12 Measure and Outer Measure

In this chapter we first consider some of the ways in which a measure can be defined on a σ-algebra. In the case of Lebesgue measure we defined measure for open sets and used this to define outer measure, from which we obtain the notion of measurable set and Lebesgue measure. Such a procedure is feasible in general. In the first section we discuss the process of deriving a measure from an outer measure, and in the second section we derive an outer measure from a measure that is defined only on an algebra of sets. The remainder of the chapter is devoted to some applications of this process.

1 Outer Measure and Measurability

By an outer measure μ^* we mean a nonnegative extended real-valued set function defined on all subsets of a space X and having the following properties:

i. $\mu^*\emptyset = 0$.

ii. $A \subset B \Rightarrow \mu^*A \le \mu^*B$.

iii. $E \subset \bigcup_{i=1}^{\infty} E_i \Rightarrow \mu^*E \le \sum_{i=1}^{\infty} \mu^*E_i$.

The second property is called monotonicity and the third countable subadditivity. In view of (i) finite subadditivity follows from (iii).

Because of (ii), property (iii) can be replaced by

iii. $E = \bigcup_{i=1}^{\infty} E_i$, E_i disjoint $\Rightarrow \mu^*E \le \sum_{i=1}^{\infty} \mu^*E_i$.

The outer measure μ^* is called finite if $\mu^*X < \infty$.

By analogy with the case of Lebesgue measure we define a set E to be **measurable** with respect to μ^* if for every set A we have

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \tilde{E}).$$

Since μ^* is subadditive, it is only necessary to show that

$$\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \cap \tilde{E})$$

for every A in order to prove that E is measurable. This inequality is trivially true when $\mu^*A = \infty$, and so we need only establish it for sets A with μ^*A finite.

1. Theorem: *The class* $\mathcal{B}$ *of* μ^**-measurable sets is a* σ*-algebra. If* $\bar{\mu}$ *is* μ^* *restricted to* $\mathcal{B}$, *then* $\bar{\mu}$ *is a complete measure on* $\mathcal{B}$.

Proof: Trivially, the empty set is measurable. The symmetry of the definition of measurability in E and $\tilde{E}$ shows that $\tilde{E}$ is measurable whenever E is.

Let E_1 and E_2 be measurable sets. From the measurability of E_2,

$$\mu^*(A) = \mu^*(A \cap E_2) + \mu^*(A \cap \tilde{E}_2)$$

and

$$\mu^*(A) = \mu^*(A \cap E_2) + \mu^*(A \cap \tilde{E}_2 \cap E_1) + \mu^*(A \cap \tilde{E}_1 \cap \tilde{E}_2)$$

by the measurability of E_1. Since

$$A \cap [E_1 \cup E_2] = [A \cap E_2] \cup [A \cap E_1 \cap \tilde{E}_2],$$

we have

$$\mu^*(A \cap [E_1 \cup E_2]) \le \mu^*(A \cap E_2) + \mu^*(A \cap \tilde{E}_2 \cap E_1)$$

by subadditivity, and so

$$\mu^*A \ge \mu^*(A \cap [E_1 \cup E_2]) + \mu^*(A \cap \tilde{E}_1 \cap \tilde{E}_2).$$

This means that $E_1 \cup E_2$ is measurable, since

$$\sim(E_1 \cup E_2) = \tilde{E}_1 \cap \tilde{E}_2.$$

Thus the union of two measurable sets is measurable, and by induction the union of any finite number of measurable sets is measurable, showing that $\mathcal{B}$ is an algebra of sets.

Assume that $E = \bigcup E_i$, where $\langle E_i \rangle$ is a disjoint sequence of measurable sets, and set

$$G_n = \bigcup_{i=1}^{n} E_i.$$

Then G_n is measurable, and

$$\mu^*(A) = \mu^*(A \cap G_n) + \mu^*(A \cap \tilde{G}_n) \geq \mu^*(A \cap G_n) + \mu^*(A \cap \tilde{E}),$$

since $\tilde{E} \subset \tilde{G}_n$. Now $G_n \cap E_n = E_n$ and $G_n \cap \tilde{E}_n = G_{n-1}$, and by the measurability of E_n we have

$$\mu^*(A \cap G_n) = \mu^*(A \cap E_n) + \mu^*(A \cap G_{n-1}).$$

By induction

$$\mu^*(A \cap G_n) = \sum_{i=1}^{n} \mu^*(A \cap E_i),$$

and so

$$\begin{aligned} \mu^*(A) &\geq \mu^*(A \cap \tilde{E}) + \sum_{i=1}^{\infty} \mu^*(A \cap E_i) \\ &\geq \mu^*(A \cap \tilde{E}) + \mu^*(A \cap E), \end{aligned}$$

since

$$A \cap E \subset \bigcup_{i=1}^{\infty} (A \cap E_i).$$

Thus E is measurable. Since the union of any sequence of sets in an algebra can be replaced by a disjoint union of sets in the algebra, it follows that $\mathfrak{B}$ is a σ-algebra.

We next demonstrate the finite additivity of $\bar{\mu}$. Let E_1 and E_2 be disjoint measurable sets. Then the measurability of E_2 implies that

$$\begin{aligned} \bar{\mu}(E_1 \cup E_2) &= \mu^*(E_1 \cup E_2) \\ &= \mu^*([E_1 \cup E_2] \cap E_2) + \mu^*([E_1 \cup E_2] \cap \tilde{E}_2) \\ &= \mu^* E_2 + \mu^* E_1. \end{aligned}$$

Finite additivity follows by induction.

If E is the disjoint union of the measurable sets $\{E_i\}$, then

$$\bar{\mu}(E) \geq \bar{\mu}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \bar{\mu}(E_i),$$

and so

$$\bar{\mu}(E) \geq \sum_{i=1}^{\infty} \bar{\mu}(E_i).$$

But

$$\bar{\mu}(E) \leq \sum_{i=1}^{\infty} \bar{\mu}(E_i)$$

by the subadditivity of μ^*. Hence $\bar{\mu}$ is countably additive and thus a measure since it is nonnegative and $\bar{\mu}\emptyset = \mu^*\emptyset = 0$. ∎

Problems

1. Prove the completeness of $\bar{\mu}$.

2. Assume that $\langle E_i \rangle$ is a sequence of disjoint measurable sets and $E = \bigcup E_i$. Then for any set A we have

$$\mu^*(A \cap E) = \sum \mu^*(A \cap E_i).$$

2 The Extension Theorem

By a **measure on an algebra** we mean a nonnegative extended real-valued set function μ defined on an algebra $\mathcal{A}$ of sets such that:

i. $\mu(\emptyset) = 0$.
ii. If $\langle A_i \rangle$ is a disjoint sequence of sets in $\mathcal{A}$ whose union is also in $\mathcal{A}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu A_i.$$

Thus a measure on an algebra $\mathcal{A}$ is a measure if and only if $\mathcal{A}$ is a σ-algebra. The purpose of this section is to show that, if we start with a measure on an algebra $\mathcal{A}$ of sets, we may extend it to a measure defined on a σ-algebra $\mathcal{B}$ containing $\mathcal{A}$. We shall do this by using the measure on the algebra to construct an outer measure μ^* and show that the measure $\bar{\mu}$ induced by μ^* is the desired extension of μ. The process by which we construct μ^* from μ is analogous to that by which we constructed Lebesgue outer measure from the lengths of intervals: We define

$$\mu^*E = \inf \sum_{i=1}^{\infty} \mu A_i, \tag{1}$$

where $\langle A_i \rangle$ ranges over all sequences from $\mathfrak{A}$ such that $E \subset \bigcup_{i=1}^{\infty} A_i$. We first establish some lemmas concerning μ^*.

2. Lemma: *If $A \in \mathfrak{A}$ and if $\langle A_i \rangle$ is any sequence of sets in $\mathfrak{A}$ such that $A \subset \bigcup_{i=1}^{\infty} A_i$, then $\mu A \leq \sum_{i=1}^{\infty} \mu A_i$.*

Proof: Set

$$B_n = A \cap A_n \cap \tilde{A}_{n-1} \cap \cdots \cap \tilde{A}_1.$$

Then $B_n \in \mathfrak{A}$ and $B_n \subset A_n$. But A is the disjoint union of the sequence $\langle B_n \rangle$, and so by countable additivity

$$\mu A = \sum_{n=1}^{\infty} \mu B_n \leq \sum_{n=1}^{\infty} \mu A_n. \quad \blacksquare$$

3. Corollary: *If $A \in \mathfrak{A}$, $\mu^* A = \mu A$.*

4. Lemma: *The set function μ^* is an outer measure.*

Proof: Since μ^* is clearly a monotone nonnegative set function defined for all sets and $\mu^*\emptyset = 0$, we have only to show that it is countably subadditive. Let $E \subset \bigcup_{i=1}^{\infty} E_i$. If $\mu^*E_i = \infty$ for any i, we have $\mu^*E \leq \sum \mu^*E_i = \infty$. If not, given $\epsilon > 0$, there is for each i a sequence $\langle A_{ij} \rangle_{j=1}^{\infty}$ of sets in $\mathfrak{A}$ such that $E_i \subset \bigcup_{j=1}^{\infty} A_{ij}$ and

$$\sum_{j=1}^{\infty} \mu A_{ij} < \mu^*E_i + \frac{\epsilon}{2^i}.$$

Then

$$\mu^*E \leq \sum_{ij} \mu A_{ij} < \sum_{i=1}^{\infty} \mu^*E_i + \epsilon.$$

Since ϵ was an arbitrary positive number,

$$\mu^*E \leq \sum_{i=1}^{\infty} \mu^*E_i,$$

and μ^* is subadditive. ∎

5. Lemma: *If $A \in \mathcal{A}$, then A is measurable with respect to μ^*.*

Proof: Let E be an arbitrary set of finite outer measure and ϵ a positive number. Then there is a sequence $\langle A_i \rangle$ from $\mathcal{A}$ such that $E \subset \bigcup A_i$ and

$$\sum \mu A_i < \mu^*E + \epsilon.$$

By the additivity of μ on $\mathcal{A}$ we have

$$\mu(A_i) = \mu(A_i \cap A) + \mu(A_i \cap \tilde{A}).$$

Hence

$$\mu^*E + \epsilon > \sum_{i=1}^{\infty} \mu(A_i \cap A) + \sum_{i=1}^{\infty} \mu(A_i \cap \tilde{A})$$
$$\geq \mu^*(E \cap A) + \mu^*(E \cap \tilde{A}),$$

since

$$E \cap A \subset \bigcup (A_i \cap A)$$

and

$$E \cap \tilde{A} \subset \bigcup (A_i \cap \tilde{A}).$$

Since ϵ was an arbitrary positive number,

$$\mu^*E \geq \mu^*(E \cap A) + \mu^*(E \cap \tilde{A}),$$

and A is measurable. ∎

The outer measure μ^* that we have defined is called the outer measure induced by μ. For a given algebra $\mathcal{A}$ of sets we use $\mathcal{A}_\sigma$ to denote those sets that are countable unions of sets of $\mathcal{A}$ and use $\mathcal{A}_{\sigma\delta}$ to denote those sets that are countable intersections of sets in $\mathcal{A}_\sigma$.

6. Proposition: *Let μ be a measure on an algebra $\mathcal{A}$, μ^* the outer measure induced by μ, and E any set. Then for $\epsilon > 0$, there is a set $A \in \mathcal{A}_\sigma$ with $E \subset A$ and*

$$\mu^*A \leq \mu^*E + \epsilon.$$

There is also a set $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*E = \mu^*B$.

Proof: By the definition of μ^* there is a sequence $\langle A_i \rangle$ from $\mathfrak{A}$ such that $E \subset \bigcup A_i$ and

$$\sum_{i=1}^{\infty} \mu A_i \leq \mu^* E + \epsilon.$$

Set $A = \bigcup A_i$. Then $\mu^* A \leq \sum \mu^* A_i = \sum \mu A_i$.

To prove the second statement, we note that for each positive integer n there is a set A_n in $\mathfrak{A}_\sigma$ with $E \subset A_n$ and $\mu^* A_n < \mu^* E + 1/n$. Let $B = \bigcap A_n$. Then $B \varepsilon \mathfrak{A}_{\sigma\delta}$ and $E \subset B$. Since $B \subset A_n$, $\mu^* B \leq \mu^* A_n < \mu^* E + 1/n$. Since n is arbitrary, $\mu^* B \leq \mu^* E$. But $E \subset B$, and so $\mu^* B \geq \mu^* E$ by monotonicity. Hence $\mu^* B = \mu^* E$. ▌

An outer measure μ^* is said to be *regular* if given any subset E of X and any $\epsilon > 0$, there is a μ^*-measurable set A with $E \subset A$ and

$$\mu^* A \leq \mu^* E + \epsilon.$$

It follows from Lemma 5 and Proposition 6 that every outer measure induced by a measure on an algebra is a regular outer measure.

If we apply this proposition in the case that E is a measurable set of finite measure, we see that E must be the difference of a set B in $\mathfrak{A}_{\sigma\delta}$ and a set of measure zero. This gives us the structure of the measurable sets of finite measure, and the next proposition extends this to the σ-finite case. It can be considered a generalization of the first principle of Littlewood. It is a key element in the proof of a number of our theorems. Other forms of this principle are given by Problems 7 and 10. Versions of Littlewood's other principles are given by Propositions 11.7 and 11.26 and by Problems 11, 11.16, and 11.21c.

7. **Proposition:** *Let μ be a σ-finite measure on an algebra $\mathfrak{A}$, and let μ^* be the outer measure generated by μ. A set E is μ^* measurable if and only if E is the proper difference $A \sim B$ of a set A in $\mathfrak{A}_{\sigma\delta}$ and a set B with $\mu^* B = 0$. Each set B with $\mu^* B = 0$ is contained in a set C in $\mathfrak{A}_{\sigma\delta}$ with $\mu^* C = 0$.*

Proof: The "if" part of the proposition follows from the fact that each set in $\mathfrak{A}_{\sigma\delta}$ must be measurable, since the measurable sets form a σ-algebra, while each set of μ^*-measure zero must be measurable, since $\bar{\mu}$ is complete.

To prove the "only if" part of the proposition, let $\{X_i\}$ be a countable disjoint collection of sets in $\mathfrak{A}$ with μX_i finite and $X = \bigcup X_i$. If E is measurable, then E is the disjoint union of the measurable sets $E_i = X_i \cap E$. By Proposition 6 we can find for each positive integer n, a set A_{ni} in $\mathfrak{A}_\sigma$ such that $E_i \subset A_{ni}$ and

$$\bar{\mu}A_{ni} \leq \bar{\mu}E_i + \frac{1}{n2^i}.$$

Set

$$A_n = \bigcup_{i=1}^{\infty} A_{ni}.$$

Then $E \subset A_n$, and $A_n \sim E \subset \bigcup_{i=1}^{\infty} [A_{ni} \sim E_i]$. Hence

$$\bar{\mu}(A_n \sim E) \leq \sum_{i=1}^{\infty} \bar{\mu}(A_{ni} \sim E_i)$$
$$\leq \sum_{i=1}^{\infty} \frac{1}{n2^i} = \frac{1}{n}.$$

Since $A_n \in \mathfrak{A}_\sigma$, the set $A = \bigcap_{n=1}^{\infty} A_n$ is in $\mathfrak{A}_{\sigma\delta}$, and for each n

$$A \sim E \subset A_n \sim E.$$

Hence

$$\bar{\mu}(A \sim E) \leq \bar{\mu}(A_n \sim E) \leq \frac{1}{n}.$$

Since this holds for each positive integer n, we must have

$$\bar{\mu}(A \sim E) = 0. \quad \blacksquare$$

We summarize the results of this section in the following theorem.

8. Theorem (Carathéodory): *Let μ be a measure on an algebra $\mathfrak{A}$, and μ^* the outer measure induced by μ. Then the restriction $\bar{\mu}$ of μ^* to the μ^*-measurable sets is an extension of μ to a σ-algebra containing $\mathfrak{A}$. If μ is finite (or σ-finite) so is $\bar{\mu}$. If μ is σ-finite, then $\bar{\mu}$ is the only measure on the smallest σ-algebra containing $\mathfrak{A}$ which is an extension of μ.*

Proof: The fact that $\bar{\mu}$ is an extension of μ from $\mathfrak{A}$ to be a measure on a σ-algebra containing $\mathfrak{A}$ follows directly from Corollary 3, Lemma 5, and Theorem 1, and it is readily verified that $\bar{\mu}$ is finite or σ-finite whenever μ is.

To show the unicity of $\bar{\mu}$ when μ is σ-finite, we let $\mathcal{B}$ be the smallest σ-algebra containing $\mathcal{A}$ and $\tilde{\mu}$ some measure on $\mathcal{B}$ that agrees with μ on $\mathcal{A}$.

Since each set in $\mathcal{A}_\sigma$ can be expressed as a disjoint countable union of sets in $\mathcal{A}$, the measure $\tilde{\mu}$ must agree with $\bar{\mu}$ on $\mathcal{A}_\sigma$. Let B be any set in $\mathcal{B}$ with finite outer measure. Then by Proposition 6 there is an A in $\mathcal{A}_\sigma$ such that $B \subset A$ and

$$\mu^*A \leq \mu^*B + \epsilon.$$

Since $B \subset A$,

$$\tilde{\mu}B \leq \tilde{\mu}A = \mu^*A \leq \mu^*B + \epsilon.$$

Since ϵ is an arbitrary positive number, we have

$$\tilde{\mu}B \leq \mu^*B$$

for each $B \in \mathcal{B}$.

Since the class of sets measurable with respect to μ^* is a σ-algebra containing $\mathcal{A}$, each B in $\mathcal{B}$ must be measurable. If B is measurable and A is in $\mathcal{A}_\sigma$ with $B \subset A$ and $\mu^*A \leq \mu^*B + \epsilon$, then

$$\mu^*A = \mu^*B + \mu^*(A \sim B),$$

and so

$$\tilde{\mu}(A \sim B) \leq \mu^*(A \sim B) \leq \epsilon,$$

if $\mu^*B < \infty$. Hence

$$\begin{aligned}\mu^*B \leq \mu^*A &= \tilde{\mu}A \\ &= \tilde{\mu}B + \tilde{\mu}(A \sim B) \\ &\leq \tilde{\mu}B + \epsilon.\end{aligned}$$

Since ϵ is arbitrary, we have

$$\mu^*B \leq \tilde{\mu}B$$

and so

$$\mu^*B = \tilde{\mu}B.$$

If μ is a σ-finite measure, let $\{X_i\}$ be a countable disjoint collection of sets in $\mathcal{A}$ with $X = \bigcup X_i$ and μX_i finite. If B is any set in $\mathcal{B}$, then

$$B = \bigcup (X_i \cap B)$$

and this is a countable disjoint union of sets in $\mathcal{B}$, and so we have

$$\tilde{\mu}B = \sum \tilde{\mu}(X_i \cap B)$$

and

$$\bar{\mu}B = \sum \bar{\mu}(X_i \cap B).$$

Since $\mu^*(X_i \cap B) < \infty$, we have

$$\bar{\mu}(X_i \cap B) = \tilde{\mu}(X_i \cap B). \quad \blacksquare$$

This extension procedure not only extends μ to a measure on the smallest σ-algebra $\mathcal{B}$ containing $\mathcal{A}$, but also completes and saturates the measure. If μ is σ-finite, the extension to $\mathcal{B}$ is already saturated, and the extension to the μ^*-measurable sets is merely the completion of the extension of $\bar{\mu}$ on $\mathcal{B}$. If μ is not σ-finite, then the extension to μ^*-measurable sets also saturates $\bar{\mu}$. It should be observed that in this case the extension of μ to $\mathcal{B}$ need not be unique (Problem 3), although any extension $\tilde{\mu}$ must agree with $\bar{\mu}$ for each set B of $\mathcal{B}$ for which $\bar{\mu}B < \infty$, and we always have $\tilde{\mu}B \le \mu^*B$. We shall return to the question of extension and unicity in Sections 6 and 7.

It is often convenient to start with a set function on a collection $\mathcal{C}$ of sets having less structure than an algebra of sets. We say that a collection $\mathcal{C}$ of subsets of X is a **semialgebra** of sets if the intersection of any two sets in $\mathcal{C}$ is again in $\mathcal{C}$ and the complement of any set in $\mathcal{C}$ is a finite disjoint union of sets in $\mathcal{C}$. If $\mathcal{C}$ is any semialgebra of sets, then the collection $\mathcal{A}$ consisting of the empty set and all finite disjoint unions of sets in $\mathcal{C}$ is an algebra of sets which is called the algebra generated by $\mathcal{C}$. If μ is a set function defined on $\mathcal{C}$, it is natural to attempt to define a finitely additive set function on $\mathcal{A}$ by setting

$$\mu A = \sum_{i=1}^{n} \mu E_i$$

whenever A is the disjoint union of the set E_i in $\mathcal{C}$. Since a set A in $\mathcal{A}$ may possibly be represented in several ways as a disjoint union of sets in $\mathcal{C}$, we must be certain that such a procedure leads to a unique value for μA. The following proposition gives conditions under which this procedure can be carried out and will give a measure on the algebra $\mathcal{A}$.

9. Proposition: *Let $\mathcal{C}$ be a semialgebra of sets and μ a nonnegative set function defined on $\mathcal{C}$ with $\mu\emptyset = 0$ (if $\emptyset \in \mathcal{C}$). Then μ has a unique* *extension to a measure on the algebra* $\mathfrak{A}$ *generated by* $\mathcal{C}$ *if the following conditions are satisfied:*

i. *If a set* C *in* $\mathcal{C}$ *is the union of a finite disjoint collection* $\{C_i\}$ *of sets in* $\mathcal{C}$, *then* $\mu C = \sum \mu C_i$.
ii. *If a set* C *in* $\mathcal{C}$ *is the union of a countable disjoint collection* $\{C_i\}$ *of sets in* $\mathcal{C}$, *then* $\mu C \leq \sum \mu C_i$.

Problems

3. Let X be the set of rational numbers and $\mathfrak{A}$ the algebra of finite unions of intervals of the form $(a, b]$ with $\mu(a, b] = \infty$ and $\mu\emptyset = 0$. The extension of μ to the smallest σ-algebra containing $\mathfrak{A}$ is not unique.

4. Prove Proposition 9 by showing:

a. Condition (i) implies that if A is the union of each of two finite disjoint collections $\{C_i\}$ and $\{D_j\}$ of sets in $\mathcal{C}$, then $\sum \mu C_i = \sum \mu D_j$. [Hint: $\mu C_i = \sum_j \mu(C_i \cap D_j)$.]

b. Condition (ii) implies that μ is countably additive on $\mathfrak{A}$ (for finite additivity and monotonicity already imply the reverse inequality).

5. Let $\mathcal{C}$ be a semialgebra of sets and $\mathfrak{A}$ the smallest algebra of sets containing $\mathcal{C}$.

a. Show that $\mathfrak{A}$ is comprised of sets of the form $A = \bigcup_{i=1}^{n} C_i$ with $C_i \in \mathcal{C}$.

b. Show $\mathfrak{A}_\sigma = \mathcal{C}_\sigma$, so that $\mathfrak{A}_\sigma$ and $\mathfrak{A}_{\sigma\delta}$ may be replaced in theorems by $\mathcal{C}_\sigma$ and $\mathcal{C}_{\sigma\delta}$, respectively.

6. Let $\mathfrak{A}$ be a collection of sets which is closed under finite unions and finite intersections; an algebra of sets, for example.

a. Show that $\mathfrak{A}_\sigma$ is closed under countable unions and finite intersections.

b. Show that each set in $\mathfrak{A}_{\sigma\delta}$ is the intersection of a *decreasing* sequence of sets in $\mathfrak{A}_\sigma$.

7. Let μ be a finite measure on an algebra $\mathfrak{A}$, and μ^* the induced outer measure. Show that a set E is measurable if and only if for each $\epsilon > 0$ there is a set $A \in \mathfrak{A}_\delta$, $A \subset E$, such that $\mu^*(E \sim A) < \epsilon$.

8. If we start with an outer measure μ^* on X and form the induced measure $\bar{\mu}$ on the μ^*-measurable sets, we can use $\bar{\mu}$ to induce an outer measure μ^+.

a. Show that for each set E we have $\mu^+ E \geq \mu^* E$.

b. For a given set E we have $\mu^+E = \mu^*E$ if and only if there is a μ^*-measurable set $A \supset E$ with $\mu^*A = \mu^*E$.

c. Show that $\mu^+E = \mu^*E$ for every E if and only if μ^* is regular.

d. Show that an outer measure μ^* is regular if and only if it is induced by a measure on an algebra.

e. Let X be a set consisting of two points. Construct an outer measure on X which is not regular.

9. Let μ^* be a regular outer measure.

a. Show that the measure $\bar{\mu}$ induced by μ^* is complete and saturated.

b. Let $(X, \mathcal{A}, \mu)$ be a complete measure space. Let $\bar{\mu}$ be the extension of μ obtained by the Carathéodory process. Then $\bar{\mu}$ is the same as the extension given in Problem 11.8c.

10. Let μ be a measure on an algebra $\mathcal{A}$ and $\bar{\mu}$ the extension of it given by the Caratheodory process. Let E be measurable with respect to $\bar{\mu}$ and $\bar{\mu}E < \infty$. Then given $\epsilon > 0$, there is an $A \in \mathcal{A}$ with

$$\bar{\mu}(A \Delta E) < \epsilon.$$

11. We say that a function φ is $\mathcal{A}$-simple if $\varphi = \sum a_i \chi_{A_i}$, where $A_I \in \mathcal{A}$. Let μ be a measure on $\mathcal{A}$ and $\bar{\mu}$ its extension.

a. Given $\epsilon > 0$ and a $\bar{\mu}$ integrable function f, there is an $\mathcal{A}$-simple function φ such that

$$\int |f - \varphi| \, d\bar{\mu} < \epsilon.$$

b. Show that the function φ in Problem 11.21c can be taken to be $\mathcal{A}$-simple.

★3 The Lebesgue–Stieltjes Integral

Let X be the set of real numbers and $\mathcal{B}$ the class of all Borel sets. A measure μ defined on $\mathcal{B}$ and finite for bounded sets is called a Baire *measure* (on the real line). To each finite Baire measure we associate a function F by setting

$$F(x) = \mu(-\infty, x].$$

The function F is called the *cumulative distribution function* of μ and is real-valued and monotone increasing. We have

$$\mu(a, b] = F(b) - F(a).$$

Since $(a, b]$ is the intersection of the sets $(a, b + 1/n]$, Proposition 11.2 implies that

$$\mu(a, b] = \lim_{n \to \infty} \mu\left(a, b + \frac{1}{n}\right],$$

and so

$$F(b) = \lim_{n \to \infty} F\left(b + \frac{1}{n}\right) = F(b+).$$

Thus a cumulative distribution function is continuous on the right. Similarly,

$$\begin{aligned}\mu\{b\} &= \lim_{n \to \infty} \mu\left(b - \frac{1}{n}, b\right] \\ &= \lim_{n \to \infty} F(b) - F\left(b - \frac{1}{n}\right) \\ &= F(b) - F(b-).\end{aligned}$$

Hence F is continuous at b if and only if the set $\{b\}$ consisting of b alone has measure zero. Since $\emptyset = \bigcap (-\infty, -n]$, we have

$$\lim_{n \to -\infty} F(n) = 0,$$

and hence

$$\lim_{x \to -\infty} F(x) = 0,$$

because of the monotonicity of F. We summarize these properties in the following lemma:

10. Lemma: *If μ is a finite Baire measure on the real line, then its cumulative distribution function F is a monotone increasing bounded function which is continuous on the right. Moreover,* $\lim_{x \to -\infty} F(x) = 0$.

Suppose that we begin with a monotone increasing function F which is continuous on the right. Then we shall show that there is a unique Baire measure μ such that

$$\mu(a, b] = F(b) - F(a) \tag{2}$$

for all intervals of the form $(a, b]$, where we define $F(\infty) = \lim_{x \to \infty} F(x)$ and $F(-\infty) = \lim_{x \to -\infty} F(x)$. We begin with the following lemma, whose proof is left to the reader (Problem 12):

11. Lemma: *Let F be a monotone increasing function continuous on the right. If $(a, b] \subset \bigcup_{i=1}^{\infty} (a_i, b_i]$, then*

$$F(b) - F(a) \leq \sum_{i=1}^{\infty} F(b_i) - F(a_i).$$

If we let $\mathcal{C}$ be the semialgebra consisting of all intervals of the form $(a, b]$ or (a, ∞) and set $\mu(a, b] = F(b) - F(a)$, then μ is easily seen to satisfy condition (i) of Proposition 9, and since Lemma 11 is precisely the second condition, we see that μ admits a unique extension to a measure on the algebra generated by $\mathcal{B}$. By Theorem 8 this μ can be extended to a σ-algebra containing $\mathcal{C}$. Since the class $\mathcal{B}$ of Borel sets is the smallest σ-algebra containing $\mathcal{C}$, we have an extension of μ to a Baire measure. The measure μ is σ-finite, since X is the union of the intervals $(n, n + 1]$ and each has finite measure. Thus the extension of μ to $\mathcal{B}$ is unique, and we have the following proposition:

12. Proposition: *Let F be a monotone increasing function which is continuous on the right. Then there is a unique Baire measure μ such that for all a and b we have*

$$\mu(a, b] = F(b) - F(a).$$

13. Corollary: *Each bounded monotone function which is continuous on the right is the cumulative distribution function of a unique finite Baire measure provided $F(-\infty) = 0$.*

If φ is a nonnegative Borel measurable function and F is a monotone increasing function which is continuous on the right, we define the Lebesgue–Stieltjes integral of φ with respect to F to be

$$\int \varphi \, dF = \int \varphi \, d\mu,$$

where μ is the Baire measure having F as its cumulative distribution function. If φ is both positive and negative, we say that it is integrable with respect to F if it is integrable with respect to μ.

If F is any monotone increasing function, then there is a unique function F^* which is monotone increasing, continuous on the right, and agrees with F wherever F is continuous on the right (Problem 13), and we define the Lebesgue–Stieltjes integral of φ with respect to F by

$$\int \varphi \, dF = \int \varphi \, dF^*.$$

If F is a monotone function, continuous on the right, then $\int_a^b \varphi \, dF$ agrees with the Riemann–Stieltjes integral whenever the latter is defined. The Lebesgue–Stieltjes integral is only defined when F is monotone (or more generally of bounded variation, as in Problem 14c), while the Riemann–Stieltjes integral can exist when F is not of bounded variation, say when F is continuous and φ is of bounded variation.

Problems

12. Prove Lemma 11. [Choose $\epsilon > 0$. By the continuity on the right of F, choose $\eta_i > 0$ so that $F(b_i + \eta_i) < F(b_i) + \epsilon 2^{-i}$, and choose $\delta > 0$ so that $F(a + \delta) < F(a) + \epsilon$. Then the open intervals $(a_i, b_i + \eta_i)$ cover the closed interval $[a + \delta, b]$, and the proof proceeds like that of Proposition 3.1. A little extra care must be taken when $(a, b]$ is infinite.]

13. Let F be a monotone increasing function, and define

$$F^*(x) = \lim_{y \to x+} F(y).$$

Then F^* is a monotone increasing function which is continuous on the right and agrees with F wherever F is continuous on the right. We have $(F^*)^* = F^*$, and if F and G are monotone increasing functions which agree wherever they are both continuous, then $F^* = G^*$.

14. a. Show that each bounded function F of bounded variation gives rise to a finite signed Baire measure ν such that

$$\nu(a, b] = F(b+) - F(a+).$$

b. Show that v^+ and v^- in the Jordan decomposition correspond to the positive and negative variations of F.

c. Extend the definition of the Lebesgue–Stieltjes integral $\int \varphi \, dF$ to functions F of bounded variation.

d. Show that if $|\varphi| \le M$ and if the total variation of F is T, then $|\int \varphi \, dF| \le MT$.

15. a. Let F be the cumulative distribution function of the Baire measure ν, and assume that F is continuous. Then for any Borel set E contained in the range of F, we have $mE = \nu[F^{-1}(E)]$, with m Lebesgue measure. [Hint: This is true for intervals, and the uniqueness part of Theorem 8 can be used to derive its truth in general.]

b. Generalize to the case of discontinuous cumulative distribution functions.

16. Let F be a continuous increasing function on $[a, b]$ with $F(a) = c$, $F(b) = d$, and let φ be a nonnegative Borel measurable function on $[c, d]$. Then $\int_a^b \varphi(F(x))\, dF(x) = \int_c^d \varphi(y)\, dy$. [Hint: Use Problem 15a to take care of the case when φ is a characteristic function and generalize first to simple φ and then to general φ.]

17. a. Show that a measure μ is absolutely continuous with respect to Lebesgue measure if and only if its cumulative distribution function is absolutely continuous.

b. If μ is absolutely continuous with respect to Lebesgue measure, then its Radon–Nikodym derivative is the derivative of its cumulative distribution function.

c. If F is absolutely continuous, then

$$\int f\, dF = \int fF'\, dx.$$

18. *Riemann's Convergence Criterion.* Let f be a nonnegative monotone decreasing function on $(0, \infty)$, g a nonnegative monotone increasing function on $(0, \infty)$, and $\langle a_n \rangle$ a nonnegative sequence. Suppose that for each $x \in (0, \infty)$ the number of n such that $a_n \geq f(x)$ is at most $g(x)$. Then we have $\sum a_n < \infty$ if $\int_b^\infty f\, dg < \infty$.

4 Product Measures

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete measure spaces, and consider the direct product $X \times Y$ of X and Y. If $A \subset X$ and $B \subset Y$, we call $A \times B$ a rectangle. If $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we call $A \times B$ a measurable rectangle. The collection $\mathcal{R}$ of measurable rectangles is a semialgebra, since

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

and

$$\sim(A \times B) = (\tilde{A} \times B) \cup (A \times \tilde{B}) \cup (\tilde{A} \times \tilde{B}).$$

If $A \times B$ is a measurable rectangle, we set

$$\lambda(A \times B) = \mu A \cdot \nu B.$$

14. Lemma: *Let $\{(A_i \times B_i)\}$ be a countable disjoint collection of measurable rectangles whose union is a measurable rectangle $A \times B$. Then*

$$\lambda(A \times B) = \sum \lambda(A_i \times B_i).$$

Proof: Fix a point $x \varepsilon A$. Then for each $y \varepsilon B$, the point $\langle x, y\rangle$ belongs to exactly one rectangle $A_i \times B_i$. Thus B is the disjoint union of those B_i such that x is in the corresponding A_i. Hence

$$\sum \nu B_i \cdot \chi_{A_i}(x) = \nu B \cdot \chi_A(x),$$

since ν is countably additive. Thus by the corollary of the Monotone Convergence Theorem (11.14), we have

$$\sum \int \nu B_i \cdot \chi_{A_i}\, d\mu = \int \nu(B) \cdot \chi_A\, d\mu$$

or

$$\sum \nu B_i \cdot \mu A_i = \nu B \cdot \mu A. \quad \blacksquare$$

The lemma implies that λ satisfies the conditions of Proposition 9 and hence has a unique extension to a measure on the algebra $\mathcal{R}'$ consisting of all finite disjoint unions of sets in $\mathcal{R}$. Theorem 8 allows us to extend λ to be a complete measure on a σ-algebra $\mathcal{S}$ containing $\mathcal{R}$. This extended measure is called the product measure of μ and ν and is denoted by $\mu \times \nu$. If μ and ν are finite (or σ-finite), so is $\mu \times \nu$. If X and Y are the real line and μ and ν are both Lebesgue measure, then $\mu \times \nu$ is called two-dimensional Lebesgue measure for the plane.

The purpose of the next few lemmas is to describe the structure of the sets which are measurable with respect to the product measure $\mu \times \nu$. If E is any subset of $X \times Y$ and x a point of X, we define the *x cross section* E_x by

$$E_x = \{y: \langle x, y\rangle \varepsilon E\},$$

and similarly for the y cross section for y in Y. The characteristic function of E_x is related to that of E by

$$\chi_{E_x}(y) = \chi_E(x, y).$$

We also have $(\tilde{E})_x = \sim(E_x)$ and $(\bigcup E_\alpha)_x = \bigcup (E_\alpha)_x$ for any collection $\{E_\alpha\}$.

15. Lemma: *Let x be a point of X and E a set in $\mathcal{R}_{\sigma\delta}$. Then E_x is a measurable subset of Y.*

Proof: The lemma is trivially true if E is in the class $\mathcal{R}$ of measurable rectangles. We next show it to be true for E in $\mathcal{R}_\sigma$. Let $E = \bigcup_{i=1}^{\infty} E_i$, where each E_i is a measurable rectangle. Then

$$\begin{aligned}\chi_{E_x}(y) &= \chi_E(x, y)\\ &= \sup_i \chi_{E_i}(x, y)\\ &= \sup_i \chi_{(E_i)_x}(y).\end{aligned}$$

Since each E_i is a measurable rectangle, $\chi_{(E_i)_x}(y)$ is a measurable function of y, and so χ_{E_x} must also be measurable, whence E_x is measurable.

Suppose now that $E = \bigcap_{i=1}^{\infty} E_i$ with $E_i \in \mathcal{R}_\sigma$. Then

$$\begin{aligned}\chi_{E_x} &= \chi_E(x, y)\\ &= \inf_i \chi_{E_i}(x, y)\\ &= \inf_i \chi_{(E_i)_x}(y),\end{aligned}$$

and we see that χ_{E_x} is measurable. Thus E_x is measurable for any $E \in \mathcal{R}_{\sigma\delta}$. ∎

16. Lemma: *Let E be a set in $\mathcal{R}_{\sigma\delta}$ with $\mu \times \nu(E) < \infty$. Then the function g defined by*

$$g(x) = \nu E_x$$

is a measurable function of x and

$$\int g \, d\mu = \mu \times \nu(E).$$

Proof: The lemma is trivially true if E is a measurable rectangle. We first note that any set in $\mathcal{R}_\sigma$ is a disjoint union of measurable rectangles. Let $\langle E_i \rangle$ be a disjoint sequence of measurable rectangles, and let $E = \bigcup E_i$. Set

$$g_i(x) = \nu[(E_i)_x].$$

Then each g_i is a nonnegative measurable function, and

$$g = \sum g_i.$$

Thus g is measurable, and by the corollary of the Monotone Convergence Theorem (11.14), we have

$$\begin{aligned}\int g\, d\mu &= \sum \int g_i\, d\mu \\ &= \sum \mu \times \nu(E_i) \\ &= \mu \times \nu(E).\end{aligned}$$

Consequently, the lemma holds for $E \in \mathcal{R}_\sigma$.

Let E be a set of finite measure in $\mathcal{R}_{\sigma\delta}$. Then there is a sequence $\langle E_i \rangle$ of sets in $\mathcal{R}_\sigma$ such that $E_{i+1} \subset E_i$ and $E = \bigcap E_i$. It follows from Proposition 6 that we may take $\mu \times \nu(E_1) < \infty$. Let $g_i(x) = \nu[(E_i)_x]$. Since

$$\int g_1\, d\mu = \mu \times \nu(E_1) < \infty,$$

we have $g_1(x) < \infty$ for almost all x. For an x with $g_1(x) < \infty$, we have $\langle (E_i)_x \rangle$ a decreasing sequence of measurable sets of finite measure whose intersection is E_x.

Thus by Proposition 11.2 we have

$$\begin{aligned}g(x) = \nu(E_x) &= \lim \nu[(E_i)_x] \\ &= \lim g_i(x).\end{aligned}$$

Hence

$$g_i \to g \text{ a.e.},$$

and so g is measurable. Since $0 \le g_i \le g_1$, the Lebesgue Convergence Theorem implies that

$$\begin{aligned}\int g\, d\mu &= \lim \int g_i\, d\mu \\ &= \lim \mu \times \nu(E_i) \\ &= \mu \times \nu(E).\end{aligned}$$

the last equality following from Proposition 11.2. ▌

17. Lemma: *Let E be a set for which $\mu \times \nu(E) = 0$. Then for almost all x we have $\nu(E_x) = 0$.*

Proof: By Proposition 6 there is a set F in $\mathcal{R}_{\sigma\delta}$ such that $E \subset F$ and $\mu \times \nu(F) = 0$. It follows from Lemma 16 that for almost all x we have $\nu(F_x) = 0$. But $E_x \subset F_x$, and so $\nu E_x = 0$ for almost all x since ν is complete. ▮

18. Proposition: *Let E be a measurable subset of $X \times Y$ such that $\mu \times \nu(E)$ is finite. Then for almost all x the set E_x is a measurable subset of Y. The function g defined by*

$$g(x) = \nu(E_x)$$

is a measurable function defined for almost all x and

$$\int g \, d\mu = \mu \times \nu(E).$$

Proof: By Proposition 6 there is a set F in $\mathcal{R}_{\sigma\delta}$ such that $E \subset F$ and $\mu \times \nu(F) = \mu \times \nu(E)$. Let $G = F \sim E$. Since E and F are measurable, so is G, and

$$\mu \times \nu(F) = \mu \times \nu(E) + \mu \times \nu(G).$$

Since $\mu \times \nu(E)$ is finite and equal to $\mu \times \nu(F)$, we have $\mu \times \nu(G) = 0$. Thus by Lemma 17 we have $\nu G_x = 0$ for almost all x. Hence

$$g(x) = \nu E_x = \nu F_x \text{ a.e.};$$

so g is a measurable function by Lemma 16. Again by Lemma 16

$$\int g \, d\mu = \mu \times \nu(F)$$
$$= \mu \times \nu(E). \quad ▮$$

The following two theorems enable us to interchange the order of integration and to calculate integrals with respect to product measures by iteration.

19. Theorem (Fubini): *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete measure spaces and f an integrable function on $X \times Y$. Then*

i. *For almost all x the function f_x defined by $f_x(y) = f(x, y)$ is an integrable function on Y.*

i′. *For almost all y the function f^y defined by $f^y(x) = f(x, y)$ is an integrable function on X.*

ii. $\int_Y f(x, y)\, d\nu(y)$ *is an integrable function on* X.

ii′. $\int_X f(x, y)\, d\mu(x)$ *is an integrable function on* Y.

iii. $$\int_X \left[\int_Y f\, d\nu\right] d\mu = \int_{X \times Y} f\, d(\mu \times \nu) = \int_Y \left[\int_X f\, d\mu\right] d\nu.$$

Proof: Because of the symmetry between x and y it suffices to prove (i), (ii), and the first half of (iii). If the conclusion of the theorem holds for each of two functions, it also holds for their difference, and hence it is sufficient to consider the case when f is nonnegative. Proposition 18 asserts that the theorem is true if f is the characteristic function of a measurable set of finite measure, and hence the theorem must be true if f is a simple function which vanishes outside a set of finite measure. Proposition 11.7 asserts that each nonnegative integrable function f is the limit of an increasing sequence $\langle \varphi_n \rangle$ of nonnegative simple functions, and, since each φ_n is integrable and simple, it must vanish outside a set of finite measure. Thus f_x is the limit of the increasing sequence $\langle (\varphi_n)_x \rangle$ and is measurable. By the Monotone Convergence Theorem

$$\int_Y f(x, y)\, d\nu(y) = \lim \int_Y \varphi_n(x, y)\, d\nu(y),$$

and so this integral is a measurable function of x. Again by the Monotone Convergence Theorem

$$\begin{aligned}\int_X \left[\int_Y f\, d\nu\right] d\mu &= \lim \int_X \left[\int_Y \varphi_n\, d\nu\right] d\mu \\ &= \lim \int_{X \times Y} \varphi_n\, d(\mu \times \nu) \\ &= \int_{X \times Y} f\, d(\mu \times \nu). \quad \blacksquare\end{aligned}$$

In order to apply the Fubini Theorem, one must first verify that f is integrable with respect to $\mu \times \nu$; that is, one must show that f is a measurable function on $X \times Y$ and that $\int |f|\, d(\mu \times \nu) < \infty$. The measurability of f on $X \times Y$ is sometimes difficult to establish, but in many cases we can establish it by topological considerations (cf. Problem 21). In the case when μ and ν are σ-finite, the integrability of f can be determined by iterated integration using the following theorem:

20. Theorem (Tonelli): *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces, and let f be a nonnegative measurable function on $X \times Y$. Then*

i. *For almost all x the function f_x defined by $f_x(y) = f(x, y)$ is a measurable function on Y.*

i′. *For almost all y the function f^y defined by $f^y(x) = f(x, y)$ is a measurable function on X.*

ii. $\int_Y f(x, y)\, d\nu(y)$ *is a measurable function on X.*

ii′. $\int_X f(x, y)\, d\mu(x)$ *is a measurable function on Y.*

iii. $$\int_X \left[\int_Y f\, d\nu\right] d\mu = \int_{X \times Y} f\, d(\mu \times \nu) = \int_Y \left[\int_X f\, d\mu\right] d\nu.$$

Proof: For a nonnegative measurable function f the only point in the proof of Theorem 19 where the integrability of f was used was to infer the existence of an increasing sequence $\langle \varphi_n \rangle$ of simple functions each vanishing outside a set of finite measure such that $f = \lim \varphi_n$. But if μ and ν are σ-finite, then so is $\mu \times \nu$, and any nonnegative measurable function on $X \times Y$ can be so approximated by Proposition 11.7. ▮

If $\mathcal{A}$ and $\mathcal{B}$ are σ-algebras on X and Y, then the smallest σ-algebra containing the measurable rectangles is denoted by $\mathcal{A} \times \mathcal{B}$. Thus the product measure is defined on a σ-algebra containing $\mathcal{A} \times \mathcal{B}$, and since $\mu \times \nu$ is obtained by the Carathéodory extension process, it is both complete and saturated. If μ and ν are both σ-finite, then the product measure on $\mathcal{A} \times \mathcal{B}$ is already saturated and the measurable sets for $\mu \times \nu$ are those which differ from sets in $\mathcal{A} \times \mathcal{B}$ by sets of measure zero.

Many authors prefer to define product measure to be the restriction of $\mu \times \nu$ to $\mathcal{A} \times \mathcal{B}$. The advantage of taking $\mu \times \nu$ to be complete, as we have done here, is that this does what we want it to for Lebesgue measure: The product of n-dimensional Lebesgue measure with m-dimensional Lebesgue measure is $(n + m)$-dimensional Lebesgue measure. Since our hypotheses for the Fubini and Tonelli theorems require only measurability with respect to the complete product measure, they are weaker than requiring measurability with respect to $\mathcal{A} \times \mathcal{B}$. The price for using these weaker hypotheses is the necessity of including the "almost all" phrases in the conclusion of the theorems. This has to be expected, since changing f arbitrarily for x in a set of measure zero does not change the measurability or integrability of f, but f_x can be arbitrary for those x. If however, f is measurable with respect to $\mathcal{A} \times \mathcal{B}$, then f_x is measurable for each x.

We have also used the completeness of μ to show that $\int f(x, y)\, d\nu(y)$ is measurable, for if μ were not complete we could only conclude that this was a function which differed on a subset of a set of measure zero from a measurable function. If, however, f is measurable with respect to $\mathcal{A} \times \mathcal{B}$, then it turns out that $\int f(x, y)\, d\nu(y)$ is measurable with respect to $\mathcal{A}$ even if μ is not complete (provided f is integrable), but the proof of this is surprisingly intricate. For a proof see Halmos [5], p. 143.

The examples in the problems show that we cannot omit the hypothesis of the integrability of f from the Fubini Theorem or the hypotheses of σ-finiteness and nonnegativity from the Tonelli Theorem. Problem 26 shows the essential role played by the measurability of f in these theorems: If we omit this assumption, even for bounded functions and finite measures, we may have the iterated integrals $\int [\int f\, d\nu]\, d\mu$ and $\int [\int f\, d\mu]\, d\nu$ well defined but unequal.

Problems

19. Let $X = Y$ be the set of positive integers, $\mathcal{A} = \mathcal{B} = \mathcal{P}(X)$, and let $\nu = \mu$ be the measure defined by setting $\mu(E)$ equal to the number of points in E if E is finite and ∞ if E is an infinite set. (This measure is called the counting measure.) State the Fubini and Tonelli Theorems explicitly for this case.

20. Let $(X, \mathcal{A}, \mu)$ be any σ-finite measure space and Y the set of positive integers with ν the counting measure (Problem 19). Then Theorem 20 and Corollary 11.14 state the same conclusion. However, Corollary 11.14 is valid even if μ is not σ-finite, and hence the Tonelli Theorem is true without σ-finiteness if $(Y, \mathcal{B}, \nu)$ is this special measure space.

21. Let $X = Y = [0, 1]$, and let $\mu = \nu$ be Lebesgue measure. Show that each open set in $X \times Y$ is measurable, and hence each Borel set in $X \times Y$ is measurable.

22. Let h and g be integrable functions on X and Y, and define $f(x, y) = h(x)g(y)$. Then f is integrable on $X \times Y$ and

$$\int_{X \times Y} f\, d(\mu \times \nu) = \int_X h\, d\mu \int_Y g\, d\nu.$$

(*Note:* We do not need to assume that μ and ν are σ-finite.)

23. Show that Tonelli's Theorem is still true if, instead of assuming μ and ν to be σ-finite, we merely assume that $\{\langle x, y\rangle : f(x, y) \neq 0\}$ is a set of σ-finite measure.

24. The following example shows that we cannot remove the hypothesis that f be nonnegative from the Tonelli Theorem or that f be integrable from the Fubini Theorem. Let $X = Y$ be the positive integers and $\mu = \nu$ be the counting measure. Let

$$f(x, y) = \begin{cases} 2 - 2^{-x} & \text{if } x = y \\ -2 + 2^{-x} & \text{if } x = y + 1 \\ 0 & \text{otherwise.} \end{cases}$$

25. The following example shows that we cannot remove the hypothesis that f be integrable from the Fubini Theorem or that μ and ν are σ-finite from the Tonelli Theorem: Let $X = Y$ be the interval $[0, 1]$, with $\mathcal{A} = \mathcal{B}$ the class of Borel sets. Let μ be Lebesgue measure and ν the counting measure. Then the diagonal $\Delta = \{\langle x, y\rangle \in X \times Y : x = y\}$ is measurable (is an $\mathcal{R}_{\sigma\delta}$, in fact), but its characteristic function fails to satisfy any of the equalities in condition (iii) of the Fubini and Tonelli Theorems.

26. The following example shows that the hypothesis that f be measurable with respect to the product measure cannot be omitted from the Fubini and Tonelli Theorems even if we assume the measurability of f^y and f_x and the integrability of $\int f(x, y)\, d\nu(y)$ and $\int f(x, y)\, d\mu(x)$. Let $X = Y =$ the set of ordinals less than or equal to the first uncountable ordinal Ω. Let $\mathcal{A} = \mathcal{B}$ be the σ-algebra consisting of all countable sets and their complements. Define $\mu = \nu$ by letting $\mu E = 0$ if E countable, $\mu E = 1$ otherwise. Define a subset S of $X \times Y$ by $S = \{\langle x, y\rangle : x < y\}$. Then S_x and S_y are measurable for each x and y, but if f is the characteristic function of S we have

$$\int \left[\int f(x, y)\, d\mu(x) \right] d\nu(y) \neq \int \left[\int f(x, y)\, d\nu(y) \right] d\mu(x).$$

If we assume the continuum hypothesis, that is, that X can be put in one-to-one correspondence with $[0, 1]$, then we can take f to be a function on the unit square such that f_x and f^y are bounded and measurable for each x and y but such that the conclusion of the Fubini and Tonelli Theorems do not hold.

27. Show that if $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$ are two σ-finite measure spaces, then $\mu \times \nu$ is the only measure on $\mathfrak{A} \times \mathfrak{B}$ which assigns the value $\mu A\ \nu B$ to each measurable rectangle $A \times B$. Show that a measure on $\mathfrak{A} \times \mathfrak{B}$ with this property need not be unique, if we do not have σ-finiteness.

28. **a.** Show that if $E \in \mathfrak{A} \times \mathfrak{B}$, then $E_x \in \mathfrak{B}$ for each x.

b. If f is measurable with respect to $\mathfrak{A} \times \mathfrak{B}$, then f_x is measurable with respect to $\mathfrak{B}$ for each x.

29. Let $X = Y = \mathbf{R}$ and let $\mu = \nu =$ Lebesgue measure. Then $\mu \times \nu$ is two-dimensional Lebesgue measure on $X \times Y = \mathbf{R}^2$. We often write $dx\,dy$ for $d(\mu \times \nu)$.

a. For each measurable subset E of $\mathbf{R}$, let

$$\sigma(E) = \{\langle x, y\rangle : x - y \in E\}.$$

Show that $\sigma(E)$ is a measurable subset of $\mathbf{R}^2$. [Hint: Consider first the cases when E open, E a G_δ, E of measure zero, and E measurable.]

b. If f is a measurable function on $\mathbf{R}$, the function F defined by $F(x, y) = f(x - y)$ is a measurable function on $\mathbf{R}^2$.

c. If f and g are integrable functions on $\mathbf{R}$, then for almost all x the function φ given by $\varphi(y) = f(x - y)g(y)$ is integrable. If we denote its integral by $h(x)$, then h is integrable and

$$\int |h| \leq \int |f| \int |g|.$$

30. Let f and g be functions in $L^1(-\infty, \infty)$, and define $f * g$ to be the function h defined by $h(y) = \int f(y - x)g(x)\,dx$.

a. Show that $f * g = g * f$.

b. Show that $(f * g) * h = f * (g * h)$.

c. For $f \in L^1$, define $\hat{f}$ by $\hat{f}(s) = \int e^{ist} f(t)\,dt$. Then $\hat{f}$ is a bounded complex function and

$$\widehat{f * g} = \hat{f}\hat{g}.$$

31. Let f be a nonnegative integrable function on $(-\infty, \infty)$, and let m_2 be two-dimensional Lebesgue measure on $\mathbf{R}^2$. Then

$$m_2\{\langle x, y\rangle : 0 \leq y \leq f(x)\} = m_2\{\langle x, y\rangle : 0 < y < f(x)\} = \int f(x)\,dx.$$

Let $\varphi(t) = m\{x : f(x) \geq t\}$. Then φ is a decreasing function and

$$\int_0^\infty \varphi(t)\,dt = \int f(x)\,dx.$$

32. If $\langle (X_i, \mathfrak{A}_i, \mu_i)\rangle_{i=1}^n$ is a finite collection of measure spaces, we can form the product measure $\mu_1 \times \cdots \times \mu_n$ on the space $X_1 \times \cdots \times X_n$ by starting with the semialgebra of rectangles of the form $R = A_1 \times \cdots \times A_n$ and $\mu(R) = \prod \mu_i A_i$, and using the Carathéodory extension procedure. Show that if we identify $(X_1 \times \cdots \times X_p) \times (X_{p+1} \times \cdots \times X_n)$ with $(X_1 \times \cdots \times X_n)$, then $(\mu_1 \times \cdots \times \mu_p) \times (\mu_{p+1} \times \cdots \times \mu_n) = \mu_1 \times \cdots \times \mu_n$.

33. A measure μ with $\mu X = 1$ is often called a *probability measure.* Let $\{(X_\lambda, \mathfrak{A}_\lambda, \mu_\lambda)\}$ be a collection of probability measure spaces. Show that we can define a probability measure

$$\mu = \prod_\lambda \mu_\lambda$$

on a suitable σ-algebra on the space $\bigtimes_\lambda X_\lambda$ so that

$$\mu A = \prod \mu_\lambda A_\lambda$$

when $A = \bigtimes_\lambda A_\lambda$. (Note that μA can only be nonzero if all but a countable number of the A_λ have $\mu A_\lambda = 1$.)

5 Integral Operators

In this section we study a class of integral operators which define linear transformations from $L^q(\nu)$ to $L^p(\mu)$. We let the letters p, q, and r stand for extended real numbers $1 \le p \le \infty$, and so on, and use p^* for the conjugate exponent $p/(p-1)$ so that $1/p + 1/p^* = 1$. We shall often denote $1/p$, $1/q$, and $1/r$ by α, β, and γ, respectively. Thus $\alpha^* = 1 - \alpha$.

Let $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$ be two σ-finite measure spaces and $k = k(x, y)$ a nonnegative measurable function on $X \times Y$. We define

$$M^*_{\gamma,\beta} = \sup \iint h(x)k(x, y)g(y)\, d(\mu \times \nu)$$

as h and g range over all functions of norm at most one in $L^r(\mu)$ and $L^q(\nu)$, where $\gamma = 1/r$ and $\beta = 1/q$. Since $k \ge 0$, it suffices to consider only nonnegative h and g. If $M^*_{\gamma,\beta} < \infty$, we say that k is an **integral kernel** of **covariant type** (r, q), and call $M^*_{\gamma,\beta}$ its covariant norm. We also say that k is an integral kernel of **operator type** (p, q), where $p = r^*$. We write

$$M_{\alpha,\beta} = M^*_{(1-\alpha),\beta} = \|k\|_{p,q}.$$

In the notion of covariant type one thinks of k as defining a bilinear form

$$[h, g] = \iint h(x)k(x, y)g(y)\, d(\mu \times \nu)$$

between the elements of $L^r(\mu)$ and $L^q(\nu)$, with $M^*_{\gamma,\beta}$ the norm of the bilinear form. The relation of this to the operator version is given by the following proposition.

21. Proposition: *Let k be a nonnegative measurable function on $X \times Y$ of covariant type (p^*, q) and $g \in L^q(\nu)$. Then for almost all $x \in X$ the integral*

$$f(x) = \int_Y k(x, y)g(y)\, d\nu$$

exists, and the function f belongs to $L^p(\mu)$ with

$$\|f\|_p \leq M_{1/p,\, 1/q} \|g\|_q .$$

Proof: Since μ is σ-finite, there is a function $h \in L^{p^*}(\mu)$ with $h(x) > 0$ everywhere. Since

$$\iint_{X \times Y} h(x)k(x, y)|g(y)|\, d(\mu \times \nu) \leq M_{\alpha,\beta} \|h\|_{p^*} \|g\|_q < \infty,$$

we see that

$$\int_Y h(x)k(x, y)|g(y)|\, d\nu = h(x) \int_Y k(x, y)|g(y)|\, d\nu$$

for almost all $x \in X$ by Tonelli's Theorem.

Thus $f(x)$ exists for almost all x. Let h be an arbitrary function in $L^{p^*}(\mu)$. Then

$$\int_X |h(x)f(x)|\, d\mu = \iint_{X \times Y} |h(x)k(x, y)g(y)|\, d(\mu \times \nu)$$

by the Fubini Theorem, since $|hkg|$ is integrable. Consequently, hf is integrable and

$$\left| \int hf\, d\mu \right| \leq M^*_{1-\alpha,\beta} \|h\|_{p^*} \|g\|_q .$$

By Lemma 7.27 we have $f \varepsilon L^p$ and

$$\|f\|_p \leq M_{\alpha,\beta}\|g\|_q. \quad \blacksquare$$

This proposition shows that we have defined a linear operator $T: L^q(\nu) \to L^p(\mu)$ by taking $Tg = f$ where

$$f(x) = \int_Y k(x, y)g(y)\, d\nu.$$

Moreover, the operator norm $\|T\|$ of T can be shown to be $M_{\alpha,\beta} = \|k\|_{p,q}$.

More generally, we call a measurable function $k(x, y)$ on $X \times Y$ an integral operator of *absolute operator type* (p, q) if $|k|$ is of operator type (p, q). The proposition can be rephrased for such kernels:

22. Corollary: *Let $k(x, y)$ be a measurable function on $X \times Y$ of absolute operator type (p, q) and $g \varepsilon L^q(\nu)$. Then for almost all $x \varepsilon X$ the integral*

$$f(x) = \int_Y k(x, y)g(y)\, d\nu$$

exists, and the function f belongs to $L^p(\mu)$ with

$$\|f\|_p \leq \|\,|k|\,\|_{p,q}\,\|g\|_q.$$

The following useful theorem is due to M. Riesz.

23. Theorem: *Let k be a nonnegative measurable function on $X \times Y$, and set*

$$M^*_{\gamma,\beta} = \sup \iint_{X \times Y} h(x)k(x, y)g(y)\, d(\mu \times \nu),$$

where f and g range over the unit balls in $L^r(\mu)$ and $L^q(\nu)$, respectively. Then the function $\log M^*_{\gamma,\beta}$ is a convex function of γ and β in the square $0 \leq \gamma \leq 1, 0 \leq \beta \leq 1$.

Proof: We have to verify that if $0 \leq \lambda \leq 1$, $\gamma = \lambda\gamma_1 + (1 - \lambda)\gamma_2$, and $\beta = \lambda\beta_1 + (1 - \lambda)\beta_2$, then

$$M^*_{\gamma,\beta} \leq (M^*_{\gamma_1,\beta_1})^{\lambda}(M^*_{\gamma_2,\beta_2})^{1-\lambda}.$$

Let h and g be arbitrary nonnegative functions in the unit balls of $L^r(\mu)$ and $L^q(\nu)$. Set

$$h_1 = h^{\gamma_1/\gamma}, \qquad h_2 = h^{\gamma_2/\gamma}, \qquad g_1 = g^{\beta_1/\beta}, \qquad g_2 = g^{\beta_2/\beta}.$$

Then h_1, h_2, g_1, and g_2 are in the unit balls of $L^{r_1}(\mu)$, $L^{r_2}(\mu)$, $L^{q_1}(\nu)$, and $L^{q_2}(\nu)$, respectively. Also,

$$\iint hkg = \iint (h_1 kg)^{\lambda}(h_2 kg_2)^{1-\lambda}$$
$$\leq \left[\iint h_1 kg_1\right]^{\lambda} \cdot \left[\iint h_2 kg\right]^{1-\lambda}$$

by the Hölder inequality for $u = 1/\lambda$ and $u^* = 1/(1 - \lambda)$. Hence

$$\iint hkg \leq (M^*_{\gamma_1, \beta_1})^{\lambda}(M^*_{\gamma_2, \beta_2})^{1-\lambda}$$

and the result follows by taking the supremum over h and g. ∎

24. Corollary: *The function* $\log M_{\alpha,\beta}$ *is a convex function of* α *and* β *in the square* $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$.

Proof: We have $M_{\alpha,\beta} = M^*_{1-\alpha,\beta}$. ∎

The preceding Theorem of Riesz requires that k be nonnegative. A deeper theorem, also due to Riesz, asserts that for a kernel of mixed sign (or even complex valued) the operator norm $M_{\alpha,\beta}$ of the corresponding integral operator is logarithmically convex on the square $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$. The interested reader will find a proof in Dunford and Schwartz [4], p. 525 or Hardy, Littlewood, and Pólya [19], p. 214.

When $(X, \mathcal{A}, \mu) = (Y, \mathcal{B}, \nu) = (\mathbf{R}^n, \mathfrak{M}, m)$, where m is Lebesgue measure, we obtain a special class of integral operators by taking $k(x, y) = k(x - y)$ for some $k \in L^r(m)$. Such operators are called *convolution operators.* It is readily verified in this case that k is of covariant types $(1, r^*)$ and $(r^*, 1)$. It follows from Proposition 21 that k is also of convariant type (p, q) when

$$\frac{1}{p} = 1 - \frac{1-\lambda}{r}, \qquad \frac{1}{q} = 1 - \frac{\lambda}{r}$$

for $0 \leq \lambda \leq 1$. This gives us the following propositions.

25. Proposition: *Let g, h, and k be functions on* $\mathbf{R}^n$ *of class* L^q, L^p, *and* L^r, *respectively, with* $1/p + 1/q + 1/r = 2$. *Then*

$$\iint_{\mathbf{R}^{2n}} |h(x)k(x - y)g(y)| \, dx \, dy \leq \|h\|_p \|k\|_r \|g\|_q.$$

26. Proposition: *Let* $g \in L^q$ *and* $k \in L^r$, *with* $1/q + 1/r > 1$. *Then the function*

$$f(x) = \int_{\mathbf{R}^n} k(x - y)g(y)\, dy$$

is defined for almost all x and

$$\|f\|_p \le \|k\|_r \|g\|_q,$$

where
$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1.$$

Problems

34. Show that for the operator T defined by Proposition 21 we have $\|T\| = M_{\alpha,\beta}$.

35. Prove Corollary 22.

36. Prove Proposition 25.

37. Prove Proposition 26.

38. Let g, h, and k be functions on $\mathbf{R}^n$ of class L^q, L^p, and L^r, with $1/p + 1/q + 1/r \le 2$. Then $h(x)k(x - y)g(y)$ belongs to L^u on $\mathbf{R}^{2n}$, where

$$\frac{2}{u} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}.$$

★6 Inner Measure

Let μ be a measure on an algebra $\mathfrak{A}$ and μ^* the induced outer measure. Then μ^*E may be thought of as the largest possible measure for E compatible with μ. We can also define an inner measure μ_* which assigns to a given set E the smallest measure compatible with μ:

Definition: *Let* μ *be a measure on an algebra* $\mathfrak{A}$ *and* μ^* *the induced outer measure. We define the inner measure* μ_* *induced by* μ *by setting*

$$\mu_* E = \sup\,[\mu A - \mu^*(A \sim E)],$$

where the supremum is taken over all sets $A \in \mathfrak{A}$ *for which* $\mu^*(A \sim E) < \infty$.