates this value by calculating the slope of a secant line. Use the data to estimate the patient's heart rate after 42 minutes using the secant line between the points with the given values of t.

(a) $t = 36$ and $t = 42$
(b) $t = 38$ and $t = 42$
(c) $t = 40$ and $t = 42$
(d) $t = 42$ and $t = 44$

What are your conclusions?

3. The point $P(1, \frac{1}{2})$ lies on the curve $y = x/(1 + x)$.

(a) If Q is the point $(x, x/(1 + x))$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:

(i) 0.5 (ii) 0.9
(iii) 0.99 (iv) 0.999
(v) 1.5 (vi) 1.1
(vii) 1.01 (viii) 1.001

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(1, \frac{1}{2})$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(1, \frac{1}{2})$.

4. The point $P(2, \ln 2)$ lies on the curve $y = \ln x$.

(a) If Q is the point $(x, \ln x)$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:

(i) 1.5 (ii) 1.9
(iii) 1.99 (iv) 1.999
(v) 2.5 (vi) 2.1
(vii) 2.01 (viii) 2.001

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, \ln 2)$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(2, \ln 2)$.

(d) Sketch the curve, two of the secant lines, and the tangent line.

5. If a ball is thrown into the air with a velocity of 40 ft/s, its height in feet after t seconds is given by $y = 40t - 16t^2$.

(a) Find the average velocity for the time period beginning when $t = 2$ and lasting

(i) 0.5 s (ii) 0.1 s
(iii) 0.05 s (iv) 0.01 s

(b) Find the instantaneous velocity when $t = 2$.

6. If an arrow is shot upward on the moon with a velocity of 58 m/s, its height in meters after t seconds is given by $h = 58t - 0.83t^2$.

(a) Find the average velocity over the given time intervals:

(i) [1, 2] (ii) [1, 1.5]
(iii) [1, 1.1] (iv) [1, 1.01]
(v) [1, 1.001]

(b) Find the instantaneous velocity after one second.

7. The displacement (in feet) of a certain particle moving in a straight line is given by $s = t^3/6$, where t is measured in seconds.

(a) Find the average velocity over the following time periods:

(i) [1, 3] (ii) [1, 2]
(iii) [1, 1.5] (iv) [1, 1.1]

(b) Find the instantaneous velocity when $t = 1$.

(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities found in part (a).

(d) Draw the tangent line whose slope is the instantaneous velocity from part (b).

8. The position of a car is given by the values in the table.

t (seconds)	0	1	2	3	4	5
s (feet)	0	10	32	70	119	178

(a) Find the average velocity for the time period beginning when $t = 2$ and lasting

(i) 3 s (ii) 2 s (iii) 1 s

(b) Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 2$.

9. The point $P(1, 0)$ lies on the curve $y = \sin(10\pi/x)$.

(a) If Q is the point $(x, \sin(10\pi/x))$, find the slope of the secant line PQ (correct to four decimal places) for $x = 2, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8,$ and 0.9. Do the slopes appear to be approaching a limit?

(b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at P.

(c) By choosing appropriate secant lines, estimate the slope of the tangent line at P.

2.2 The Limit of a Function

Having seen in the preceding section how limits arise when we want to find the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and methods for computing them.

Let's investigate the behavior of the function f defined by $f(x) = x^2 - x + 2$ for values of x near 2. The following table gives values of $f(x)$ for values of x close to 2, but not equal to 2.

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

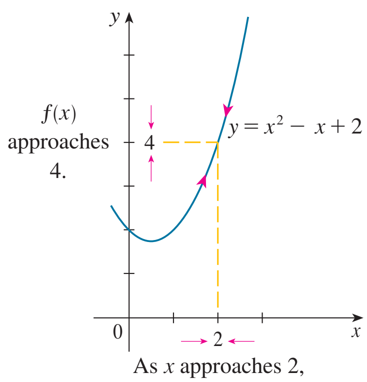


FIGURE 1

From the table and the graph of f (a parabola) shown in Figure 1 we see that when x is close to 2 (on either side of 2), $f(x)$ is close to 4. In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2. We express this by saying "the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4." The notation for this is

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

In general, we use the following notation.

1 Definition We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Roughly speaking, this says that the values of $f(x)$ become closer and closer to the number L as x approaches the number a (from either side of a) but $x \neq a$.

An alternative notation for

$$\lim_{x \to a} f(x) = L$$

is

$$f(x) \to L \qquad \text{as} \qquad x \to a$$

which is usually read "$f(x)$ approaches L as x approaches a."

Notice the phrase "but $x \neq a$" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near* a.

Figure 2 shows the graphs of three functions. Note that in part (c), $f(a)$ is not defined and in part (b), $f(a) \neq L$. But in each case, regardless of what happens at a, $\lim_{x \to a} f(x) = L$.

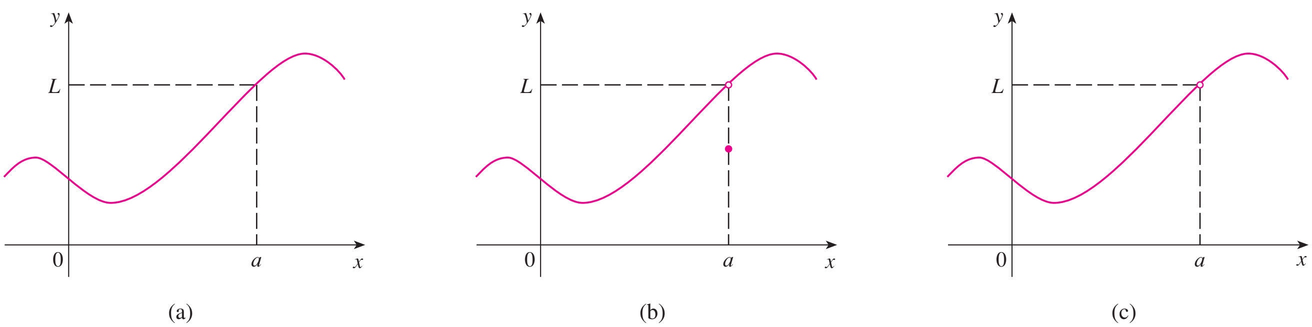


FIGURE 2 $\lim_{x \to a} f(x) = L$ in all three cases

EXAMPLE 1 Guess the value of $\lim_{x \to 1} \dfrac{x - 1}{x^2 - 1}$.

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

SOLUTION Notice that the function $f(x) = (x - 1)/(x^2 - 1)$ is not defined when $x = 1$, but that doesn't matter because the definition of $\lim_{x \to a} f(x)$ says that we consider values of x that are close to a but not equal to a. The tables at the left give values of $f(x)$ (correct to six decimal places) for values of x that approach 1 (but are not equal to 1). On the basis of the values in the table, we make the guess that

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = 0.5$$

Example 1 is illustrated by the graph of f in Figure 3. Now let's change f slightly by giving it the value 2 when $x = 1$ and calling the resulting function g:

$$g(x) = \begin{cases} \dfrac{x - 1}{x^2 - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

This new function g still has the same limit as x approaches 1 (see Figure 4).

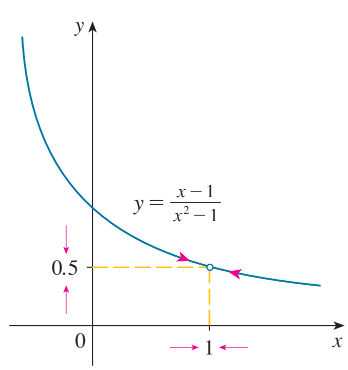


FIGURE 3

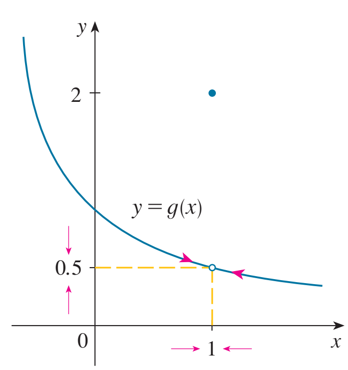


FIGURE 4

EXAMPLE 2 Find $\displaystyle\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

SOLUTION The table lists values of the function for several values of t near 0.

t	$\dfrac{\sqrt{t^2 + 9} - 3}{t^2}$
± 1.0	0.16228
± 0.5	0.16553
± 0.1	0.16662
± 0.05	0.16666
± 0.01	0.16667

As t approaches 0, the values of the function seem to approach 0.1666666 . . . and so we guess that

$$\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} = \frac{1}{6}$$

t	$\dfrac{\sqrt{t^2 + 9} - 3}{t^2}$
± 0.0005	0.16800
± 0.0001	0.20000
± 0.00005	0.00000
± 0.00001	0.00000

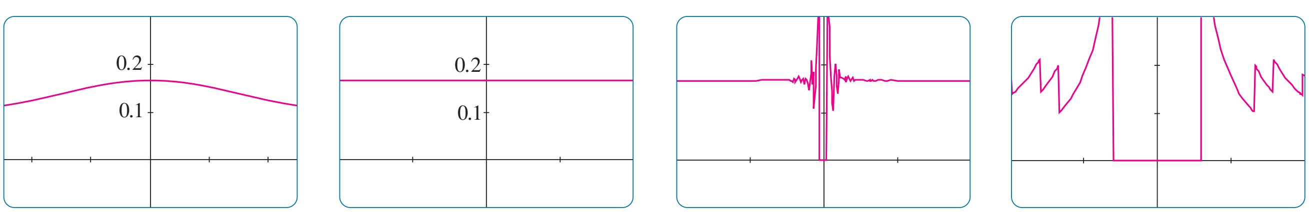
In Example 2 what would have happened if we had taken even smaller values of t? The table in the margin shows the results from one calculator; you can see that something strange seems to be happening.

If you try these calculations on your own calculator you might get different values, but eventually you will get the value 0 if you make t sufficiently small. Does this mean that the answer is really 0 instead of $\frac{1}{6}$? No, the value of the limit is $\frac{1}{6}$, as we will show in the next section. The problem is that the calculator gave false values because $\sqrt{t^2 + 9}$ is very close to 3 when t is small. (In fact, when t is sufficiently small, a calculator's value for $\sqrt{t^2 + 9}$ is 3.000. . . to as many digits as the calculator is capable of carrying.)

Something similar happens when we try to graph the function

$$f(t) = \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

of Example 2 on a graphing calculator or computer. Parts (a) and (b) of Figure 5 show quite accurate graphs of f and when we use the trace mode (if available), we can estimate easily that the limit is about $\frac{1}{6}$. But if we zoom in too far, as in parts (c) and (d), then we get inaccurate graphs, again because of problems with subtraction.



(a) $[-5, 5]$ by $[-0.1, 0.3]$ (b) $[-0.1, 0.1]$ by $[-0.1, 0.3]$ (c) $[-10^{-6}, 10^{-6}]$ by $[-0.1, 0.3]$ (d) $[-10^{-7}, 10^{-7}]$ by $[-0.1, 0.3]$

FIGURE 5

EXAMPLE 3 Find $\lim_{x \to 0} \frac{\sin x}{x}$.

SOLUTION Again the function $f(x) = (\sin x)/x$ is not defined when $x = 0$. Using a calculator (and remembering that, if $x \in \mathbb{R}$, $\sin x$ means the sine of the angle whose *radian* measure is x), we construct the following table of values correct to eight decimal places. From the table and the graph in Figure 6 we guess that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

This guess is in fact correct, as will be proved in Section 3.4 using a geometric argument.

x	$\frac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

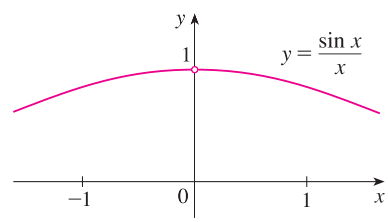


FIGURE 6

EXAMPLE 4 Find $\lim_{x \to 0} \sin \frac{\pi}{x}$.

SOLUTION Once again the function $f(x) = \sin(\pi/x)$ is undefined at 0. Evaluating the function for some small values of x, we get

$$f(1) = \sin \pi = 0 \qquad f\left(\tfrac{1}{2}\right) = \sin 2\pi = 0$$

$$f\left(\tfrac{1}{3}\right) = \sin 3\pi = 0 \qquad f\left(\tfrac{1}{4}\right) = \sin 4\pi = 0$$

$$f(0.1) = \sin 10\pi = 0 \qquad f(0.01) = \sin 100\pi = 0$$

Similarly, $f(0.001) = f(0.0001) = 0$. On the basis of this information we might be tempted to guess that

$$\lim_{x \to 0} \sin \frac{\pi}{x} = 0$$

but this time our guess is wrong. Note that although $f(1/n) = \sin n\pi = 0$ for any integer n, it is also true that $f(x) = 1$ for infinitely many values of x that approach 0. [In fact, $\sin(\pi/x) = 1$ when

$$\frac{\pi}{x} = \frac{\pi}{2} + 2n\pi$$

and, solving for x, we get $x = 2/(4n + 1)$.] The graph of f is given in Figure 7.

▲ **Computer Algebra Systems**
Computer algebra systems (CAS) have commands that compute limits. In order to avoid the types of pitfalls demonstrated in Examples 2, 4, and 5, they don't find limits by numerical experimentation. Instead, they use more sophisticated techniques such as computing infinite series. If you have access to a CAS, use the limit command to compute the limits in the examples of this section and to check your answers in the exercises of this chapter.

Listen to the sound of this function trying to approach a limit.

Resources / Module 2
/ Basics of Limits
/ Sound of a Limit that Does Not Exist

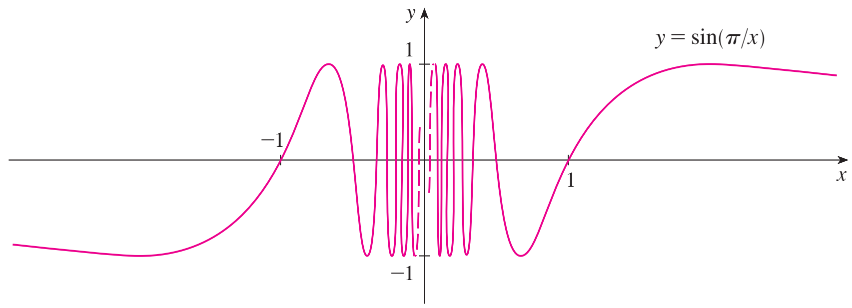


FIGURE 7

TEC Module 2.2 helps you explore limits at points where graphs exhibit unusual behavior.

The broken lines indicate that the values of $\sin(\pi/x)$ oscillate between 1 and -1 infinitely often as x approaches 0. (Use a graphing device to graph f and zoom in toward the origin several times. What do you observe?)

Since the values of $f(x)$ do not approach a fixed number as x approaches 0,

$$\lim_{x \to 0} \sin \frac{\pi}{x} \quad \text{does not exist}$$

EXAMPLE 5 Find $\displaystyle \lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10{,}000} \right)$.

SOLUTION As before, we construct a table of values.

x	$x^3 + \dfrac{\cos 5x}{10{,}000}$
1	1.000028
0.5	0.124920
0.1	0.001088
0.05	0.000222
0.01	0.000101

From the table it appears that

$$\lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10{,}000} \right) = 0$$

But if we persevere with smaller values of x, the second table suggests that

$$\lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10{,}000} \right) = 0.000100 = \frac{1}{10{,}000}$$

x	$x^3 + \dfrac{\cos 5x}{10{,}000}$
0.005	0.00010009
0.001	0.00010000

Later we will see that $\lim_{x \to 0} \cos 5x = 1$ and then it follows that the limit is 0.0001.

Examples 4 and 5 illustrate some of the pitfalls in guessing the value of a limit. It is easy to guess the wrong value if we use inappropriate values of x, but it is difficult to know when to stop calculating values. And, as the discussion after Example 2 shows, sometimes calculators and computers give the wrong values. Later, however, we will develop foolproof methods for calculating limits.

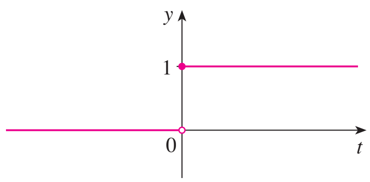


FIGURE 8

EXAMPLE 6 The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

[This function is named after the electrical engineer Oliver Heaviside (1850–1925) and can be used to describe an electric current that is switched on at time $t = 0$.] Its graph is shown in Figure 8.

As t approaches 0 from the left, $H(t)$ approaches 0. As t approaches 0 from the right, $H(t)$ approaches 1. There is no single number that $H(t)$ approaches as t approaches 0. Therefore, $\lim_{t \to 0} H(t)$ does not exist. ■

One-Sided Limits

We noticed in Example 6 that $H(t)$ approaches 0 as t approaches 0 from the left and $H(t)$ approaches 1 as t approaches 0 from the right. We indicate this situation symbolically by writing

$$\lim_{t \to 0^-} H(t) = 0 \qquad \text{and} \qquad \lim_{t \to 0^+} H(t) = 1$$

The symbol "$t \to 0^-$" indicates that we consider only values of t that are less than 0. Likewise, "$t \to 0^+$" indicates that we consider only values of t that are greater than 0.

2 Definition We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ as close to L as we like by taking x to be sufficiently close to a and x less than a.

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a. Similarly, if we require that x be greater than a, we get "the **right-hand limit of $f(x)$ as x approaches a** is equal to L" and we write

$$\lim_{x \to a^+} f(x) = L$$

Thus, the symbol "$x \to a^+$" means that we consider only $x > a$. These definitions are illustrated in Figure 9.

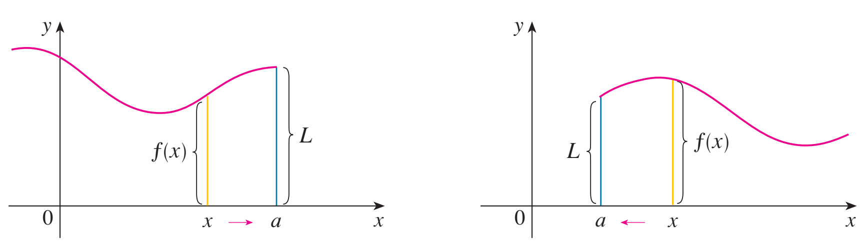


FIGURE 9 (a) $\lim_{x \to a^-} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$

By comparing Definition l with the definitions of one-sided limits, we see that the following is true.

3 $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$

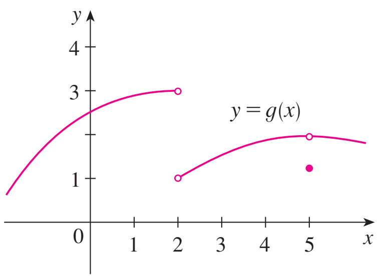


FIGURE 10

EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$ (c) $\lim_{x\to 2} g(x)$

(d) $\lim_{x\to 5^-} g(x)$ (e) $\lim_{x\to 5^+} g(x)$ (f) $\lim_{x\to 5} g(x)$

SOLUTION From the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore

$$\text{(a)}\ \lim_{x\to 2^-} g(x) = 3 \quad \text{and} \quad \text{(b)}\ \lim_{x\to 2^+} g(x) = 1$$

(c) Since the left and right limits are different, we conclude from (3) that $\lim_{x\to 2} g(x)$ does not exist.

The graph also shows that

$$\text{(d)}\ \lim_{x\to 5^-} g(x) = 2 \quad \text{and} \quad \text{(e)}\ \lim_{x\to 5^+} g(x) = 2$$

(f) This time the left and right limits are the same and so, by (3), we have

$$\lim_{x\to 5} g(x) = 2$$

Despite this fact, notice that $g(5) \neq 2$. ■

EXAMPLE 8 Find $\lim_{x\to 0} \dfrac{1}{x^2}$ if it exists.

x	$\dfrac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

SOLUTION As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large. (See the table at the left.) In fact, it appears from the graph of the function $f(x) = 1/x^2$ shown in Figure 11 that the values of $f(x)$ can be made arbitrarily large by taking x close enough to 0. Thus, the values of $f(x)$ do not approach a number, so $\lim_{x\to 0} (1/x^2)$ does not exist.

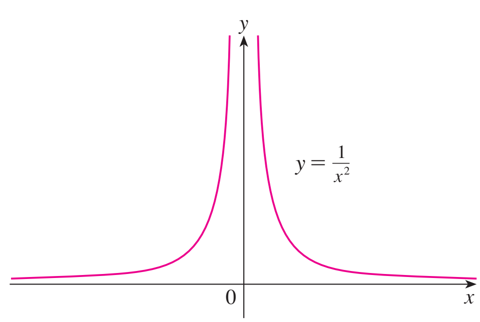


FIGURE 11 ■

At the beginning of this section we considered the function $f(x) = x^2 - x + 2$ and, based on numerical and graphical evidence, we saw that

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

According to Definition 1, this means that the values of $f(x)$ can be made as close to 4 as we like, provided that we take x sufficiently close to 2. In the following example we use graphical methods to determine just how close is sufficiently close.

EXAMPLE 9 If $f(x) = x^2 - x + 2$, how close to 2 does x have to be to ensure that $f(x)$ is within a distance 0.1 of the number 4?

SOLUTION If the distance from $f(x)$ to 4 is less than 0.1, then $f(x)$ lies between 3.9 and 4.1, so the requirement is that

$$3.9 < x^2 - x + 2 < 4.1$$

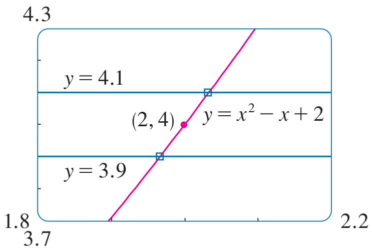


FIGURE 12

Thus, we need to determine the values of x such that the curve $y = x^2 - x + 2$ lies between the horizontal lines $y = 3.9$ and $y = 4.1$. We graph the curve and lines near the point $(2, 4)$ in Figure 12. With the cursor, we estimate that the x-coordinate of the point of intersection of the line $y = 3.9$ and the curve $y = x^2 - x + 2$ is about 1.966. Similarly, the curve intersects the line $y = 4.1$ when $x \approx 2.033$. So, rounding to be safe, we conclude that

$$3.9 < x^2 - x + 2 < 4.1 \quad \text{when} \quad 1.97 < x < 2.03$$

Therefore, $f(x)$ is within a distance 0.1 of 4 when x is within a distance 0.03 of 2. ■

The idea behind Example 9 can be used to formulate the precise definition of a limit that is discussed in Appendix D.

2.2 Exercises

1. Explain in your own words what is meant by the equation

$$\lim_{x \to 2} f(x) = 5$$

Is it possible for this statement to be true and yet $f(2) = 3$? Explain.

2. Explain what it means to say that

$$\lim_{x \to 1^-} f(x) = 3 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 7$$

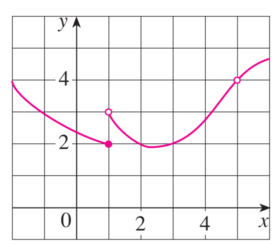
In this situation is it possible that $\lim_{x \to 1} f(x)$ exists? Explain.

3. Use the given graph of f to state the value of the given quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 1^-} f(x)$ (b) $\lim_{x \to 1^+} f(x)$

(c) $\lim_{x \to 1} f(x)$ (d) $\lim_{x \to 5} f(x)$

(e) $f(5)$

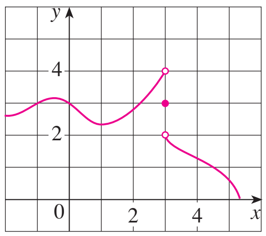


4. For the function f whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 0} f(x)$ (b) $\lim_{x \to 3^-} f(x)$

(c) $\lim_{x\to 3^+} f(x)$ (d) $\lim_{x\to 3} f(x)$

(e) $f(3)$

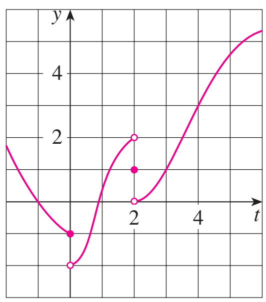


5. For the function g whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{t\to 0^-} g(t)$ (b) $\lim_{t\to 0^+} g(t)$ (c) $\lim_{t\to 0} g(t)$

(d) $\lim_{t\to 2^-} g(t)$ (e) $\lim_{t\to 2^+} g(t)$ (f) $\lim_{t\to 2} g(t)$

(g) $g(2)$ (h) $\lim_{t\to 4} g(t)$



6. Sketch the graph of the following function and use it to determine the values of a for which $\lim_{x\to a} f(x)$ exists:

$$f(x) = \begin{cases} 2 - x & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x < 1 \\ (x-1)^2 & \text{if } x \geqslant 1 \end{cases}$$

7. Use the graph of the function $f(x) = 1/(1 + e^{1/x})$ to state the value of each limit, if it exists. If it does not exist, explain why.

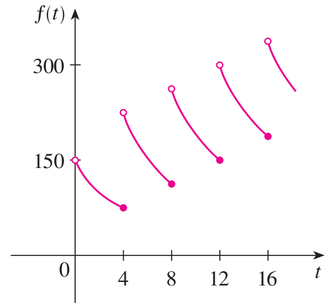
(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 0} f(x)$

8. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. (Later we will be able to compute the dosage and time interval to ensure that the concentration of the drug does not reach a harmful level.) Find

$$\lim_{t\to 12^-} f(t) \quad \text{and} \quad \lim_{t\to 12^+} f(t)$$

and explain the significance of these one-sided limits.



9–10 ■ Sketch the graph of an example of a function f that satisfies all of the given conditions.

9. $\lim_{x\to 3^+} f(x) = 4$, $\lim_{x\to 3^-} f(x) = 2$, $\lim_{x\to -2} f(x) = 2$,
$f(3) = 3$, $f(-2) = 1$

10. $\lim_{x\to 0^-} f(x) = 1$, $\lim_{x\to 0^+} f(x) = -1$, $\lim_{x\to 2^-} f(x) = 0$
$\lim_{x\to 2^+} f(x) = 1$, $f(2) = 1$, $f(0)$ is undefined

11–14 ■ Evaluate the function at the given numbers (correct to six decimal places). Use the results to guess the value of the limit, or explain why it does not exist.

11. $g(x) = \dfrac{x-1}{x^3-1}$;
$x = 0.2, 0.4, 0.6, 0.8, 0.9, 0.99, 1.8, 1.6, 1.4, 1.2, 1.1, 1.01$;
$\lim_{x\to 1} \dfrac{x-1}{x^3-1}$

12. $F(t) = \dfrac{\sqrt[3]{t}-1}{\sqrt{t}-1}$;
$t = 1.5, 1.2, 1.1, 1.01, 1.001$;
$\lim_{t\to 1} \dfrac{\sqrt[3]{t}-1}{\sqrt{t}-1}$

13. $f(x) = \dfrac{e^x - 1 - x}{x^2}$;
$x = \pm 1, \pm 0.5, \pm 0.1, \pm 0.05, \pm 0.01$;
$\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$

14. $g(x) = x \ln(x + x^2)$;
$x = 1, 0.5, 0.1, 0.05, 0.01, 0.005, 0.001$;
$\lim_{x\to 0^+} x \ln(x + x^2)$

15. (a) By graphing the function $f(x) = (\tan 4x)/x$ and zooming in toward the point where the graph crosses the y-axis, estimate the value of $\lim_{x\to 0} f(x)$.
(b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

16. (a) Estimate the value of

$$\lim_{x\to 0} \frac{6^x - 2^x}{x}$$

by graphing the function $y = (6^x - 2^x)/x$. State your answer correct to two decimal places.

(b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

17. (a) Estimate the value of the limit $\lim_{x\to 0} (1 + x)^{1/x}$ to five decimal places. Does this number look familiar?

(b) Illustrate part (a) by graphing the function $y = (1 + x)^{1/x}$.

18. The slope of the tangent line to the graph of the exponential function $y = 2^x$ at the point $(0, 1)$ is $\lim_{x\to 0}(2^x - 1)/x$. Estimate the slope to three decimal places.

19. (a) Evaluate the function $f(x) = x^2 - (2^x/1000)$ for $x = 1$, 0.8, 0.6, 0.4, 0.2, 0.1, and 0.05, and guess the value of

$$\lim_{x\to 0}\left(x^2 - \frac{2^x}{1000}\right)$$

(b) Evaluate $f(x)$ for $x = 0.04$, 0.02, 0.01, 0.005, 0.003, and 0.001. Guess again.

20. (a) Evaluate $h(x) = (\tan x - x)/x^3$ for $x = 1$, 0.5, 0.1, 0.05, 0.01, and 0.005.

(b) Guess the value of $\lim_{x\to 0} \dfrac{\tan x - x}{x^3}$.

(c) Evaluate $h(x)$ for successively smaller values of x until you finally reach 0 values for $h(x)$. Are you still confident that your guess in part (b) is correct? Explain why you eventually obtained 0 values. (In Section 4.5 a method for evaluating the limit will be explained.)

(d) Graph the function h in the viewing rectangle $[-1, 1]$ by $[0, 1]$. Then zoom in toward the point where the graph crosses the y-axis to estimate the limit of $h(x)$ as x approaches 0. Continue to zoom in until you observe distortions in the graph of h. Compare with the results of part (c).

21. Use a graph to determine how close to 0 we have to take x to ensure that e^x is within a distance 0.2 of the number 1. What if we insist that e^x be within 0.1 of 1?

22. (a) Use numerical and graphical evidence to guess the value of the limit

$$\lim_{x\to 1} \frac{x^3 - 1}{\sqrt{x} - 1}$$

(b) How close to 1 does x have to be to ensure that the function in part (a) is within a distance 0.5 of its limit?

2.3 Calculating Limits Using the Limit Laws

In Section 2.2 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answer. In this section we use the following properties of limits, called the *Limit Laws,* to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$

4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0$

These five laws can be stated verbally as follows:

Sum Law

1. The limit of a sum is the sum of the limits.

Difference Law

2. The limit of a difference is the difference of the limits.

Constant Multiple Law

3. The limit of a constant times a function is the constant times the limit of the function.

Product Law

4. The limit of a product is the product of the limits.

Quotient Law

5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

It is easy to believe that these properties are true. For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$. This gives us an intuitive basis for believing that Law 1 is true. All of these laws can be proved using the precise definition of a limit. In Appendix E we give the proof of Law 1.

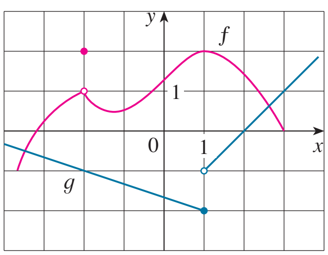


FIGURE 1

EXAMPLE 1 Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \dfrac{f(x)}{g(x)}$

SOLUTION

(a) From the graphs of f and g we see that

$$\lim_{x \to -2} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to -2} g(x) = -1$$

Therefore, we have

$$\begin{aligned} \lim_{x \to -2} [f(x) + 5g(x)] &= \lim_{x \to -2} f(x) + \lim_{x \to -2} [5g(x)] && \text{(by Law 1)} \\ &= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x) && \text{(by Law 3)} \\ &= 1 + 5(-1) = -4 \end{aligned}$$

(b) We see that $\lim_{x \to 1} f(x) = 2$. But $\lim_{x \to 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x \to 1^-} g(x) = -2 \qquad \lim_{x \to 1^+} g(x) = -1$$

So we can't use Law 4. The given limit does not exist since the left limit is not equal to the right limit.

(c) The graphs show that

$$\lim_{x \to 2} f(x) \approx 1.4 \qquad \text{and} \qquad \lim_{x \to 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5. The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number. ■

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

In applying these six limit laws we need to use two special limits:

7. $\lim_{x \to a} c = c$ **8.** $\lim_{x \to a} x = a$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$).

If we now put $f(x) = x$ in Law 6 and use Law 8, we get another useful special limit.

9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

A similar limit holds for roots as follows.

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

More generally, we have the following law.

Root Law

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

$\left[\text{If } n \text{ is even, we assume that } \lim_{x \to a} f(x) > 0.\right]$

Explore limits like these interactively.

Resources / Module 2 / The Essential Examples / Examples D and E

EXAMPLE 2 Evaluate the following limits and justify each step.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$ (b) $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

SOLUTION

(a)
$$\begin{aligned}
\lim_{x \to 5} (2x^2 - 3x + 4) &= \lim_{x \to 5} (2x^2) - \lim_{x \to 5} (3x) + \lim_{x \to 5} 4 && \text{(by Laws 2 and 1)} \\
&= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + \lim_{x \to 5} 4 && \text{(by 3)} \\
&= 2(5^2) - 3(5) + 4 && \text{(by 9, 8, and 7)} \\
&= 39
\end{aligned}$$

(b) We start by using Law 5, but its use is fully justified only at the final stage when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x \to -2} (x^3 + 2x^2 - 1)}{\lim_{x \to -2} (5 - 3x)} \qquad \text{(by Law 5)}$$

$$= \frac{\lim_{x \to -2} x^3 + 2 \lim_{x \to -2} x^2 - \lim_{x \to -2} 1}{\lim_{x \to -2} 5 - 3 \lim_{x \to -2} x} \qquad \text{(by 1, 2, and 3)}$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} \qquad \text{(by 9, 8, and 7)}$$

$$= -\frac{1}{11}$$

NOTE • If we let $f(x) = 2x^2 - 3x + 4$, then $f(5) = 39$. In other words, we would have gotten the correct answer in Example 2(a) by substituting 5 for x. Similarly, direct substitution provides the correct answer in part (b). The functions in Example 2 are a polynomial and a rational function, respectively, and similar use of the Limit Laws proves that direct substitution always works for such functions (see Exercises 39 and 40). We state this fact as follows.

> **Direct Substitution Property** If f is a polynomial or a rational function and a is in the domain of f, then
>
> $$\lim_{x \to a} f(x) = f(a)$$

Functions with the Direct Substitution Property are called *continuous at a* and will be studied in Section 2.4. However, not all limits can be evaluated by direct substitution, as the following examples show.

EXAMPLE 3 Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

SOLUTION Let $f(x) = (x^2 - 1)/(x - 1)$. We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined. Nor can we apply the Quotient Law because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$

The numerator and denominator have a common factor of $x - 1$. When we take the limit as x approaches 1, we have $x \neq 1$ and so $x - 1 \neq 0$. Therefore, we can cancel the common factor and compute the limit as follows:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1)$$

$$= 1 + 1 = 2$$

▲ Newton and Limits

Isaac Newton was born on Christmas Day in 1642, the year of Galileo's death. When he entered Cambridge University in 1661 Newton didn't know much mathematics, but he learned quickly by reading Euclid and Descartes and by attending the lectures of Isaac Barrow. Cambridge was closed because of the plague in 1665 and 1666, and Newton returned home to reflect on what he had learned. Those two years were amazingly productive for at that time he made four of his major discoveries: (1) his representation of functions as sums of infinite series, including the binomial theorem; (2) his work on differential and integral calculus; (3) his laws of motion and law of universal gravitation; and (4) his prism experiments on the nature of light and color. Because of a fear of controversy and criticism, he was reluctant to publish his discoveries and it wasn't until 1687, at the urging of the astronomer Halley, that Newton published *Principia Mathematica*. In this work, the greatest scientific treatise ever written, Newton set forth his version of calculus and used it to investigate mechanics, fluid dynamics, and wave motion, and to explain the motion of planets and comets.

The beginnings of calculus are found in the calculations of areas and volumes by ancient Greek scholars such as Eudoxus and Archimedes. Although aspects of the idea of a limit are implicit in their "method of exhaustion," Eudoxus and Archimedes never explicitly formulated the concept of a limit. Likewise, mathematicians such as Cavalieri, Fermat, and Barrow, the immediate precursors of Newton in the development of calculus, did not actually use limits. It was Isaac Newton who was the first to talk explicitly about limits. He explained that the main idea behind limits is that quantities "approach nearer than by any given difference." Newton stated that the limit was the basic concept in calculus, but it was left to later mathematicians like Cauchy to clarify his ideas about limits.

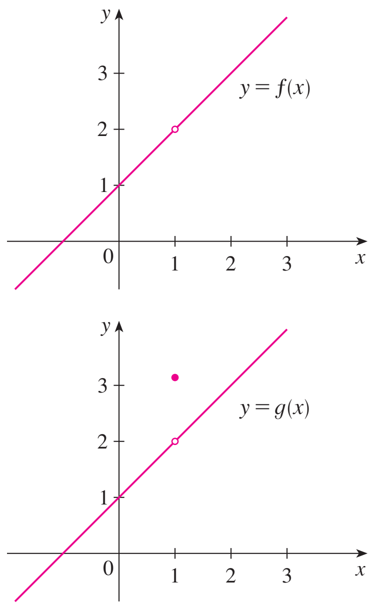


FIGURE 2
The graphs of the functions f (from Example 3) and g (from Example 4)

The limit in this example arose in Section 2.1 when we were trying to find the tangent to the parabola $y = x^2$ at the point $(1, 1)$. ■

EXAMPLE 4 Find $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$$

SOLUTION Here g is defined at $x = 1$ and $g(1) = \pi$, but the value of a limit as x approaches 1 does not depend on the value of the function at 1. Since $g(x) = x + 1$ for $x \neq 1$, we have

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} (x + 1) = 2$$ ■

Note that the values of the functions in Examples 3 and 4 are identical except when $x = 1$ (see Figure 2) and so they have the same limit as x approaches 1.

EXAMPLE 5 Evaluate $\lim_{h \to 0} \dfrac{(3 + h)^2 - 9}{h}$.

SOLUTION If we define

$$F(h) = \frac{(3 + h)^2 - 9}{h}$$

then, as in Example 3, we can't compute $\lim_{h \to 0} F(h)$ by letting $h = 0$ since $F(0)$ is undefined. But if we simplify $F(h)$ algebraically, we find that

$$F(h) = \frac{(9 + 6h + h^2) - 9}{h} = \frac{6h + h^2}{h} = 6 + h$$

(Recall that we consider only $h \neq 0$ when letting h approach 0.) Thus

$$\lim_{h \to 0} \frac{(3 + h)^2 - 9}{h} = \lim_{h \to 0} (6 + h) = 6$$ ■

Explore a limit like this one interactively.
Resources / Module 2
/ The Essential Examples
/ Example C

EXAMPLE 6 Find $\lim_{t \to 0} \dfrac{\sqrt{t^2 + 9} - 3}{t^2}$.

SOLUTION We can't apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned} \lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} &= \lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3} \\ &= \lim_{t \to 0} \frac{(t^2 + 9) - 9}{t^2\left(\sqrt{t^2 + 9} + 3\right)} = \lim_{t \to 0} \frac{t^2}{t^2\left(\sqrt{t^2 + 9} + 3\right)} \\ &= \lim_{t \to 0} \frac{1}{\sqrt{t^2 + 9} + 3} = \frac{1}{\sqrt{\lim_{t \to 0} (t^2 + 9)} + 3} = \frac{1}{3 + 3} = \frac{1}{6} \end{aligned}$$

This calculation confirms the guess that we made in Example 2 in Section 2.2. ■

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 2.2. It says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

When computing one-sided limits we use the fact that the Limit Laws also hold for one-sided limits.

EXAMPLE 7 Show that $\lim_{x \to 0} |x| = 0$.

SOLUTION Recall that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

▲ The result of Example 7 looks plausible from Figure 3.

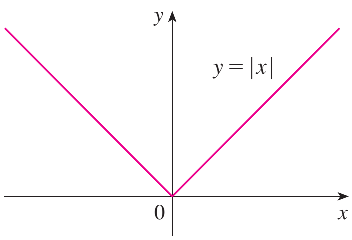


FIGURE 3

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ and so

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Therefore, by Theorem 1,

$$\lim_{x \to 0} |x| = 0$$

EXAMPLE 8 Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

SOLUTION

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = -1$$

Since the right- and left-hand limits are different, it follows from Theorem 1 that $\lim_{x \to 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ is shown in Figure 4 and supports the limits that we found.

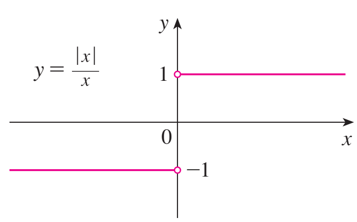


FIGURE 4

EXAMPLE 9 The **greatest integer function** is defined by $[\![x]\!]$ = the largest integer that is less than or equal to x. (For instance, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}]\!] = 1$, $[\![-\frac{1}{2}]\!] = -1$.) Show that $\lim_{x \to 3} [\![x]\!]$ does not exist.

▲ Other notations for $[\![x]\!]$ are $[x]$ and $\lfloor x \rfloor$.

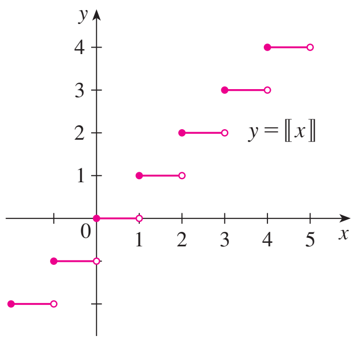


FIGURE 5
Greatest integer function

SOLUTION The graph of the greatest integer function is shown in Figure 5. Since $[\![x]\!] = 3$ for $3 \leqslant x < 4$, we have

$$\lim_{x \to 3^+} [\![x]\!] = \lim_{x \to 3^+} 3 = 3$$

Since $[\![x]\!] = 2$ for $2 \leqslant x < 3$, we have

$$\lim_{x \to 3^-} [\![x]\!] = \lim_{x \to 3^-} 2 = 2$$

Because these one-sided limits are not equal, $\lim_{x \to 3} [\![x]\!]$ does not exist by Theorem 1.

The next two theorems give two additional properties of limits. Both can be proved using the precise definition of a limit in Appendix D.

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $$\lim_{x \to a} g(x) = L$$

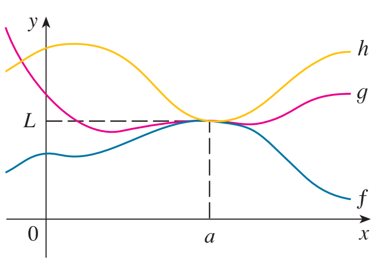


FIGURE 6

The Squeeze Theorem, sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 6. It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

EXAMPLE 10 Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

SOLUTION First note that we *cannot* use

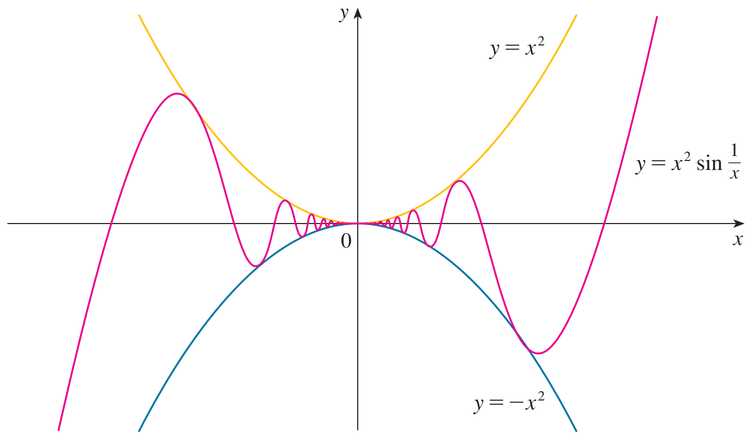
$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}$$

because $\lim_{x \to 0} \sin(1/x)$ does not exist (see Example 4 in Section 2.2). However, since

$$-1 \leqslant \sin \frac{1}{x} \leqslant 1$$

we have, as illustrated by Figure 7,

$$-x^2 \leqslant x^2 \sin \frac{1}{x} \leqslant x^2$$



Watch an animation of a similar limit.
Resources / Module 2
/ Basics of Limits
/ Sound of a Limit that Exists

FIGURE 7

We know that

$$\lim_{x \to 0} x^2 = 0 \quad \text{and} \quad \lim_{x \to 0} -x^2 = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

2.3 Exercises

1. Given that

$$\lim_{x \to a} f(x) = -3 \qquad \lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} h(x) = 8$$

find the limits that exist. If the limit does not exist, explain why.

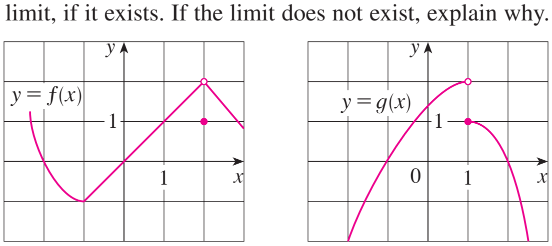
(a) $\lim_{x \to a} [f(x) + h(x)]$ (b) $\lim_{x \to a} [f(x)]^2$

(c) $\lim_{x \to a} \sqrt[3]{h(x)}$ (d) $\lim_{x \to a} \dfrac{1}{f(x)}$

(e) $\lim_{x \to a} \dfrac{f(x)}{h(x)}$ (f) $\lim_{x \to a} \dfrac{g(x)}{f(x)}$

(g) $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ (h) $\lim_{x \to a} \dfrac{2f(x)}{h(x) - f(x)}$

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.



(a) $\lim_{x \to 2} [f(x) + g(x)]$ (b) $\lim_{x \to 1} [f(x) + g(x)]$

(c) $\lim_{x \to 0} [f(x)g(x)]$ (d) $\lim_{x \to -1} \dfrac{f(x)}{g(x)}$

(e) $\lim_{x \to 2} x^3 f(x)$ (f) $\lim_{x \to 1} \sqrt{3 + f(x)}$

3–7 ■ Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x \to 4} (5x^2 - 2x + 3)$

4. $\lim_{x \to 2} \dfrac{2x^2 + 1}{x^2 + 6x - 4}$

5. $\lim_{t \to -2} (t + 1)^9 (t^2 - 1)$

6. $\lim_{u \to -2} \sqrt{u^4 + 3u + 6}$

7. $\lim_{x \to 1} \left(\dfrac{1 + 3x}{1 + 4x^2 + 3x^4} \right)^3$

8. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} (x + 3)$$

is correct.

9–20 ■ Evaluate the limit, if it exists.

9. $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x - 2}$

10. $\lim_{x \to -4} \dfrac{x^2 + 5x + 4}{x^2 + 3x - 4}$

11. $\lim_{x \to 2} \dfrac{x^2 - x + 6}{x - 2}$

12. $\lim_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1}$

13. $\lim_{t \to -3} \dfrac{t^2 - 9}{2t^2 + 7t + 3}$

14. $\lim_{h \to 0} \dfrac{\sqrt{1 + h} - 1}{h}$

15. $\lim_{h \to 0} \dfrac{(2 + h)^3 - 8}{h}$

16. $\lim_{x \to 2} \dfrac{x^4 - 16}{x - 2}$

17. $\lim_{x \to 7} \dfrac{\sqrt{x + 2} - 3}{x - 7}$

18. $\lim_{h \to 0} \dfrac{(3 + h)^{-1} - 3^{-1}}{h}$

19. $\lim_{x \to -4} \dfrac{\frac{1}{4} + \frac{1}{x}}{4 + x}$

20. $\lim_{t \to 0} \left[\dfrac{1}{t} - \dfrac{1}{t^2 + t} \right]$

21. (a) Estimate the value of

$$\lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/(\sqrt{1 + 3x} - 1)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

22. (a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

to estimate the value of $\lim_{x \to 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

23. Use the Squeeze Theorem to show that $\lim_{x \to 0} x^2 \cos 20\pi x = 0$. Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$ on the same screen.

24. Use the Squeeze Theorem to show that

$$\lim_{x \to 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x} = 0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

25. If $1 \leqslant f(x) \leqslant x^2 + 2x + 2$ for all x, find $\lim_{x \to -1} f(x)$.

26. If $3x \leqslant f(x) \leqslant x^3 + 2$ for $0 \leqslant x \leqslant 2$, evaluate $\lim_{x \to 1} f(x)$.

27. Prove that $\lim_{x \to 0} x^4 \cos \dfrac{2}{x} = 0$.

28. Prove that $\lim_{x \to 0^+} \sqrt{x}\, e^{\sin(\pi/x)} = 0$.

29–32 ■ Find the limit, if it exists. If the limit does not exist, explain why.

29. $\lim_{x \to -4} |x + 4|$

30. $\lim_{x \to 2} \dfrac{|x - 2|}{x - 2}$

31. $\lim_{x \to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|} \right)$

32. $\lim_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{|x|} \right)$

33. Let

$$g(x) = \begin{cases} -x & \text{if } x \leqslant -1 \\ 1 - x^2 & \text{if } -1 < x < 1 \\ x - 1 & \text{if } x > 1 \end{cases}$$

(a) Evaluate each of the following limits, if it exists.

(i) $\lim_{x \to 1^+} g(x)$ (ii) $\lim_{x \to 1} g(x)$ (iii) $\lim_{x \to 0} g(x)$

(iv) $\lim_{x \to -1^-} g(x)$ (v) $\lim_{x \to -1^+} g(x)$ (vi) $\lim_{x \to -1} g(x)$

(b) Sketch the graph of g.

34. Let $F(x) = \dfrac{x^2 - 1}{|x - 1|}$.

(a) Find

(i) $\lim_{x \to 1^+} F(x)$ (ii) $\lim_{x \to 1^-} F(x)$

(b) Does $\lim_{x \to 1} F(x)$ exist?

(c) Sketch the graph of F.

35. (a) If the symbol $[\![\ ]\!]$ denotes the greatest integer function defined in Example 9, evaluate

(i) $\lim_{x \to -2^+} [\![x]\!]$ (ii) $\lim_{x \to -2} [\![x]\!]$ (iii) $\lim_{x \to -2.4} [\![x]\!]$

(b) If n is an integer, evaluate

(i) $\lim_{x \to n^-} [\![x]\!]$ (ii) $\lim_{x \to n^+} [\![x]\!]$

(c) For what values of a does $\lim_{x \to a} [\![x]\!]$ exist?

36. Let $f(x) = x - [\![x]\!]$.

(a) Sketch the graph of f.

(b) If n is an integer, evaluate

(i) $\lim_{x \to n^-} f(x)$ (ii) $\lim_{x \to n^+} f(x)$

(c) For what values of a does $\lim_{x \to a} f(x)$ exist?

37. If $f(x) = [\![x]\!] + [\![-x]\!]$, show that $\lim_{x \to 2} f(x)$ exists but is not equal to $f(2)$.

38. In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

39. If p is a polynomial, show that $\lim_{x \to a} p(x) = p(a)$.

40. If r is a rational function, use Exercise 39 to show that $\lim_{x \to a} r(x) = r(a)$ for every number a in the domain of r.

41. Show by means of an example that $\lim_{x\to a}[f(x)+g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

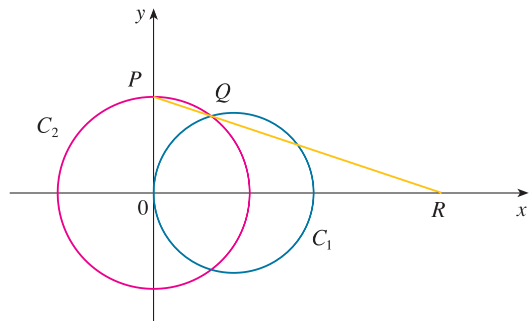
42. Show by means of an example that $\lim_{x\to a}[f(x)g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

43. Is there a number a such that

$$\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}$$

exists? If so, find the value of a and the value of the limit.

44. The figure shows a fixed circle C_1 with equation $(x-1)^2+y^2=1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?



Continuity

Explore continuous functions interactively.

Resources / Module 2 / Continuity / Start of Continuity

We noticed in Section 2.3 that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called *continuous at a*. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

1 Definition A function f is **continuous at a number *a*** if

$$\lim_{x\to a} f(x) = f(a)$$

If f is not continuous at a, we say f is **discontinuous at *a***, or f has a **discontinuity** at a. Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

▲ As illustrated in Figure 1, if f is continuous, then the points $(x, f(x))$ on the graph of f approach the point $(a, f(a))$ on the graph. So there is no gap in the curve.

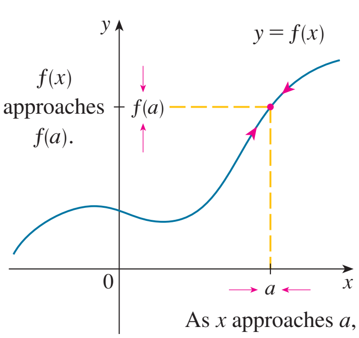


FIGURE 1

The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a. Thus, a continuous function f has the property that a small change in x produces only a small change in $f(x)$. In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents. [See Example 6 in Section 2.2, where the Heaviside function is discontinuous at 0 because $\lim_{t\to 0} H(t)$ does not exist.]

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper.

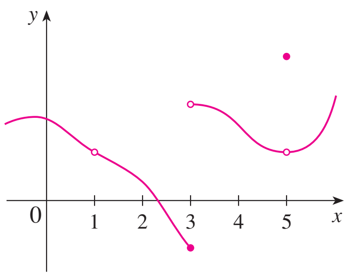


FIGURE 2

EXAMPLE 1 Figure 2 shows the graph of a function f. At which numbers is f discontinuous? Why?

SOLUTION It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The official reason that f is discontinuous at 1 is that $f(1)$ is not defined.

The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x\to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3.

What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x\to 5} f(x)$ exists (because the left and right limits are the same). But

$$\lim_{x\to 5} f(x) \neq f(5)$$

So f is discontinuous at 5. ■

Now let's see how to detect discontinuities when a function is defined by a formula.

Resources / Module 2
/ Continuity
/ Problems and Tests

EXAMPLE 2 Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ (b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ (d) $f(x) = [\![x]\!]$

SOLUTION

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2.

(b) Here $f(0) = 1$ is defined but

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2}$$

does not exist. (See Example 8 in Section 2.2.) So f is discontinuous at 0.

(c) Here $f(2) = 1$ is defined and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x\to 2} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x\to 2} (x + 1) = 3$$

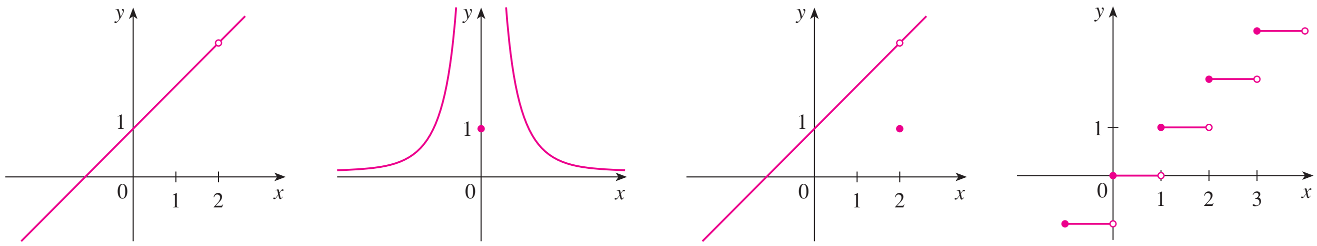
exists. But

$$\lim_{x\to 2} f(x) \neq f(2)$$

so f is not continuous at 2.

(d) The greatest integer function $f(x) = [\![x]\!]$ has discontinuities at all of the integers because $\lim_{x\to n} [\![x]\!]$ does not exist if n is an integer. (See Example 9 and Exercise 35 in Section 2.3.) ■

Figure 3 shows the graphs of the functions in Example 2. In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.] The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.



(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ (b) $f(x) = \begin{cases} 1/x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ (c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ (d) $f(x) = [\![x]\!]$

FIGURE 3
Graphs of the functions in Example 2

2 Definition A function f is **continuous from the right at a number a** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \to a^-} f(x) = f(a)$$

EXAMPLE 3 At each integer n, the function $f(x) = [\![x]\!]$ shown in Figure 3(d) is continuous from the right but discontinuous from the left because

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [\![x]\!] = n = f(n)$$

but
$$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [\![x]\!] = n - 1 \neq f(n)$$

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left.*)

EXAMPLE 4 Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

SOLUTION If $-1 < a < 1$, then using the Limit Laws, we have

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right) \\
&= 1 - \lim_{x \to a} \sqrt{1 - x^2} && \text{(by Laws 2 and 7)} \\
&= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} && \text{(by 11)} \\
&= 1 - \sqrt{1 - a^2} && \text{(by 2, 7, and 9)} \\
&= f(a)
\end{aligned}$$

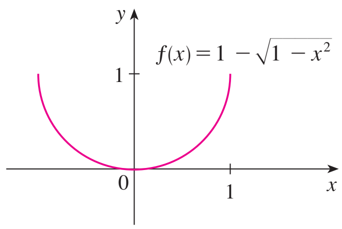


FIGURE 4

Thus, by Definition 1, f is continuous at a if $-1 < a < 1$. Similar calculations show that

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) \qquad \text{and} \qquad \lim_{x \to 1^-} f(x) = 1 = f(1)$$

so f is continuous from the right at -1 and continuous from the left at 1. Therefore, according to Definition 3, f is continuous on $[-1, 1]$.

The graph of f is sketched in Figure 4. It is the lower half of the circle

$$x^2 + (y - 1)^2 = 1$$

Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function as we did in Example 4, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones.

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

Proof Each of the five parts of this theorem follows from the corresponding Limit Law in Section 2.3. For instance, we give the proof of part 1. Since f and g are continuous at a, we have

$$\lim_{x \to a} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to a} g(x) = g(a)$$

Therefore

$$\begin{aligned} \lim_{x \to a} (f + g)(x) &= \lim_{x \to a} [f(x) + g(x)] \\ &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \qquad \text{(by Law 1)} \\ &= f(a) + g(a) \\ &= (f + g)(a) \end{aligned}$$

This shows that $f + g$ is continuous at a.

It follows from Theorem 4 and Definition 3 that if f and g are continuous on an interval, then so are the functions $f + g$, $f - g$, cf, fg, and (if g is never 0) f/g. The following theorem was stated in Section 2.3 as the Direct Substitution Property.

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Proof

(a) A polynomial is a function of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \ldots, c_n$ are constants. We know that

$$\lim_{x \to a} c_0 = c_0 \qquad \text{(by Law 7)}$$

and
$$\lim_{x \to a} x^m = a^m \qquad m = 1, 2, \ldots, n \qquad \text{(by 9)}$$

This equation is precisely the statement that the function $f(x) = x^m$ is a continuous function. Thus, by part 3 of Theorem 4, the function $g(x) = cx^m$ is continuous. Since P is a sum of functions of this form and a constant function, it follows from part 1 of Theorem 4 that P is continuous.

(b) A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain of f is $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$. We know from part (a) that P and Q are continuous everywhere. Thus, by part 5 of Theorem 4, f is continuous at every number in D. ■

As an illustration of Theorem 5, observe that the volume of a sphere varies continuously with its radius because the formula $V(r) = \frac{4}{3}\pi r^3$ shows that V is a polynomial function of r. Likewise, if a ball is thrown vertically into the air with a velocity of 50 ft/s, then the height of the ball in feet after t seconds is given by the formula $h = 50t - 16t^2$. Again this is a polynomial function, so the height is a continuous function of the elapsed time.

Knowledge of which functions are continuous enables us to evaluate some limits very quickly, as the following example shows. Compare it with Example 2(b) in Section 2.3.

EXAMPLE 5 Find $\displaystyle\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$.

SOLUTION The function

$$f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

is rational, so by Theorem 5 it is continuous on its domain, which is $\left\{x \mid x \neq \frac{5}{3}\right\}$. Therefore

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \lim_{x \to -2} f(x) = f(-2)$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}$$ ■

It turns out that most of the familiar functions are continuous at every number in their domains. For instance, Limit Law 10 (page 112) is exactly the statement that root functions are continuous.

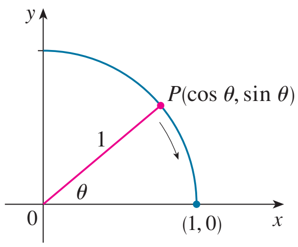


FIGURE 5

▲ Another way to establish the limits in (6) is to use the Squeeze Theorem with the inequality $\sin\theta < \theta$ (for $\theta > 0$), which is proved in Section 3.4.

From the appearance of the graphs of the sine and cosine functions (Figure 18 in Section 1.2), we would certainly guess that they are continuous. We know from the definitions of $\sin\theta$ and $\cos\theta$ that the coordinates of the point P in Figure 5 are $(\cos\theta, \sin\theta)$. As $\theta \to 0$, we see that P approaches the point $(1, 0)$ and so $\cos\theta \to 1$ and $\sin\theta \to 0$. Thus

6 $$\lim_{\theta\to 0} \cos\theta = 1 \qquad \lim_{\theta\to 0} \sin\theta = 0$$

Since $\cos 0 = 1$ and $\sin 0 = 0$, the equations in (6) assert that the cosine and sine functions are continuous at 0. The addition formulas for cosine and sine can then be used to deduce that these functions are continuous everywhere (see Exercises 43 and 44).

It follows from part 5 of Theorem 4 that

$$\tan x = \frac{\sin x}{\cos x}$$

is continuous except where $\cos x = 0$. This happens when x is an odd integer multiple of $\pi/2$, so $y = \tan x$ has infinite discontinuities when $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$, and so on (see Figure 6).

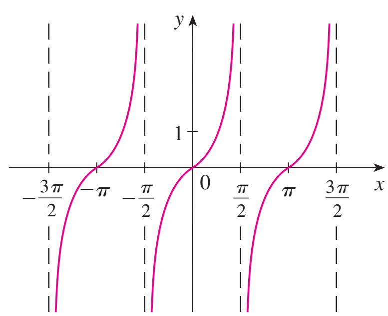


FIGURE 6
$y = \tan x$

▲ The inverse trigonometric functions are reviewed in Appendix C.

The inverse function of any continuous function is also continuous. (The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$. So if the graph of f has no break in it, neither does the graph of f^{-1}.) Thus, the inverse trigonometric functions are continuous.

In Section 1.5 we defined the exponential function $y = a^x$ so as to fill in the holes in the graph of $y = a^x$ where x is rational. In other words, the very definition of $y = a^x$ makes it a continuous function on $\mathbb{R}$. Therefore, its inverse function $y = \log_a x$ is continuous on $(0, \infty)$.

7 Theorem The following types of functions are continuous at every number in their domains:

polynomials	rational functions	root functions
trigonometric functions	inverse trigonometric functions	
exponential functions	logarithmic functions	

EXAMPLE 6 Where is the function $f(x) = \dfrac{\ln x + \tan^{-1}x}{x^2 - 1}$ continuous?

SOLUTION We know from Theorem 7 that the function $y = \ln x$ is continuous for $x > 0$ and $y = \tan^{-1}x$ is continuous on $\mathbb{R}$. Thus, by part 1 of Theorem 4, $y = \ln x + \tan^{-1}x$ is continuous on $(0, \infty)$. The denominator, $y = x^2 - 1$, is a polynomial, so it is continuous everywhere. Therefore, by part 5 of Theorem 4, f is continuous at all positive numbers x except where $x^2 - 1 = 0$. So f is continuous on the intervals $(0, 1)$ and $(1, \infty)$. ■

EXAMPLE 7 Evaluate $\displaystyle\lim_{x \to \pi} \frac{\sin x}{2 + \cos x}$.

SOLUTION Theorem 7 tells us that $y = \sin x$ is continuous. The function in the denominator, $y = 2 + \cos x$, is the sum of two continuous functions and is therefore continuous. Notice that this function is never 0 because $\cos x \geqslant -1$ for all x and so $2 + \cos x > 0$ everywhere. Thus, the ratio

$$f(x) = \frac{\sin x}{2 + \cos x}$$

is continuous everywhere. Hence, by definition of a continuous function,

$$\lim_{x \to \pi} \frac{\sin x}{2 + \cos x} = \lim_{x \to \pi} f(x) = f(\pi) = \frac{\sin \pi}{2 + \cos \pi} = \frac{0}{2 - 1} = 0$$ ■

Another way of combining continuous functions f and g to get a new continuous function is to form the composite function $f \circ g$. This fact is a consequence of the following theorem.

▲ This theorem says that a limit symbol can be moved through a function symbol if the function is continuous and the limit exists. In other words, the order of these two symbols can be reversed.

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Intuitively, this theorem is reasonable because if x is close to a, then $g(x)$ is close to b, and since f is continuous at b, if $g(x)$ is close to b, then $f(g(x))$ is close to $f(b)$.

EXAMPLE 8 Evaluate $\displaystyle\lim_{x \to 1} \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right)$.

SOLUTION Because arcsin is a continuous function, we can apply Theorem 8:

$$\begin{aligned}
\lim_{x \to 1} \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right) &= \arcsin\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}\right) \\
&= \arcsin\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})}\right) \\
&= \arcsin\left(\lim_{x \to 1} \frac{1}{1 + \sqrt{x}}\right) \\
&= \arcsin \frac{1}{2} = \frac{\pi}{6}
\end{aligned}$$ ■

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

This theorem is often expressed informally by saying "a continuous function of a continuous function is a continuous function."

Proof Since g is continuous at a, we have

$$\lim_{x \to a} g(x) = g(a)$$

Since f is continuous at $b = g(a)$, we can apply Theorem 8 to obtain

$$\lim_{x \to a} f(g(x)) = f(g(a))$$

which is precisely the statement that the function $h(x) = f(g(x))$ is continuous at a; that is, $f \circ g$ is continuous at a. ■

EXAMPLE 9 Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$ (b) $F(x) = \ln(1 + \cos x)$

SOLUTION

(a) We have $h(x) = f(g(x))$, where

$$g(x) = x^2 \qquad \text{and} \qquad f(x) = \sin x$$

Now g is continuous on $\mathbb{R}$ since it is a polynomial, and f is also continuous everywhere. Thus, $h = f \circ g$ is continuous on $\mathbb{R}$ by Theorem 9.

(b) We know from Theorem 7 that $f(x) = \ln x$ is continuous and $g(x) = 1 + \cos x$ is continuous (because both $y = 1$ and $y = \cos x$ are continuous). Therefore, by Theorem 9, $F(x) = f(g(x))$ is continuous wherever it is defined. Now $\ln(1 + \cos x)$ is defined when $1 + \cos x > 0$. So it is undefined when $\cos x = -1$, and this happens when $x = \pm\pi, \pm 3\pi, \ldots$. Thus, F has discontinuities when x is an odd multiple of π and is continuous on the intervals between these values (see Figure 7). ■

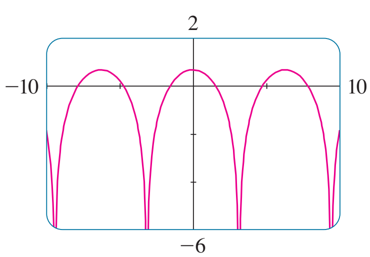


FIGURE 7

An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 8. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].

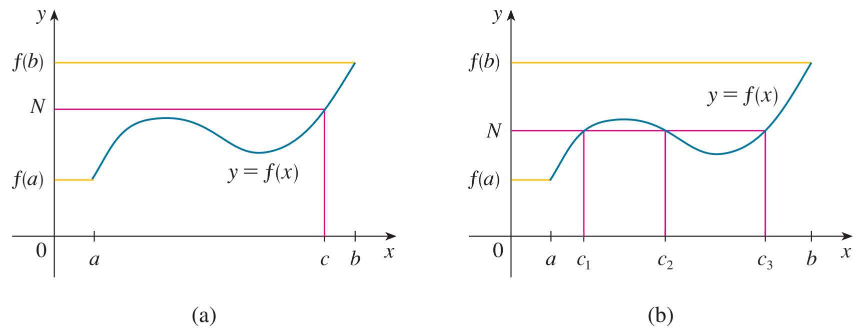


FIGURE 8

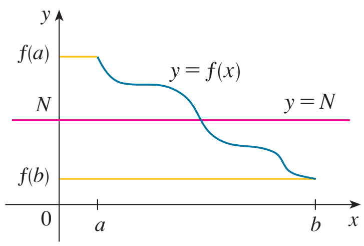


FIGURE 9

If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true. In geometric terms it says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ as in Figure 9, then the graph of f can't jump over the line. It must intersect $y = N$ somewhere.

It is important that the function f in Theorem 10 be continuous. The Intermediate Value Theorem is not true in general for discontinuous functions (see Exercise 32).

One use of the Intermediate Value Theorem is in locating roots of equations as in the following example.

EXAMPLE 10 Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

SOLUTION Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that $f(c) = 0$. Therefore, we take $a = 1$, $b = 2$, and $N = 0$ in Theorem 10. We have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

and

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Thus $f(1) < 0 < f(2)$, that is, $N = 0$ is a number between $f(1)$ and $f(2)$. Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that $f(c) = 0$. In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval $(1, 2)$.

In fact, we can locate a root more precisely by using the Intermediate Value Theorem again. Since

$$f(1.2) = -0.128 < 0 \qquad \text{and} \qquad f(1.3) = 0.548 > 0$$

a root must lie between 1.2 and 1.3. A calculator gives, by trial and error,

$$f(1.22) = -0.007008 < 0 \qquad \text{and} \qquad f(1.23) = 0.056068 > 0$$

so a root lies in the interval $(1.22, 1.23)$. ■

We can use a graphing calculator or computer to illustrate the use of the Intermediate Value Theorem in Example 10. Figure 10 shows the graph of f in the viewing rectangle $[-1, 3]$ by $[-3, 3]$ and you can see the graph crossing the x-axis between 1 and 2. Figure 11 shows the result of zooming in to the viewing rectangle $[1.2, 1.3]$ by $[-0.2, 0.2]$.

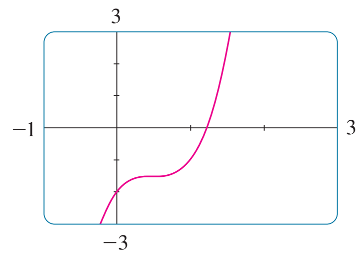


FIGURE 10

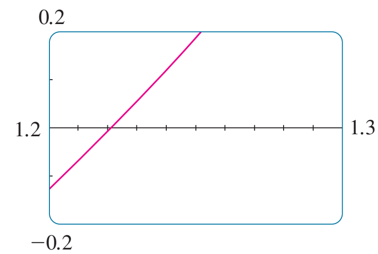


FIGURE 11

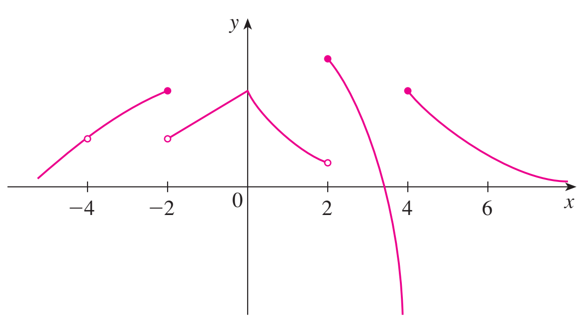
In fact, the Intermediate Value Theorem plays a role in the very way these graphing devices work. A computer calculates a finite number of points on the graph and turns on the pixels that contain these calculated points. It assumes that the function is continuous and takes on all the intermediate values between two consecutive points. The computer therefore connects the pixels by turning on the intermediate pixels.

2.4 Exercises

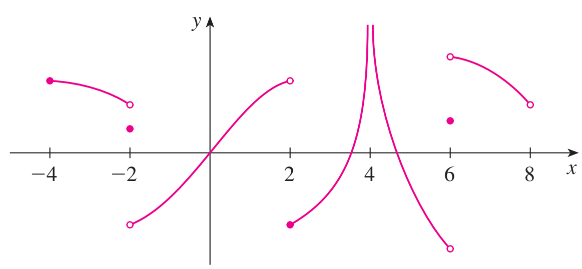
1. Write an equation that expresses the fact that a function f is continuous at the number 4.

2. If f is continuous on $(-\infty, \infty)$, what can you say about its graph?

3. (a) From the graph of f, state the numbers at which f is discontinuous and explain why.
(b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



4. From the graph of g, state the intervals on which g is continuous.



5. Sketch the graph of a function that is continuous everywhere except at $x = 3$ and is continuous from the left at 3.

6. Sketch the graph of a function that has a jump discontinuity at $x = 2$ and a removable discontinuity at $x = 4$, but is continuous elsewhere.

7. A parking lot charges \$3 for the first hour (or part of an hour) and \$2 for each succeeding hour (or part), up to a daily maximum of \$10.
(a) Sketch a graph of the cost of parking at this lot as a function of the time parked there.

(b) Discuss the discontinuities of this function and their significance to someone who parks in the lot.

8. Explain why each function is continuous or discontinuous.
(a) The temperature at a specific location as a function of time
(b) The temperature at a specific time as a function of the distance due west from New York City
(c) The altitude above sea level as a function of the distance due west from New York City
(d) The cost of a taxi ride as a function of the distance traveled
(e) The current in the circuit for the lights in a room as a function of time

9. If f and g are continuous functions with $f(3) = 5$ and $\lim_{x\to 3}[2f(x) - g(x)] = 4$, find $g(3)$.

10–11 ■ Use the definition of continuity and the properties of limits to show that the function is continuous at the given number.

10. $f(x) = x^2 + \sqrt{7 - x}, \quad a = 4$

11. $f(x) = (x + 2x^3)^4, \quad a = -1$

12. Use the definition of continuity and the properties of limits to show that the function $f(x) = x\sqrt{16 - x^2}$ is continuous on the interval $[-4, 4]$.

13–16 ■ Explain why the function is discontinuous at the given number. Sketch the graph of the function.

13. $f(x) = \ln|x - 2| \qquad a = 2$

14. $f(x) = \begin{cases} \dfrac{1}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases} \qquad a = 1$

15. $f(x) = \begin{cases} \dfrac{x^2 - x - 12}{x + 3} & \text{if } x \neq -3 \\ -5 & \text{if } x = -3 \end{cases} \qquad a = -3$

16. $f(x) = \begin{cases} 1 + x^2 & \text{if } x < 1 \\ 4 - x & \text{if } x \geq 1 \end{cases} \qquad a = 1$

17–22 ■ Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

17. $F(x) = \dfrac{x}{x^2 + 5x + 6}$

18. $f(t) = 2t + \sqrt{25 - t^2}$

19. $f(x) = e^x \sin 5x$

20. $F(x) = \sin^{-1}(x^2 - 1)$

21. $G(t) = \ln(t^4 - 1)$

22. $H(x) = \cos(e^{\sqrt{x}})$

23–24 ■ Locate the discontinuities of the function and illustrate by graphing.

23. $y = \dfrac{1}{1 + e^{1/x}}$

24. $y = \ln(\tan^2 x)$

25–28 ■ Use continuity to evaluate the limit.

25. $\lim_{x\to 4} \dfrac{5 + \sqrt{x}}{\sqrt{5 + x}}$

26. $\lim_{x\to \pi} \sin(x + \sin x)$

27. $\lim_{x\to 1} e^{x^2 - x}$

28. $\lim_{x\to 2} \arctan\left(\dfrac{x^2 - 4}{3x^2 - 6x}\right)$

29. Find the numbers at which the function

$$f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

is discontinuous. At which of these points is f continuous from the right, from the left, or neither? Sketch the graph of f.

30. The gravitational force exerted by Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\ \dfrac{GM}{r^2} & \text{if } r \geq R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

31. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx + 1 & \text{if } x \leq 3 \\ cx^2 - 1 & \text{if } x > 3 \end{cases}$$

32. Suppose that a function f is continuous on [0, 1] except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

33. If $f(x) = x^3 - x^2 + x$, show that there is a number c such that $f(c) = 10$.

34. Use the Intermediate Value Theorem to prove that there is a positive number c such that $c^2 = 2$. (This proves the existence of the number $\sqrt{2}$.)

35–38 ■ Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

35. $x^3 - 3x + 1 = 0, \quad (0, 1)$

36. $x^2 = \sqrt{x + 1}$, $(1, 2)$

37. $\cos x = x$, $(0, 1)$

38. $\ln x = e^{-x}$, $(1, 2)$

39–40 ■ (a) Prove that the equation has at least one real root. (b) Use your calculator to find an interval of length 0.01 that contains a root.

39. $e^x = 2 - x$

40. $x^5 - x^2 + 2x + 3 = 0$

41–42 ■ (a) Prove that the equation has at least one real root. (b) Use your graphing device to find the root correct to three decimal places.

41. $100e^{-x/100} = 0.01x^2$ **42.** $\arctan x = 1 - x$

43. To prove that sine is continuous we need to show that $\lim_{x \to a} \sin x = \sin a$ for every real number a. If we let $h = x - a$, then $x = a + h$ and $x \to a \iff h \to 0$. So an equivalent statement is that

$$\lim_{h \to 0} \sin(a + h) = \sin a$$

Use (6) to show that this is true.

44. Prove that cosine is a continuous function.

45. Is there a number that is exactly 1 more than its cube?

46. (a) Show that the absolute value function $F(x) = |x|$ is continuous everywhere.
(b) Prove that if f is a continuous function on an interval, then so is $|f|$.
(c) Is the converse of the statement in part (b) also true? In other words, if $|f|$ is continuous, does it follow that f is continuous? If so, prove it. If not, find a counterexample.

47. A Tibetan monk leaves the monastery at 7:00 A.M. and takes his usual path to the top of the mountain, arriving at 7:00 P.M. The following morning, he starts at 7:00 A.M. at the top and takes the same path back, arriving at the monastery at 7:00 P.M. Use the Intermediate Value Theorem to show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

2.5 Limits Involving Infinity

In this section we investigate the global behavior of functions and, in particular, whether their graphs approach asymptotes, vertical or horizontal.

Infinite Limits

In Example 8 in Section 2.2 we concluded that

$$\lim_{x \to 0} \frac{1}{x^2} \quad \text{does not exist}$$

by observing from the table of values and the graph of $y = 1/x^2$ in Figure 1, that the values of $1/x^2$ can be made arbitrarily large by taking x close enough to 0. Thus, the values of $f(x)$ do not approach a number, so $\lim_{x \to 0} (1/x^2)$ does not exist.

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

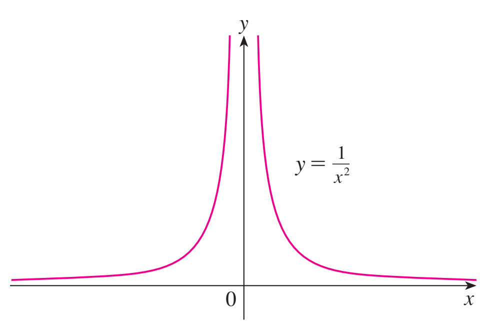


FIGURE 1

To indicate this kind of behavior we use the notation

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

⊘ This does not mean that we are regarding ∞ as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist: $1/x^2$ can be made as large as we like by taking x close enough to 0.

In general, we write symbolically

$$\lim_{x \to a} f(x) = \infty$$

to indicate that the values of $f(x)$ become larger and larger (or "increase without bound") as x approaches a.

▲ A more precise version of Definition 1 is given in Appendix D, Exercise 16.

1 Definition The notation

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a (on either side of a) but not equal to a.

Another notation for $\lim_{x \to a} f(x) = \infty$ is

$$f(x) \to \infty \qquad \text{as} \qquad x \to a$$

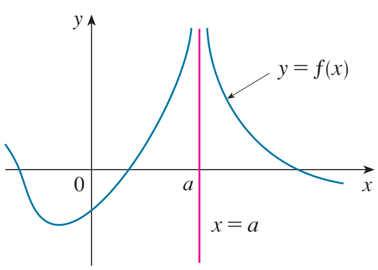


FIGURE 2
$\lim_{x \to a} f(x) = \infty$

Again, the symbol ∞ is not a number, but the expression $\lim_{x \to a} f(x) = \infty$ is often read as

"the limit of $f(x)$, as x approaches a, is infinity"

or "$f(x)$ becomes infinite as x approaches a"

or "$f(x)$ increases without bound as x approaches a"

This definition is illustrated graphically in Figure 2.

Similarly, as shown in Figure 3,

$$\lim_{x \to a} f(x) = -\infty$$

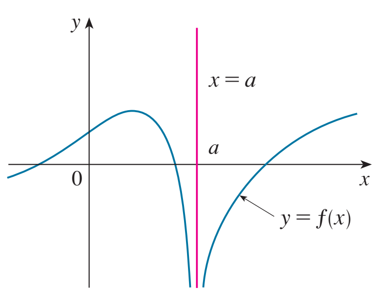


FIGURE 3
$\lim_{x \to a} f(x) = -\infty$

means that the values of $f(x)$ are as large negative as we like for all values of x that are sufficiently close to a, but not equal to a.

The symbol $\lim_{x \to a} f(x) = -\infty$ can be read as "the limit of $f(x)$, as x approaches a, is negative infinity" or "$f(x)$ decreases without bound as x approaches a." As an example we have

$$\lim_{x \to 0} \left(-\frac{1}{x^2} \right) = -\infty$$

Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

remembering that "$x \to a^-$" means that we consider only values of x that are less than a, and similarly "$x \to a^+$" means that we consider only $x > a$. Illustrations of these four cases are given in Figure 4.

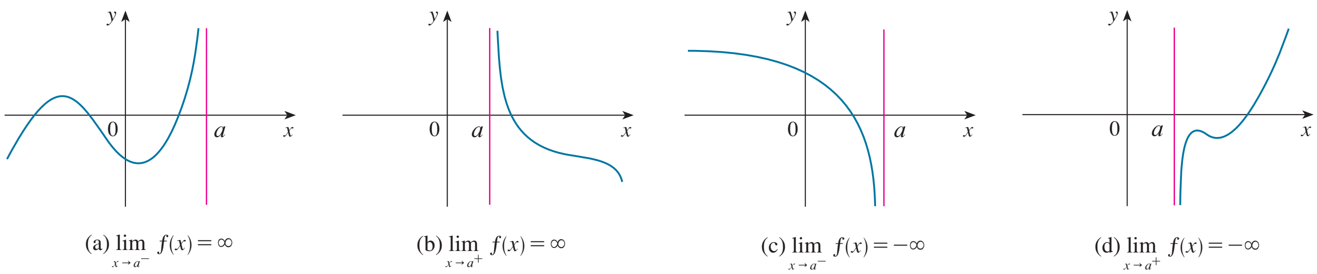


FIGURE 4

2 Definition The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

For instance, the y-axis is a vertical asymptote of the curve $y = 1/x^2$ because $\lim_{x \to 0} (1/x^2) = \infty$. In Figure 4 the line $x = a$ is a vertical asymptote in each of the four cases shown.

EXAMPLE 1 Find $\lim_{x \to 3^+} \dfrac{2x}{x - 3}$ and $\lim_{x \to 3^-} \dfrac{2x}{x - 3}$.

SOLUTION If x is close to 3 but larger than 3, then the denominator $x - 3$ is a small positive number and $2x$ is close to 6. So the quotient $2x/(x - 3)$ is a large *positive* number. Thus, intuitively we see that

$$\lim_{x \to 3^+} \frac{2x}{x - 3} = \infty$$

Likewise, if x is close to 3 but smaller than 3, then $x - 3$ is a small negative number but $2x$ is still a positive number (close to 6). So $2x/(x - 3)$ is a numerically large *negative* number. Thus

$$\lim_{x \to 3^-} \frac{2x}{x - 3} = -\infty$$

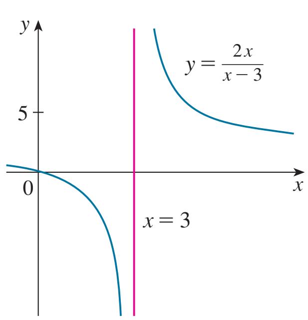


FIGURE 5

The graph of the curve $y = 2x/(x - 3)$ is given in Figure 5. The line $x = 3$ is a vertical asymptote. ■

Two familiar functions whose graphs have vertical asymptotes are $y = \tan x$ and $y = \ln x$. From Figure 6 we see that

3 $$\lim_{x \to 0^+} \ln x = -\infty$$

and so the line $x = 0$ (the y-axis) is a vertical asymptote. In fact, the same is true for $y = \log_a x$ provided that $a > 1$. (See Figures 11 and 12 in Section 1.6.)

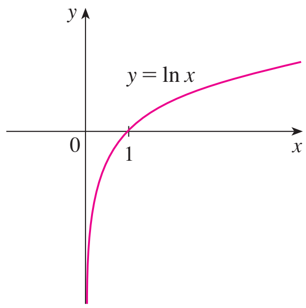


FIGURE 6

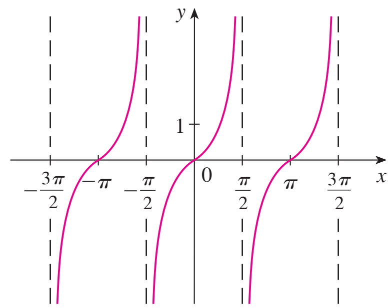


FIGURE 7
$y = \tan x$

Figure 7 shows that

$$\lim_{x \to (\pi/2)^-} \tan x = \infty$$

and so the line $x = \pi/2$ is a vertical asymptote. In fact, the lines $x = (2n + 1)\pi/2$, n an integer, are all vertical asymptotes of $y = \tan x$.

EXAMPLE 2 Find $\lim_{x \to 0} \ln(\tan^2 x)$.

■ The problem-solving strategy for Example 2 is *Introduce Something Extra* (see page 88). Here, the something extra, the auxiliary aid, is the new variable t.

SOLUTION We introduce a new variable, $t = \tan^2 x$. Then $t \geqslant 0$ and $t = \tan^2 x \to \tan^2 0 = 0$ as $x \to 0$ because tan is a continuous function. So, by (3), we have

$$\lim_{x \to 0} \ln(\tan^2 x) = \lim_{t \to 0^+} \ln t = -\infty$$

Limits at Infinity

In computing infinite limits, we let x approach a number and the result was that the values of y became arbitrarily large (positive or negative). Here we let x become arbitrarily large (positive or negative) and see what happens to y.

Let's begin by investigating the behavior of the function f defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as x becomes large. The table at the left gives values of this function correct to six decimal places, and the graph of f has been drawn by a computer in Figure 8.

x	$f(x)$
0	−1
±1	0
±2	0.600000
±3	0.800000
±4	0.882353
±5	0.923077
±10	0.980198
±50	0.999200
±100	0.999800
±1000	0.999998

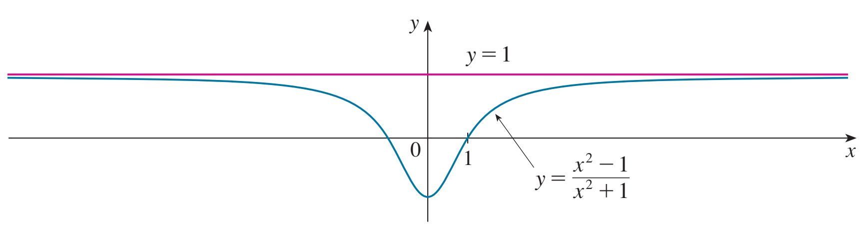


FIGURE 8

As x grows larger and larger you can see that the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x\to\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, we use the notation

$$\lim_{x\to\infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

4 Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made as close to L as we like by taking x sufficiently large.

▲ A more precise version of Definition 4 is given in Appendix D.

Another notation for $\lim_{x\to\infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

The symbol ∞ does not represent a number. Nonetheless, the expression $\lim\limits_{x\to\infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches infinity, is L"

or "the limit of $f(x)$, as x becomes infinite, is L"

or "the limit of $f(x)$, as x increases without bound, is L"

The meaning of such phrases is given by Definition 4.

Geometric illustrations of Definition 4 are shown in Figure 9. Notice that there are many ways for the graph of f to approach the line $y = L$ (which is called a *horizontal asymptote*).

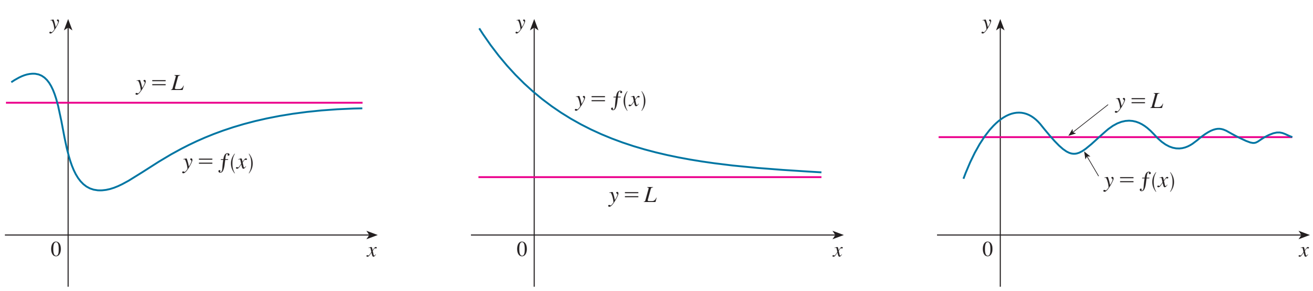


FIGURE 9
Examples illustrating $\lim\limits_{x\to\infty} f(x) = L$

Referring back to Figure 8, we see that for numerically large negative values of x, the values of $f(x)$ are close to 1. By letting x decrease through negative values with-

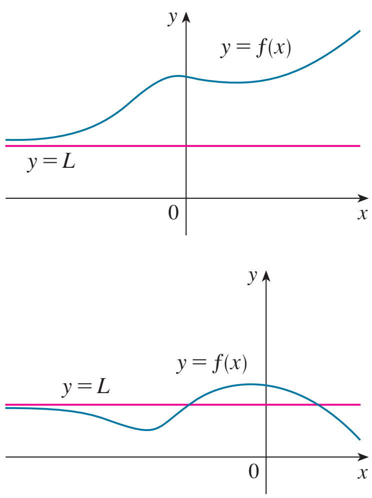


FIGURE 10
Examples illustrating $\lim_{x \to -\infty} f(x) = L$

out bound, we can make $f(x)$ as close to 1 as we like. This is expressed by writing

$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, as shown in Figure 10, the notation

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

Again, the symbol $-\infty$ does not represent a number, but the expression $\lim_{x \to -\infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches negative infinity, is L"

5 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

For instance, the curve illustrated in Figure 8 has the line $y = 1$ as a horizontal asymptote because

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

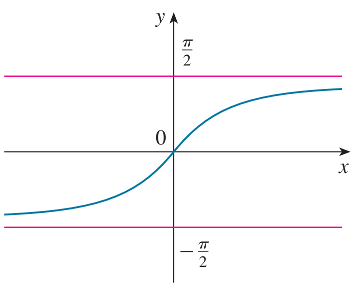


FIGURE 11
$y = \tan^{-1}x$

An example of a curve with two horizontal asymptotes is $y = \tan^{-1}x$. (See Figure 11.) In fact,

6
$$\lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2} \qquad \lim_{x \to \infty} \tan^{-1}x = \frac{\pi}{2}$$

so both of the lines $y = -\pi/2$ and $y = \pi/2$ are horizontal asymptotes. (This follows from the fact that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan.)

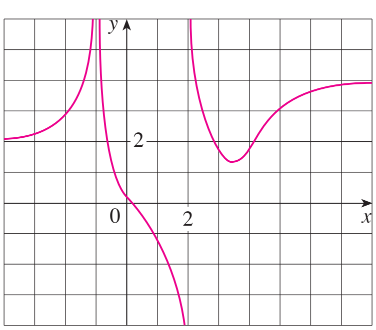


FIGURE 12

EXAMPLE 3 Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in Figure 12.

SOLUTION We see that the values of $f(x)$ become large as $x \to -1$ from both sides, so

$$\lim_{x \to -1} f(x) = \infty$$

Notice that $f(x)$ becomes large negative as x approaches 2 from the left, but large positive as x approaches 2 from the right. So

$$\lim_{x \to 2^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x \to 2^+} f(x) = \infty$$

Thus, both of the lines $x = -1$ and $x = 2$ are vertical asymptotes.

As x becomes large, we see that $f(x)$ approaches 4. But as x decreases through negative values, $f(x)$ approaches 2. So

$$\lim_{x\to\infty} f(x) = 4 \qquad \text{and} \qquad \lim_{x\to-\infty} f(x) = 2$$

This means that both $y = 4$ and $y = 2$ are horizontal asymptotes. ■

EXAMPLE 4 Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.

SOLUTION Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore, according to Definition 4, we have

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is an equilateral hyperbola; see Figure 13.) ■

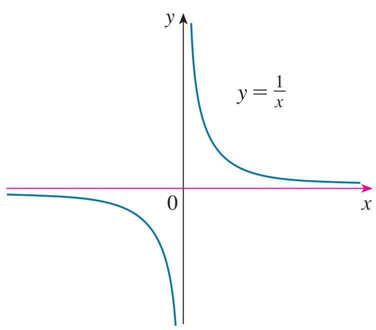


FIGURE 13

$\lim_{x\to\infty} \frac{1}{x} = 0, \ \lim_{x\to-\infty} \frac{1}{x} = 0$

Most of the Limit Laws that were given in Section 2.3 also hold for limits at infinity. It can be proved that the *Limit Laws listed in Section 2.3 (with the exception of Laws 9 and 10) are also valid if "$x \to a$" is replaced by "$x \to \infty$" or "$x \to -\infty$."* In particular, if we combine Law 6 with the results of Example 4 we obtain the following important rule for calculating limits.

7 If n is a positive integer, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

EXAMPLE 5 Evaluate

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

SOLUTION To evaluate the limit at infinity of a rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of x.)

In this case the highest power of x is x^2, and so, using the Limit Laws, we have

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x \to \infty} \frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}} = \lim_{x \to \infty} \frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}}$$

$$= \frac{\lim\limits_{x \to \infty}\left(3 - \dfrac{1}{x} - \dfrac{2}{x^2}\right)}{\lim\limits_{x \to \infty}\left(5 + \dfrac{4}{x} + \dfrac{1}{x^2}\right)}$$

$$= \frac{\lim\limits_{x \to \infty} 3 - \lim\limits_{x \to \infty} \dfrac{1}{x} - 2\lim\limits_{x \to \infty} \dfrac{1}{x^2}}{\lim\limits_{x \to \infty} 5 + 4\lim\limits_{x \to \infty} \dfrac{1}{x} + \lim\limits_{x \to \infty} \dfrac{1}{x^2}}$$

$$= \frac{3 - 0 - 0}{5 + 0 + 0} \qquad \text{[by (7)]}$$

$$= \frac{3}{5}$$

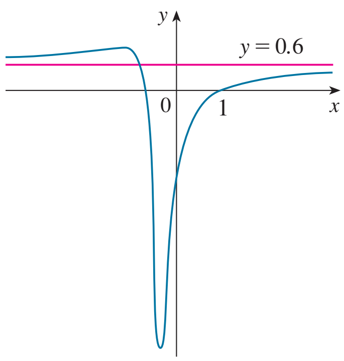


FIGURE 14
$y = \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

A similar calculation shows that the limit as $x \to -\infty$ is also $\frac{3}{5}$. Figure 14 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5}$. ■

EXAMPLE 6 Compute $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + 1} - x\right)$.

SOLUTION We first multiply numerator and denominator by the conjugate radical:

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 1} - x\right) = \lim_{x \to \infty} \left(\sqrt{x^2 + 1} - x\right) \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1} + x}$$

$$= \lim_{x \to \infty} \frac{(x^2 + 1) - x^2}{\sqrt{x^2 + 1} + x} = \lim_{x \to \infty} \frac{1}{\sqrt{x^2 + 1} + x}$$

The Squeeze Theorem could be used to show that this limit is 0. But an easier method is to divide numerator and denominator by x. Doing this and remembering that $x = \sqrt{x^2}$ for $x > 0$, we obtain

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 1} - x\right) = \lim_{x \to \infty} \frac{1}{\sqrt{x^2 + 1} + x} = \lim_{x \to \infty} \frac{\dfrac{1}{x}}{\dfrac{\sqrt{x^2 + 1} + x}{x}}$$

$$= \lim_{x \to \infty} \frac{\dfrac{1}{x}}{\sqrt{1 + \dfrac{1}{x^2}} + 1} = \frac{0}{\sqrt{1 + 0} + 1} = 0$$

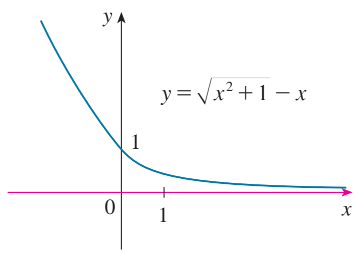


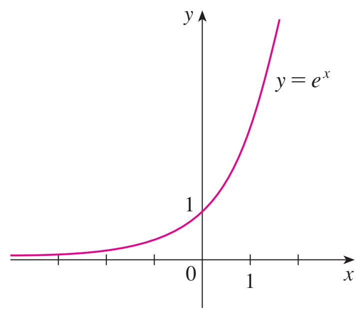
FIGURE 15

Figure 15 illustrates this result. ■

The graph of the natural exponential function $y = e^x$ has the line $y = 0$ (the x-axis) as a horizontal asymptote. (The same is true of any exponential function with base $a > 1$.) In fact, from the graph in Figure 16 and the corresponding table of values, we see that

8 $$\lim_{x \to -\infty} e^x = 0$$

Notice that the values of e^x approach 0 very rapidly.



x	e^x
0	1.00000
−1	0.36788
−2	0.13534
−3	0.04979
−5	0.00674
−8	0.00034
−10	0.00005

FIGURE 16

EXAMPLE 7 Evaluate $\lim_{x \to 0^-} e^{1/x}$.

SOLUTION If we let $t = 1/x$, we know from Example 4 that $t \to -\infty$ as $x \to 0^-$. Therefore, by (8),

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

EXAMPLE 8 Evaluate $\lim_{x \to \infty} \sin x$.

SOLUTION As x increases, the values of $\sin x$ oscillate between 1 and -1 infinitely often. Thus, $\lim_{x \to \infty} \sin x$ does not exist.

Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

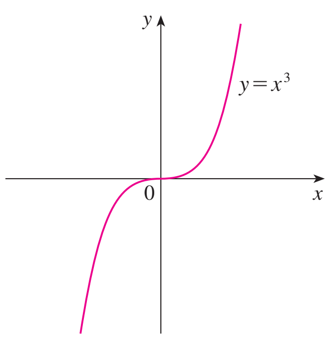


FIGURE 17

From Figures 16 and 17 we see that

$$\lim_{x \to \infty} e^x = \infty \qquad \lim_{x \to \infty} x^3 = \infty \qquad \lim_{x \to -\infty} x^3 = -\infty$$

but, as Figure 18 demonstrates, $y = e^x$ becomes large as $x \to \infty$ at a much faster rate than $y = x^3$.

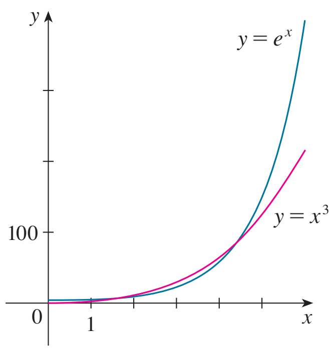


FIGURE 18

EXAMPLE 9 Find $\lim_{x \to \infty} (x^2 - x)$.

SOLUTION Note that we *cannot* write

$$\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x^2 - \lim_{x \to \infty} x$$
$$= \infty - \infty$$

The Limit Laws can't be applied to infinite limits because ∞ is not a number ($\infty - \infty$ can't be defined). However, we can write

$$\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x(x - 1) = \infty$$

because both x and $x - 1$ become arbitrarily large. ■

EXAMPLE 10 Find $\lim_{x \to \infty} \frac{x^2 + x}{3 - x}$.

SOLUTION We divide numerator and denominator by x (the highest power of x that occurs in the denominator):

$$\lim_{x \to \infty} \frac{x^2 + x}{3 - x} = \lim_{x \to \infty} \frac{x + 1}{\dfrac{3}{x} - 1} = -\infty$$

because $x + 1 \to \infty$ and $3/x - 1 \to -1$ as $x \to \infty$. ■

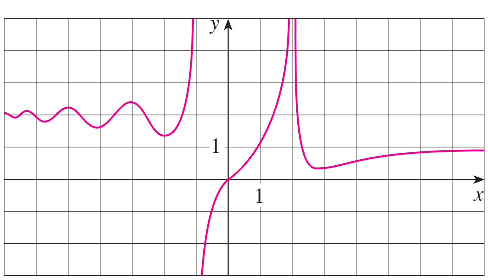
2.5 Exercises

1. Explain in your own words the meaning of each of the following.
(a) $\lim_{x \to 2} f(x) = \infty$ (b) $\lim_{x \to 1^+} f(x) = -\infty$
(c) $\lim_{x \to \infty} f(x) = 5$ (d) $\lim_{x \to -\infty} f(x) = 3$

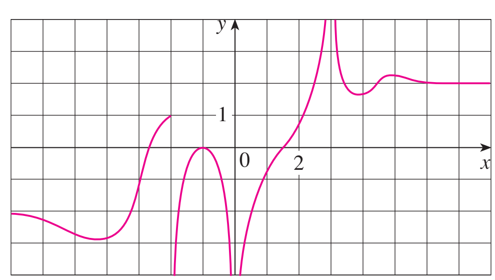
2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.
(b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. For the function f whose graph is given, state the following.

(a) $\lim_{x \to 2} f(x)$ (b) $\lim_{x \to -1^-} f(x)$
(c) $\lim_{x \to -1^+} f(x)$ (d) $\lim_{x \to \infty} f(x)$
(e) $\lim_{x \to -\infty} f(x)$ (f) The equations of the asymptotes



4. For the function g whose graph is given, state the following.
(a) $\lim_{x \to \infty} g(x)$ (b) $\lim_{x \to -\infty} g(x)$
(c) $\lim_{x \to 3} g(x)$ (d) $\lim_{x \to 0} g(x)$
(e) $\lim_{x \to -2^+} g(x)$ (f) The equations of the asymptotes



5–8 ■ Sketch the graph of an example of a function f that satisfies all of the given conditions.

5. $f(0) = 0, \quad f(1) = 1, \quad \lim_{x\to\infty} f(x) = 0, \quad f$ is odd

6. $\lim_{x\to 0^+} f(x) = \infty, \quad \lim_{x\to 0^-} f(x) = -\infty, \quad \lim_{x\to\infty} f(x) = 1,$ $\lim_{x\to -\infty} f(x) = 1$

7. $\lim_{x\to 2} f(x) = -\infty, \quad \lim_{x\to\infty} f(x) = \infty, \quad \lim_{x\to -\infty} f(x) = 0,$ $\lim_{x\to 0^+} f(x) = \infty, \quad \lim_{x\to 0^-} f(x) = -\infty$

8. $\lim_{x\to -2} f(x) = \infty, \quad \lim_{x\to -\infty} f(x) = 3, \quad \lim_{x\to\infty} f(x) = -3$

9. Guess the value of the limit

$$\lim_{x\to\infty} \frac{x^2}{2^x}$$

by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Then use a graph of f to support your guess.

10. Determine $\lim_{x\to 1^-} \frac{1}{x^3 - 1}$ and $\lim_{x\to 1^+} \frac{1}{x^3 - 1}$

(a) by evaluating $f(x) = 1/(x^3 - 1)$ for values of x that approach 1 from the left and from the right,
(b) by reasoning as in Example 1, and
(c) from a graph of f.

11. Use a graph to estimate all the vertical and horizontal asymptotes of the curve

$$y = \frac{x^3}{x^3 - 2x + 1}$$

12. (a) Use a graph of

$$f(x) = \left(1 - \frac{2}{x}\right)^x$$

to estimate the value of $\lim_{x\to\infty} f(x)$ correct to two decimal places.
(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

13–29 ■ Find the limit.

13. $\lim_{x\to -3^+} \frac{x + 2}{x + 3}$

14. $\lim_{x\to 5^-} \frac{e^x}{(x - 5)^3}$

15. $\lim_{x\to 1} \frac{2 - x}{(x - 1)^2}$

16. $\lim_{x\to 5^+} \ln(x - 5)$

17. $\lim_{x\to(-\pi/2)^-} \sec x$

18. $\lim_{x\to\infty} \frac{3x + 5}{x - 4}$

19. $\lim_{x\to\infty} \frac{x^3 + 5x}{2x^3 - x^2 + 4}$

20. $\lim_{t\to -\infty} \frac{t^2 + 2}{t^3 + t^2 - 1}$

21. $\lim_{u\to\infty} \frac{4u^4 + 5}{(u^2 - 2)(2u^2 - 1)}$

22. $\lim_{x\to\infty} \frac{x + 2}{\sqrt{9x^2 + 1}}$

23. $\lim_{x\to\infty} \left(\sqrt{9x^2 + x} - 3x\right)$

24. $\lim_{x\to\infty} \frac{\sin^2 x}{x^2}$

25. $\lim_{x\to\infty} \cos x$

26. $\lim_{x\to\infty} \tan^{-1}(x^4 - x^2)$

27. $\lim_{x\to\infty} \frac{x^7 - 1}{x^6 + 1}$

28. $\lim_{x\to\infty} e^{-x^2}$

29. $\lim_{x\to -\infty} (x^3 - 5x^2)$

30. (a) Graph the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

How many horizontal and vertical asymptotes do you observe? Use the graph to estimate the values of the limits

$$\lim_{x\to\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \quad \text{and} \quad \lim_{x\to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

(b) By calculating values of $f(x)$, give numerical estimates of the limits in part (a).
(c) Calculate the exact values of the limits in part (a). Did you get the same value or different values for these two limits? [In view of your answer to part (a), you might have to check your calculation for the second limit.]

31–32 ■ Find the horizontal and vertical asymptotes of each curve. Check your work by graphing the curve and estimating the asymptotes.

31. $y = \frac{2x^2 + x - 1}{x^2 + x - 2}$

32. $y = \frac{x - 9}{\sqrt{4x^2 + 3x + 2}}$

33. (a) Estimate the value of

$$\lim_{x\to -\infty} \left(\sqrt{x^2 + x + 1} + x\right)$$

by graphing the function $f(x) = \sqrt{x^2 + x + 1} + x$.
(b) Use a table of values of $f(x)$ to guess the value of the limit.
(c) Prove that your guess is correct.

34. (a) Use a graph of

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$$

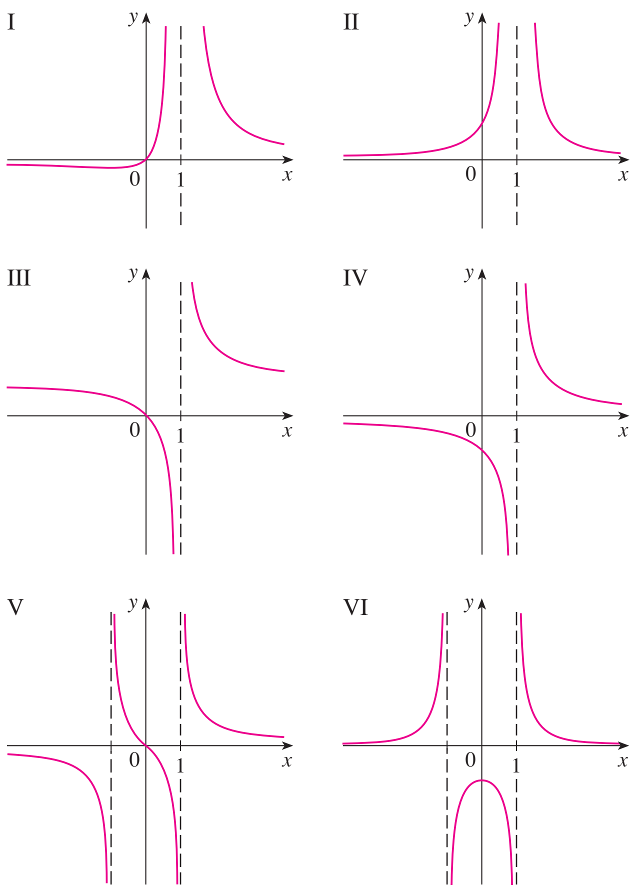
to estimate the value of $\lim_{x\to\infty} f(x)$ to one decimal place.
(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.
(c) Find the exact value of the limit.

35. Match each function in (a)–(f) with its graph (labeled I–VI). Give reasons for your choices.

(a) $y = \dfrac{1}{x - 1}$ (b) $y = \dfrac{x}{x - 1}$

(c) $y = \dfrac{1}{(x - 1)^2}$ (d) $y = \dfrac{1}{x^2 - 1}$

(e) $y = \dfrac{x}{(x - 1)^2}$ (f) $y = \dfrac{x}{x^2 - 1}$



36. Find a formula for a function that has vertical asymptotes $x = 1$ and $x = 3$ and horizontal asymptote $y = 1$.

37. Find a formula for a function f that satisfies the following conditions:

$\lim_{x \to \pm\infty} f(x) = 0, \quad \lim_{x \to 0} f(x) = -\infty, \quad f(2) = 0,$

$\lim_{x \to 3^-} f(x) = \infty, \quad \lim_{x \to 3^+} f(x) = -\infty$

38. By the *end behavior* of a function we mean a description of what happens to its values as $x \to \infty$ and as $x \to -\infty$.

(a) Describe and compare the end behavior of the functions

$$P(x) = 3x^5 - 5x^3 + 2x \qquad Q(x) = 3x^5$$

by graphing both functions in the viewing rectangles $[-2, 2]$ by $[-2, 2]$ and $[-10, 10]$ by $[-10{,}000, 10{,}000]$.

(b) Two functions are said to have the *same end behavior* if their ratio approaches 1 as $x \to \infty$. Show that P and Q have the same end behavior.

39. Let P and Q be polynomials. Find

$$\lim_{x \to \infty} \frac{P(x)}{Q(x)}$$

if the degree of P is (a) less than the degree of Q and (b) greater than the degree of Q.

40. Make a rough sketch of the curve $y = x^n$ (n an integer) for the following five cases:

(i) $n = 0$ (ii) $n > 0$, n odd
(iii) $n > 0$, n even (iv) $n < 0$, n odd
(v) $n < 0$, n even

Then use these sketches to find the following limits.

(a) $\lim_{x \to 0^+} x^n$ (b) $\lim_{x \to 0^-} x^n$

(c) $\lim_{x \to \infty} x^n$ (d) $\lim_{x \to -\infty} x^n$

41. Find $\lim_{x \to \infty} f(x)$ if

$$\frac{4x - 1}{x} < f(x) < \frac{4x^2 + 3x}{x^2}$$

for all $x > 5$.

42. In the theory of relativity, the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light. What happens as $v \to c^-$?

43. (a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200 + t}$$

(b) What happens to the concentration as $t \to \infty$?

44. In Chapter 7 we will be able to show, under certain assumptions, that the velocity $v(t)$ of a falling raindrop at time t is

$$v(t) = v^*(1 - e^{-gt/v^*})$$

where g is the acceleration due to gravity and v^* is the terminal velocity of the raindrop.

(a) Find $\lim_{t \to \infty} v(t)$.

(b) Graph $v(t)$ if $v^* = 1$ m/s and $g = 9.8$ m/s^2. How long does it take for the velocity of the raindrop to reach 99% of its terminal velocity?

45. (a) Show that $\lim_{x\to\infty} e^{-x/10} = 0$.
(b) By graphing $y = e^{-x/10}$ and $y = 0.1$ on a common screen, discover how large you need to make x so that $e^{-x/10} < 0.1$.
(c) Can you solve part (b) without using a graphing device?

46. (a) Show that $\displaystyle\lim_{x\to\infty} \frac{4x^2 - 5x}{2x^2 + 1} = 2$.
(b) By graphing the function in part (a) and the line $y = 1.9$ on a common screen, find a number N such that

$$\frac{4x^2 - 5x}{2x^2 + 1} > 1.9 \qquad \text{when} \qquad x > N$$

What if 1.9 is replaced by 1.99?

2.6 Tangents, Velocities, and Other Rates of Change

In Section 2.1 we guessed the values of slopes of tangent lines and velocities on the basis of numerical evidence. Now that we have defined limits and have learned techniques for computing them, we return to the tangent and velocity problems with the ability to calculate slopes of tangents, velocities, and other rates of change.

Tangents

If a curve C has equation $y = f(x)$ and we want to find the tangent to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a. If m_{PQ} approaches a number m, then we define the *tangent* t to be the line through P with slope m. (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P. See Figure 1.)

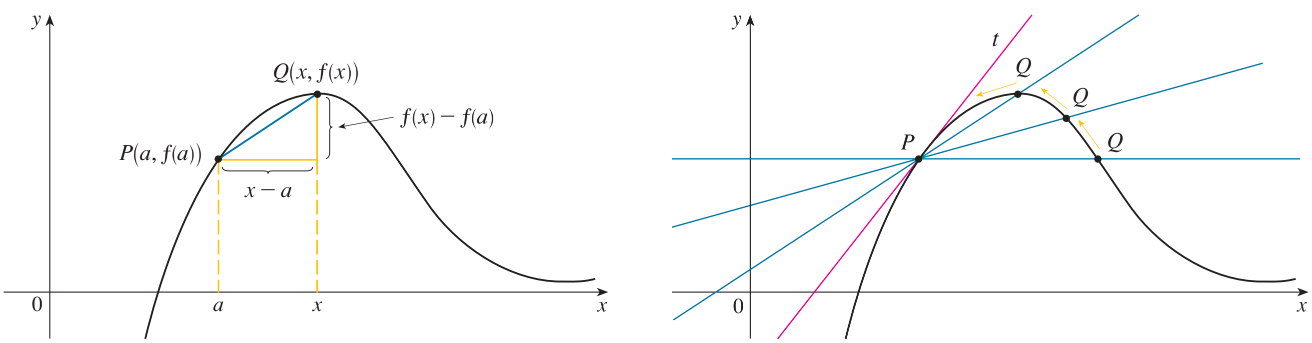


FIGURE 1

1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

In our first example we confirm the guess we made in Example 1 in Section 2.1.

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

▲ Point-slope form for a line through the point (x_1, y_1) with slope m:

$$y - y_1 = m(x - x_1)$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at $(1, 1)$ is

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$ ■

We sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point. The idea is that if we zoom in far enough toward the point, the curve looks almost like a straight line. Figure 2 illustrates this procedure for the curve $y = x^2$ in Example 1. The more we zoom in, the more the parabola looks like a line. In other words, the curve becomes almost indistinguishable from its tangent line.

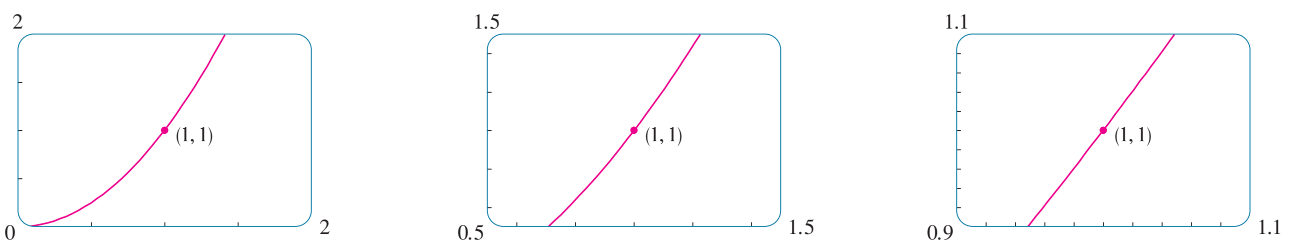


FIGURE 2
Zooming in toward the point $(1, 1)$ on the parabola $y = x^2$

There is another expression for the slope of a tangent line that is sometimes easier to use. Let

$$h = x - a$$

Then

$$x = a + h$$

so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

(See Figure 3 where the case $h > 0$ is illustrated and Q is to the right of P. If it happened that $h < 0$, however, Q would be to the left of P.)

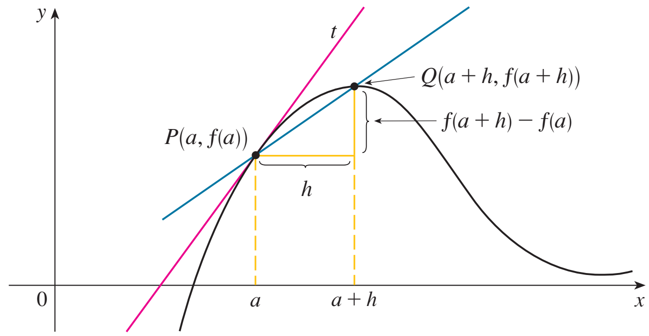


FIGURE 3

Notice that as x approaches a, h approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 1 becomes

2
$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

EXAMPLE 2 Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point (3, 1).

SOLUTION Let $f(x) = 3/x$. Then the slope of the tangent at (3, 1) is

$$m = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{3}{3 + h} - 1}{h} = \lim_{h \to 0} \frac{\dfrac{3 - (3 + h)}{3 + h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(3 + h)} = \lim_{h \to 0} -\frac{1}{3 + h} = -\frac{1}{3}$$

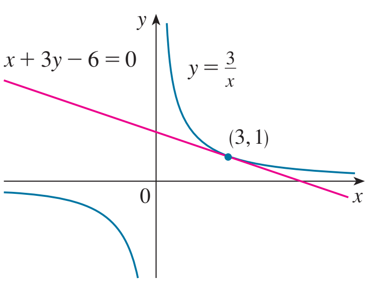


FIGURE 4

Therefore, an equation of the tangent at the point (3, 1) is

$$y - 1 = -\tfrac{1}{3}(x - 3)$$

which simplifies to
$$x + 3y - 6 = 0$$

The hyperbola and its tangent are shown in Figure 4. ■

Velocities

Learn about average and instantaneous velocity by comparing falling objects.
Resources / Module 3 / Derivative at a Point / The Falling Robot

In Section 2.1 we investigated the motion of a ball dropped from the CN Tower and defined its velocity to be the limiting value of average velocities over shorter and shorter time periods.

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the **position**

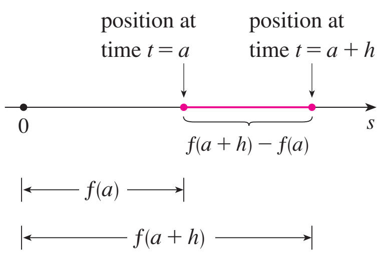


FIGURE 5

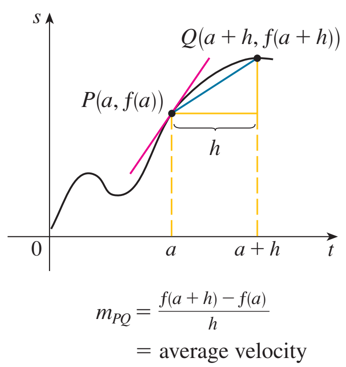


FIGURE 6

function of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$. (See Figure 5.) The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

which is the same as the slope of the secant line PQ in Figure 6.

Now suppose we compute the average velocities over shorter and shorter time intervals $[a, a + h]$. In other words, we let h approach 0. As in the example of the falling ball, we define the **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ to be the limit of these average velocities:

3
$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This means that the velocity at time $t = a$ is equal to the slope of the tangent line at P (compare Equations 2 and 3).

Now that we know how to compute limits, let's reconsider the problem of the falling ball.

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

▲ Recall from Section 2.1: The distance (in meters) fallen after t seconds is $4.9t^2$.

SOLUTION We first use the equation of motion $s = f(t) = 4.9t^2$ to find the velocity $v(a)$ after a seconds:

$$\begin{aligned} v(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{4.9(a+h)^2 - 4.9a^2}{h} \\ &= \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2 - a^2)}{h} = \lim_{h \to 0} \frac{4.9(2ah + h^2)}{h} \\ &= \lim_{h \to 0} 4.9(2a + h) = 9.8a \end{aligned}$$

(a) The velocity after 5 s is $v(5) = (9.8)(5) = 49$ m/s.

(b) Since the observation deck is 450 m above the ground, the ball will hit the ground at the time t_1 when $s(t_1) = 450$, that is,

$$4.9t_1^2 = 450$$

This gives

$$t_1^2 = \frac{450}{4.9} \qquad \text{and} \qquad t_1 = \sqrt{\frac{450}{4.9}} \approx 9.6 \text{ s}$$

The velocity of the ball as it hits the ground is therefore

$$v(t_1) = 9.8t_1 = 9.8\sqrt{\frac{450}{4.9}} \approx 94 \text{ m/s}$$

■

Other Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus, y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

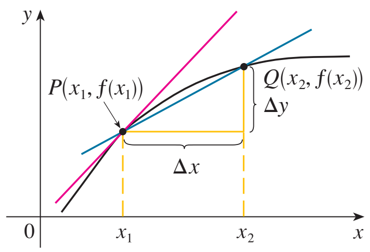
and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of *y* with respect to *x*** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 7.



average rate of change $= m_{PQ}$

instantaneous rate of change = slope of tangent at P

FIGURE 7

By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0. The limit of these average rates of change is called the (**instantaneous**) **rate of change of *y* with respect to *x*** at $x = x_1$, which is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$:

$$\boxed{4} \qquad \text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

x (h)	T (°C)	x (h)	T (°C)
0	6.5	13	16.0
1	6.1	14	17.3
2	5.6	15	18.2
3	4.9	16	18.8
4	4.2	17	17.6
5	4.0	18	16.0
6	4.0	19	14.1
7	4.8	20	11.5
8	6.1	21	10.2
9	8.3	22	9.0
10	10.0	23	7.9
11	12.1	24	7.0
12	14.3		

EXAMPLE 4 Temperature readings T (in degrees Celsius) were recorded every hour starting at midnight on a day in April in Whitefish, Montana. The time x is measured in hours from midnight. The data are given in the table at the left.

(a) Find the average rate of change of temperature with respect to time
  (i) from noon to 3 P.M.
  (ii) from noon to 2 P.M.
  (iii) from noon to 1 P.M.

(b) Estimate the instantaneous rate of change at noon.

SOLUTION

(a) (i) From noon to 3 P.M. the temperature changes from 14.3 °C to 18.2 °C, so

$$\Delta T = T(15) - T(12) = 18.2 - 14.3 = 3.9 \text{ °C}$$

while the change in time is $\Delta x = 3$ h. Therefore, the average rate of change of temperature with respect to time is

$$\frac{\Delta T}{\Delta x} = \frac{3.9}{3} = 1.3 \text{ °C/h}$$

(ii) From noon to 2 P.M. the average rate of change is

$$\frac{\Delta T}{\Delta x} = \frac{T(14) - T(12)}{14 - 12} = \frac{17.3 - 14.3}{2} = 1.5 \text{ °C/h}$$

▲ **A Note on Units**

The units for the average rate of change $\Delta T/\Delta x$ are the units for ΔT divided by the units for Δx, namely, degrees Celsius per hour. The instantaneous rate of change is the limit of the average rates of change, so it is measured in the same units: degrees Celsius per hour.

(iii) From noon to 1 P.M. the average rate of change is

$$\frac{\Delta T}{\Delta x} = \frac{T(13) - T(12)}{13 - 12}$$

$$= \frac{16.0 - 14.3}{1} = 1.7\ ^\circ\mathrm{C/h}$$

(b) We plot the given data in Figure 8 and use them to sketch a smooth curve that approximates the graph of the temperature function. Then we draw the tangent at the point P where $x = 12$ and, after measuring the sides of triangle ABC, we estimate that the slope of the tangent line is

▲ Another method is to average the slopes of two secant lines. See Example 2 in Section 2.1.

$$\frac{|BC|}{|AC|} = \frac{10.3}{5.5} \approx 1.9$$

Therefore, the instantaneous rate of change of temperature with respect to time at noon is about 1.9 °C/h.

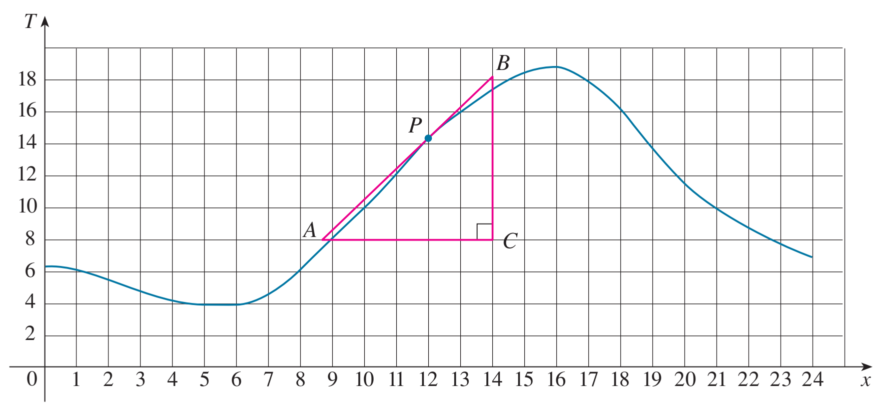


FIGURE 8

The velocity of a particle is the rate of change of displacement with respect to time. Physicists are interested in other rates of change as well—for instance, the rate of change of work with respect to time (which is called *power*). Chemists who study a chemical reaction are interested in the rate of change in the concentration of a reactant with respect to time (called the *rate of reaction*). A steel manufacturer is interested in the rate of change of the cost of producing x tons of steel per day with respect to x (called the *marginal cost*). A biologist is interested in the rate of change of the population of a colony of bacteria with respect to time. In fact, the computation of rates of change is important in all of the natural sciences, in engineering, and even in the social sciences. Further examples will be given in Section 3.3.

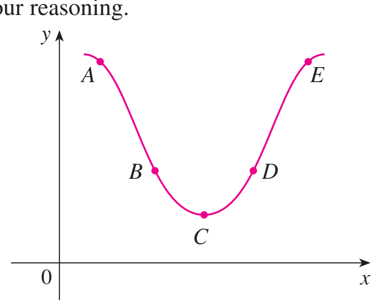
All these rates of change can be interpreted as slopes of tangents. This gives added significance to the solution of the tangent problem. Whenever we solve a problem involving tangent lines, we are not just solving a problem in geometry. We are also implicitly solving a great variety of problems involving rates of change in science and engineering.

2.6 Exercises

1. A curve has equation $y = f(x)$.
(a) Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.
(b) Write an expression for the slope of the tangent line at P.

2. Suppose an object moves with position function $s = f(t)$.
(a) Write an expression for the average velocity of the object in the time interval from $t = a$ to $t = a + h$.
(b) Write an expression for the instantaneous velocity at time $t = a$.

3. Consider the slope of the given curve at each of the five points shown. List these five slopes in decreasing order and explain your reasoning.



4. Graph the curve $y = e^x$ in the viewing rectangles $[-1, 1]$ by $[0, 2]$, $[-0.5, 0.5]$ by $[0.5, 1.5]$, and $[-0.1, 0.1]$ by $[0.9, 1.1]$. What do you notice about the curve as you zoom in toward the point $(0, 1)$?

5. (a) Find the slope of the tangent line to the parabola $y = x^2 + 2x$ at the point $(-3, 3)$
(i) using Definition 1
(ii) using Equation 2
(b) Find an equation of the tangent line in part (a).
(c) Graph the parabola and the tangent line. As a check on your work, zoom in toward the point $(-3, 3)$ until the parabola and the tangent line are indistinguishable.

6. (a) Find the slope of the tangent line to the curve $y = x^3$ at the point $(-1, -1)$
(i) using Definition 1
(ii) using Equation 2
(b) Find an equation of the tangent line in part (a).
(c) Graph the curve and the tangent line in successively smaller viewing rectangles centered at $(-1, -1)$ until the curve and the line appear to coincide.

7–10 ■ Find an equation of the tangent line to the curve at the given point.

7. $y = (x - 1)/(x - 2)$, $(3, 2)$

8. $y = 2x^3 - 5x$, $(-1, 3)$

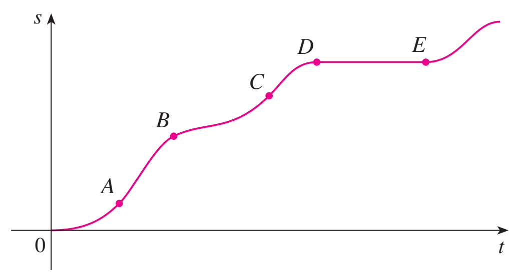
9. $y = \sqrt{x}$, $(1, 1)$

10. $y = 2x/(x + 1)^2$, $(0, 0)$

11. (a) Find the slope of the tangent to the curve $y = x^3 - 4x + 1$ at the point where $x = a$.
(b) Find equations of the tangent lines at the points $(1, -2)$ and $(2, 1)$.
(c) Graph the curve and both tangents on a common screen.

12. (a) Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point where $x = a$.
(b) Find equations of the tangent lines at the points $(1, 1)$ and $\left(4, \frac{1}{2}\right)$.
(c) Graph the curve and both tangents on a common screen.

13. The graph shows the position function of a car. Use the shape of the graph to explain your answers to the following questions.
(a) What was the initial velocity of the car?
(b) Was the car going faster at B or at C?
(c) Was the car slowing down or speeding up at A, B, and C?
(d) What happened between D and E?



14. Valerie is driving along a highway. Sketch the graph of the position function of her car if she drives in the following manner: At time $t = 0$, the car is at mile marker 15 and is traveling at a constant speed of 55 mi/h. She travels at this speed for exactly an hour. Then the car slows gradually over a 2-minute period as Valerie comes to a stop for dinner. Dinner lasts 26 min; then she restarts the car, gradually speeding up to 65 mi/h over a 2-minute period. She drives at a constant 65 mi/h for two hours and then over a 3-minute period gradually slows to a complete stop.

15. If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$.

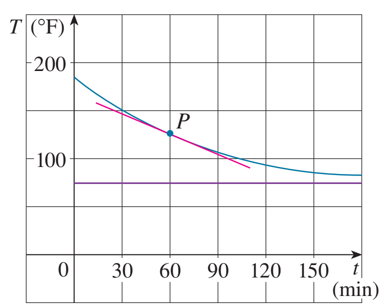
16. If an arrow is shot upward on the moon with a velocity of 58 m/s, its height (in meters) after t seconds is given by $H = 58t - 0.83t^2$.
(a) Find the velocity of the arrow after one second.
(b) Find the velocity of the arrow when $t = a$.
(c) When will the arrow hit the moon?
(d) With what velocity will the arrow hit the moon?

17. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 4t^3 + 6t + 2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.

18. The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds.
(a) Find the average velocities over the following time intervals:
(i) $[3, 4]$ (ii) $[3.5, 4]$
(iii) $[4, 5]$ (iv) $[4, 4.5]$
(b) Find the instantaneous velocity when $t = 4$.
(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a) and the tangent line whose slope is the instantaneous velocity in part (b).

19. A warm can of soda is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Is the initial rate of change of temperature greater or less than the rate of change after an hour?

20. A roast turkey is taken from an oven when its temperature has reached 185 °F and is placed on a table in a room where the temperature is 75 °F. The graph shows how the temperature of the turkey decreases and eventually approaches room temperature. (In Section 7.4 we will be able to use Newton's Law of Cooling to find an equation for T as a function of time.) By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



21. (a) Use the data in Example 4 to find the average rate of change of temperature with respect to time
(i) from 8 P.M. to 11 P.M.
(ii) from 8 P.M. to 10 P.M.
(iii) from 8 P.M. to 9 P.M.
(b) Estimate the instantaneous rate of change of T with respect to time at 8 P.M. by measuring the slope of a tangent.

22. The population P (in thousands) of Belgium from 1992 to 2000 is shown in the table. (Midyear estimates are given.)

Year	1992	1994	1996	1998	2000
P	10,036	10,109	10,152	10,175	10,186

(a) Find the average rate of growth
(i) from 1992 to 1996
(ii) from 1994 to 1996
(iii) from 1996 to 1998
In each case, include the units.
(b) Estimate the instantaneous rate of growth in 1996 by taking the average of two average rates of change. What are its units?
(c) Estimate the instantaneous rate of growth in 1996 by measuring the slope of a tangent.

23. The number N (in thousands) of cellular phone subscribers in Malaysia is shown in the table. (Midyear estimates are given.)

Year	1993	1994	1995	1996	1997
N	304	572	873	1513	2461

(a) Find the average rate of growth
(i) from 1995 to 1997
(ii) from 1995 to 1996
(iii) from 1994 to 1995
In each case, include the units.
(b) Estimate the instantaneous rate of growth in 1995 by taking the average of two average rates of change. What are its units?
(c) Estimate the instantaneous rate of growth in 1995 by measuring the slope of a tangent.

24. The number N of locations of a popular coffeehouse chain is given in the table. (The number of locations as of June 30 are given.)

Year	1994	1995	1996	1997	1998
N	425	676	1015	1412	1886

(a) Find the average rate of growth
(i) from 1996 to 1998
(ii) from 1996 to 1997
(iii) from 1995 to 1996
In each case, include the units.
(b) Estimate the instantaneous rate of growth in 1996 by taking the average of two average rates of change. What are its units?
(c) Estimate the instantaneous rate of growth in 1996 by measuring the slope of a tangent.

25. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.
(a) Find the average rate of change of C with respect to x when the production level is changed
(i) from $x = 100$ to $x = 105$
(ii) from $x = 100$ to $x = 101$
(b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the *marginal cost.* Its significance will be explained in Section 3.3.)

26. If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in an hour, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V(t) = 100{,}000\left(1 - \frac{t}{60}\right)^2 \qquad 0 \leqslant t \leqslant 60$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) as a function of t. What are its units? For times $t = 0$, 10, 20, 30, 40, 50, and 60 min, find the flow rate and the amount of water remaining in the tank. Summarize your findings in a sentence or two. At what time is the flow rate the greatest? The least?

2.7 Derivatives

In Section 2.6 we defined the slope of the tangent to a curve with equation $y = f(x)$ at the point where $x = a$ to be

1 $$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

We also saw that the velocity of an object with position function $s = f(t)$ at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

In fact, limits of the form

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics. Since this type of limit occurs so widely, it is given a special name and notation.

2 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

▲ $f'(a)$ is read "f prime of a."

If we write $x = a + h$, then $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore, an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines, is

3 $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

EXAMPLE 1 Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

SOLUTION From Definition 2 we have

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \\
&= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\
&= \lim_{h\to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h\to 0} (2a + h - 8) \\
&= 2a - 8
\end{aligned}
$$

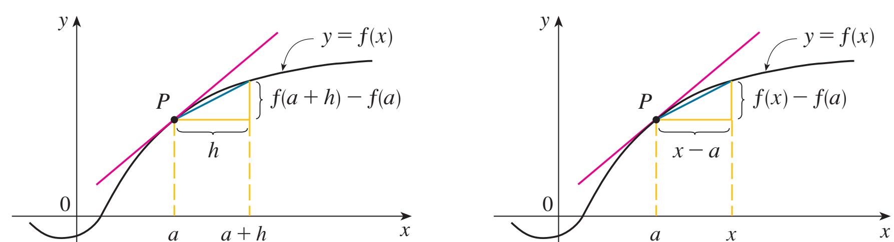
Try problems like this one.
Resources / Module 3
/ Derivative at a Point
/ Problem Wizard

Interpretation of the Derivative as the Slope of a Tangent

In Section 2.6 we defined the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ to be the line that passes through P and has slope m given by Equation 1. Since, by Definition 2, this is the same as the derivative $f'(a)$, we can now say the following.

> The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

Thus, the geometric interpretation of a derivative [as defined by either (2) or (3)] is as shown in Figure 1.



(a) $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$
= slope of tangent at P
= slope of curve at P

(b) $f'(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x - a}$
= slope of tangent at P
= slope of curve at P

FIGURE 1
Geometric interpretation of the derivative

If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

EXAMPLE 2 Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

SOLUTION From Example 1 we know that the derivative of $f(x) = x^2 - 8x + 9$ at the number a is $f'(a) = 2a - 8$. Therefore, the slope of the tangent line at $(3, -6)$ is

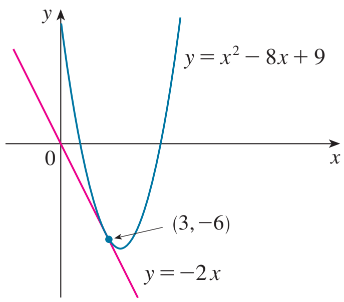


FIGURE 2

$f'(3) = 2(3) - 8 = -2$. Thus, an equation of the tangent line, shown in Figure 2, is

$$y - (-6) = (-2)(x - 3) \qquad \text{or} \qquad y = -2x$$

EXAMPLE 3 Let $f(x) = 2^x$. Estimate the value of $f'(0)$ in two ways:
(a) By using Definition 2 and taking successively smaller values of h.
(b) By interpreting $f'(0)$ as the slope of a tangent and using a graphing calculator to zoom in on the graph of $y = 2^x$.

SOLUTION
(a) From Definition 2 we have

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{2^h - 1}{h}$$

h	$\frac{2^h - 1}{h}$
0.1	0.718
0.01	0.696
0.001	0.693
0.0001	0.693
−0.1	0.670
−0.01	0.691
−0.001	0.693
−0.0001	0.693

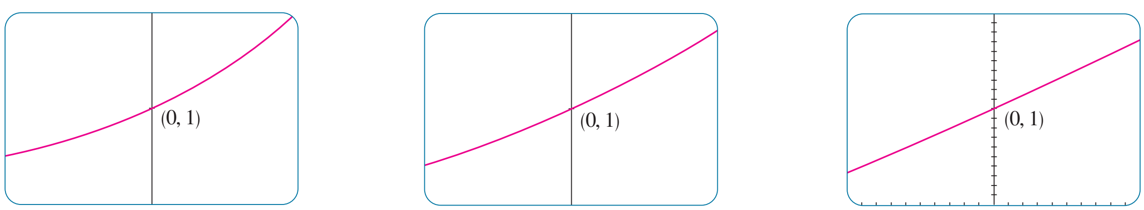
Since we are not yet able to evaluate this limit exactly, we use a calculator to approximate the values of $(2^h - 1)/h$. From the numerical evidence in the table at the left we see that as h approaches 0, these values appear to approach a number near 0.69. So our estimate is

$$f'(0) \approx 0.69$$

(b) In Figure 3 we graph the curve $y = 2^x$ and zoom in toward the point (0, 1). We see that the closer we get to (0, 1), the more the curve looks like a straight line. In fact, in Figure 3(c) the curve is practically indistinguishable from its tangent line at (0, 1). Since the x-scale and the y-scale are both 0.01, we estimate that the slope of this line is

$$\frac{0.14}{0.20} = 0.7$$

So our estimate of the derivative is $f'(0) \approx 0.7$. In Section 3.5 we will show that, correct to six decimal places, $f'(0) \approx 0.693147$.



(a) $[-1, 1]$ by $[0, 2]$ (b) $[-0.5, 0.5]$ by $[0.5, 1.5]$ (c) $[-0.1, 0.1]$ by $[0.9, 1.1]$

FIGURE 3 Zooming in on the graph of $y = 2^x$ near (0, 1)

Interpretation of the Derivative as a Rate of Change

In Section 2.6 we defined the instantaneous rate of change of $y = f(x)$ with respect to x at $x = x_1$ as the limit of the average rates of change over smaller and smaller intervals. If the interval is $[x_1, x_2]$, then the change in x is $\Delta x = x_2 - x_1$, the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

and

$$\boxed{4} \qquad \text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

From Equation 3 we recognize this limit as being the derivative of f at x_1, that is, $f'(x_1)$. This gives a second interpretation of the derivative:

> The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

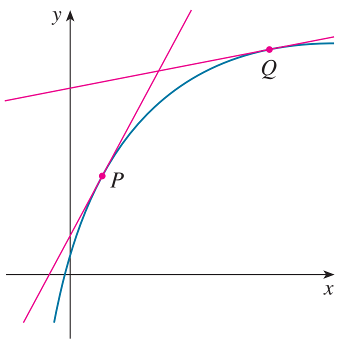


FIGURE 4
The y-values are changing rapidly at P and slowly at Q.

The connection with the first interpretation is that if we sketch the curve $y = f(x)$, then the instantaneous rate of change is the slope of the tangent to this curve at the point where $x = a$. This means that when the derivative is large (and therefore the curve is steep, as at the point P in Figure 4), the y-values change rapidly. When the derivative is small, the curve is relatively flat and the y-values change slowly.

In particular, if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the rate of change of the displacement s with respect to the time t. In other words, $f'(a)$ *is the velocity of the particle at time* $t = a$ (see Section 2.6). The *speed* of the particle is the absolute value of the velocity, that is, $|f'(a)|$.

TEC In Module 2.7 you are asked to compare and order the slopes of tangent and secant lines at several points on a curve.

EXAMPLE 4 The position of a particle is given by the equation of motion $s = f(t) = 1/(1 + t)$, where t is measured in seconds and s in meters. Find the velocity and the speed after 2 seconds.

SOLUTION The derivative of f when $t = 2$ is

$$f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{\dfrac{1}{1 + (2 + h)} - \dfrac{1}{1 + 2}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1}{3 + h} - \dfrac{1}{3}}{h} = \lim_{h \to 0} \frac{\dfrac{3 - (3 + h)}{3(3 + h)}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{3(3 + h)h} = \lim_{h \to 0} \frac{-1}{3(3 + h)} = -\frac{1}{9}$$

Thus, the velocity after 2 seconds is $f'(2) = -\frac{1}{9}$ m/s, and the speed is $|f'(2)| = |-\frac{1}{9}| = \frac{1}{9}$ m/s. ■

EXAMPLE 5 A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ dollars.

(a) What is the meaning of the derivative $f'(x)$? What are its units?

(b) In practical terms, what does it mean to say that $f'(1000) = 9$?

(c) Which do you think is greater, $f'(50)$ or $f'(500)$? What about $f'(5000)$?

SOLUTION

(a) The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x; that is, $f'(x)$ means the rate of change of the production cost with respect to the number of yards produced. (Economists call this rate of change the *marginal cost.* This idea is discussed in more detail in Sections 3.3 and 4.7.)

Because

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$$

the units for $f'(x)$ are the same as the units for the difference quotient $\Delta C/\Delta x$. Since ΔC is measured in dollars and Δx in yards, it follows that the units for $f'(x)$ are dollars per yard.

(b) The statement that $f'(1000) = 9$ means that, after 1000 yards of fabric have been manufactured, the rate at which the production cost is increasing is \$9/yard. (When $x = 1000$, C is increasing 9 times as fast as x.)

Since $\Delta x = 1$ is small compared with $x = 1000$, we could use the approximation

▲ Here we are assuming that the cost function is well behaved; in other words, $C(x)$ doesn't oscillate rapidly near $x = 1000$.

$$f'(1000) \approx \frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \Delta C$$

and say that the cost of manufacturing the 1000th yard (or the 1001st) is about \$9.

(c) The rate at which the production cost is increasing (per yard) is probably lower when $x = 500$ than when $x = 50$ (the cost of making the 500th yard is less than the cost of the 50th yard) because of economies of scale. (The manufacturer makes more efficient use of the fixed costs of production.) So

$$f'(50) > f'(500)$$

But, as production expands, the resulting large-scale operation might become inefficient and there might be overtime costs. Thus, it is possible that the rate of increase of costs will eventually start to rise. So it may happen that

$$f'(5000) > f'(500)$$ ■

The following example shows how to estimate the derivative of a tabular function, that is, a function defined not by a formula but by a table of values.

t	$P(t)$
1992	255,002,000
1994	260,292,000
1996	265,253,000
1998	270,002,000
2000	274,634,000

EXAMPLE 6 Let $P(t)$ be the population of the United States at time t. The table at the left gives approximate values of this function by providing midyear population estimates from 1992 to 2000. Interpret and estimate the value of $P'(1996)$.

SOLUTION The derivative $P'(1996)$ means the rate of change of P with respect to t when $t = 1996$, that is, the rate of increase of the population in 1996.

According to Equation 3,

$$P'(1996) = \lim_{t \to 1996} \frac{P(t) - P(1996)}{t - 1996}$$

So we compute and tabulate values of the difference quotient (the average rates of change) as follows.

t	$\dfrac{P(t) - P(1996)}{t - 1996}$
1992	2,562,750
1994	2,480,500
1998	2,374,500
2000	2,345,250

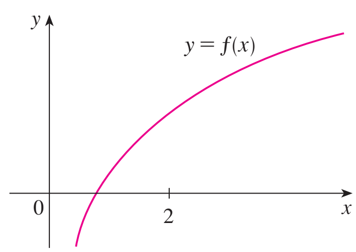
▲ Another method is to plot the population function and estimate the slope of the tangent line when $t = 1996$. (See Example 4 in Section 2.6.)

From this table we see that $P'(1996)$ lies somewhere between 2,480,500 and 2,374,500. [Here we are making the reasonable assumption that the population didn't fluctuate wildly between 1992 and 2000.] We estimate that the rate of increase of the population of the United States in 1996 was the average of these two numbers, namely

$$P'(1996) \approx 2.4 \text{ million people/year}$$

2.7 Exercises

1. On the given graph of f, mark lengths that represent $f(2)$, $f(2 + h)$, $f(2 + h) - f(2)$, and h. (Choose $h > 0$.) What line has slope $\dfrac{f(2 + h) - f(2)}{h}$?

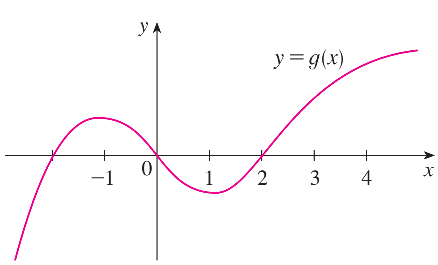


2. For the function f whose graph is shown in Exercise 1, arrange the following numbers in increasing order and explain your reasoning:

$$0 \qquad f'(2) \qquad f(3) - f(2) \qquad \tfrac{1}{2}[f(4) - f(2)]$$

3. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

$$0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$$



4. If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.

5. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.

6. Sketch the graph of a function g for which $g(0) = 0$, $g'(0) = 3$, $g'(1) = 0$, and $g'(2) = 1$.

7. If $f(x) = 3x^2 - 5x$, find $f'(2)$ and use it to find an equation of the tangent line to the parabola $y = 3x^2 - 5x$ at the point $(2, 2)$.

8. If $g(x) = 1 - x^3$, find $g'(0)$ and use it to find an equation of the tangent line to the curve $y = 1 - x^3$ at the point $(0, 1)$.

9. (a) If $F(x) = x^3 - 5x + 1$, find $F'(1)$ and use it to find an equation of the tangent line to the curve $y = x^3 - 5x + 1$ at the point $(1, -3)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

10. (a) If $G(x) = x/(1 + 2x)$, find $G'(a)$ and use it to find an equation of the tangent line to the curve $y = x/(1 + 2x)$ at the point $\left(-\tfrac{1}{4}, -\tfrac{1}{2}\right)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

11. Let $f(x) = 3^x$. Estimate the value of $f'(1)$ in two ways:
(a) By using Definition 2 and taking successively smaller values of h.
(b) By zooming in on the graph of $y = 3^x$ and estimating the slope.

12. Let $g(x) = \tan x$. Estimate the value of $g'(\pi/4)$ in two ways:
(a) By using Definition 2 and taking successively smaller values of h.
(b) By zooming in on the graph of $y = \tan x$ and estimating the slope.

13–18 ■ Find $f'(a)$.

13. $f(x) = 3 - 2x + 4x^2$

14. $f(t) = t^4 - 5t$

15. $f(t) = \dfrac{2t + 1}{t + 3}$

16. $f(x) = \dfrac{x^2 + 1}{x - 2}$

17. $f(x) = \dfrac{1}{\sqrt{x + 2}}$

18. $f(x) = \sqrt{3x + 1}$

19–24 ■ Each limit represents the derivative of some function f at some number a. State f and a in each case.

19. $\displaystyle\lim_{h \to 0} \frac{(1 + h)^{10} - 1}{h}$

20. $\displaystyle\lim_{h \to 0} \frac{\sqrt[4]{16 + h} - 2}{h}$

21. $\displaystyle\lim_{x \to 5} \frac{2^x - 32}{x - 5}$

22. $\displaystyle\lim_{x \to \pi/4} \frac{\tan x - 1}{x - \pi/4}$

23. $\displaystyle\lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$

24. $\displaystyle\lim_{t \to 1} \frac{t^4 + t - 2}{t - 1}$

25–26 ■ A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity when $t = 2$.

25. $f(t) = t^2 - 6t - 5$

26. $f(t) = 2t^3 - t + 1$

27. The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars.
(a) What is the meaning of the derivative $f'(x)$? What are its units?
(b) What does the statement $f'(800) = 17$ mean?
(c) Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

28. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.
(a) What is the meaning of the derivative $f'(5)$? What are its units?
(b) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited, would that affect your conclusion? Explain.

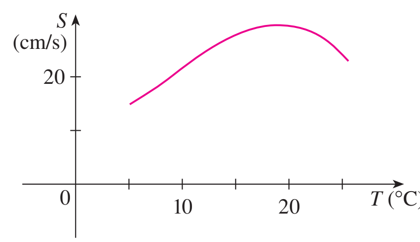
29. The fuel consumption (measured in gallons per hour) of a car traveling at a speed of v miles per hour is $c = f(v)$.
(a) What is the meaning of the derivative $f'(v)$? What are its units?
(b) Write a sentence (in layman's terms) that explains the meaning of the equation $f'(20) = -0.05$.

30. The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$.
(a) What is the meaning of the derivative $f'(8)$? What are its units?
(b) Is $f'(8)$ positive or negative? Explain.

31. Let $T(t)$ be the temperature (in °C) in Cairo, Egypt, t hours after midnight on July 21, 1999. The table shows values of this function recorded every two hours. What is the meaning of $T'(6)$? Estimate its value.

t	0	2	4	6	8	10	12	14
T	23	26	29	32	33	33	32	32

32. The graph shows the influence of the temperature T on the maximum sustainable swimming speed S of Coho salmon.
(a) What is the meaning of the derivative $S'(T)$? What are its units?
(b) Estimate the values of $S'(15)$ and $S'(25)$ and interpret them.



33. Let $C(t)$ be the amount of U.S. cash per capita in circulation at time t. The table, supplied by the Treasury Department, gives values of $C(t)$ as of June 30 of the specified year. Interpret and estimate the value of $C'(1980)$.

t	1960	1970	1980	1990
$C(t)$	\$177	\$265	\$571	\$1063

34. Life expectancy improved dramatically in the 20th century. The table gives values of $E(t)$, the life expectancy at birth (in years) of a male born in the year t in the United States. Interpret and estimate the values of $E'(1910)$ and $E'(1950)$.

t	$E(t)$	t	$E(t)$
1900	48.3	1950	65.6
1910	51.1	1960	66.6
1920	55.2	1970	67.1
1930	57.4	1980	70.0
1940	62.5	1990	71.8

35–36 ■ Determine whether or not $f'(0)$ exists.

35. $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

36. $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Writing Project

Early Methods for Finding Tangents

The first person to formulate explicitly the ideas of limits and derivatives was Sir Isaac Newton in the 1660s. But Newton acknowledged that "if I have seen farther than other men, it is because I have stood on the shoulders of giants." Two of those giants were Pierre Fermat (1601–1665) and Newton's teacher at Cambridge, Isaac Barrow (1630–1677). Newton was familiar with the methods that these men used to find tangent lines, and their methods played a role in Newton's eventual formulation of calculus.

The following references contain explanations of these methods. Read one or more of the references and write a report comparing the methods of either Fermat or Barrow to modern methods. In particular, use the method of Section 2.7 to find an equation of the tangent line to the curve $y = x^3 + 2x$ at the point (1, 3) and show how either Fermat or Barrow would have solved the same problem. Although you used derivatives and they did not, point out similarities between the methods.

1. Carl Boyer and Uta Merzbach, *A History of Mathematics* (New York: John Wiley, 1989), pp. 389, 432.
2. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 124, 132.
3. Howard Eves, *An Introduction to the History of Mathematics*, 6th ed. (New York: Saunders, 1990), pp. 391, 395.
4. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 344, 346.

2.8 The Derivative as a Function

In the preceding section we considered the derivative of a function f at a fixed number a:

1 $$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Here we change our point of view and let the number a vary. If we replace a in Equation 1 by a variable x, we obtain

2 $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we can regard f' as a new function, called the **derivative of f** and defined by Equation 2. We know that the value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

The function f' is called the derivative of f because it has been "derived" from f by the limiting operation in Equation 2. The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$ and may be smaller than the domain of f.

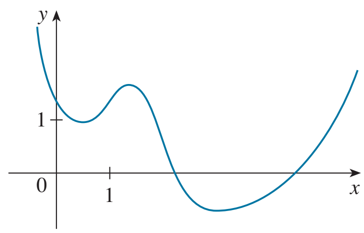


FIGURE 1

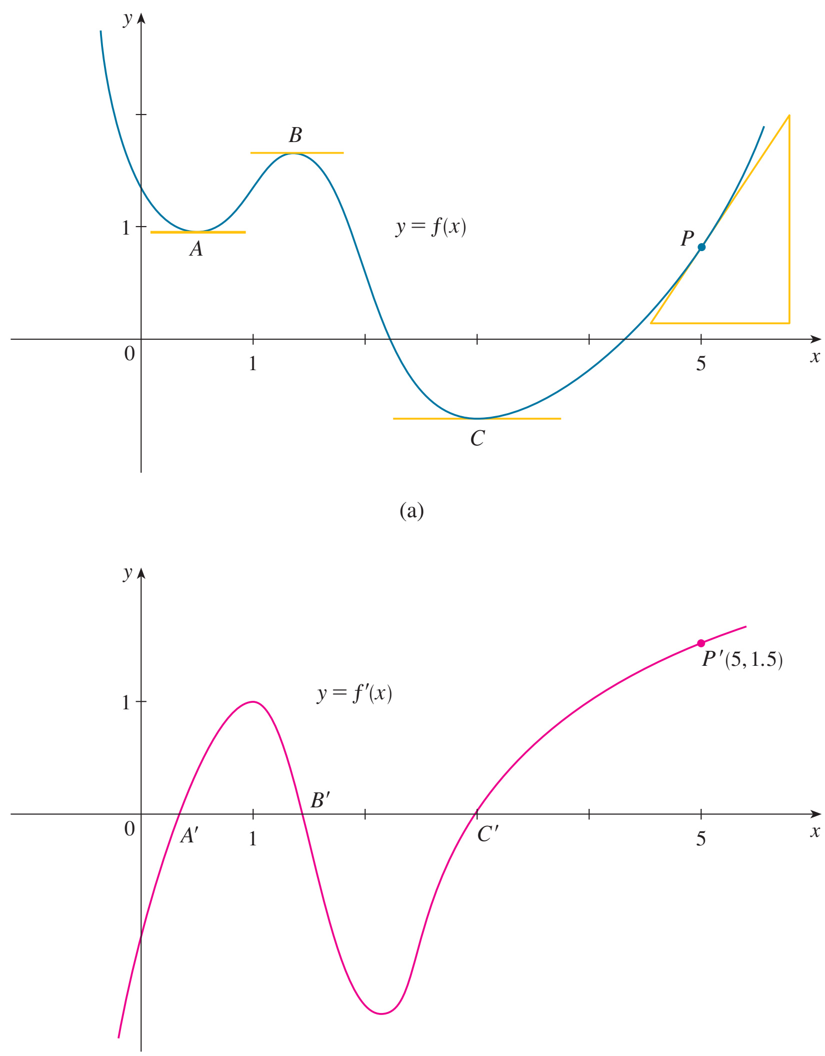
EXAMPLE 1 The graph of a function f is given in Figure 1. Use it to sketch the graph of the derivative f'.

SOLUTION We can estimate the value of the derivative at any value of x by drawing the tangent at the point $(x, f(x))$ and estimating its slope. For instance, for $x = 5$ we draw the tangent at P in Figure 2(a) and estimate its slope to be about $\frac{3}{2}$, so $f'(5) \approx 1.5$. This allows us to plot the point $P'(5, 1.5)$ on the graph of f' directly beneath P. Repeating this procedure at several points, we get the graph shown in Figure 2(b). Notice that the tangents at A, B, and C are horizontal, so the derivative is 0 there and the graph of f' crosses the x-axis at the points A', B', and C', directly beneath A, B, and C. Between A and B the tangents have positive slope, so $f'(x)$ is positive there. But between B and C the tangents have negative slope, so $f'(x)$ is negative there.

Watch an animation of the relation between a function and its derivative.

Resources / Module 3 / Derivatives as Functions / Mars Rover

Resources / Module 3 / Slope-a-Scope / Derivative of a Cubic



(b)

FIGURE 2

If a function is defined by a table of values, then we can construct a table of approximate values of its derivative, as in the next example.

t	$B(t)$
1980	9,847
1982	9,856
1984	9,855
1986	9,862
1988	9,884
1990	9,962
1992	10,036
1994	10,109
1996	10,152
1998	10,175
2000	10,186

EXAMPLE 2 Let $B(t)$ be the population of Belgium at time t. The table at the left gives midyear values of $B(t)$, in thousands, from 1980 to 2000. Construct a table of values for the derivative of this function.

SOLUTION We assume that there were no wild fluctuations in the population between the stated values. Let's start by approximating $B'(1988)$, the rate of increase of the population of Belgium in mid-1988. Since

$$B'(1988) = \lim_{h \to 0} \frac{B(1988 + h) - B(1988)}{h}$$

we have

$$B'(1988) \approx \frac{B(1988 + h) - B(1988)}{h}$$

for small values of h.

For $h = 2$, we get

$$B'(1988) \approx \frac{B(1990) - B(1988)}{2} = \frac{9962 - 9884}{2} = 39$$

(This is the average rate of increase between 1988 and 1990.) For $h = -2$, we have

$$B'(1988) \approx \frac{B(1986) - B(1988)}{-2} = \frac{9862 - 9884}{-2} = 11$$

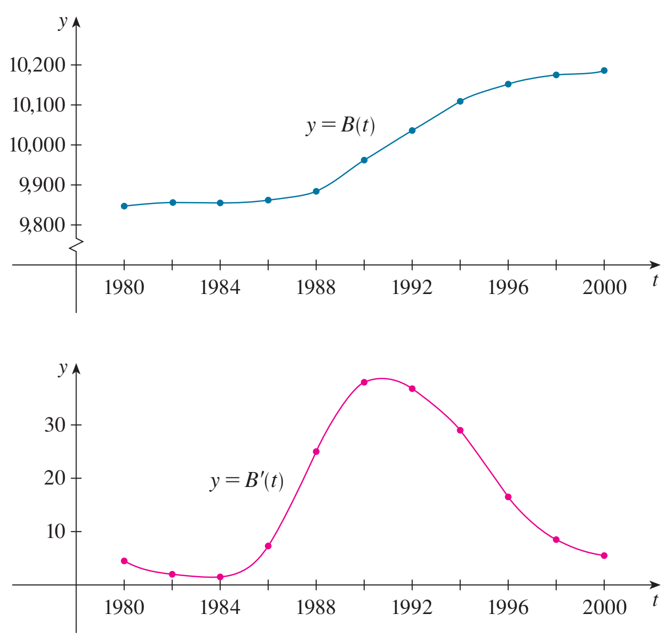
which is the average rate of increase between 1986 and 1988. We get a more accurate approximation if we take the average of these rates of change:

$$B'(1988) \approx \tfrac{1}{2}(39 + 11) = 25$$

This means that in 1988 the population was increasing at a rate of about 25,000 people per year.

Making similar calculations for the other values (except at the endpoints), we get the table of approximate values for the derivative.

t	$B'(t)$
1980	4.5
1982	2.0
1984	1.5
1986	7.3
1988	25.0
1990	38.0
1992	36.8
1994	29.0
1996	16.5
1998	8.5
2000	5.5



▲ Figure 3 illustrates Example 2 by showing graphs of the population function $B(t)$ and its derivative $B'(t)$. Notice how the rate of population growth increases to a maximum in 1990 and decreases thereafter.

FIGURE 3

EXAMPLE 3
(a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.
(b) Illustrate by comparing the graphs of f and f'.

SOLUTION
(a) When using Equation 2 to compute a derivative, we must remember that the variable is h and that x is temporarily regarded as a constant during the calculation of the limit.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1$$

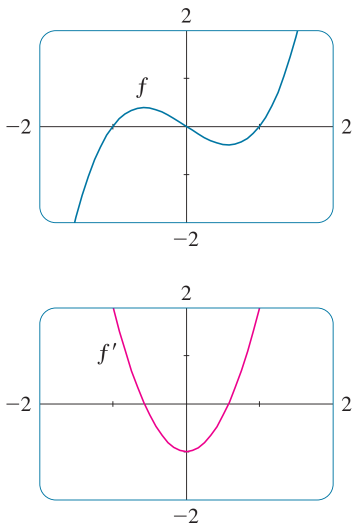


FIGURE 4

(b) We use a graphing device to graph f and f' in Figure 4. Notice that $f'(x) = 0$ when f has horizontal tangents and $f'(x)$ is positive when the tangents have positive slope. So these graphs serve as a check on our work in part (a). ■

EXAMPLE 4 If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

See more problems like these.
Resources / Module 3
/ How to Calculate
/ The Essential Examples

SOLUTION

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Here we rationalize the numerator.

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

We see that $f'(x)$ exists if $x > 0$, so the domain of f' is $(0, \infty)$. This is smaller than the domain of f, which is $[0, \infty)$. ■

Let's check to see that the result of Example 4 is reasonable by looking at the graphs of f and f' in Figure 5. When x is close to 0, $\sqrt{x}$ is also close to 0, so $f'(x) = 1/\left(2\sqrt{x}\right)$ is very large and this corresponds to the steep tangent lines near $(0, 0)$ in Figure 5(a) and the large values of $f'(x)$ just to the right of 0 in Figure 5(b). When x is large, $f'(x)$ is very small and this corresponds to the flatter tangent lines at the far right of the graph of f and the horizontal asymptote of the graph of f'.

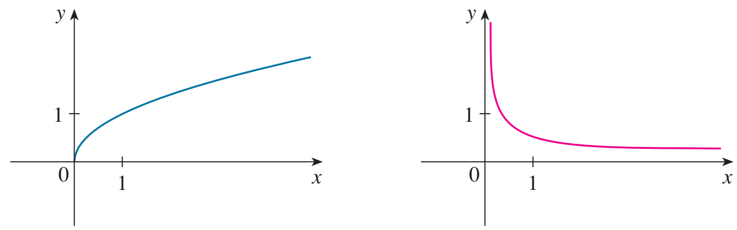


FIGURE 5 (a) $f(x) = \sqrt{x}$ (b) $f'(x) = \dfrac{1}{2\sqrt{x}}$

EXAMPLE 5 Find f' if $f(x) = \dfrac{1 - x}{2 + x}$.

SOLUTION

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1 - (x + h)}{2 + (x + h)} - \dfrac{1 - x}{2 + x}}{h}$$

$$\frac{\dfrac{a}{b} - \dfrac{c}{d}}{e} = \frac{ad - bc}{bd} \cdot \frac{1}{e}$$

$$= \lim_{h \to 0} \frac{(1 - x - h)(2 + x) - (1 - x)(2 + x + h)}{h(2 + x + h)(2 + x)}$$

$$= \lim_{h \to 0} \frac{(2 - x - 2h - x^2 - xh) - (2 - x + h - x^2 - xh)}{h(2 + x + h)(2 + x)}$$

$$= \lim_{h \to 0} \frac{-3h}{h(2 + x + h)(2 + x)}$$

$$= \lim_{h \to 0} \frac{-3}{(2 + x + h)(2 + x)} = -\frac{3}{(2 + x)^2}$$

Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

The symbol dy/dx, which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is simply a synonym for $f'(x)$. Nonetheless, it is a very useful and suggestive notation, especially when used in conjunction with increment notation. Referring to Equation 2.7.4, we can rewrite the definition of derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

▲ Gottfried Wilhelm Leibniz was born in Leipzig in 1646 and studied law, theology, philosophy, and mathematics at the university there, graduating with a bachelor's degree at age 17. After earning his doctorate in law at age 20, Leibniz entered the diplomatic service and spent most of his life traveling to the capitals of Europe on political missions. In particular, he worked to avert a French military threat against Germany and attempted to reconcile the Catholic and Protestant churches.

His serious study of mathematics did not begin until 1672 while he was on a diplomatic mission in Paris. There he built a calculating machine and met scientists, like Huygens, who directed his attention to the latest developments in mathematics and science. Leibniz sought to develop a symbolic logic and system of notation that would simplify logical reasoning. In particular, the version of calculus that he published in 1684 established the notation and the rules for finding derivatives that we use today.

Unfortunately, a dreadful priority dispute arose in the 1690s between the followers of Newton and those of Leibniz as to who had invented calculus first. Leibniz was even accused of plagiarism by members of the Royal Society in England. The truth is that each man invented calculus independently. Newton arrived at his version of calculus first but, because of his fear of controversy, did not publish it immediately. So Leibniz's 1684 account of calculus was the first to be published.

If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$.

3 Definition A function f is **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

EXAMPLE 6 Where is the function $f(x) = |x|$ differentiable?

SOLUTION If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$ we have

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{(x + h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

and so f is differentiable for any $x > 0$.

Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small enough that $x + h < 0$ and so $|x + h| = -(x + h)$. Therefore, for $x < 0$,

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{-(x + h) - (-x)}{h} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

and so f is differentiable for any $x < 0$.

For $x = 0$ we have to investigate

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0 + h| - |0|}{h} \qquad \text{(if it exists)}$$

Let's compute the left and right limits separately:

$$\lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

and $$\lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

Since these limits are different, $f'(0)$ does not exist. Thus, f is differentiable at all x except 0.

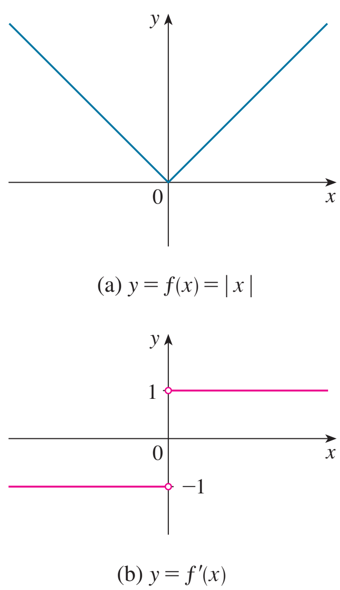


FIGURE 6

A formula for f' is given by

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and its graph is shown in Figure 6(b). The fact that $f'(0)$ does not exist is reflected geometrically in the fact that the curve $y = |x|$ does not have a tangent line at (0, 0). [See Figure 6(a).]

Both continuity and differentiability are desirable properties for a function to have. The following theorem shows how these properties are related.

4 Theorem If f is differentiable at a, then f is continuous at a.

Proof To prove that f is continuous at a, we have to show that $\lim_{x \to a} f(x) = f(a)$. We do this by showing that the difference $f(x) - f(a)$ approaches 0.

The given information is that f is differentiable at a, that is,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists (see Equation 2.7.3). To connect the given and the unknown, we divide and multiply $f(x) - f(a)$ by $x - a$ (which we can do when $x \neq a$):

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}\,(x - a)$$

Thus, using the Product Law and (2.7.3), we can write

$$\begin{aligned} \lim_{x \to a} [f(x) - f(a)] &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}\,(x - a) \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} (x - a) \\ &= f'(a) \cdot 0 = 0 \end{aligned}$$

To use what we have just proved, we start with $f(x)$ and add and subtract $f(a)$:

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} [f(a) + (f(x) - f(a))] \\ &= \lim_{x \to a} f(a) + \lim_{x \to a} [f(x) - f(a)] \\ &= f(a) + 0 = f(a) \end{aligned}$$

Therefore, f is continuous at a. ■

NOTE • The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

(See Example 7 in Section 2.3.) But in Example 6 we showed that f is not differentiable at 0.

How Can a Function Fail to be Differentiable?

We saw that the function $y = |x|$ in Example 6 is not differentiable at 0 and Figure 6(a) shows that its graph changes direction abruptly when $x = 0$. In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]

Theorem 4 gives another way for a function not to have a derivative. It says that if f is not continuous at a, then f is not differentiable at a. So at any discontinuity (for instance, a jump discontinuity) f fails to be differentiable.

A third possibility is that the curve has a **vertical tangent line** when $x = a$, that is, f is continuous at a and

$$\lim_{x \to a} |f'(x)| = \infty$$

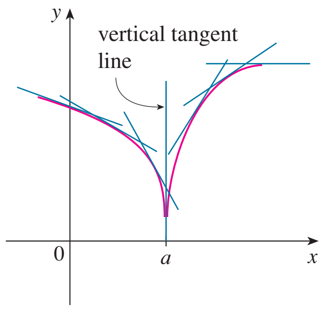
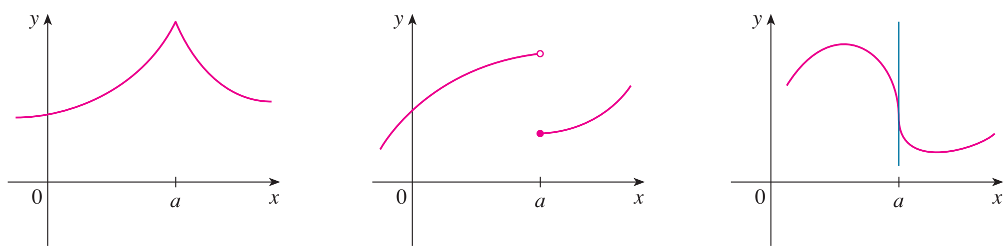


FIGURE 7

This means that the tangent lines become steeper and steeper as $x \to a$. Figure 7 shows one way that this can happen; Figure 8(c) shows another. Figure 8 illustrates the three possibilities that we have discussed.



(a) A corner (b) A discontinuity (c) A vertical tangent

FIGURE 8
Three ways for f not to be differentiable at a

A graphing calculator or computer provides another way of looking at differentiability. If f is differentiable at a, then when we zoom in toward the point $(a, f(a))$ the graph straightens out and appears more and more like a line. (See Figure 9. We saw a specific example of this in Figure 3 in Section 2.7.) But no matter how much we zoom in toward a point like the ones in Figures 7 and 8(a), we can't eliminate the sharp point or corner (see Figure 10).

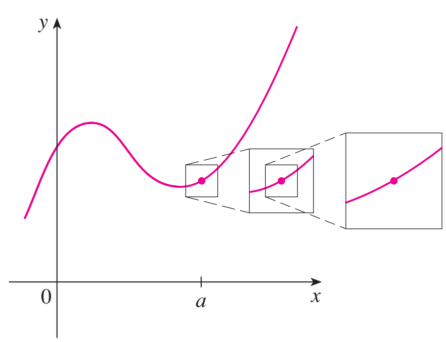


FIGURE 9
f is differentiable at a.

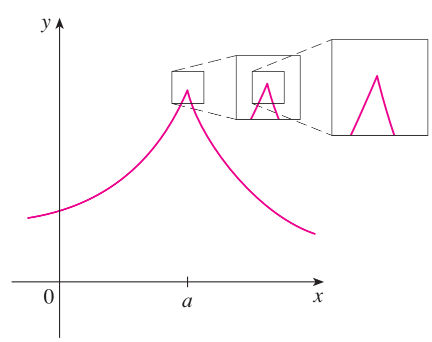


FIGURE 10
f is not differentiable at a.

The Second Derivative

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

EXAMPLE 7 If $f(x) = x^3 - x$, find and interpret $f''(x)$.

SOLUTION In Example 3 we found that the first derivative is $f'(x) = 3x^2 - 1$. So the second derivative is

$$\begin{aligned} f''(x) &= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h \to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h} \\ &= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h} \\ &= \lim_{h \to 0} (6x + 3h) = 6x \end{aligned}$$

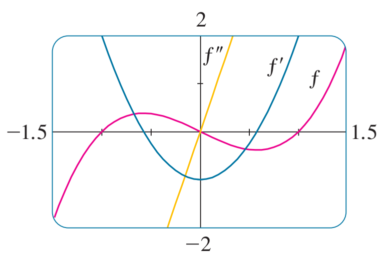


FIGURE 11

TEC Module 2.8A guides you in determining properties of the derivative f' by examining the graphs of a variety of functions f.

TEC In Module 2.8B you can see how changing the coefficients of a polynomial f affects the appearance of the graphs of f, f', and f''.

The graphs of f, f', f'' are shown in Figure 11.

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

Notice from Figure 11 that $f''(x)$ is negative when $y = f'(x)$ has negative slope and positive when $y = f'(x)$ has positive slope. So the graphs serve as a check on our calculations.

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration,* which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus, the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

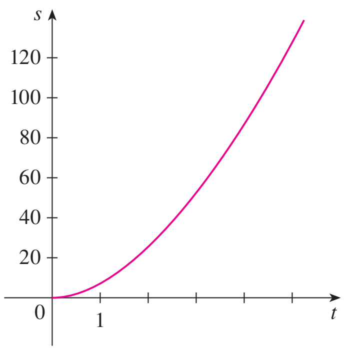


FIGURE 12
Position function of a car

▲ The units for acceleration are feet per second per second, written as ft/s^2.

EXAMPLE 8 A car starts from rest and the graph of its position function is shown in Figure 12, where s is measured in feet and t in seconds. Use it to graph the velocity and acceleration of the car. What is the acceleration at $t = 2$ seconds?

SOLUTION By measuring the slope of the graph of $s = f(t)$ at $t = 0, 1, 2, 3, 4,$ and 5, and using the method of Example 1, we plot the graph of the velocity function $v = f'(t)$ in Figure 13. The acceleration when $t = 2$ s is $a = f''(2)$, the slope of the tangent line to the graph of f' when $t = 2$. We estimate the slope of this tangent line to be

$$a(2) = f''(2) = v'(2) \approx \frac{27}{3} = 9 \text{ ft/s}^2$$

Similar measurements enable us to graph the acceleration function in Figure 14.

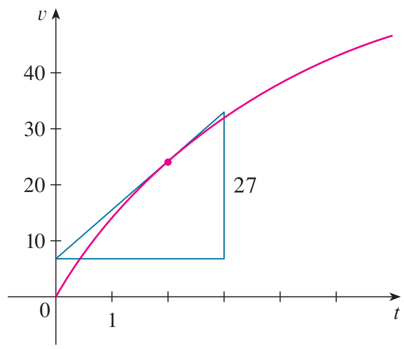


FIGURE 13
Velocity function

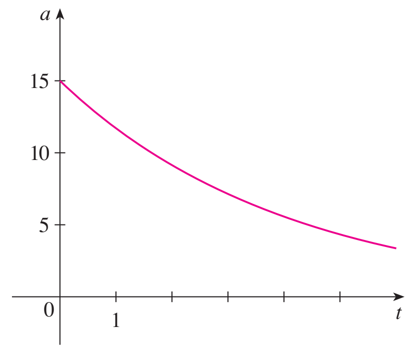


FIGURE 14
Acceleration function ■

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$. If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

EXAMPLE 9 If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

SOLUTION In Example 7 we found that $f''(x) = 6x$. The graph of the second derivative has equation $y = 6x$ and so it is a straight line with slope 6. Since the derivative $f'''(x)$ is the slope of $f''(x)$, we have

$$f'''(x) = 6$$

for all values of x. So f''' is a constant function and its graph is a horizontal line. Therefore, for all values of x,

$$f^{(4)}(x) = 0$$

■

We can interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line. Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

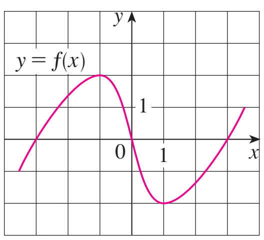
Thus, the jerk j is the rate of change of acceleration. It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

We have seen that one application of second and third derivatives occurs in analyzing the motion of objects using acceleration and jerk. We will investigate another application of second derivatives in Section 2.10, where we show how knowledge of f'' gives us information about the shape of the graph of f. In Section 8.9 we will see how second and higher derivatives enable us to obtain more accurate approximations of functions than linear approximations and also to represent functions as sums of infinite series.

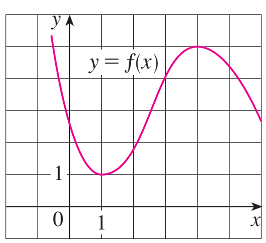
2.8 Exercises

1–2 ■ Use the given graph to estimate the value of each derivative. Then sketch the graph of f'.

1. (a) $f'(-3)$
(b) $f'(-2)$
(c) $f'(-1)$
(d) $f'(0)$
(e) $f'(1)$
(f) $f'(2)$
(g) $f'(3)$

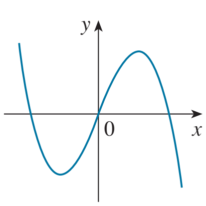


2. (a) $f'(0)$
(b) $f'(1)$
(c) $f'(2)$
(d) $f'(3)$
(e) $f'(4)$
(f) $f'(5)$

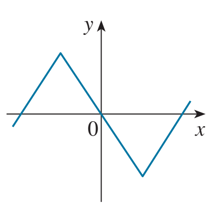


3. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.

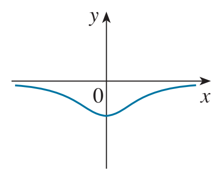
(a)



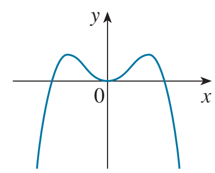
(b)



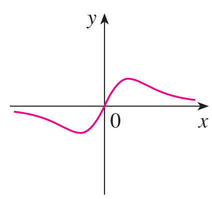
(c)



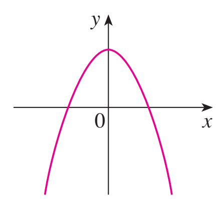
(d)



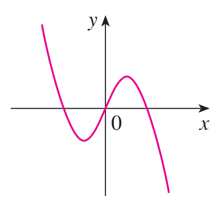
I



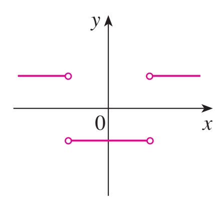
II



III

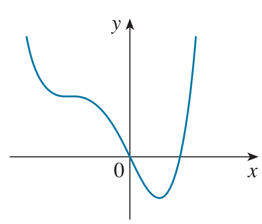


IV

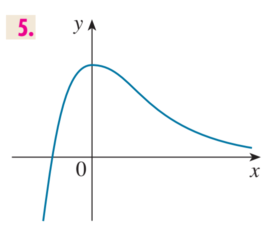


4–11 ■ Trace or copy the graph of the given function f. (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of f' below it.

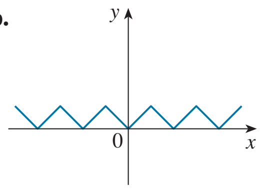
4.



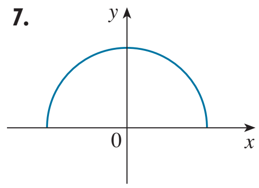
5.



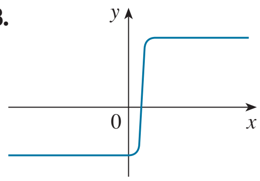
6.



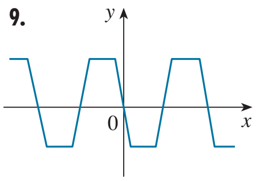
7.



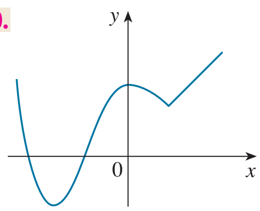
8.



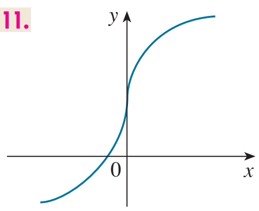
9.



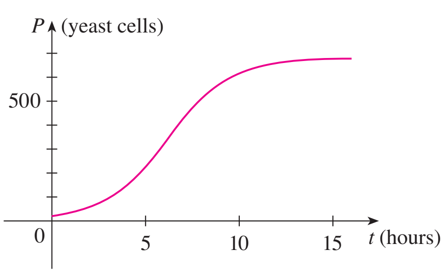
10.



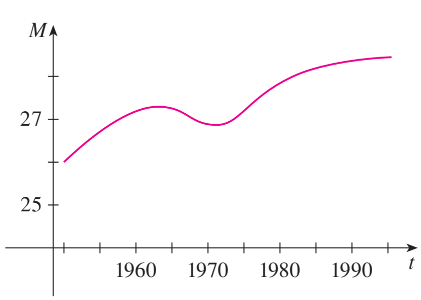
11.



12. Shown is the graph of the population function $P(t)$ for yeast cells in a laboratory culture. Use the method of Example 1 to graph the derivative $P'(t)$. What does the graph of P' tell us about the yeast population?



13. The graph shows how the average age of first marriage of Japanese men has varied in the last half of the 20th century. Sketch the graph of the derivative function $M'(t)$. During which years was the derivative negative?



14–16 ■ Make a careful sketch of the graph of f and below it sketch the graph of f' in the same manner as in Exercises 4–11. Can you guess a formula for $f'(x)$ from its graph?

14. $f(x) = \sin x$

15. $f(x) = e^x$

16. $f(x) = \ln x$

17. Let $f(x) = x^2$.

(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, and $f'(2)$ by using a graphing device to zoom in on the graph of f.

(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, and $f'(-2)$.

(c) Use the results from parts (a) and (b) to guess a formula for $f'(x)$.

(d) Use the definition of a derivative to prove that your guess in part (c) is correct.

18. Let $f(x) = x^3$.

(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, $f'(2)$, and $f'(3)$ by using a graphing device to zoom in on the graph of f.

(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.

(c) Use the values from parts (a) and (b) to graph f'.

(d) Guess a formula for $f'(x)$.

(e) Use the definition of a derivative to prove that your guess in part (d) is correct.

19–25 ■ Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

19. $f(x) = 4 - 7x$

20. $f(x) = 5 - 4x + 3x^2$

21. $f(x) = x^3 - 3x + 5$

22. $f(x) = x + \sqrt{x}$

23. $g(x) = \sqrt{1 + 2x}$

24. $f(x) = \dfrac{3 + x}{1 - 3x}$

25. $G(t) = \dfrac{4t}{t + 1}$

26. (a) Sketch the graph of $f(x) = \sqrt{6 - x}$ by starting with the graph of $y = \sqrt{x}$ and using the transformations of Section 1.3.

(b) Use the graph from part (a) to sketch the graph of f'.

(c) Use the definition of a derivative to find $f'(x)$. What are the domains of f and f'?

(d) Use a graphing device to graph f' and compare with your sketch in part (b).

27. (a) If $f(x) = x - (2/x)$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

28. (a) If $f(t) = 6/(1 + t^2)$, find $f'(t)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

29. The unemployment rate $U(t)$ varies with time. The table (from the Bureau of Labor Statistics) gives the percentage of unemployed in the U.S. labor force from 1989 to 1998.

t	$U(t)$	t	$U(t)$
1989	5.3	1994	6.1
1990	5.6	1995	5.6
1991	6.8	1996	5.4
1992	7.5	1997	4.9
1993	6.9	1998	4.5

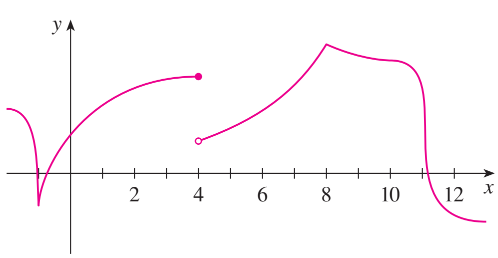
(a) What is the meaning of $U'(t)$? What are its units?
(b) Construct a table of values for $U'(t)$.

30. Let the smoking rate among high-school seniors at time t be $S(t)$. The table (from the Institute of Social Research, University of Michigan) gives the percentage of seniors who reported that they had smoked one or more cigarettes per day during the past 30 days.

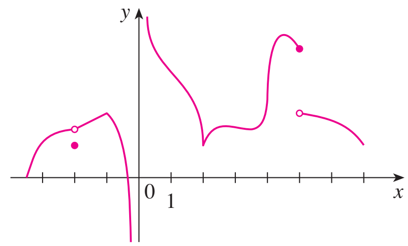
t	$S(t)$	t	$S(t)$
1980	21.4	1990	19.1
1982	21.0	1992	17.2
1984	18.7	1994	19.4
1986	18.7	1996	22.2
1988	18.1	1998	22.4

(a) What is the meaning of $S'(t)$? What are its units?
(b) Construct a table of values for $S'(t)$.
(c) Graph S and S'.
(d) How would it be possible to get more accurate values for $S'(t)$?

31. The graph of f is given. State, with reasons, the numbers at which f is not differentiable.



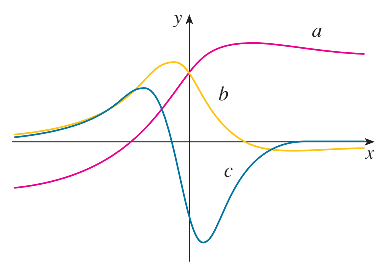
32. The graph of g is given.
(a) At what numbers is g discontinuous? Why?
(b) At what numbers is g not differentiable? Why?



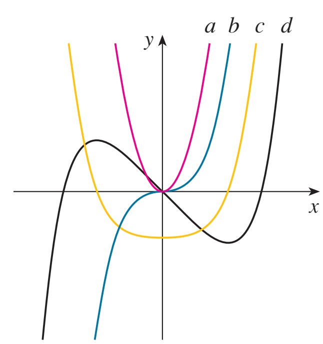
33. Graph the function $f(x) = x + \sqrt{|x|}$. Zoom in repeatedly, first toward the point $(-1, 0)$ and then toward the origin. What is different about the behavior of f in the vicinity of these two points? What do you conclude about the differentiability of f?

34. Zoom in toward the points $(1, 0)$, $(0, 1)$, and $(-1, 0)$ on the graph of the function $g(x) = (x^2 - 1)^{2/3}$. What do you notice? Account for what you see in terms of the differentiability of g.

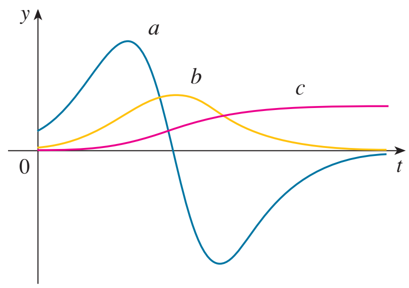
35. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



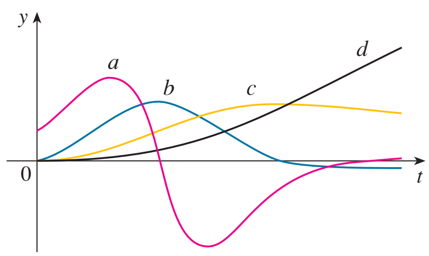
36. The figure shows graphs of f, f', f'', and f'''. Identify each curve, and explain your choices.



37. The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.



38. The figure shows the graphs of four functions. One is the position function of a car, one is the velocity of the car, one is its acceleration, and one is its jerk. Identify each curve, and explain your choices.



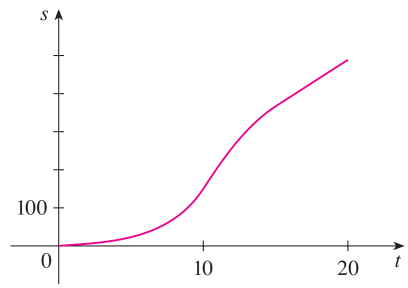
39–40 ■ Use the definition of a derivative to find $f'(x)$ and $f''(x)$. Then graph f, f', and f'' on a common screen and check to see if your answers are reasonable.

39. $f(x) = 1 + 4x - x^2$

40. $f(x) = 1/x$

41. If $f(x) = 2x^2 - x^3$, find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$. Graph f, f', f'', and f''' on a common screen. Are the graphs consistent with the geometric interpretations of these derivatives?

42. (a) The graph of a position function of a car is shown, where s is measured in feet and t in seconds. Use it to graph the velocity and acceleration of the car. What is the acceleration at $t = 10$ seconds?



(b) Use the acceleration curve from part (a) to estimate the jerk at $t = 10$ seconds. What are the units for jerk?

43. Let $f(x) = \sqrt[3]{x}$.
(a) If $a \neq 0$, use Equation 2.7.3 to find $f'(a)$.
(b) Show that $f'(0)$ does not exist.
(c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at $(0, 0)$. (Recall the shape of the graph of f. See Figure 13 in Section 1.2.)

44. (a) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.
(b) If $a \neq 0$, find $g'(a)$.
(c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0, 0)$.
(d) Illustrate part (c) by graphing $y = x^{2/3}$.

45. Show that the function $f(x) = |x - 6|$ is not differentiable at 6. Find a formula for f' and sketch its graph.

46. Where is the greatest integer function $f(x) = [\![x]\!]$ not differentiable? Find a formula for f' and sketch its graph.

47. Recall that a function f is called *even* if $f(-x) = f(x)$ for all x in its domain and *odd* if $f(-x) = -f(x)$ for all such x. Prove each of the following.
(a) The derivative of an even function is an odd function.
(b) The derivative of an odd function is an even function.

48. When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running.
(a) Sketch a possible graph of T as a function of the time t that has elapsed since the faucet was turned on.
(b) Describe how the rate of change of T with respect to t varies as t increases.
(c) Sketch a graph of the derivative of T.

49. Let ℓ be the tangent line to the parabola $y = x^2$ at the point $(1, 1)$. The *angle of inclination* of ℓ is the angle ϕ that ℓ makes with the positive direction of the x-axis. Calculate ϕ correct to the nearest degree.

2.9 Linear Approximations

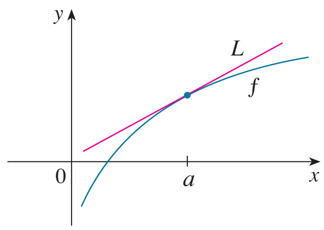


FIGURE 1

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. (See Figure 2 in Section 2.6 and Figure 3 in Section 2.7.) This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f. So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. (See Figure 1.) The following example illustrates the method.

EXAMPLE 1 Use a linear approximation to estimate the values of $2^{0.1}$ and $2^{0.4}$.

SOLUTION The desired values are values of the function $f(x) = 2^x$ near $a = 0$. From Example 3 in Section 2.7 we know that the slope of the tangent line to the curve $y = 2^x$ at the point (0, 1) is $f'(0) \approx 0.69$. So an equation of the tangent line is approximately

$$y - 1 = 0.69(x - 0) \qquad \text{or} \qquad y = 1 + 0.69x$$

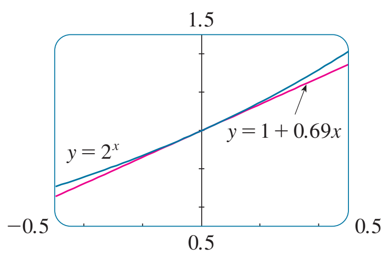


FIGURE 2

Because the tangent line lies close to the curve when $x = 0.1$ (see Figure 2), the value of the function is almost the same as the height of the tangent line when $x = 0.1$. Thus

$$2^{0.1} = f(0.1) \approx 1 + 0.69(0.1) = 1.069$$

Similarly,

$$2^{0.4} = f(0.4) \approx 1 + 0.69(0.4) = 1.276$$

It appears from Figure 2 that our estimate for $2^{0.1}$ is better than our estimate for $2^{0.4}$ and that both estimates are less than the true values because the tangent line lies below the curve. In fact, this is correct because the true values of these numbers are

$$2^{0.1} = 1.07177\ldots \qquad 2^{0.4} = 1.31950\ldots$$

In general, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

▲ We will see in Sections 3.8 and 8.9 that linear approximations are very useful in physics for the purpose of simplifying a calculation or even an entire theory. Sometimes it is easier to measure the derivative of a function than to measure the function itself. Then the derivative measurement can be used in the linear approximation to estimate the function.

EXAMPLE 2 Find the linear approximation for the function $f(x) = \sqrt{x}$ at $a = 1$. Then use it to approximate the numbers $\sqrt{0.99}$, $\sqrt{1.01}$, and $\sqrt{1.05}$. Are these approximations overestimates or underestimates?

SOLUTION We first have to find $f'(1)$, the slope of the tangent line to $y = \sqrt{x}$ when $x = 1$. We could estimate $f'(1)$ using numerical or graphical methods as in Section 2.7, or we could find the value exactly using the definition of a derivative. In fact, in Example 4 in Section 2.8, we found that

$$f'(x) = \frac{1}{2\sqrt{x}}$$

and so $f'(1) = \frac{1}{2}$. Therefore, an equation of the tangent line at (1, 1) is

$$y - 1 = \tfrac{1}{2}(x - 1) \qquad \text{or} \qquad y = \tfrac{1}{2}x + \tfrac{1}{2}$$

and the linear approximation is

$$\sqrt{x} \approx L(x) = \tfrac{1}{2}x + \tfrac{1}{2}$$

In particular, we have

$$\sqrt{0.99} \approx L(0.99) = \tfrac{1}{2}(0.99) + \tfrac{1}{2} = 0.995$$

$$\sqrt{1.01} \approx L(1.01) = \tfrac{1}{2}(1.01) + \tfrac{1}{2} = 1.005$$

$$\sqrt{1.05} \approx L(1.05) = \tfrac{1}{2}(1.05) + \tfrac{1}{2} = 1.025$$

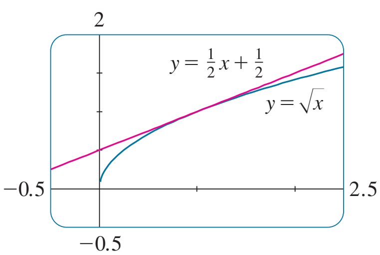


FIGURE 3

In Figure 3 we graph the root function $y = \sqrt{x}$ and its linear approximation $L(x) = \frac{1}{2}x + \frac{1}{2}$. We see that our approximations are overestimates because the tangent line lies above the curve.

In the following table we compare the estimates from the linear approximation with the true values. Notice from this table, and also from Figure 3, that the tangent line approximation gives good estimates when x is close to 1 but the accuracy of the approximation deteriorates when x is farther away from 1.

	From $L(x)$	Actual value
$\sqrt{0.99}$	0.995	0.99498743 . . .
$\sqrt{1.001}$	1.0005	1.00049987 . . .
$\sqrt{1.01}$	1.005	1.00498756 . . .
$\sqrt{1.05}$	1.025	1.02469507 . . .
$\sqrt{1.1}$	1.05	1.04880884 . . .
$\sqrt{1.5}$	1.25	1.22474487 . . .
$\sqrt{2}$	1.5	1.41421356 . . .

Of course, a calculator can give us better approximations than the linear approximations we found in Examples 1 and 2. But a linear approximation gives an approximation over an entire *interval* and that is the reason that scientists often use such approximations. (See Sections 3.8 and 8.9.)

The following example is typical of situations in which we use linear approximation to predict the future behavior of a function given by empirical data.

EXAMPLE 3 Suppose that after you stuff a turkey its temperature is 50 °F and you then put it in a 325 °F oven. After an hour the meat thermometer indicates that the temperature of the turkey is 93 °F and after two hours it indicates 129 °F. Predict the temperature of the turkey after three hours.

SOLUTION If $T(t)$ represents the temperature of the turkey after t hours, we are given that $T(0) = 50$, $T(1) = 93$, and $T(2) = 129$. In order to make a linear approximation with $a = 2$, we need an estimate for the derivative $T'(2)$. Because

$$T'(2) = \lim_{t \to 2} \frac{T(t) - T(2)}{t - 2}$$

we could estimate $T'(2)$ by the difference quotient with $t = 1$:

$$T'(2) \approx \frac{T(1) - T(2)}{1 - 2} = \frac{93 - 129}{-1} = 36$$

This amounts to approximating the instantaneous rate of temperature change by the average rate of change between $t = 1$ and $t = 2$, which is 36 °F/h. With this estimate, the linear approximation for the temperature after 3 h is

$$\begin{aligned} T(3) &\approx T(2) + T'(2)(3 - 2) \\ &\approx 129 + 36 \cdot 1 = 165 \end{aligned}$$

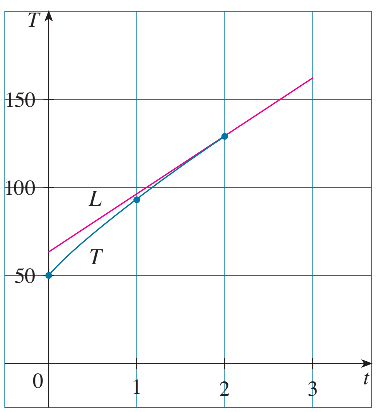


FIGURE 4

So the predicted temperature after three hours is 165 °F.

We obtain a more accurate estimate for $T'(2)$ by plotting the given data, as in Figure 4, and estimating the slope of the tangent line at $t = 2$ to be

$$T'(2) \approx 33$$

Then our linear approximation becomes

$$T(3) \approx T(2) + T'(2) \cdot 1 \approx 129 + 33 = 162$$

and our improved estimate for the temperature is 162 °F.

Because the temperature curve lies below the tangent line, it appears that the actual temperature after three hours will be somewhat less than 162 °F, perhaps closer to 160 °F. ■

2.9 Exercises

1. (a) If $f(x) = 3^x$, estimate the value of $f'(0)$ either numerically or graphically.
   (b) Use the tangent line to the curve $y = 3^x$ at $(0, 1)$ to find approximate values for $3^{0.05}$ and $3^{0.1}$.
   (c) Graph the curve and its tangent line. Are the approximations in part (b) less than or greater than the true values? Why?

2. (a) If $f(x) = \ln x$, estimate the value of $f'(1)$ graphically.
   (b) Use the tangent line to the curve $y = \ln x$ at $(1, 0)$ to estimate the values of ln 0.9 and ln 1.3.
   (c) Graph the curve and its tangent line. Are the estimates in part (b) less than or greater than the true values? Why?

3. (a) If $f(x) = \sqrt[3]{x}$, estimate the value of $f'(1)$.
   (b) Find the linear approximation for f at $a = 1$.
   (c) Use part (b) to estimate the cube roots of the numbers 0.5, 0.9, 0.99, 1.01, 1.1, 1.5, and 2. Compare these estimates with the values of the cube roots from your calculator. Did you obtain underestimates or overestimates? Which of your estimates are the most accurate?
   (d) Graph the curve $y = \sqrt[3]{x}$ and its tangent line at $(1, 1)$. Use these graphs to explain your results from part (c).

4. (a) If $f(x) = \cos x$, estimate the value of $f'(\pi/3)$.
   (b) Find the linear approximation for f at $a = \pi/3$.

(c) Use the linear approximation to estimate the values of cos 1, cos 1.1, cos 1.5, and cos 2. Are these underestimates or overestimates? Which of your estimates are the most accurate?

(d) Graph the curve $y = \cos x$ and its tangent line at $(\pi/3, \frac{1}{2})$. Use these graphs to explain your results from part (c).

5–6 ■

(a) Use the definition of a derivative to compute $f'(1)$.

(b) Use the linear approximation for f at $a = 1$ to estimate $f(x)$ for $x = 0.9, 0.95, 0.99, 1.01, 1.05$, and 1.1. How do these estimates compare with the actual values?

(c) Graph f and its tangent line at (1, 1). Do the graphs support your comments in part (b)?

5. $f(x) = x^2$ **6.** $f(x) = x^3$

7. The turkey in Example 3 is removed from the oven when its temperature reaches 185 °F and is placed on a table in a room where the temperature is 75 °F. After 10 minutes the temperature of the turkey is 172 °F and after 20 minutes it is 160 °F. Use a linear approximation to predict the temperature of the turkey after half an hour. Do you think your prediction is an overestimate or an underestimate? Why?

8. Atmospheric pressure P decreases as altitude h increases. At a temperature of 15 °C, the pressure is 101.3 kilopascals (kPa) at sea level, 87.1 kPa at $h = 1$ km, and 74.9 kPa at $h = 2$ km. Use a linear approximation to estimate the atmospheric pressure at an altitude of 3 km.

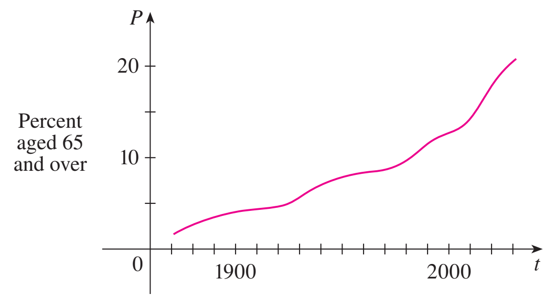
9. The table lists the amount of U.S. cash per capita in circulation as of June 30 in the given year. Use a linear approximation to estimate the amount of cash per capita in circulation in the year 2000. Is your prediction an underestimate or an overestimate? Why?

t	1960	1970	1980	1990
$C(t)$	\$177	\$265	\$571	\$1063

10. The table shows the population of Nepal (in millions) as of June 30 of the given year. Use a linear approximation to estimate the population at midyear in 1984. Use another linear approximation to predict the population in 2006.

t	1980	1985	1990	1995	2000
$N(t)$	15.0	17.0	19.3	22.0	24.9

11. The graph indicates how Australia's population is aging by showing the past and projected percentage of the population aged 65 and over. Use a linear approximation to predict the percentage of the population that will be 65 and over in the years 2040 and 2050. Do you think your predictions are too high or too low? Why?

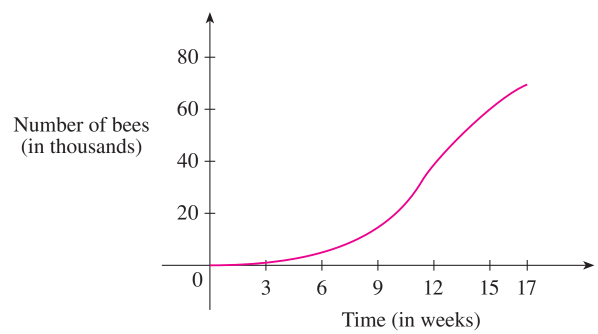


12. The figure shows the graph of a population of Cyprian honeybees raised in an apiary.

(a) Use a linear approximation to predict the bee population after 18 weeks and after 20 weeks.

(b) Are your predictions underestimates or overestimates? Why?

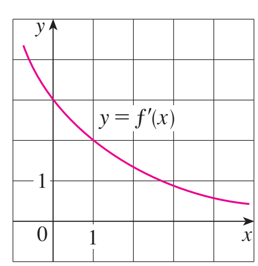
(c) Which of your predictions do you think is the more accurate? Why?



13. Suppose that the only information we have about a function f is that $f(1) = 5$ and the graph of its *derivative* is as shown.

(a) Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.

(b) Are your estimates in part (a) too large or too small? Explain.



14. Suppose that we don't have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x) = \sqrt{x^2 + 5}$ for all x.

(a) Use a linear approximation to estimate $g(1.95)$ and $g(2.05)$.

(b) Are your estimates in part (a) too large or too small? Explain.

2.10 What Does f' Say About f?

Many of the applications of calculus depend on our ability to deduce facts about a function f from information concerning its derivatives. Because $f'(x)$ represents the slope of the curve $y = f(x)$ at the point $(x, f(x))$, it tells us the direction in which the curve proceeds at each point. So it is reasonable to expect that information about $f'(x)$ will provide us with information about $f(x)$.

In particular, to see how the derivative of f can tell us where a function is increasing or decreasing, look at Figure 1. (Increasing functions and decreasing functions were defined in Section 1.1.) Between A and B and between C and D, the tangent lines have positive slope and so $f'(x) > 0$. Between B and C, the tangent lines have negative slope and so $f'(x) < 0$. Thus, it appears that f increases when $f'(x)$ is positive and decreases when $f'(x)$ is negative.

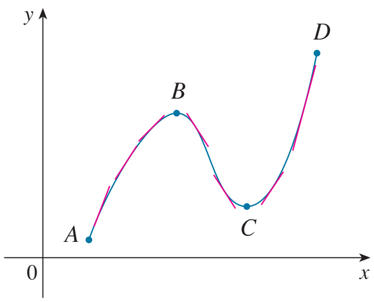


FIGURE 1

It turns out, as we will see in Chapter 4, that what we observed for the function graphed in Figure 1 is always true. We state the general result as follows.

> If $f'(x) > 0$ on an interval, then f is increasing on that interval.
> If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

EXAMPLE 1

(a) If it is known that the graph of the derivative f' of a function is as shown in Figure 2, what can we say about f?
(b) If it is known that $f(0) = 0$, sketch a possible graph of f.

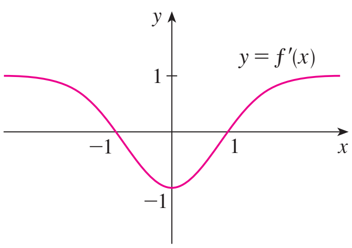


FIGURE 2

SOLUTION

(a) We observe from Figure 2 that $f'(x)$ is negative when $-1 < x < 1$, so the original function f must be decreasing on the interval $(-1, 1)$. Similarly, $f'(x)$ is positive for $x < -1$ and for $x > 1$, so f is increasing on the intervals $(-\infty, -1)$ and $(1, \infty)$. Also note that, since $f'(-1) = 0$ and $f'(1) = 0$, the graph of f has horizontal tangents when $x = \pm 1$.

(b) We use the information from part (a), and the fact that the graph passes through the origin, to sketch a possible graph of f in Figure 3. Notice that $f'(0) = -1$, so we have drawn the curve $y = f(x)$ passing through the origin with a slope of -1. Notice also that $f'(x) \to 1$ as $x \to \pm\infty$ (from Figure 2). So the slope of the curve $y = f(x)$ approaches 1 as x becomes large (positive or negative). That is why we have drawn the graph of f in Figure 3 progressively straighter as $x \to \pm\infty$. ■

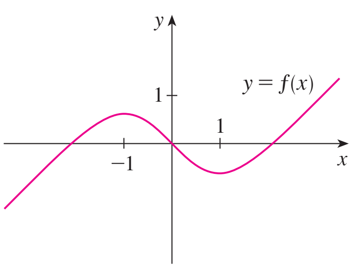


FIGURE 3

TEC In Module 2.10 you can practice using graphical information about f' to determine the shape of the graph of f.

We say that the function f in Example 1 has a **local maximum** at -1 because near $x = -1$ the values of $f(x)$ are at least as big as the neighboring values. Note that $f'(x)$ is positive to the left of -1 and negative just to the right of -1. Similarly, f has a **local minimum** at 1, where the derivative changes from negative to positive. In Chapter 4 we will develop these observations into a general method for finding optimal values of functions.

What Does f'' Say about f?

Let's see how the sign of $f''(x)$ affects the appearance of the graph of f. Since $f'' = (f')'$, we know that if $f''(x)$ is positive, then f' is an increasing function. This says that the slopes of the tangent lines of the curve $y = f(x)$ increase from left to right. Figure 4 shows the graph of such a function. The slope of this curve becomes progressively larger as x increases and we observe that, as a consequence, the curve bends upward. Such a curve is called **concave upward**. In Figure 5, however, $f''(x)$ is negative, which means that f' is decreasing. Thus, the slopes of f decrease from left to right and the curve bends downward. This curve is called **concave downward**. We summarize our discussion as follows. (Concavity is discussed in greater detail in Section 4.3.)

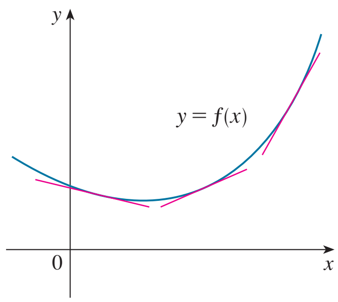


FIGURE 4
Since $f''(x) > 0$, the slopes increase and f is concave upward.

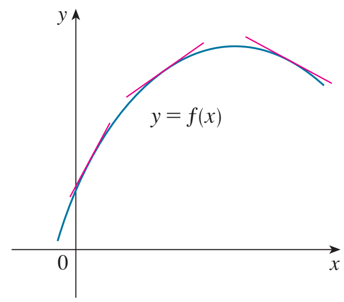


FIGURE 5
Since $f''(x) < 0$, the slopes decrease and f is concave downward.

> If $f''(x) > 0$ on an interval, then f is concave upward on that interval
> If $f''(x) < 0$ on an interval, then f is concave downward on that interval.

EXAMPLE 2 Figure 6 shows a population graph for Cyprian honeybees raised in an apiary. How does the rate of population increase change over time? When is this rate highest? Over what intervals is P concave upward or concave downward?

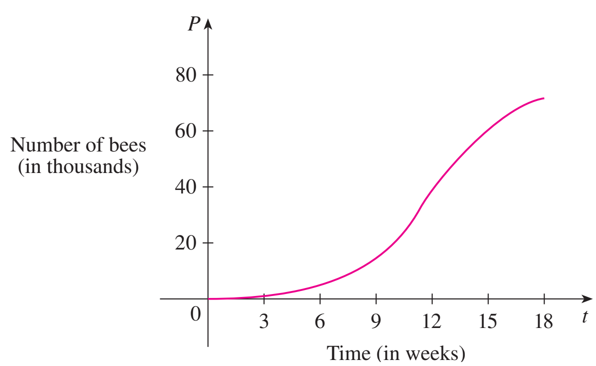


FIGURE 6

SOLUTION By looking at the slope of the curve as t increases, we see that the rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t = 12$ weeks, and decreases as the population begins to level off. As the population approaches its maximum value of about 75,000 (called the *carrying capacity*), the rate of increase, $P'(t)$, approaches 0. The curve appears to be concave upward on (0, 12) and concave downward on (12, 18). ■

In Example 2, the population curve changed from concave upward to concave downward at approximately the point (12, 38,000). This point is called an *inflection* *point* of the curve. The significance of this point is that the rate of population increase has its maximum value there. In general, an **inflection point** is a point where a curve changes its direction of concavity.

EXAMPLE 3 Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$

(ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$

(iii) $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$

SOLUTION Condition (i) tells us that f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. Condition (ii) says that f is concave upward on $(-\infty, -2)$ and $(2, \infty)$, and concave downward on $(-2, 2)$. From condition (iii) we know that the graph of f has two horizontal asymptotes: $y = -2$ and $y = 0$.

We first draw the horizontal asymptote $y = -2$ as a dashed line (see Figure 7). We then draw the graph of f approaching this asymptote at the far left, increasing to its maximum point at $x = 1$ and decreasing toward the x-axis as $x \to \infty$. We also make sure that the graph has inflection points when $x = -2$ and 2. Notice that the curve bends upward for $x < -2$ and $x > 2$, and bends downward when x is between -2 and 2.

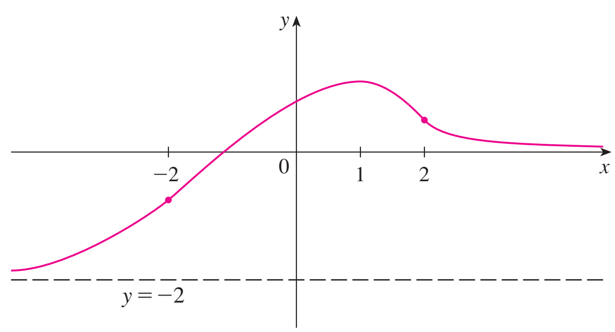


FIGURE 7

Antiderivatives

In many problems in mathematics and its applications, we are given a function f and we are required to find a function F whose derivative is f. If such a function F exists, we call it an *antiderivative* of f. In other words, an **antiderivative** of f is a function F such that $F' = f$. (In Example 1 we sketched an antiderivative f of the function f'.)

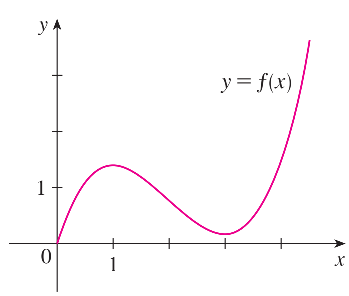


FIGURE 8

EXAMPLE 4 Let F be an antiderivative of the function f whose graph is shown in Figure 8.

(a) Where is F increasing or decreasing?

(b) Where is F concave upward or concave downward?

(c) At what values of x does F have an inflection point?

(d) If $F(0) = 1$, sketch the graph of F.

(e) How many antiderivatives does f have?

SOLUTION

(a) We see from Figure 8 that $f(x) > 0$ for all $x > 0$. Since F is an antiderivative of f, we have $F'(x) = f(x)$ and so $F'(x)$ is positive when $x > 0$. This means that F is increasing on $(0, \infty)$.

(b) F is concave upward when $F''(x) > 0$. But $F''(x) = f'(x)$, so F is concave upward when $f'(x) > 0$, that is, when f is increasing. From Figure 8 we see that f is increasing when when $0 < x < 1$ and when $x > 3$. So F is concave upward on $(0, 1)$ and $(3, \infty)$. F is concave downward when $F''(x) = f'(x) < 0$, that is, when f is decreasing. So F is concave downward on $(1, 3)$.

(c) F has an inflection point when the direction of concavity changes. From part (b) we know that F changes from concave upward to concave downward at $x = 1$, so F has an inflection point there. F changes from concave downward to concave upward when $x = 3$, so F has another inflection point when $x = 3$.

(d) In sketching the graph of F, we use the information from parts (a), (b), and (c). But, for finer detail, we also bear in mind the meaning of an antiderivative: Because $F'(x) = f(x)$, the slope of $y = F(x)$ at any value of x is equal to the height of $y = f(x)$. (Of course, this is the exact opposite of the procedure we used in Example 1 in Section 2.8 to sketch a derivative.)

Therefore, since $f(0) = 0$, we start drawing the graph of F at the given point $(0, 1)$ with slope 0, always increasing, with upward concavity to $x = 1$, downward concavity to $x = 3$, and upward concavity when $x > 3$. (See Figure 9.) Notice that $f(3) \approx 0.2$, so $y = F(x)$ has a gentle slope at the second inflection point. But we see that the slope becomes steeper when $x > 3$.

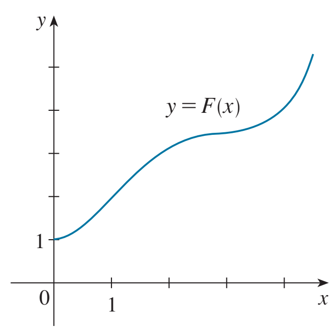


FIGURE 9
An antiderivative of f

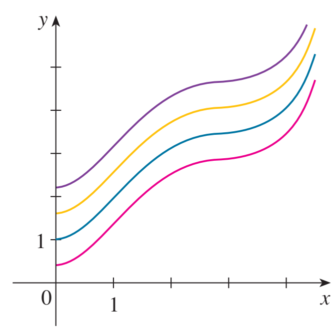


FIGURE 10
Members of the family of antiderivatives of f

(e) The antiderivative of f that we sketched in Figure 9 satisfies $F(0) = 1$, so its graph starts at the point $(0, 1)$. But there are many other antiderivatives, whose graphs start at other points on the y-axis. In fact, f has infinitely many antiderivatives; their graphs are obtained from the graph of F by shifting upward or downward as in Figure 10. ■

2.10 Exercises

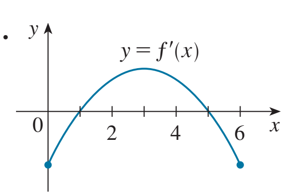
1–2 ■ The graph of the *derivative* f' of a function f is shown.

(a) On what intervals is f increasing or decreasing?

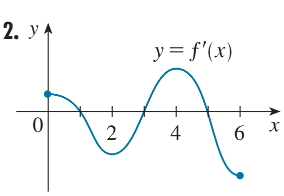
(b) At what values of x does f have a local maximum or minimum?

(c) If it is known that $f(0) = 0$, sketch a possible graph of f.

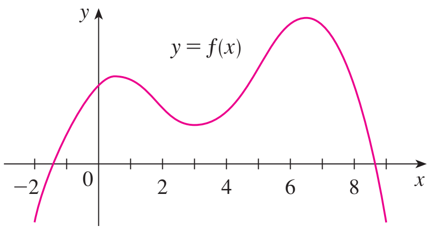
1.



2.



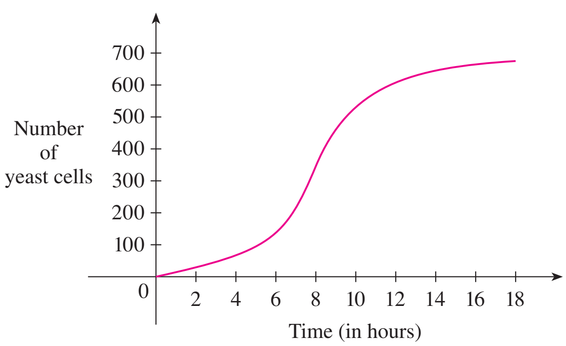
3. Use the given graph of f to estimate the intervals on which the derivative f' is increasing or decreasing.



4. (a) Sketch a curve whose slope is always positive and increasing.
(b) Sketch a curve whose slope is always positive and decreasing.
(c) Give equations for curves with these properties.

5. The president announces that the national deficit is increasing, but at a decreasing rate. Interpret this statement in terms of a function and its derivatives.

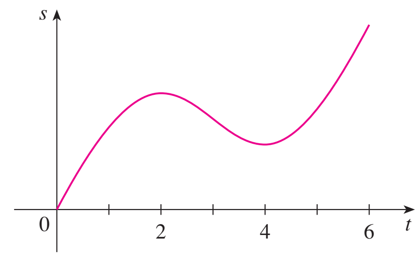
6. A graph of a population of yeast cells in a new laboratory culture as a function of time is shown.
(a) Describe how the rate of population increase varies.
(b) When is this rate highest?
(c) On what intervals is the population function concave upward or downward?
(d) Estimate the coordinates of the inflection point.



7. The table gives population densities for ring-necked pheasants (in number of pheasants per acre) on Pelee Island, Ontario.
(a) Describe how the rate of change of population varies.
(b) Estimate the inflection points of the graph. What is the significance of these points?

t	1927	1930	1932	1934	1936	1938	1940
$P(t)$	0.1	0.6	2.5	4.6	4.8	3.5	3.0

8. A particle is moving along a horizontal straight line. The graph of its position function (the distance to the right of a fixed point as a function of time) is shown.
(a) When is the particle moving toward the right and when is it moving toward the left?
(b) When does the particle have positive acceleration and when does it have negative acceleration?

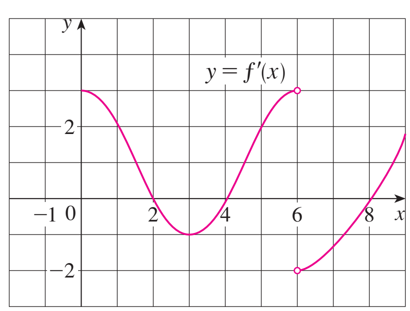


9. Let $K(t)$ be a measure of the knowledge you gain by studying for a test for t hours. Which do you think is larger, $K(8) - K(7)$ or $K(3) - K(2)$? Is the graph of K concave upward or concave downward? Why?

10. Coffee is being poured into the mug shown in the figure at a constant rate (measured in volume per unit time). Sketch a rough graph of the depth of the coffee in the mug as a function of time. Account for the shape of the graph in terms of concavity. What is the significance of the inflection point?

11–12 ■ The graph of the derivative f' of a continuous function f is shown.
(a) On what intervals is f increasing or decreasing?
(b) At what values of x does f have a local maximum or minimum?
(c) On what intervals is f concave upward or downward?
(d) State the x-coordinate(s) of the point(s) of inflection.
(e) Assuming that $f(0) = 0$, sketch a graph of f.

11.



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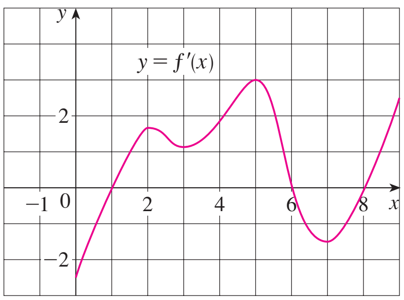


13. Sketch the graph of a function whose first and second derivatives are always negative.

14. Sketch the graph of a function whose first derivative is always negative and whose second derivative is always positive.

15–20 ■ Sketch the graph of a function that satisfies all of the given conditions.

15. $f'(x) > 0$ if $x < 2$, $f'(x) > 0$ if $x > 2$, $f'(2) = 0$

16. $f''(x) < 0$ if $x < 2$, $f''(x) < 0$ if $x > 2$, f is not differentiable at 2

17. $f'(-1) = f'(1) = 0$, $f'(x) < 0$ if $|x| < 1$, $f'(x) > 0$ if $|x| > 1$, $f(-1) = 4$, $f(1) = 0$, $f''(x) < 0$ if $x < 0$, $f''(x) > 0$ if $x > 0$

18. $f'(-1) = 0$, $f'(1)$ does not exist, $f'(x) < 0$ if $|x| < 1$, $f'(x) > 0$ if $|x| > 1$, $f(-1) = 4$, $f(1) = 0$, $f''(x) < 0$ if $x \neq 1$

19. $f'(2) = 0$, $f(2) = -1$, $f(0) = 0$, $f'(x) < 0$ if $0 < x < 2$, $f'(x) > 0$ if $x > 2$, $f''(x) < 0$ if $0 \leqslant x < 1$ or if $x > 4$, $f''(x) > 0$ if $1 < x < 4$, $\lim_{x \to \infty} f(x) = 1$, $f(-x) = f(x)$ for all x

20. $\lim_{x \to 3} f(x) = -\infty$, $f''(x) < 0$ if $x \neq 3$, $f'(0) = 0$, $f'(x) > 0$ if $x < 0$ or $x > 3$, $f'(x) < 0$ if $0 < x < 3$

21. Suppose $f'(x) = xe^{-x^2}$.
(a) On what interval is f increasing? On what interval is f decreasing?
(b) Does f have a maximum or minimum value?

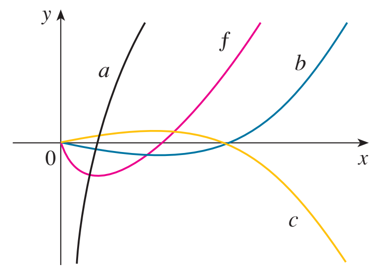
22. If $f'(x) = e^{-x^2}$, what can you say about f?

23. Let $f(x) = x^3 - x$. In Examples 3 and 7 in Section 2.8, we showed that $f'(x) = 3x^2 - 1$ and $f''(x) = 6x$. Use these facts to find the following.
(a) The intervals on which f is increasing or decreasing.
(b) The intervals on which f is concave upward or downward.
(c) The inflection point of f.

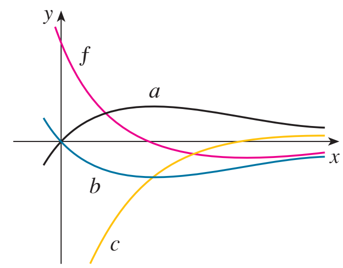
24. Let $f(x) = x^4 - 2x^2$.
(a) Use the definition of a derivative to find $f'(x)$ and $f''(x)$.
(b) On what intervals is f increasing or decreasing?
(c) On what intervals is f concave upward or concave downward?

25–26 ■ The graph of a function f is shown. Which graph is an antiderivative of f and why?

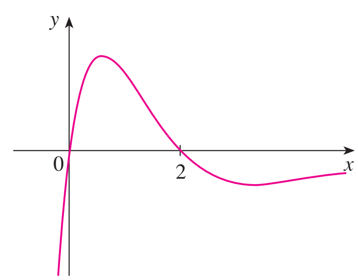
25.



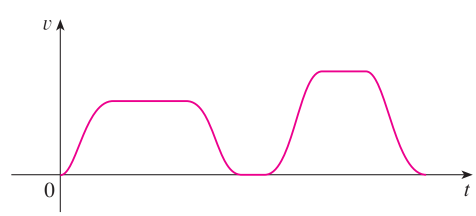
26.



27. The graph of a function is shown in the figure. Make a rough sketch of an antiderivative F, given that $F(0) = 0$.



28. The graph of the velocity function of a car is shown in the figure. Sketch the graph of the position function.



29–30 ■ Draw a graph of f and use it to make a rough sketch of the antiderivative that passes through the origin.

29. $f(x) = \sin(x^2)$, $0 \leqslant x \leqslant 4$

30. $f(x) = 1/(x^4 + 1)$

2 Review

CONCEPT CHECK

1. Explain what each of the following means and illustrate with a sketch.
   (a) $\lim_{x \to a} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$
   (c) $\lim_{x \to a^-} f(x) = L$ (d) $\lim_{x \to a} f(x) = \infty$
   (e) $\lim_{x \to \infty} f(x) = L$

2. Describe several ways in which a limit can fail to exist. Illustrate with sketches.

3. State the following Limit Laws.
   (a) Sum Law (b) Difference Law
   (c) Constant Multiple Law (d) Product Law
   (e) Quotient Law (f) Power Law
   (g) Root Law

4. What does the Squeeze Theorem say?

5. (a) What does it mean to say that the line $x = a$ is a vertical asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.
   (b) What does it mean to say that the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.

6. Which of the following curves have vertical asymptotes? Which have horizontal asymptotes?
   (a) $y = x^4$ (b) $y = \sin x$ (c) $y = \tan x$
   (d) $y = \tan^{-1}x$ (e) $y = e^x$ (f) $y = \ln x$
   (g) $y = 1/x$ (h) $y = \sqrt{x}$

7. (a) What does it mean for f to be continuous at a?
   (b) What does it mean for f to be continuous on the interval $(-\infty, \infty)$? What can you say about the graph of such a function?

8. What does the Intermediate Value Theorem say?

9. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

10. Suppose an object moves along a straight line with position $f(t)$ at time t. Write an expression for the instantaneous velocity of the object at time $t = a$. How can you interpret this velocity in terms of the graph of f?

11. If $y = f(x)$ and x changes from x_1 to x_2, write expressions for the following.
    (a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$.
    (b) The instantaneous rate of change of y with respect to x at $x = x_1$.

12. Define the derivative $f'(a)$. Discuss two ways of interpreting this number.

13. Define the second derivative of f. If $f(t)$ is the position function of a particle, how can you interpret the second derivative?

14. (a) What does it mean for f to be differentiable at a?
    (b) What is the relation between the differentiability and continuity of a function?

15. (a) What does the sign of $f'(x)$ tell us about f?
    (b) What does the sign of $f''(x)$ tell us about f?

16. (a) Define the linear approximation to f at a.
    (b) Define an antiderivative of f.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\lim_{x \to 4} \left(\frac{2x}{x-4} - \frac{8}{x-4} \right) = \lim_{x \to 4} \frac{2x}{x-4} - \lim_{x \to 4} \frac{8}{x-4}$

2. $\lim_{x \to 1} \frac{x^2 + 6x - 7}{x^2 + 5x - 6} = \frac{\lim_{x \to 1}(x^2 + 6x - 7)}{\lim_{x \to 1}(x^2 + 5x - 6)}$

3. $\lim_{x \to 1} \frac{x - 3}{x^2 + 2x - 4} = \frac{\lim_{x \to 1}(x - 3)}{\lim_{x \to 1}(x^2 + 2x - 4)}$

4. If $\lim_{x \to 5} f(x) = 2$ and $\lim_{x \to 5} g(x) = 0$, then $\lim_{x \to 5} [f(x)/g(x)]$ does not exist.

5. If $\lim_{x \to 5} f(x) = 0$ and $\lim_{x \to 5} g(x) = 0$, then $\lim_{x \to 5} [f(x)/g(x)]$ does not exist.

6. If $\lim_{x \to 6} f(x)g(x)$ exists, then the limit must be $f(6)g(6)$.

7. If p is a polynomial, then $\lim_{x \to b} p(x) = p(b)$.

8. If $\lim_{x \to 0} f(x) = \infty$ and $\lim_{x \to 0} g(x) = \infty$, then $\lim_{x \to 0} [f(x) - g(x)] = 0$.

9. A function can have two different horizontal asymptotes.

10. If f has domain $[0, \infty)$ and has no horizontal asymptote, then $\lim_{x \to \infty} f(x) = \infty$ or $\lim_{x \to \infty} f(x) = -\infty$.

11. If the line $x = 1$ is a vertical asymptote of $y = f(x)$, then f is not defined at 1.

12. If $f(1) > 0$ and $f(3) < 0$, then there exists a number c between 1 and 3 such that $f(c) = 0$.

13. If f is continuous at 5 and $f(5) = 2$ and $f(4) = 3$, then $\lim_{x \to 2} f(4x^2 - 11) = 2$.

14. If f is continuous on $[-1, 1]$ and $f(-1) = 4$ and $f(1) = 3$, then there exists a number r such that $|r| < 1$ and $f(r) = \pi$.

15. If f is continuous at a, then f is differentiable at a.

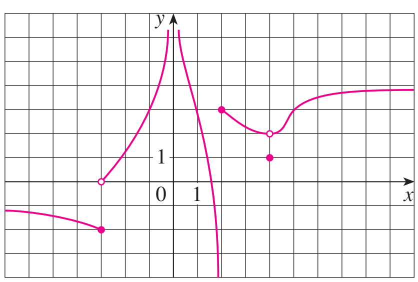
16. If $f'(r)$ exists, then $\lim_{x \to r} f(x) = f(r)$.

17. $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$

18. If $f(x) > 1$ for all x and $\lim_{x \to 0} f(x)$ exists, then $\lim_{x \to 0} f(x) > 1$.

EXERCISES

1. The graph of f is given.
(a) Find each limit, or explain why it does not exist.
(i) $\lim_{x \to 2^+} f(x)$ (ii) $\lim_{x \to -3^+} f(x)$
(iii) $\lim_{x \to -3} f(x)$ (iv) $\lim_{x \to 4} f(x)$
(v) $\lim_{x \to 0} f(x)$ (vi) $\lim_{x \to 2^-} f(x)$
(vii) $\lim_{x \to \infty} f(x)$ (viii) $\lim_{x \to -\infty} f(x)$
(b) State the equations of the horizontal asymptotes.
(c) State the equations of the vertical asymptotes.
(d) At what numbers is f discontinuous? Explain.



2. Sketch the graph of a function f that satisfies all of the following conditions:
$\lim_{x \to 0^+} f(x) = -2$, $\lim_{x \to 0^-} f(x) = 1$, $f(0) = -1$,
$\lim_{x \to 2^-} f(x) = \infty$, $\lim_{x \to 2^+} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = 3$,
$\lim_{x \to -\infty} f(x) = 4$

3–16 ■ Find the limit.

3. $\lim_{x \to 1} e^{x^3 - x}$

4. $\lim_{x \to 3} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

5. $\lim_{x \to -3} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

6. $\lim_{x \to 1^+} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

7. $\lim_{h \to 0} \dfrac{(h - 1)^3 + 1}{h}$

8. $\lim_{t \to 2} \dfrac{t^2 - 4}{t^3 - 8}$

9. $\lim_{r \to 9} \dfrac{\sqrt{r}}{(r - 9)^4}$

10. $\lim_{v \to 4^+} \dfrac{4 - v}{|4 - v|}$

11. $\lim_{x \to \infty} e^{-3x}$

12. $\lim_{x \to 10^-} \ln(100 - x^2)$

13. $\lim_{x \to 0} \dfrac{1 - \sqrt{1 - x^2}}{x}$

14. $\lim_{x \to -\infty} \dfrac{5x^3 - x^2 + 2}{2x^3 + x - 3}$

15. $\lim_{x \to \infty} \dfrac{\sqrt{3x^2 - 1}}{x - 1}$

16. $\lim_{x \to \infty} \arctan(x^3 - x)$

17–18 ■ Use graphs to discover the asymptotes of the curve. Then prove what you have discovered.

17. $y = \dfrac{\cos^2 x}{x^2}$

18. $y = \sqrt{x^2 + x + 1} - \sqrt{x^2 - x}$

19. If $2x - 1 \leqslant f(x) \leqslant x^2$ for $0 < x < 3$, find $\lim_{x \to 1} f(x)$.

20. Prove that $\lim_{x \to 0} x^2 \cos(1/x^2) = 0$.

21. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leqslant x < 3 \\ (x - 3)^2 & \text{if } x > 3 \end{cases}$$

(a) Evaluate each limit, if it exists.
(i) $\lim_{x \to 0^+} f(x)$ (ii) $\lim_{x \to 0^-} f(x)$ (iii) $\lim_{x \to 0} f(x)$
(iv) $\lim_{x \to 3^-} f(x)$ (v) $\lim_{x \to 3^+} f(x)$ (vi) $\lim_{x \to 3} f(x)$
(b) Where is f discontinuous?
(c) Sketch the graph of f.

22. Show that each function is continuous on its domain. State the domain.

(a) $g(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 2}$ (b) $h(x) = xe^{\sin x}$

23–24 ■ Use the Intermediate Value Theorem to show that there is a root of the equation in the given interval.

23. $2x^3 + x^2 + 2 = 0$, $(-2, -1)$

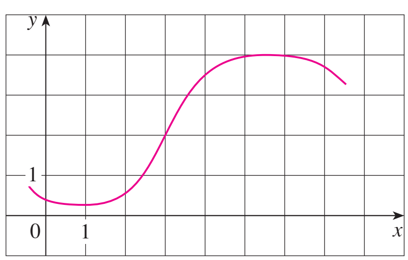
24. $e^{-x^2} = x$, $(0, 1)$

25. The displacement (in meters) of an object moving in a straight line is given by $s = 1 + 2t + t^2/4$, where t is measured in seconds.
(a) Find the average velocity over the following time periods.
(i) $[1, 3]$ (ii) $[1, 2]$
(iii) $[1, 1.5]$ (iv) $[1, 1.1]$
(b) Find the instantaneous velocity when $t = 1$.

26. According to Boyle's Law, if the temperature of a confined gas is held fixed, then the product of the pressure P and the volume V is a constant. Suppose that, for a certain gas, $PV = 800$, where P is measured in pounds per square inch and V is measured in cubic inches.
(a) Find the average rate of change of P as V increases from 200 in^3 to 250 in^3.
(b) Express V as a function of P and show that the instantaneous rate of change of V with respect to P is inversely proportional to the square of P.

27. For the function f whose graph is shown, arrange the following numbers in increasing order:

$$0 \qquad 1 \qquad f'(2) \qquad f'(3) \qquad f'(5) \qquad f''(5)$$



28. (a) Use the definition of a derivative to find $f'(2)$, where $f(x) = x^3 - 2x$.
(b) Find an equation of the tangent line to the curve $y = x^3 - 2x$ at the point $(2, 4)$.
(c) Illustrate part (b) by graphing the curve and the tangent line on the same screen.

29. (a) If $f(x) = e^{-x^2}$, estimate the value of $f'(1)$ graphically and numerically.
(b) Find an approximate equation of the tangent line to the curve $y = e^{-x^2}$ at the point where $x = 1$.
(c) Illustrate part (b) by graphing the curve and the tangent line on the same screen.

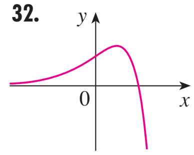
30. Find a function f and a number a such that

$$\lim_{h \to 0} \frac{(2 + h)^6 - 64}{h} = f'(a)$$

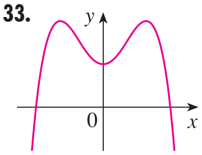
31. The total cost of repaying a student loan at an interest rate of r% per year is $C = f(r)$.
(a) What is the meaning of the derivative $f'(r)$? What are its units?
(b) What does the statement $f'(10) = 1200$ mean?
(c) Is $f'(r)$ always positive or does it change sign?

32–34 ■ Trace or copy the graph of the function. Then sketch a graph of its derivative directly beneath.

32.



33.



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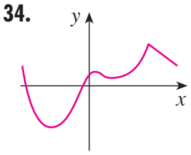
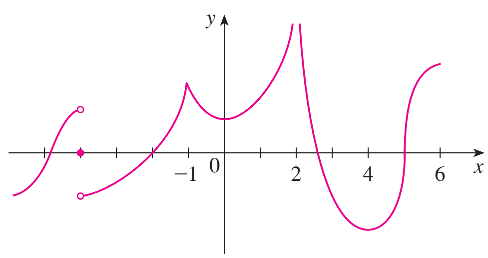


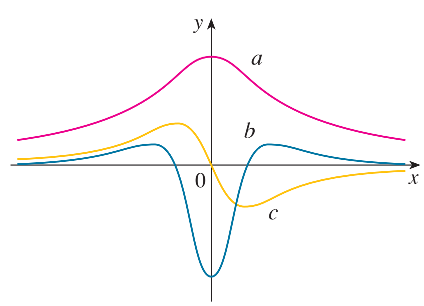
35. (a) If $f(x) = \sqrt{3 - 5x}$, use the definition of a derivative to find $f'(x)$.
(b) Find the domains of f and f'.
(c) Graph f and f' on a common screen. Compare the graphs to see whether your answer to part (a) is reasonable.

36. (a) Find the asymptotes of the graph of $f(x) = (4 - x)/(3 + x)$ and use them to sketch the graph.
(b) Use your graph from part (a) to sketch the graph of f'.
(c) Use the definition of a derivative to find $f'(x)$.
(d) Use a graphing device to graph f' and compare with your sketch in part (b).

37. The graph of f is shown. State, with reasons, the numbers at which f is not differentiable.



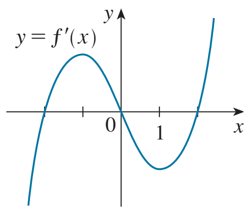
38. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



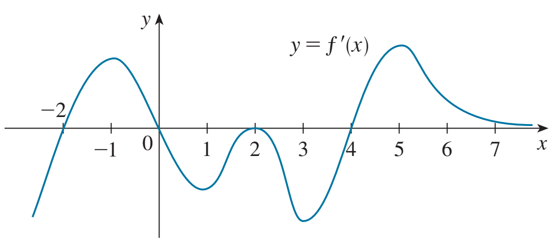
39. (a) If $f(x) = e^x$, what is the value of $f'(0)$?
(b) Find the linear approximation for f at $a = 0$.
(c) Use the linear approximation to estimate the values of $e^{-0.2}$, $e^{-0.1}$, $e^{-0.01}$, $e^{0.01}$, $e^{0.1}$, and $e^{0.2}$.
(d) Are your approximations overestimates or underestimates? Which of your estimates are the most accurate?

40. The cost of living continues to rise, but at a slower rate. In terms of a function and its derivatives, what does this statement mean?

41. The graph of the *derivative* f' of a function f is given.
(a) On what intervals is f increasing or decreasing?
(b) At what values of x does f have a local maximum or minimum?
(c) Where is f concave upward or downward?
(d) If $f(0) = 0$, sketch a possible graph of f.



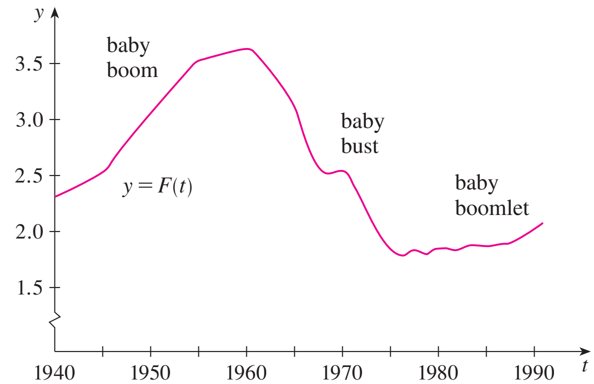
42. The figure shows the graph of the *derivative* f' of a function f.
(a) Sketch the graph of f''.
(b) Sketch a possible graph of f.



43. Sketch the graph of a function that satisfies the given conditions:

$f(0) = 0, \quad f'(-2) = f'(1) = f'(9) = 0,$
$\lim_{x\to\infty} f(x) = 0, \quad \lim_{x\to 6} f(x) = -\infty,$
$f'(x) < 0$ on $(-\infty, -2)$, $(1, 6)$, and $(9, \infty)$,
$f'(x) > 0$ on $(-2, 1)$ and $(6, 9)$,
$f''(x) > 0$ on $(-\infty, 0)$ and $(12, \infty)$,
$f''(x) < 0$ on $(0, 6)$ and $(6, 12)$

44. The *total fertility rate* at time t, denoted by $F(t)$, is an estimate of the average number of children born to each woman (assuming that current birth rates remain constant). The graph of the total fertility rate in the United States shows the fluctuations from 1940 to 1990.



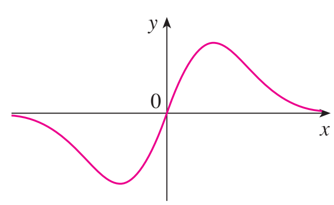
(a) Estimate the values of $F'(1950)$, $F'(1965)$, and $F'(1987)$.
(b) What are the meanings of these derivatives?
(c) Can you suggest reasons for the values of these derivatives?

45. A car starts from rest and its distance traveled is recorded in the table in 2-second intervals.

t (seconds)	s (feet)	t (seconds)	s (feet)
0	0	8	180
2	8	10	260
4	40	12	319
6	95	14	373

(a) Estimate the speed after 6 seconds.
(b) Estimate the coordinates of the inflection point of the graph of the position function.
(c) What is the significance of the inflection point?

46. The graph of the function is shown. Sketch the graph of an antiderivative F, given that $F(0) = 0$.



Focus on Problem Solving

In our discussion of the principles of problem solving we considered the problem-solving strategy of *introducing something extra* (see page 88). In the following example we show how this principle is sometimes useful when we evaluate limits. The idea is to change the variable—to introduce a new variable that is related to the original variable—in such a way as to make the problem simpler. Later, in Section 5.5, we will make more extensive use of this general idea.

EXAMPLE 1 Evaluate $\lim_{x \to 0} \dfrac{\sqrt[3]{1 + cx} - 1}{x}$, where c is a constant.

SOLUTION As it stands, this limit looks challenging. In Section 2.3 we evaluated several limits in which both numerator and denominator approached 0. There our strategy was to perform some sort of algebraic manipulation that led to a simplifying cancellation, but here it's not clear what kind of algebra is necessary.

So we introduce a new variable t by the equation

$$t = \sqrt[3]{1 + cx}$$

We also need to express x in terms of t, so we solve this equation:

$$t^3 = 1 + cx \qquad x = \frac{t^3 - 1}{c}$$

Notice that $x \to 0$ is equivalent to $t \to 1$. This allows us to convert the given limit into one involving the variable t:

$$\begin{aligned} \lim_{x \to 0} \frac{\sqrt[3]{1 + cx} - 1}{x} &= \lim_{t \to 1} \frac{t - 1}{(t^3 - 1)/c} \\ &= \lim_{t \to 1} \frac{c(t - 1)}{t^3 - 1} \end{aligned}$$

The change of variable allowed us to replace a relatively complicated limit by a simpler one of a type that we have seen before. Factoring the denominator as a difference of cubes, we get

$$\begin{aligned} \lim_{t \to 1} \frac{c(t - 1)}{t^3 - 1} &= \lim_{t \to 1} \frac{c(t - 1)}{(t - 1)(t^2 + t + 1)} \\ &= \lim_{t \to 1} \frac{c}{t^2 + t + 1} = \frac{c}{3} \end{aligned}$$

▲ Before you look at Example 2, cover up the solution and try it yourself first.

EXAMPLE 2 How many lines are tangent to both of the parabolas $y = -1 - x^2$ and $y = 1 + x^2$? Find the coordinates of the points at which these tangents touch the parabolas.

SOLUTION To gain insight into this problem it is essential to draw a diagram. So we sketch the parabolas $y = 1 + x^2$ (which is the standard parabola $y = x^2$ shifted 1 unit upward) and $y = -1 - x^2$ (which is obtained by reflecting the first parabola about the x-axis). If we try to draw a line tangent to both parabolas, we soon discover that there are only two possibilities, as illustrated in Figure 1.

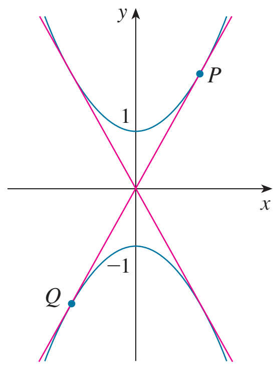


FIGURE 1

Let P be a point at which one of these tangents touches the upper parabola and let a be its x-coordinate. (The choice of notation for the unknown is important. Of course we could have used b or c or x_0 or x_1 instead of a. However, it's not advisable to use x in place of a because that x could be confused with the variable x in the equation of the parabola.) Then, since P lies on the parabola $y = 1 + x^2$, its y-coordinate must be $1 + a^2$. Because of the symmetry shown in Figure 1, the coordinates of the point Q where the tangent touches the lower parabola must be $(-a, -(1 + a^2))$.

To use the given information that the line is a tangent, we equate the slope of the line PQ to the slope of the tangent line at P. We have

$$m_{PQ} = \frac{1 + a^2 - (-1 - a^2)}{a - (-a)} = \frac{1 + a^2}{a}$$

If $f(x) = 1 + x^2$, then the slope of the tangent line at P is $f'(a)$. Using the definition of the derivative as in Section 2.7, we find that $f'(a) = 2a$. Thus, the condition that we need to use is that

$$\frac{1 + a^2}{a} = 2a$$

Solving this equation, we get $1 + a^2 = 2a^2$, so $a^2 = 1$ and $a = \pm 1$. Therefore, the points are $(1, 2)$ and $(-1, -2)$. By symmetry, the two remaining points are $(-1, 2)$ and $(1, -2)$. ■

The following problems are meant to test and challenge your problem-solving skills. Some of them require a considerable amount of time to think through, so don't be discouraged if you can't solve them right away. If you get stuck, you might find it helpful to refer to the discussion of the principles of problem solving on page 88.

• Problems

1. Evaluate $\displaystyle\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$.

2. Find numbers a and b such that $\displaystyle\lim_{x \to 0} \frac{\sqrt{ax + b} - 2}{x} = 1$.

3. Evaluate $\displaystyle\lim_{x \to 0} \frac{|2x - 1| - |2x + 1|}{x}$.

4. The figure shows a point P on the parabola $y = x^2$ and the point Q where the perpendicular bisector of OP intersects the y-axis. As P approaches the origin along the parabola, what happens to Q? Does it have a limiting position? If so, find it.

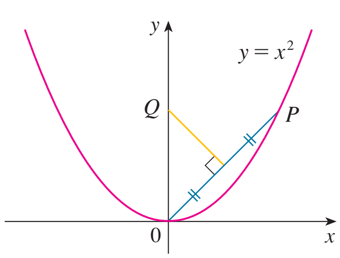


FIGURE FOR PROBLEM 4

5. If $[\![x]\!]$ denotes the greatest integer function, find $\lim_{x \to \infty} x/[\![x]\!]$.

6. Sketch the region in the plane defined by each of the following equations.
(a) $[\![x]\!]^2 + [\![y]\!]^2 = 1$ (b) $[\![x]\!]^2 - [\![y]\!]^2 = 3$ (c) $[\![x + y]\!]^2 = 1$ (d) $[\![x]\!] + [\![y]\!] = 1$

7. Find all values of a such that f is continuous on $\mathbb{R}$:

$$f(x) = \begin{cases} x + 1 & \text{if } x \leqslant a \\ x^2 & \text{if } x > a \end{cases}$$

8. A **fixed point** of a function f is a number c in its domain such that $f(c) = c$. (The function doesn't move c; it stays fixed.)

(a) Sketch the graph of a continuous function with domain $[0, 1]$ whose range also lies in $[0, 1]$. Locate a fixed point of f.

(b) Try to draw the graph of a continuous function with domain $[0, 1]$ and range in $[0, 1]$ that does *not* have a fixed point. What is the obstacle?

(c) Use the Intermediate Value Theorem to prove that any continuous function with domain $[0, 1]$ and range in $[0, 1]$ must have a fixed point.

9. (a) If we start from 0° latitude and proceed in a westerly direction, we can let $T(x)$ denote the temperature at the point x at any given time. Assuming that T is a continuous function of x, show that at any fixed time there are at least two diametrically opposite points on the equator that have exactly the same temperature.

(b) Does the result in part (a) hold for points lying on any circle on Earth's surface?

(c) Does the result in part (a) hold for barometric pressure and for altitude above sea level?

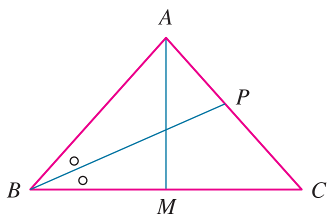


FIGURE FOR PROBLEM 10

10. (a) The figure shows an isosceles triangle ABC with $\angle B = \angle C$. The bisector of angle B intersects the side AC at the point P. Suppose that the base BC remains fixed but the altitude $|AM|$ of the triangle approaches 0, so A approaches the midpoint M of BC. What happens to P during this process? Does it have a limiting position? If so, find it.

(b) Try to sketch the path traced out by P during this process. Then find an equation of this curve and use this equation to sketch the curve.

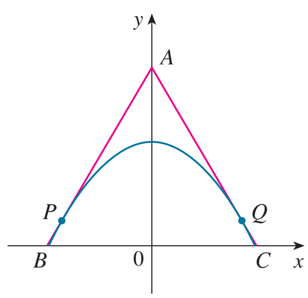


FIGURE FOR PROBLEM 11

11. Find points P and Q on the parabola $y = 1 - x^2$ so that the triangle ABC formed by the x-axis and the tangent lines at P and Q is an equilateral triangle. (See the figure.)

12. Water is flowing at a constant rate into a spherical tank. Let $V(t)$ be the volume of water in the tank and $H(t)$ be the height of the water in the tank at time t.

(a) What are the meanings of $V'(t)$ and $H'(t)$? Are these derivatives positive, negative, or zero?

(b) Is $V''(t)$ positive, negative, or zero? Explain.

(c) Let t_1, t_2, and t_3 be the times when the tank is one-quarter full, half full, and three-quarters full, respectively. Are the values $H''(t_1)$, $H''(t_2)$, and $H''(t_3)$ positive, negative, or zero? Why?

13. Suppose f is a function that satisfies the equation

$$f(x + y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose also that

$$\lim_{x \to 0} \frac{f(x)}{x} = 1$$

(a) Find $f(0)$. (b) Find $f'(0)$. (c) Find $f'(x)$.

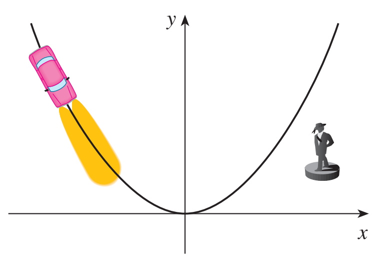


FIGURE FOR PROBLEM 14

14. A car is traveling at night along a highway shaped like a parabola with its vertex at the origin. The car starts at a point 100 m west and 100 m north of the origin and travels in an easterly direction. There is a statue located 100 m east and 50 m north of the origin. At what point on the highway will the car's headlights illuminate the statue?

15. If $\lim_{x \to a} [f(x) + g(x)] = 2$ and $\lim_{x \to a} [f(x) - g(x)] = 1$, find $\lim_{x \to a} f(x)g(x)$.

16. If f is a differentiable function and $g(x) = xf(x)$, use the definition of a derivative to show that $g'(x) = xf'(x) + f(x)$.

17. Suppose f is a function with the property that $|f(x)| \leq x^2$ for all x. Show that $f(0) = 0$. Then show that $f'(0) = 0$.