6 Different payoffs

Summary

Most of the concrete examples of options considered so far have been the standard examples of calls and puts. Such options have liquid markets, their prices are fairly well determined and margins are competitive. Any option that is not one of these *vanilla* calls or puts is called an exotic option. Such options are introduced to extend a bank's product range or to meet hedging and speculative needs of clients. There are usually no markets in these options and they are bought and sold purely 'over the counter'. Although the principles of pricing and hedging exotics are exactly the same as for vanillas, risk management requires care. Not only are these exotic products much less liquid than standard options, but they often have discontinuous payoffs and so can have huge 'deltas' close to the expiry time making them difficult to hedge.

This chapter is devoted to examples of exotic options. The simplest exotics to price and hedge are *packages*, that is, options for which the payoff is a combination of our standard 'vanilla' options and the underlying asset. We already encountered such options in §1.1. We relegate their valuation to the exercises. The next simplest examples are European options, meaning options whose payoff is a function of the stock price at the maturity time. The payoffs considered in §6.1 are discontinuous and we discover potential hedging problems. In §6.2 we turn our attention to multistage options. Such options allow decisions to be made or stipulate conditions at intermediate dates during their lifetime. The rest of the chapter is devoted to path-dependent options. In §6.3 we use our work of §3.3 to price lookback and barrier options. Asian options, whose payoff depends on the average of the stock price over the lifetime of the option, are discussed briefly in §6.4 and finally §6.5 is a very swift introduction to pricing American options in continuous time.

6.1 European options with discontinuous payoffs

We work in the basic Black–Scholes framework. That is, our market consists of a riskless cash bond whose value at time t is $B_t = e^{rt}$ and a single risky asset whose price, $\{S_t\}_{t\geq 0}$, follows a geometric Brownian motion.

In §5.2 we established explicit formulae for both the price and the hedging portfolio for European options within this framework. Specifically, if the payoff of the option at the maturity time T is $C_T = f(S_T)$ then for $0 \le t \le T$ the value of the option at time t is

$$\begin{aligned} V_t &= F(t, S_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} f(S_T) \middle| \mathcal{F}_t \right] \\ &= e^{-r(T-t)} \int_{-\infty}^{\infty} f\left(S_t \exp\left(\left(r - \frac{\sigma^2}{2} \right)(T-t) + \sigma y \sqrt{T-t} \right) \right) \\ &\qquad \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2} \right) dy, \end{aligned} \tag{6.1}$$

where $\mathbb{Q}$ is the martingale measure, and the claim $f(S_T)$ can be replicated by a portfolio consisting at time t of ϕ_t units of stock and $\psi_t = e^{-rt}(V_t - \phi_t S_t)$ cash bonds where

$$\phi_t = \frac{\partial F}{\partial x}(t, x)\bigg|_{x=S_t}. \tag{6.2}$$

Mathematically, other than the issue of actually *evaluating* the integrals, that would appear to be the end of the story. However, as we shall see, rather more careful consideration of our assumptions might lead us to doubt the usefulness of these formulae when the payoff is a discontinuous function of S_T.

Digitals and pin risk

Example 6.1.1 (Digital options) *The payoff of a* digital option*, also sometimes called a* binary option *or a* cash-or-nothing option*, is given by a Heaviside function. For example, a digital call option with strike price K at time T has payoff*

$$C_T = \begin{cases} 1 & \text{if } S_T \ge K, \\ 0 & \text{if } S_T < K \end{cases}$$

at maturity. Find the price and the hedge for such an option.

Solution: In order to implement the formula (6.1) we must establish the range of y for which

$$S_t \exp\left(\left(r - \frac{\sigma^2}{2} \right)(T-t) + \sigma y \sqrt{T-t} \right) > K.$$

Rearranging we see that this holds for $y > d$ where

$$d = \frac{1}{\sigma\sqrt{T-t}} \left(\log\left(\frac{K}{S_t} \right) - \left(r - \frac{\sigma^2}{2} \right)(T-t) \right).$$

Writing Φ for the normal distribution function and substituting in equation (6.1) we obtain

$$\begin{aligned} V_t &= e^{-r(T-t)} \int_d^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = e^{-r(T-t)} \int_{-\infty}^{-d} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\ &= e^{-r(T-t)} \Phi(-d) = e^{-r(T-t)} \Phi(d_2), \end{aligned}$$

where

$$d_2 = \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right),$$

as in Example 5.2.2.

Now we turn to the hedge. By (6.2), the stock holding in our replicating portfolio at time t is

$$\begin{aligned}\phi_t &= e^{-r(T-t)}\frac{1}{S_t}\frac{1}{\sqrt{2\pi(T-t)}\sigma} \\ &\quad\times \exp\left(-\frac{1}{2(T-t)\sigma^2}\left(\log\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)^2\right).\end{aligned}$$

Now as $t \uparrow T$, this converges to $1/K$ times the delta function concentrated on $S_T = K$. Consider what this means for the replicating portfolio as $t \uparrow T$. Away from $S_t = K$, ϕ_t is close to zero, but if S_t is close to K the stock holding in the portfolio will be very large. Now if near expiry the asset price is close to K, there is a high probability that its value will cross the value $S_t = K$ many times before expiry. But if the asset price oscillates around the strike price close to expiry our prescription for the hedging portfolio will tell us to rapidly buy and sell large numbers of the underlying asset. Since markets are not the perfect objects envisaged in our Black–Scholes model and we cannot instantaneously buy and sell, risk from small asset price changes (not to mention transaction costs) can easily outweigh the maximum liability that we are exposed to by having sold the digital. This is known as the *pin risk* associated with the option. □

If we can overcome our misgivings about the validity of the Black–Scholes price for digitals, then we can use them as building blocks for other exotics. Indeed, since the option with payoff $\mathbf{1}_{[K_1,K_2]}(S_T)$ at time T can be replicated by buying a digital with strike K_2 and maturity T and selling a digital with strike K_1 and maturity T, in theory we could price any European option by replicating it by (possibly infinite) linear combinations of digitals.

6.2 Multistage options

Some options either allow decisions to be made or stipulate conditions at intermediate dates during their lifetime. An example is the forward start option of Exercise 3 of Chapter 2. To illustrate the procedure for valuation of multistage options, we find the Black–Scholes price of a forward start.

Example 6.2.1 (Forward start option) *Recall that a forward start option is a contract in which the holder receives, at time T_0, at no extra cost, an option with expiry date $T_1 > T_0$ and strike price equal to S_{T_0}. If the risk-free rate is r find the Black–Scholes price, V_t, of such an option at times $t < T_1$.*

Solution: First suppose that $t \in [T_0, T_1]$. Then by time t we know S_{T_0} and so the value of the option is just that of a European call option with strike S_{T_0} and maturity T_1, namely

$$V_t = e^{-r(T_1-t)}\mathbb{E}^{\mathbb{Q}}\left[\left(S_{T_1} - S_{T_0}\right)_+ \middle| \mathcal{F}_t\right],$$

where $\mathbb{Q}$ is a probability measure under which the discounted price of the underlying is a martingale. In particular, at time T_0, using Example 5.2.2,

$$V_{T_0} = S_{T_0}\Phi(d_1) - S_{T_0}e^{-r(T_1-T_0)}\Phi(d_2)$$

where

$$d_1 = \frac{\left(r + \frac{\sigma^2}{2}\right)(T_1 - T_0)}{\sigma\sqrt{T_1 - T_0}} \quad \text{and} \quad d_2 = \frac{\left(r - \frac{\sigma^2}{2}\right)(T_1 - T_0)}{\sigma\sqrt{T_1 - T_0}}.$$

In other words

$$\begin{aligned} V_{T_0} &= S_{T_0}\left\{\Phi\left(\left(r + \frac{\sigma^2}{2}\right)\frac{\sqrt{T_1 - T_0}}{\sigma}\right) - e^{-r(T_1-T_0)}\Phi\left(\left(r - \frac{\sigma^2}{2}\right)\frac{\sqrt{T_1 - T_0}}{\sigma}\right)\right\} \\ &= cS_{T_0} \end{aligned}$$

where $c = c(r, \sigma, T_0, T_1)$ is independent of the asset price.

To find the price at time $t < T_0$, observe that the portfolio consisting of c units of the underlying over the time interval $0 \le t \le T_0$ exactly replicates the option at time T_0. Thus for $t < T_0$, the price is given by cS_t. In particular, the time zero price of the option is

$$V_0 = S_0\left\{\Phi\left(\left(r + \frac{\sigma^2}{2}\right)\frac{\sqrt{T_1 - T_0}}{\sigma}\right) - e^{-r(T_1-T_0)}\Phi\left(\left(r - \frac{\sigma^2}{2}\right)\frac{\sqrt{T_1 - T_0}}{\sigma}\right)\right\}.$$

□

General strategy

Notice that, in order to price the forward start option, we worked our way back from time T_1. This reflects a general strategy. For a multistage option with maturity T_1 and conditions stipulated at an intermediate time T_0, we invoke the following procedure.

Valuing multistage options:
1 Find the payoff at time T_1.
2 Use Black–Scholes to value the option for $t \in [T_0, T_1]$.
3 Apply the contract conditions at time T_0.
4 Use Black–Scholes to value the option for $t \in [0, T_0]$.

We put this into action for two more examples.

Example 6.2.2 (Ratio derivative) *A ratio derivative can be described as follows. Two times $0 < T_0 < T_1$ are fixed. The derivative matures at time T_1 when its payoff is S_{T_1}/S_{T_0}. Find the value of the option at times $t < T_1$.*

Solution: First suppose that $t \in [T_0, T_1]$. At such times S_{T_0} is known and so

$$V_t = \frac{1}{S_{T_0}} \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T_1 - t)} S_{T_1} \middle| \mathcal{F}_t \right]$$

where, under $\mathbb{Q}$, the discounted asset price is a martingale. Hence $V_t = S_t / S_{T_0}$. In particular, $V_{T_0} = 1$. Evidently the value of the option for $t < T_0$ is therefore $e^{-r(T_0 - t)}$. □

Both forward start options and ratio derivatives, in which the strike price is set to be a function of the stock price at some intermediate time T_0, are examples of *cliquets*.

Compound options

A rather more complex class of examples is provided by the *compound options*. These are 'options on options', that is options in which the rôle of the underlying is itself played by an option. There are four basic types of compound option: call-on-call, call-on-put, put-on-call and put-on-put.

Example 6.2.3 (Call-on-call option) *To describe the call-on-call option we must specify two exercise prices, K_0 and K_1, and two maturity times $T_0 < T_1$. The 'underlying' option is a European call with strike price K_1 and maturity T_1. The call-on-call contract gives the holder the right to buy the underlying option for price K_0 at time T_0. Find the value of such an option for $t < T_0$.*

Solution: We know how to price the underlying call. Its value at time T_0 is given by the Black–Scholes formula as

$$\begin{aligned} C\left(S_{T_0}, T_0; K_1, T_1\right) &= S_{T_0} \Phi\left(d_1\left(S_{T_0}, T_1 - T_0, K_1\right)\right) \\ &\quad - K_0 e^{-r(T_1 - T_0)} \Phi\left(d_2\left(S_{T_0}, T_1 - T_0, K_1\right)\right) \end{aligned}$$

where

$$d_1\left(S_{t_0}, T_1 - T_0, K_1\right) = \frac{\log\left(\frac{S_{T_0}}{K_1}\right) + \left(r + \frac{\sigma^2}{2}\right)(T_1 - T_0)}{\sigma\sqrt{T_1 - T_0}}$$

and $d_2\left(S_{t_0}, T_1 - T_0, K_1\right) = d_1\left(S_{t_0}, T_1 - T_0, K_1\right) - \sigma\sqrt{T_1 - T_0}$. The value of the compound option at time T_0 is then

$$V\left(T_0, S_{T_0}\right) = \left(C\left(S_{T_0}, T_0; K_1, T_1\right) - K_0\right)_+ .$$

Now we apply Black–Scholes again. The value of the option at times $t < T_0$ is

$$V(t, S_t) = e^{-r(T_0 - t)} \mathbb{E}^{\mathbb{Q}} \left[\left(C(S_{T_0}, T_0, K_1, T_1) - K_0\right)_+ \middle| \mathcal{F}_t^S \right] \tag{6.3}$$

where the discounted asset price is a martingale under $\mathbb{Q}$. Using that

$$S_{T_0} = S_t \exp\left(\sigma Z \sqrt{T_0 - t} + \left(r - \frac{1}{2}\sigma^2\right)(T_0 - t)\right),$$

where, under $\mathbb{Q}$, $Z \sim N(0, 1)$, equation (6.3) now gives an analytic expression for the value in terms of the cumulative distribution function of a bivariate normal random variable. We write

$$f(y) = S_0 \exp\left(\sigma y \sqrt{T_0 - t} + \left(r - \frac{1}{2}\sigma^2\right)(T_0 - t)\right)$$

and define x_0 implicitly by

$$x_0 = \inf\{y \in \mathbb{R} : C(f(y), T_0; K_1, T_1) \geq K_0\}.$$

Now

$$\log\left(\frac{f(y)}{K_1}\right) = \log\left(\frac{S_0}{K_1}\right) + \sigma y \sqrt{T_0 - t} + \left(r - \frac{1}{2}\sigma^2\right)(T_0 - t)$$

and so writing

$$\hat{d}_1(y) = \frac{\log(S_0/K_1) + \sigma y \sqrt{T_0 - t} + rT_1 - \sigma^2 T_0 + \frac{1}{2}\sigma^2 T_1}{\sigma\sqrt{T_1 - T_0}}$$

and

$$\hat{d}_2(y) = \frac{\log(S_0/K_1) + \sigma y \sqrt{T_0 - t} + rT_1 - \frac{1}{2}\sigma^2 T_1}{\sigma\sqrt{T_1 - T_0}}$$

we obtain

$$\begin{aligned} V(t, S_t) &= e^{-r(T_0 - t)} \int_{x_0}^{\infty} \left(f(y)\Phi(\hat{d}_1(y)) - K_0 e^{-r(T_1 - T_0)}\Phi(\hat{d}_2(y)) - K_0\right) \\ &\qquad \times \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy. \end{aligned}$$

□

6.3 Lookbacks and barriers

We now turn to our first example of path-dependent options, that is options for which the history of the asset price over the duration of the contract determines the payout at expiry.

As usual we use $\{S_t\}_{0 \leq t \leq T}$ to denote the price of the underlying asset over the duration of the contract. In this section we shall consider options whose payoff at maturity depends on S_T and one or both of the maximum and minimum values taken by the asset price over $[0, T]$.

Notation: We write

$$S_*(t) = \min\{S_u : 0 \leq u \leq t\},$$

$$S^*(t) = \max\{S_u : 0 \leq u \leq t\}.$$

Definition 6.3.1 (Lookback call) A lookback call *gives the holder the right to buy a unit of stock at time T for a price equal to the minimum achieved by the stock up to time T. That is the payoff is*

$$C_T = S_T - S_*(T).$$

Definition 6.3.2 (Barrier options) A barrier option *is one that is activated or deactivated if the asset price crosses a preset barrier. There are two basic types:*

1 **knock-ins**

(a) *the barrier is* up-and-in *if the option is only active if the barrier is hit from below,*
(b) *the barrier is* down-and-in *if the option is only active if the barrier is hit from above;*

2 **knock-outs**

(a) *the barrier is* up-and-out *if the option is worthless if the barrier is hit from below,*
(b) *the barrier is* down-and-out *if the option is worthless if the barrier is hit from above.*

Example 6.3.3 A down-and-in call option *pays out $(S_T - K)_+$ only if the stock price fell below some preagreed level c some time before T, otherwise it is worthless. That is, the payoff is*

$$C_T = \mathbf{1}_{\{S_*(T) \leq c\}}(S_T - K)_+.$$

As always we can express the value of such an option as a discounted expected value under the martingale measure $\mathbb{Q}$. Thus the value at time zero can be written as

$$V(0, S_0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[C_T] \tag{6.4}$$

where r is the riskless borrowing rate and the discounted stock price is a $\mathbb{Q}$-martingale. However, in order to actually evaluate the expectation in (6.4) for barrier options we need to know the joint distribution of $(S_T, S_*(T))$ and $(S_T, S^*(T))$ under the martingale measure $\mathbb{Q}$. Fortunately we did most of the work in Chapter 3.

Joint distribution of the stock price and its minimum

In Lemma 3.3.4 we found the joint distribution of Brownian motion and its maximum. Specifically, if $\{W_t\}_{t\geq 0}$ is a standard $\mathbb{P}$-Brownian motion, writing $M_t = \max_{0\leq s\leq t} W_s$, for $a > 0$ and $x \leq a$

$$\mathbb{P}[M_t \geq a, W_t \leq x] = 1 - \Phi\left(\frac{2a - x}{\sqrt{t}}\right).$$

By symmetry, writing $m_t = \min_{0\leq s\leq t} W_s$, for $a < 0$ and $x \geq a$,

$$\mathbb{P}[m_t \leq a, W_t \geq x] = 1 - \Phi\left(\frac{-2a + x}{\sqrt{t}}\right),$$

or, differentiating, if $a < 0$ and $x \geq a$

$$\mathbb{P}[m_T \leq a, W_T \in dx] = p_T(0, -2a + x)dx = p_T(2a, x)dx$$

where

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-|x-y|^2/2t\right).$$

Combining these results with (two applications of) the Girsanov Theorem will allow us to calculate the joint distribution of $(S_T, S^*(T))$ and of $(S_T, S_*(T))$ under the martingale measure $\mathbb{Q}$.

As usual, under the market measure $\mathbb{P}$,

$$S_t = S_0 \exp(\nu t + \sigma W_t)$$

where $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion. Let us suppose, temporarily, that $\nu = 0$ so that $S_t = S_0 \exp(\sigma W_t)$ and moreover $S_*(t) = S_0 \exp(\sigma m_t)$ and $S^*(t) = S_0 \exp(\sigma M_t)$. In this special case then the joint distribution of the stock price and its minimum (resp. maximum) can be deduced from that of (W_t, m_t) (resp. (W_t, M_t)). Of course, in general, ν will not be zero either under the market measure $\mathbb{P}$ or under the martingale measure $\mathbb{Q}$. Our strategy will be to use the Girsanov Theorem not only to switch to the martingale measure but also to switch, temporarily, to an equivalent measure under which $S_t = S_0 \exp(\sigma W_t)$.

Lemma 6.3.4 *Let $\{Y_t\}_{t\geq 0}$ be given by $Y_t = bt + X_t$ where b is a constant and $\{X_t\}_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion. Writing $Y_*(t) = \min\{Y_u : 0 \leq u \leq t\}$,*

$$\mathbb{Q}\left[Y_*(T) \leq a, Y_T \in dx\right] = \begin{cases} p_T(bT, x)dx & \text{if } x < a, \\ e^{2ab} p_T(2a + bT, x)dx & \text{if } x \geq a, \end{cases}$$

where, as above, $p_t(x, y)$ is the Brownian transition density function.

Proof: By the Girsanov Theorem, there is a measure $\mathbb{P}$, equivalent to $\mathbb{Q}$, under which $\{Y_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion and

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_r} = \exp\left(-bX_T - \frac{1}{2}b^2 T\right).$$

Notice that this depends on $\{X_t\}_{0\leq t\leq T}$ only through X_T. The $\mathbb{Q}$-probability of the event $\{Y_*(T) \leq a, Y_T \in dx\}$ will be the $\mathbb{P}$-probability of that event multiplied by $\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_r}$ evaluated at $Y_T = x$. Now

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(bX_T + \frac{1}{2}b^2 T\right) = \exp\left(bY_T - \frac{1}{2}b^2 T\right)$$

and so for $a < 0$ and $x \geq a$

$$\begin{aligned} \mathbb{Q}\left[Y_*(T) \leq a, Y_T \in dx\right] &= \mathbb{P}\left[Y_*(T) \leq a, Y_T \in dx\right] \exp\left(bx - \frac{1}{2}b^2 T\right) \\ &= p_T(2a, x) \exp\left(bx - \frac{1}{2}b^2 T\right) dx \\ &= e^{2ab} p_T(2a + bT, x)dx. \end{aligned} \tag{6.5}$$

Evidently for $x \leq a$, $\{Y_*(T) \leq a, Y_T \in dx\} = \{Y_T \in dx\}$ and so for $x \leq a$

$$\begin{aligned}
\mathbb{Q}[Y_*(T) \leq a, Y_T \in dx] &= \mathbb{Q}[Y_T \in dx] \\
&= \mathbb{Q}[bT + X_T \in dx] \\
&= p_T(bT, x)dx
\end{aligned}$$

and the proof is complete. □

Differentiating (6.5) with respect to a, we see that, in terms of joint densities, for $a < 0$

$$\mathbb{Q}[Y_*(T) \in da, Y_T \in dx] = \frac{2e^{2ab}}{T}|x - 2a|p_T(2a + bT, x)dx\,da \quad \text{for } x \geq a.$$

The joint density evidently vanishes if $x < a$ or $a > 0$. In Exercise 13 you are asked to find the joint distribution of Y_T and $Y^*(T)$ under $\mathbb{Q}$.

An expression for the price

From Chapter 5, under the martingale measure $\mathbb{Q}$, $S_t = S_0 \exp(\sigma Y_t)$ where

$$Y_t = \frac{(r - \frac{1}{2}\sigma^2)}{\sigma}t + X_t$$

and $\{X_t\}_{t \geq 0}$ is a $\mathbb{Q}$-Brownian motion. So by applying these results with $b = (r - \frac{1}{2}\sigma^2)/\sigma$ we can now evaluate the price of any option maturing at time T whose payoff depends just on the stock price at time T and its minimum (or maximum) value over the lifetime of the contract. If the payoff is $C_T = g(S_*(T), S_T)$ and r is the riskless borrowing rate then the value of the option at time zero is

$$\begin{aligned}
V(0, S_0) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[g(S_*(T), S_T)] \\
&= e^{-rT}\int_{a=-\infty}^{0}\int_{x=a}^{\infty} g\left(S_0 e^{\sigma x}, S_0 e^{\sigma a}\right)\mathbb{Q}[Y_*(T) \in da, Y_T \in dx].
\end{aligned}$$

Example 6.3.5 (Down-and-in call option) *Find the time zero price of a down-and-in call option whose payoff at time T is*

$$C_T = \mathbf{1}_{\{S_*(T) \leq c\}}(S_T - K)_+$$

where c is a (positive) preagreed constant less than K.

Solution: Using $S_t = S_0 \exp(\sigma Y_t)$ we rewrite the payoff as

$$C_T = \mathbf{1}_{\{Y_*(T) \leq \frac{1}{\sigma}\log(c/S_0)\}}\left(S_0 e^{\sigma Y_T} - K\right)_+.$$

Writing $b = (r - \frac{1}{2}\sigma^2)/\sigma$, $a = \frac{1}{\sigma}\log(c/S_0)$ and $x_0 = \frac{1}{\sigma}\log(K/S_0)$ we obtain

$$V(0, S_0) = e^{-rT}\int_{x_0}^{\infty}\left(S_0 e^{\sigma x} - K\right)\mathbb{Q}(Y_*(T) \leq a, Y_T \in dx).$$

Using the expression for the joint distribution of $(Y_*(T), Y_T)$ obtained above yields

$$V(0, S_0) = e^{-rT} \int_{x_0}^{\infty} \left(S_0 e^{\sigma x} - K\right) e^{2ab} p_T(2a + bT, x) dx.$$

We have used the fact that, since $c < K$, $x_0 \geq a$. First observe that

$$\begin{aligned} e^{-rT} \int_{x_0}^{\infty} K e^{2ab} p_T(2a + bT, x) dx &= K e^{-rT} e^{2ab} \int_{(x_0 - 2a - bT)/\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\ &= K e^{-rT} e^{2ab} \int_{-\infty}^{(2a + bT - x_0)/\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\ &= K e^{-rT} \left(\frac{c}{S_0}\right)^{\frac{2r}{\sigma^2} - 1} \Phi\left(\frac{2a + bT - x_0}{\sqrt{T}}\right) \\ &= K e^{-rT} \left(\frac{c}{S_0}\right)^{\frac{2r}{\sigma^2} - 1} \Phi\left(\frac{\log(F/K) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \end{aligned}$$

where $F = e^{rT} c^2 / S_0$.

Similarly,

$$\begin{aligned} & e^{-rT} \int_{x_0}^{\infty} S_0 e^{\sigma x} e^{2ab} p_T(2a + bT, x) dx \\ &= S_0 e^{-rT} e^{2ab} \int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{(x - (2a + bT))^2 - 2\sigma x T}{2T}\right) dx \\ &= S_0 e^{-rT} e^{2ab} \int_{(x_0 - (2a + bT) - \sigma T)/\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\ &\quad \times \exp\left(\frac{1}{2}\sigma^2 T + 2a\sigma + b\sigma T\right) \\ &= e^{-rT} \left(\frac{c}{S_0}\right)^{\frac{2r}{\sigma^2} - 1} F \Phi\left(\frac{\log(F/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right). \end{aligned}$$

Comparing this with Example 5.2.2

$$V(0, S_0) = \left(\frac{c}{S_0}\right)^{\frac{2r}{\sigma^2} - 1} C\left(\frac{c^2}{S_0}, 0; K, T\right),$$

where $C(x, t; K, T)$ is the price at time t of a European call option with strike K and maturity T if the stock price at time t is x. □

The price of a barrier option can also be expressed as the solution of a partial differential equation.

Example 6.3.6 (Down-and-out call) *A down-and-out call has the same payoff as a European call option, $(S_T - K)_+$, unless during the lifetime of the contract the price of the underlying asset has fallen below some preagreed barrier, c, in which case the option is 'knocked out' worthless.*

Writing $V(t,x)$ for the value of such an option at time t if $S_t = x$ and assuming that $K > c$, $V(t,x)$ solves the Black–Scholes equation for $(t,x) \in [0,T] \times [c,\infty)$ subject to the boundary conditions

$$\begin{aligned} V(T, S_T) &= (S_T - K)_+, \\ V(t,c) &= 0, \quad t \in [0,T], \\ \frac{V(t,x)}{x} &\to 1, \quad \text{as } x \to \infty. \end{aligned}$$

The last boundary condition follows since as $S_t \to \infty$, the probability of the asset price hitting level c before time T tends to zero.

Exercise 16 provides a method for solving the Black–Scholes partial differential equation with these boundary conditions.

Of course more and more complicated barrier options can be dreamt up. For example, a *double knock-out option* is worthless if the stock price leaves some interval $[c_1, c_2]$ during the lifetime of the contract. The probabilistic pricing formula for such a contract then requires the joint distribution of the triple $(S_T, S_*(T), S^*(T))$. As in the case of a single barrier, the trick is to use Girsanov's Theorem to deduce the joint distribution from that of (W_T, m_T, M_T) where $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion and $\{m_t\}_{t\geq 0}$, $\{M_t\}_{t\geq 0}$ are its running minimum and maximum respectively. This in turn is given by

$$\mathbb{P}[W_T \in dy, a < m_T, M_T < b] = \sum_{n\in\mathbb{Z}} \Big\{ p_T(2n(a-b), y) - p(2n(b-a), y-2a) \Big\} dy;$$

see Freedman (1971) for a proof. An explicit pricing formula will then be in the form of an infinite sum. In Exercise 20 you obtain the pricing formula by directly solving the Black–Scholes differential equation.

Probability or pde?

As we have seen in Exercise 7 of Chapter 5 and we see again in the exercises at the end of this chapter, the Black–Scholes partial differential equation can be solved by first transforming it to the heat equation (with appropriate boundary conditions). This is entirely parallel to our probabilistic technique of transforming the expectation price to an expectation of a function of Brownian motion.

6.4 Asian options

The payoff of an Asian option is a function of the average of the asset price over the lifetime of the contract. For example, the payoff of an *Asian call* with strike price K and maturity time T is

$$C_T = \left(\frac{1}{T} \int_0^T S_t dt - K \right)_+ .$$

Evidently $C_T \in \mathcal{F}_T$ and so our Black–Scholes analysis of Chapter 5 gives the value of such an option at time zero as

$$V_0 = \mathbb{E}^{\mathbb{Q}} \left[e^{-rT} \left(\frac{1}{T} \int_0^T S_t dt - K \right)_+ \right]. \tag{6.6}$$

However, evaluation of this integral is a highly non-trivial matter and we do not obtain the nice explicit formulae of the previous sections.

There are many variants on this theme. For example, we might want to value a claim with payoff

$$C_T = f\left(S_T, \frac{1}{T}\int_0^T S_t dt\right).$$

In §7.2 we shall develop the technology to express the price of such claims (and indeed slightly more complex claims) as solutions to a multidimensional version of the Black–Scholes equation. Moreover (see Exercise 12 of Chapter 7) one can also find an explicit expression for the hedging portfolio in terms of the solution to this equation. However, multidimensional versions of the Black–Scholes equation are much harder to solve than their one-dimensional counterpart and generally one must resort to numerical techniques.

The main difficulty with evaluating (6.6) directly is that, although there are explicit formulae for all the moments of the average process $\frac{1}{T}\int_0^T S_t dt$, in contrast to the lognormal distribution of S_T, we do not have an expression for the distribution function. A number of approaches have been suggested to overcome this, including simply approximating the distribution of the average process by a lognormal distribution with suitably chosen parameters.

A very natural approach is to replace the continuous average by a discrete analogue obtained by sampling the price of the process at agreed times $t_1, \dots, t_n$ and averaging the result. This also makes sense from a practical point of view as calculating the continuous average for a real asset can be a difficult process. Many contracts actually specify that the average be calculated from such a discrete sample – for example from daily closing prices. Mathematically, the continuous average $\frac{1}{T}\int_0^T S_t dt$ is replaced by $\frac{1}{n}\sum_{i=1}^n S_{t_i}$. Options based on a discrete sample can be treated in the same way as multistage options, although evaluation of the price rapidly becomes impractical (see Exercise 21).

A further approximation is to replace the arithmetic average by a *geometric* average. That is, in place of $\frac{1}{n}\sum_{i=1}^n S_{t_i}$ we consider $\left(\prod_{i=1}^n S_{t_i}\right)^{1/n}$. This quantity has a lognormal distribution (Exercise 22) and so the corresponding approximate pricing formula for the Asian option can be evaluated exactly. (You are asked to find the pricing formula for an Asian call option based on a continuous version of the geometric average in Exercise 23.) Of course the arithmetic mean of a collection of positive numbers always dominates their geometric mean and so it is no surprise that this approximation consistently under-prices the Asian call option.

6.5 American options

A full treatment of American options is beyond our scope here. Explicit formulae for the prices of American options only exist in a few special cases and so one must employ numerical techniques. One approach is to use our discrete (binomial tree) models of Chapter 2. An alternative is to reformulate the price as a solution to a partial differential equation. We do not give a rigorous derivation of this equation, but instead we use the results of Chapter 2 to give a heuristic explanation of its form.

The discrete case

As we saw in Chapter 2, the price of an American call option on non-dividend-paying stock is the same as that of a European call and so we concentrate on the *American put*. This option gives the holder the right to buy one unit of stock for price K at any time before the maturity time T.

As we illustrated in §2.2, in our discrete time model, if $V(n, S_n)$ is the value of the option at time $n\delta t$ given that the asset price at time $n\delta t$ is S_n then

$$V(n, S_n) = \max\left\{(K - S_n)_+, \mathbb{E}^{\mathbb{Q}}\left[e^{-r\delta t} V(n+1, S_{n+1}) \middle| \mathcal{F}_n\right]\right\},$$

where $\mathbb{Q}$ is the martingale measure. In particular, $V(n, S_n) \geq (K - S_n)_+$ everywhere. We saw that for each fixed n the possible values of S_n are separated into two ranges by a boundary value that we shall denote by $S_f(n)$: if $S_n > S_f(n)$ then it is optimal to hold the option whereas if $S_n \leq S_f(n)$ it is optimal to exercise. We call $\{S_f(n)\}_{0\leq n\leq N}$ the *exercise boundary*.

In Example 2.4.7 we found a characterisation of the exercise boundary. We showed that the discounted option price can be written as $\tilde{V}_n = \tilde{M}_n - \tilde{A}_n$ where $\{\tilde{M}_n\}_{0\leq n\leq N}$ is a $\mathbb{Q}$-martingale and $\{\tilde{A}_n\}_{0\leq n\leq N}$ is a non-decreasing predictable process. The option is exercised at the first time $n\delta t$ when $\tilde{A}_{n+1} \neq 0$. In summary, within the exercise region $\tilde{A}_{n+1} \neq 0$ and $V_n = (K - S_n)_+$, whereas away from the exercise region, that is when $S_n > S_f(n)$, $V(n, S_n) = M_n$.

The strategy of exercising the option at the first time when $\tilde{A}_{n+1} \neq 0$ is *optimal* in the sense that if we write $\mathcal{T}_N$ for the set of all possible stopping times taking values in $\{0, 1, \ldots, N\}$ then

$$V(0, S_0) = \sup_{\tau\in\mathcal{T}_N} \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}(K - S_\tau)_+ \middle| \mathcal{F}_0\right].$$

Since the exercise time of any permissible strategy must be a stopping time, this says that as holder of the option one can't do better by choosing any other exercise strategy. That this optimality characterises the fair price follows from a now familiar arbitrage argument that you are asked to provide in Exercise 24.

Continuous time

Now suppose that we formally pass to the continuous limit as in §2.6. We expect that in the limit too $V(t, S_t) \geq (K - S_t)_+$ everywhere and that for each t we can define $S_f(t)$ so that if $S_t > S_f(t)$ it is optimal to hold on to the option, whereas if $S_t \leq S_f(t)$ it is optimal to exercise. In the exercise region $V(t, S_t) = (K - S_t)_+$ whereas away from the exercise region $V(t, S_t) = M_t$ where the discounted process $\{\tilde{M}_t\}_{0\leq t\leq T}$ is a $\mathbb{Q}$-martingale and $\mathbb{Q}$ is the measure, equivalent to $\mathbb{P}$, under which the discounted stock price is a martingale. Since $\{\tilde{M}_t\}_{0\leq t\leq T}$ can be thought of as the discounted value of a European option, this tells us that away from the exercise region, $V(t, x)$ must satisfy the Black–Scholes differential equation.

We guess then that for $\{(t, x) : x > S_f(t)\}$ the price $V(t, x)$ must satisfy the Black–Scholes equation whereas outside this region $V(t, x) = (K - x)_+$. This can be extended to a characterisation of $V(t, x)$ if we specify appropriate boundary conditions on S_f. This is complicated by the fact that $S_f(t)$ is a *free boundary* – we don't know its location a priori.

An arbitrage argument (Exercise 25) says that the price of an American put option should be continuous. We have checked already that $V(t, S_f(t)) = (K - S_f(t))_+$. Since it is clearly not optimal to exercise at a time $t < T$ if the value of the option is zero, in fact we have $V(t, S_f(t)) = K - S_f(t)$. Let us suppose now that $V(t, x)$ is continuously differentiable with respect to x as we cross the exercise boundary (we shall omit the proof of this). Then, since

$$\begin{aligned} V(t, x) &= (K - x) \quad \text{for } x \leq S_f \text{ and} \\ V(t, x) &\geq (K - x) \quad \text{for } x > S_f, \end{aligned}$$

we must have that at the exercise boundary $\frac{\partial V}{\partial x} \geq -1$. Suppose that $\frac{\partial V}{\partial x} > -1$ at some point of the exercise boundary. Then by reducing the value of the stock price at which we choose to exercise from S_f to S_f^* we can actually *increase* the value of the option at $(t, S_f(t))$. This contradicts the optimality of our exercise strategy. It must be that $\frac{\partial V}{\partial x} = -1$ at the exercise boundary.

We can now fully characterise $V(t, x)$ as a solution to a free boundary value problem:

Proposition 6.5.1 (The value of an American put) *We write $V(t, x)$ for the value of an American put option with strike price K and maturity time T and r for the riskless borrowing rate. $V(t, x)$ can be characterised as follows. For each time $t \in [0, T]$ there is a number $S_f(t) \in (0, \infty)$ such that for $0 \leq x \leq S_f(t)$ and $0 \leq t \leq T$,*

$$V(t, x) = K - x \quad \textit{and} \quad \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} - rV < 0.$$

For $t \in [0, T]$ and $S_f(t) < x < \infty$

$$V(t, x) > (K - x)_+ \quad \textit{and} \quad \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} - rV = 0.$$

The boundary conditions at $x = S_f(t)$ are that the option price process is continuously differentiable with respect to x, is continuous in time and

$$V(t, S_f(t)) = (K - S_f(t))_+, \quad \frac{\partial V}{\partial x}(t, S_f(t)) = -1.$$

In addition, V satisfies the terminal condition

$$V(T, S_T) = (K - S_T)_+.$$

The free boundary problem of Proposition 6.5.1 is easier to analyse as a *linear complementarity problem*. If we use the notation

$$\mathcal{L}_{BS} f = \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + rx\frac{\partial f}{\partial x} - rf,$$

then the free boundary value problem can be restated as

$$\mathcal{L}_{BS}V(t,x)\,(V(t,x)-(K-x)_+)=0,$$

subject to $\mathcal{L}_{BS}V(t,x)\le 0$, $V(t,x)-(K-x)_+\ge 0$, $V(T,x)=(K-x)_+$, $V(t,x)\to\infty$ as $x\to\infty$ and $V(t,x)$, $\frac{\partial V}{\partial x}(t,x)$ are continuous.

Notice that this reformulation has removed explicit dependence on the free boundary. Variational techniques can be applied to solve the problem and then the boundary is recovered from that solution. This is beyond our scope here. See Wilmott, Howison & Dewynne (1995) for more detail.

An explicit solution

We finish this chapter with one of the rare examples of an American option for which the price can be obtained explicitly.

Example 6.5.2 (Perpetual American put) *Find the value of a perpetual American put option on non-dividend-paying stock, that is a contract that the holder can choose to exercise at any time t in which case the payoff is $(K-S_t)_+$.*

Solution(s): We sketch *two* possible solutions to this problem, first via the free boundary problem of Proposition 6.5.1 and second via the expectation price.

Since the time to expiry of the contract is always infinite, $V(t,x)$ is a function of x alone and the exercise boundary must be of the form $S_f(t)=\alpha$ for *all* $t>0$ and some constant α. The option will be exercised as soon as $S_t\le\alpha$. The Black–Scholes equation reduces to an *ordinary* differential equation:

$$\frac{1}{2}\sigma^2x^2\frac{d^2V}{dx^2}+rx\frac{dV}{dx}-rV=0,\qquad\text{for all } x\in(\alpha,\infty).\tag{6.7}$$

The general solution to equation (6.7) is of the form $v(x)=c_1x^{d_1}+c_2x^{d_2}$ for some constants c_1, c_2, d_1 and d_2. Fitting the boundary conditions

$$V(\alpha)=K-\alpha,\qquad \lim_{x\downarrow\alpha}\frac{dV}{dx}=-1\quad\text{and}\quad\lim_{x\to\infty}V(x)=0$$

gives

$$V(x)=\begin{cases}(K-\alpha)\left(\frac{\alpha}{x}\right)^{2r\sigma^{-2}}, & x\in(\alpha,\infty),\\ (K-x), & x\in[0,\alpha],\end{cases}$$

where

$$\alpha=\frac{2r\sigma^{-2}K}{2r\sigma^{-2}+1}.$$

An alternative approach to this problem would be to apply the results of §3.3. As we argued above, the option will be exercised when the stock price first hits level α for some $\alpha>0$. This means that the value will be of the form

$$V(0,S_0)=\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau_\alpha}(K-\alpha)_+\right],$$

where $\tau_\alpha = \inf\{t > 0 : S_t \le \alpha\}$. We rewrite this stopping time in terms of the time that it takes a $\mathbb{Q}$-Brownian motion to hit a sloping line. Since

$$S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t + \sigma X_t\right)$$

where $\{X_t\}_{t\ge 0}$ is a standard Brownian motion under the martingale measure $\mathbb{Q}$, the event $\{S_t \le \alpha\}$ is the same as the event

$$\left\{-\sigma X_t - \left(r - \frac{1}{2}\sigma^2\right) t \ge \log\left(\frac{S_0}{\alpha}\right)\right\}.$$

The process $\{-X_t\}_{t\ge 0}$ is also a standard $\mathbb{Q}$-Brownian motion and so, in the notation of §3.3, the time τ_α is given by $T_{a,b}$ with

$$a = \frac{1}{\sigma}\log\left(\frac{S_0}{\alpha}\right), \quad b = \frac{r - \frac{1}{2}\sigma^2}{\sigma}.$$

We can then read off $\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau_\alpha}\right]$ from Proposition 3.3.5 and maximise over α to yield the result. □

Exercises

1 Let K_1 and K_2 be fixed real numbers with $0 < K_1 < K_2$. A *collar option* has payoff

$$C_T = \min\{\max\{S_T, K_1\}, K_2\}.$$

Find the Black–Scholes price for such an option.

2 What is the maximum potential loss associated with taking the long position in a forward contract? And with taking the short position?
Consider the derivative whose payoff at expiry to the holder of the long position is

$$C_T = \min\{S_T, F\} - K,$$

where F is the standard forward price for the underlying stock and K is a constant. Such a contract is constructed so as to have zero value at the time at which it is struck. Find an expression for the value of K that should be written into such a contract. What is the maximum potential loss for the holder of the long or short position now?

3 The *digital put option* with strike K at time T has payoff

$$C_T = \begin{cases} 0, & S_T \ge K, \\ 1, & S_T < K. \end{cases}$$

Find the Black–Scholes price for a digital put. What is the put–call parity for digital options?

4 *Digital call option* In Example 6.1.1 we calculated the price of a digital call. Here is an alternative approach:

(a) Use the Feynman–Kac stochastic representation to find the partial differential equation satisfied by the value of a digital call with strike K and maturity T.
(b) Show that the delta of a standard European call option solves the partial differential equation that you have found in (a).
(c) Hence or otherwise solve the equation in (a) to find the value of the digital.

5 An *asset-or-nothing* call option with strike K and maturity T has payoff

$$C_T = \begin{cases} S_T, & S_T \geq K, \\ 0, & S_T < K. \end{cases}$$

Find the Black–Scholes price and hedge for such an option. What happens to the stock holding in the replicating portfolio if the asset price is near K at times close to T? Comment.

6 Construct a portfolio consisting entirely of cash-or-nothing and asset-or-nothing options whose value at time T is exactly that of a European call option with strike K at maturity T.

7 In §6.1 we have seen that for certain options with discontinuous payoffs at maturity, the stock holding in the replicating portfolio can oscillate wildly close to maturity. Do you see this phenomenon if the payoff is continuous?

8 *Pay-later option* This option, also known as a *contingent premium option*, is a standard European option except that the buyer pays the premium only at maturity of the option and then only if the option is in the money. The premium is chosen so that the value of the option at time zero is zero. This option is equivalent to a portfolio consisting of one standard European call option with strike K and maturity T and $-V$ digital call options with maturity T where V is the premium for the option.

(a) What is the value of holding such a portfolio at time zero?
(b) Find an expression for V.
(c) If a speculator enters such a contract, what does this suggest about her market view?

9 *Ratchet option* A two-leg ratchet call option can be described as follows. At time zero an initial strike price K is set. At time $T_0 > 0$ the strike is *reset* to S_{T_0}, the value of the underlying at time T_0. At the maturity time $T_1 > T_0$ the holder receives the payoff of the call with strike S_{T_0} plus $S_{T_1} - S_{T_0}$ if this is positive. That is, the payoff is $(S_{T_1} - S_{T_0})_+ + (S_{T_0} - K)_+$.
If $(S_{T_0} - K)$ is positive, then the intermediate profit $(S_{T_0} - K)_+$ is said to be 'locked in'. Why? Value this option for $0 < t < T_1$.

10 *Chooser option* A chooser option is specified by two strike prices, K_0 and K_1, and two maturity dates, $T_0 < T_1$. At time T_0 the holder has the right to buy, for price K_0, *either* a call *or* a put with strike K_1 and maturity T_1.

What is the value of the option at time T_0? In the special case $K_0 = 0$ use put–call parity to express this as the sum of the value of a call and a put with suitably chosen strike prices and maturity dates and hence find the value of the option at time zero.

11 *Options on futures* In our simple model where the riskless rate of borrowing is deterministic, forward and futures prices coincide. A European call option with strike price K and maturity T_0 written on an underlying futures contract with delivery date $T_1 > T_0$ delivers to the holder, at time T_0, a long position in the futures contract and an amount of money $(F(T_0, T_1) - K)_+$, where $F(T_0, T_1)$ is the value of the futures contract at time T_0. Find the value of such an option at time zero.

12 Use the method of Example 6.2.3 to find the value of a put-on-put option.

By considering the portfolio obtained by buying one call-on-put and selling one put-on-put (with the same strikes and maturities) obtain a put–call parity relation for compound options. Hence write down prices for all four classes of compound option.

13 Let $\{Y_t\}_{t\geq 0}$ be given by $Y_t = bt + X_t$ where b is a constant and $\{X_t\}_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion. Writing $Y^*(t) = \max\{Y_u : 0 \leq u \leq t\}$, find the joint distribution of $(Y_T, Y^*(T))$ under $\mathbb{Q}$.

14 What is the value of a portfolio consisting of one down-and-in call and one down-and-out call with the same strike price and maturity?

15 Find the value of a down-and-out call with barrier c and strike K at maturity T if $c > K$.

16 One approach to finding the value of the down-and-out call of Example 6.3.6 is to express it as an expectation under the martingale measure and exploit our knowledge of the joint distribution of Brownian motion and its minimum. Alternatively one can solve the partial differential equation directly and that is the purpose of this exercise.

(a) Use the method of Exercise 7 of Chapter 5 to transform the equation for the price into the heat equation. What are the boundary conditions for this heat equation?

(b) Solve the heat equation that you have obtained using, for example, the ‘method of images’. (If you are unfamiliar with this technique, then try Wilmott, Howison & Dewynne (1995).)

(c) Undo the transformation to obtain the solution to the partial differential equation.

17 An *American cash-or-nothing call option* can be exercised at any time $t \in [0, T]$. If exercised at time t its payoff is

$$\begin{matrix} 1 & \text{if } S_t \geq K, \\ 0 & \text{if } S_t < K. \end{matrix}$$

When will such an option be exercised? Find its value.

18 Suppose that the down-and-in call option of Example 6.3.5 is modified so that if the option is never activated, that is the stock price never crosses the barrier, then the holder receives a rebate of Z. Find the price of this modified option.

19 A *perpetual option* is one with no expiry time. For example, a perpetual American cash-or-nothing call option can be exercised at any time. If exercised at time t, its payoff is 1 if $S_t \geq K$ and 0 if $S_t < K$. What is the probability that such an option is never exercised?

20 Formulate the price of a double knock-out call option as a solution to a partial differential equation with suitably chosen boundary conditions. Mimic your approach in Exercise 16 to see that this too leads to an expression for the price as an infinite sum.

21 Calculate the value of an Asian call option, with strike price K, in which the average of the stock price is calculated on the basis of just two sampling times, 0 and T, where T is the maturity time of the contract.
Find an expression for the value of the corresponding contract when there are three sampling times, 0, $T/2$ and T.

22 Suppose that $\{S_t\}_{t\geq 0}$ is a geometric Brownian motion under $\mathbb{P}$. Let $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ be fixed times and define

$$G_n = \left(\prod_{i=1}^{n} S_{t_i} \right)^{1/n}.$$

Show that G_n has a lognormal distribution under $\mathbb{P}$.

23 An asset price $\{S_t\}_{t\geq 0}$ is a geometric Brownian motion under the market measure $\mathbb{P}$. Define

$$Y_T = \exp\left(\frac{1}{T} \int_0^T \log S_t \, dt \right).$$

Suppose that an Asian call option has payoff $(Y_T - K)_+$ at time T. Find an explicit formula for the price of such an option at time zero.

24 Use an arbitrage argument to show that if $V(0, S_0)$ is the fair price of an American put option on non-dividend-paying stock with strike price K and maturity T, then writing $\mathcal{T}_T$ for the set of all possible stopping times taking values in $[0, T]$

$$V(0, S_0) = \sup_{\tau \in \mathcal{T}_T} \mathbb{E}^{\mathbb{Q}} \left[e^{-r\tau} (K - S_\tau)_+ \middle| \mathcal{F}_0 \right].$$

25 Consider the value of an American put on non-dividend-paying stock. Show that if there were a discontinuity in the option value (as a function of stock price) that persisted for more than an infinitesimal time then a portfolio consisting entirely of options would offer an arbitrage opportunity.
Remark: This does not mean that *all* option prices are continuous. If there is an instantaneous change in the conditions of a contract (as in multistage options) then discontinuities certainly can occur.

26 Find the value of a perpetual American call option on non-dividend-paying stock.

7 Bigger models

Summary

Having applied our basic Black–Scholes model to the pricing of some exotic options, we now turn to more general market models.

In §7.1 we replace the (constant) parameters that characterised our basic Black–Scholes model by previsible processes. Under appropriate boundedness assumptions, we then repeat our analysis of Chapter 5 to obtain the fair price of an option as the discounted expected value of the claim under a martingale measure. In general this expectation must be evaluated numerically. We also make the connection with a generalised Black–Scholes equation via the Feynman–Kac Stochastic Representation Theorem.

Our models so far have assumed that the market consists of a single stock and a riskless cash bond. More complex equity products can depend on the behaviour of several separate securities and, in general, the prices of these securities will not evolve independently. In §7.2 we extend some of the fundamental results of Chapter 4 to allow us to manipulate systems of stochastic differential equations driven by *correlated* Brownian motions. For markets consisting of many assets we have much more freedom in our choice of 'reference asset' or numeraire and so we revisit this issue before illustrating the application of the 'multifactor' theory by pricing a 'quanto' product.

We still have no satisfactory justification for the geometric Brownian motion model. Indeed, there is considerable evidence that it does not capture all features of stock price evolution. A first objection is that stock prices occasionally 'jump' at unpredictable times. In §7.3 we introduce a Poisson process of jumps into our Black–Scholes model and investigate the implications for option pricing. This approach is popular in the analysis of credit risk. In §1.5 we saw that, if a model is to be free from arbitrage and complete, there must be a balance between the number of sources of randomness and the number of independent stocks. We reiterate this here. We see more evidence that the Black–Scholes model does not reflect the true behaviour of the market in §7.4. It seems a little late in the day to condemn the model that has been the subject of all our efforts so far and so we ask how much it matters

if we use the wrong model. We also very briefly discuss models with stochastic volatility that have the potential to better reflect true market behaviour.

This chapter is intended to do no more than indicate some of the topics that might be addressed in a *second* course in financial calculus. Much more detail can be found in some of the suggestions for further reading in the bibliography.

7.1 General stock model

In our classical Black–Scholes framework we assume that the riskless borrowing rate is constant and that the returns of the stock follow a Brownian motion with constant drift. In this section we consider much more general models to which we can apply the Black–Scholes analysis although, in practice, even for vanilla options the prices that we obtain must now be evaluated numerically. The key assumption that we retain is that there is only one source of randomness in the market, the Brownian motion that drives the stock price (cf. §7.3).

The model

Writing $\{\mathcal{F}_t\}_{t\geq 0}$ for the filtration generating the driving Brownian motion, we replace the riskless borrowing rate, r, the drift μ and the volatility σ in our basic Black–Scholes model by $\{\mathcal{F}_t\}_{t\geq 0}$-predictable processes $\{r_t\}_{t\geq 0}$, $\{\mu_t\}_{t\geq 0}$ and $\{\sigma_t\}_{t\geq 0}$. In particular, r_t, μ_t and σ_t can depend on the whole history of the market before time t. Our market model is then as follows.

General stock model: The market consists of a riskless cash bond, $\{B_t\}_{t\geq 0}$, and a single risky asset with price process $\{S_t\}_{t\geq 0}$ governed by

$$\begin{aligned} dB_t &= r_t B_t dt, \quad B_0 = 1, \\ dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t, \end{aligned}$$

where $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion generating the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\{r_t\}_{t\geq 0}$, $\{\mu_t\}_{t\geq 0}$ and $\{\sigma_t\}_{t\geq 0}$ are $\{\mathcal{F}_t\}_{t\geq 0}$-predictable processes.

Evidently a solution to these equations should take the form

$$B_t = \exp\left(\int_0^t r_s ds\right), \tag{7.1}$$

$$S_t = S_0 \exp\left(\int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s dW_s\right), \tag{7.2}$$

but we need to make some boundedness assumptions if these expressions are to make sense. So to ensure the existence of the integrals in equations (7.1) and (7.2) we assume that $\int_0^T |r_t|dt$, $\int_0^T |\mu_t|dt$ and $\int_0^T \sigma_t^2 dt$ are all finite with $\mathbb{P}$-probability one.

A word of warning is in order. In order to 'calibrate' such a model to the market we must choose the parameters $\{r_t\}_{t\geq 0}$, $\{\mu\}_{t\geq 0}$ and $\{\sigma_t\}_{t\geq 0}$ from an infinite-dimensional space. Unless we restrict the possible forms of these processes, this presents a major obstacle to implementation. In §7.4 we examine the effect of model misspecification on pricing and hedging strategies. Now, however, we set this worry aside and repeat the Black–Scholes analysis for our general class of market models.

A martingale measure

We must mimic the three steps to replication that we followed in the classical setting. The first of these is to find an equivalent probability measure, $\mathbb{Q}$, under which the discounted stock price, $\{\tilde{S}_t\}_{t\geq 0}$, is a martingale.

Exactly as before, we use the Girsanov Theorem to find a measure, $\mathbb{Q}$, under which the process $\{\tilde{W}_t\}_{t\geq 0}$ defined by

$$\tilde{W}_t = W_t + \int_0^t \gamma_s ds$$

is a standard Brownian motion. The discounted stock price, $\{\tilde{S}_t\}_{t\geq 0}$ defined as $\tilde{S}_t = S_t/B_t$, is governed by the stochastic differential equation

$$\begin{aligned} d\tilde{S}_t &= (\mu_t - r_t)\,\tilde{S}_t dt + \sigma_t \tilde{S}_t dW_t \\ &= (\mu_t - r_t - \sigma_t\gamma_t)\,\tilde{S}_t dt + \sigma_t \tilde{S}_t d\tilde{W}_t, \end{aligned}$$

and so we choose $\gamma_t = (\mu_t - r_t)/\sigma_t$. To ensure that $\{\tilde{S}_t\}_{t\geq 0}$ really is a $\mathbb{Q}$-martingale we make two further boundedness assumptions. First, in order to apply the Girsanov Theorem, we insist that

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_0^T \frac{1}{2}\gamma_t^2 dt\right)\right] < \infty.$$

Second we require that $\{\tilde{S}_t\}_{t\geq 0}$ is a $\mathbb{Q}$-martingale (not just a local martingale) and so we assume a second Novikov condition:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^T \frac{1}{2}\sigma_t^2 dt\right)\right] < \infty.$$

Under these extra boundedness assumptions $\{\tilde{S}_t\}_{t\geq 0}$ then is a martingale under the measure $\mathbb{Q}$ defined by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t^{(\gamma)} = \exp\left(-\int_0^t \gamma_s dW_s - \int_0^t \frac{1}{2}\gamma_s^2 ds\right).$$

Second step to replication

That completes the first step in our replication strategy. The second is to form the $(\mathbb{Q}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale $\{M_t\}_{t\geq 0}$ given by

$$M_t = \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} C_T \middle| \mathcal{F}_t\right].$$

Replicating a claim

The third step is to show that our market is complete, that is any claim C_T can be replicated. First we invoke the martingale representation theorem to write

$$M_t = M_0 + \int_0^t \theta_u d\tilde{W}_u$$

and consequently, provided that σ_t never vanishes,

$$M_t = M_0 + \int_0^t \phi_s d\tilde{S}_s,$$

where $\{\phi_t\}_{t\geq 0}$ is $\{\mathcal{F}_t\}_{t\geq 0}$-predictable.

Guided by our previous work we guess that a replicating portfolio should consist of ϕ_t units of stock and $\psi_t = M_t - \phi_t S_t$ units of cash bond at time t. In Exercise 1 it is checked that such a portfolio is self-financing. Its value at time t is

$$V_t = \phi_t S_t + \psi_t B_t = B_t M_t.$$

In particular, at time T, $V_T = B_T M_T = C_T$, and so we have a self-financing, replicating portfolio. The usual arbitrage argument tells us that the fair value of the claim at time t is V_t, that is the arbitrage price of the option at time t is

$$V_t = B_t \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} C_T \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u du} C_T \middle| \mathcal{F}_t \right].$$

The generalised Black–Scholes equation

In general such an expectation must be evaluated numerically. If r_t, μ_t and σ_t depend only on (t, S_t) then one approach to this is first to express the price as the solution to a generalised Black–Scholes partial differential equation. This is achieved with the Feynman–Kac Stochastic Representation Theorem. Specifically, using Example 4.8.6, $V_t = F(t, S_t)$ where $F(t, x)$ solves

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2(t,x)x^2\frac{\partial^2 F}{\partial x^2}(t,x) + r(t,x)x\frac{\partial F}{\partial x}(t,x) - r(t,x)F(t,x) = 0,$$

subject to the terminal condition corresponding to the claim C_T, at least provided

$$\int_0^T \mathbb{E}^{\mathbb{Q}}\left[\left(\sigma(t,x)\frac{\partial F}{\partial x}(t,x) \right)^2 \right] ds < \infty.$$

For vanilla options, in the special case when r, μ and σ are functions of t alone, the partial differential equation can be solved explicitly. As is shown in Exercise 3 the procedure is exactly that used to solve the usual Black–Scholes equation. The price can be found from the classical Black–Scholes price via the following simple rule: for the value of the option at time t replace r and σ^2 by

$$\frac{1}{T-t}\int_t^T r(s)ds \quad \text{and} \quad \frac{1}{T-t}\int_t^T \sigma^2(s)ds$$

respectively.

7.2 Multiple stock models

So far we have assumed that the market consists of a riskless cash bond and a single 'risky' asset. However, the need to model whole portfolios of options or more complex equity products leads us to seek models describing several securities simultaneously. Such models must encode the *interdependence* between different security prices.

Correlated security prices

Suppose that we are modelling the evolution of n risky assets and, as ever, a single risk-free cash bond. We assume that it is not possible to exactly replicate one of the assets by a portfolio composed entirely of the others. In the most natural extension of the classical Black–Scholes model, considered individually the price of each risky asset follows a geometric Brownian motion, and interdependence of different asset prices is achieved by taking the driving Brownian motions to be correlated. Equivalently, we take a set of n independent Brownian motions and drive the asset prices by linear combinations of these; see Exercise 2. This suggests the following market model.

Multiple asset model: Our market consists of a cash bond $\{B_t\}_{0\leq t\leq T}$ and n different securities with prices $\{S_t^1, S_t^2, \dots, S_t^n\}_{0\leq t\leq T}$, governed by the system of stochastic differential equations

$$dB_t = rB_t dt,$$

$$dS_t^i = S_t^i \left(\sum_{j=1}^{n} \sigma_{ij}(t) dW_t^j + \mu_i(t) dt \right), \quad i = 1, 2, \dots, n, \tag{7.3}$$

where $\{W_t^j\}_{t\geq 0}$, $j = 1, \dots, n$, are independent Brownian motions. We assume that the matrix $\sigma = (\sigma_{ij})$ is invertible.

Remarks:

1. This model is called an n-factor model as there are n sources of randomness. If there are fewer sources of randomness than stocks then there is redundancy in the model as we can replicate one of the stocks by a portfolio composed of the others. On the other hand, if we are to be able to hedge any claim in the market, then, roughly speaking, we need as many 'independent' stocks as sources of randomness. This mirrors Proposition 1.6.5.
2. Notice that the volatility of each stock in this model is really a *vector*. Since the Brownian motions $\{W_t^j\}_{t\geq 0}$, $j = 1, \dots, n$, are independent, the total volatility of the process $\{S_t^i\}_{t\geq 0}$ is $\left\{\sqrt{\sum_{j=1}^n \sigma_{ij}^2(t)}\right\}_{t\geq 0}$. □

Of course we haven't checked that this model really makes sense. That is, we need to know that the system of stochastic differential equations (7.3) has a solution. In order to verify this and to analyse such multifactor market models we need multidimensional analogues of the key results of Chapter 4.

Multifactor Itô formula

The most basic tool will be an n-factor version of the Itô formula. In the same way as we used the one-factor Itô formula to find a description (in the form of a stochastic differential equation) of models constructed as functions of Brownian motion, here we shall build new multifactor models from old. Our basic building blocks will be solutions to systems of stochastic differential equations of the form

$$dX_t^i = \mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW_t^j, \quad i = 1, \ldots, n, \tag{7.4}$$

where $\{W_t^j\}_{t\geq 0}$, $j = 1, \ldots, n$, are independent Brownian motions. We write $\{\mathcal{F}_t\}_{t\geq 0}$ for the σ-algebra generated by $\{W_t^j\}_{t\geq 0}$, $j = 1, \ldots, n$. Our work of Chapter 4 gives a rigorous meaning to (the integrated version of) the system (7.4) provided $\{\mu_i(t)\}_{t\geq 0}$ and $\{\sigma_{ij}(t)\}_{t\geq 0}$, $1 \leq i \leq n$, $1 \leq j \leq n$, are $\{\mathcal{F}_t\}_{t\geq 0}$-predictable processes with

$$\mathbb{E}\left[\int_0^t \left(\sum_{j=1}^{n} \left(\sigma_{ij}(s)\right)^2 + |\mu_i(s)|\right) ds\right] < \infty, \quad t > 0, i = 1, \ldots, n.$$

Let us write $\{X_t\}_{t\geq 0}$ for the vector of processes $\{X_t^1, X_t^2, \ldots, X_t^n\}_{t\geq 0}$ and define a new stochastic process by $Z_t = f(t, X_t)$. Here we suppose that $f(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is sufficiently smooth that we can apply Taylor's Theorem, just as in §4.3, to find the stochastic differential equation governing $\{Z_t\}_{t\geq 0}$. Writing $x = (x_1, \ldots, x_n)$, we obtain

$$dZ_t = \frac{\partial f}{\partial t}(t, X_t)dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X_t)dX_t^i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)dX_t^i dX_t^j + \cdots . \tag{7.5}$$

Since the Brownian motions $\{W_t^i\}_{t\geq 0}$ are *independent* we have the multiplication table

$\times$	dW_t^i	dW_t^j	dt
dW_t^i	dt	0	0
dW_t^j	0	dt	0
dt	0	0	0

for $i \neq j$ (7.6)

and this gives $dX_t^i dX_t^j = \sum_{k=1}^{n} \sigma_{ik}\sigma_{jk}dt$. The same multiplication table tells us that $dX_t^i dX_t^j dX_t^k$ is $o(dt)$ and so substituting into equation (7.5) we have provided a heuristic justification of the following result.

Theorem 7.2.1 (Multifactor Itô formula) *Let* $\{X_t\}_{t\geq 0} = \{X_t^1, X_t^2, \dots, X_t^n\}_{t\geq 0}$ *solve*

$$dX_t^i = \mu_i(t)dt + \sum_{j=1}^n \sigma_{ij}(t)dW_t^j, \qquad i = 1, 2, \dots, n,$$

where $\{W_t^i\}_{t\geq 0}, i = 1, \dots, n$, *are independent* $\mathbb{P}$*-Brownian motions. Further suppose that the real-valued function* $f(t, x)$ *on* $\mathbb{R}_+ \times \mathbb{R}^n$ *is continuously differentiable with respect to* t *and twice continuously differentiable in the* x*-variables. Then defining* $Z_t = f(t, X_t)$ *we have*

$$dZ_t = \frac{\partial f}{\partial t}(t, X_t)dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t, X_t)dX_t^i + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)C_{ij}(t)dt$$

where $C_{ij}(t) = \sum_{k=1}^n \sigma_{ik}(t)\sigma_{jk}(t)$.

Remark: Notice that if we write σ for the matrix (σ_{ij}) then $C_{ij} = \left(\sigma\sigma^t\right)_{ij}$ where σ^t is the transpose of σ. □

We can now check that there *is* a solution to the system of equations (7.3).

Example 7.2.2 (Multiple asset model) *Let* $\{W_t^i\}_{t\geq 0}, i = 1, \dots, n$, *be independent Brownian motions. Define* $\{S_t^1, S_t^2, \dots, S_t^n\}_{t\geq 0}$ *by*

$$S_t^i = S_0^i \exp\left(\int_0^t \left(\mu_i(s) - \frac{1}{2}\sum_{k=1}^n \sigma_{ik}^2(s)\right) ds + \int_0^t \sum_{j=1}^n \sigma_{ij}(s)dW_s^j\right);$$

then $\{S_t^1, S_t^2, \dots, S_t^n\}_{t\geq 0}$ *solves the system (7.3).*

Justification: Defining the processes $\{X_t^i\}_{t\geq 0}$ for $i = 1, 2, \dots, n$ by

$$dX_t^i = \left(\mu_i(t) - \frac{1}{2}\sum_{k=1}^n \sigma_{ik}^2(t)\right) dt + \sum_{j=1}^n \sigma_{ij}(t)dW_t^j$$

we see that $S_t^i = f^i(t, X_t)$ where, writing $x = (x_1, \dots, x_n)$, $f^i(t, x) \triangleq S_0^i e^{x_i}$. Applying Theorem 7.2.1 gives

$$\begin{aligned} dS_t^i &= S_0^i \exp(X_t^i)dX_t^i + \frac{1}{2}S_0^i \exp(X_t^i)C_{ii}(t)dt \\ &= S_t^i\left\{\left(\mu_i(t) - \frac{1}{2}\sum_{k=1}^n \sigma_{ik}^2(t)\right)dt + \sum_{j=1}^n \sigma_{ij}(t)dW_t^j + \frac{1}{2}\sum_{k=1}^n \sigma_{ik}(t)\sigma_{ik}(t)dt\right\} \\ &= S_t^i\left\{\mu_i(t)dt + \sum_{j=1}^n \sigma_{ij}(t)dW_t^j\right\} \end{aligned}$$

as required. □

Remark: Exactly as in the single factor models, although we can write down arbitrarily complicated systems of stochastic differential equations, existence and uniqueness of solutions are far from guaranteed. If the coefficients are bounded and uniformly Lipschitz then a unique solution does exist, but such results are beyond our scope here. Instead, once again, we refer to Chung & Williams (1990) or Ikeda & Watanabe (1989). □

Integration by parts

We can also use the multiplication table (7.6) to write down an n-factor version of the integration by parts formula.

Lemma 7.2.3 *If*

$$dX_t = \mu(t, X_t)dt + \sum_{i=1}^{n} \sigma_i(t, X_t)dW_t^i$$

and

$$dY_t = \nu(t, Y_t)dt + \sum_{i=1}^{n} \rho_i(t, Y_t)dW_t^i$$

then

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + \sum_{i=1}^{n} \sigma_i(t, X_t)\rho_i(t, Y_t)dt.$$

Change of measure

Pricing and hedging in the multiple stock model will follow a familiar pattern. First we find an equivalent probability measure under which *all* of the discounted stock prices $\{\tilde{S}_t^i\}_{t\geq 0}, i = 1, \ldots, n$, given by $\tilde{S}_t^i = e^{-rt}S_t^i$, are martingales. We then use a multifactor version of the Martingale Representation Theorem to construct a replicating portfolio.

Construction of the martingale measure is, of course, via a multifactor version of the Girsanov Theorem.

Theorem 7.2.4 (Multifactor Girsanov Theorem) *Let $\{W_t^i\}_{t\geq 0}, i = 1, \ldots, n$, be independent Brownian motions under the measure $\mathbb{P}$ generating the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and let $\{\theta_i(t)\}_{t\geq 0}, i = 1, \ldots, n$, be $\{\mathcal{F}_t\}_{t\geq 0}$-previsible processes such that*

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \sum_{i=1}^{n} \theta_i^2(s)ds\right)\right] ds < \infty. \tag{7.7}$$

Define

$$L_t = \exp\left(-\sum_{i=1}^{n}\left(\int_0^T \theta_i(s)dW_s^i + \frac{1}{2}\int_0^T \theta_i^2(s)ds\right)\right)$$

and let $\mathbb{P}^{(L)}$ be the probability measure defined by

$$\left.\frac{d\mathbb{P}^{(L)}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t.$$

Then under $\mathbb{P}^{(L)}$ *the processes* $\{X_t^i\}_{t\geq 0}$, $i = 1, \dots, n$, *defined by*

$$X_t^i = W_t^i + \int_0^t \theta_i(s)ds$$

are all martingales.

Sketch of proof: The proof mimics that in the one-factor case. It is convenient to write $L_t = \prod_{i=1}^n L_t^i$ where

$$L_t^i = \exp\left(-\int_0^t \theta_i(s)dW_s^i - \frac{1}{2}\int_0^t \theta_i^2(s)ds\right).$$

That $\{L_t\}_{t\geq 0}$ defines a martingale follows from (7.7) and the independence of the Brownian motions $\{W_t^i\}_{t\geq 0}$, $i = 1, \dots, n$.

To check that $\{X_t^i\}_{t\geq 0}$ is a (local) $\mathbb{P}^{(L)}$-martingale we find the stochastic differential equation satisfied by $\{X_t^i L_t\}_{t\geq 0}$. Since

$$dL_t^i = -\theta_i(t)L_t^i dW_t^i,$$

repeated application of our product rule gives

$$dL_t = -L_t \sum_{i=1}^n \theta_i(t)dW_t^i.$$

Moreover,

$$dX_t^i = dW_t^i + \theta_i(t)dt,$$

and so another application of our product rule gives

$$\begin{aligned} d(X_t^i L_t) &= X_t^i dL_t + L_t dW_t^i + L_t\theta_i(t)dt - L_t\theta_i(t)dt \\ &= -X_t^i L_t \sum_{i=1}^n \theta_i(t)dW_t^i + L_t dW_t^i. \end{aligned}$$

Combined with the boundedness condition (7.7), this proves that $\{X_t^i L_t\}_{t\geq 0}$ is a $\mathbb{P}$-martingale and hence $\{X_t^i\}_{t\geq 0}$ is a $\mathbb{P}^{(L)}$-martingale. $\mathbb{P}^{(L)}$ is equivalent to $\mathbb{P}$ so $\{X_t^i\}_{t\geq 0}$ has quadratic variation $[X^i]_t = t$ with $\mathbb{P}^{(L)}$-probability one and once again Lévy's characterisation of Brownian motion confirms that $\{X_t^i\}_{t\geq 0}$ is a $\mathbb{P}^{(L)}$-Brownian motion as required. □

A martingale measure

As promised we now use this to find a measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, under which the discounted stock price processes $\{\tilde{S}_t^i\}_{t\geq 0}$, $i = 1, \dots, n$, are all martingales. The measure $\mathbb{Q}$ will be one of the measures $\mathbb{P}^{(L)}$ of Theorem 7.2.4. We just need to identify the appropriate drifts $\{\theta_i\}_{t\geq 0}$.

The discounted stock price $\{\tilde{S}_t^i\}_{t\geq 0}$, defined by $\tilde{S}_t^i = B_t^{-1}S_t^i$, is governed by the stochastic differential equation

$$\begin{aligned} d\tilde{S}_t^i &= \tilde{S}_t^i\left(\mu_i(t) - r\right)dt + \tilde{S}_t^i \sum_{j=1}^n \sigma_{ij}(t)dW_t^j \\ &= \tilde{S}_t^i\left(\mu_i(t) - r - \sum_{j=1}^n \theta_j(t)\sigma_{ij}(t)\right)dt + \tilde{S}_t^i \sum_{j=1}^n \sigma_{ij}(t)dX_t^j, \end{aligned}$$

where as in Theorem 7.2.4

$$dX_t^j = \theta_j(t)dt + dW_t^j.$$

The discounted stock price processes will (simultaneously) be (local) martingales under $\mathbb{Q} = \mathbb{P}^{(L)}$ if we can make all the drift terms vanish. That is, if we can find $\{\theta_j(t)\}_{t\geq 0}$, $j = 1, \ldots, n$, such that

$$\mu_i(t) - r - \sum_{j=1}^{n} \theta_j(t)\sigma_{ij}(t) = 0 \quad \text{for all } i = 1, \ldots, n.$$

Dropping the dependence on t in our notation and writing

$$\mu = (\mu_1, \ldots, \mu_n), \quad \theta = (\theta_1, \ldots, \theta_n), \quad \mathbf{1} = (1, \ldots, 1) \quad \text{and } \sigma = \left(\sigma_{ij}\right),$$

this becomes

$$\mu - r\mathbf{1} = \theta\sigma. \tag{7.8}$$

A solution certainly exists if the matrix σ is invertible, an assumption that we made in setting up our multiple asset model.

In order to guarantee that the discounted price processes are martingales, not just local martingales, once again we impose a Novikov condition:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^t \frac{1}{2}\sum_{j=1}^{n}\sigma_{ij}^2(t)dt\right)\right] < \infty \quad \text{for each } i.$$

Replicating the claim

At this point we guess, correctly, that the value of a claim $C_T \in \mathcal{F}_T$ at time $t < T$ is its discounted expected value under the measure $\mathbb{Q}$. To prove this we show that there is a self-financing replicating portfolio and this we infer from a multifactor version of the Martingale Representation Theorem.

Theorem 7.2.5 (Multifactor Martingale Representation Theorem) *Let*

$$\{W_t^i\}_{t\geq 0}, \qquad i = 1, \ldots, n,$$

be independent $\mathbb{P}$*-Brownian motions generating the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$*. Let* $\{M_t^1, \ldots, M_t^n\}_{t\geq 0}$ *be given by*

$$dM_t^i = \sum_{j=1}^{n} \sigma_{ij}(t)dW_t^j,$$

where

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \sum_{j=1}^{n}\sigma_{ij}(t)^2 dt\right)\right] < \infty.$$

Suppose further that the volatility matrix $\left(\sigma_{ij}(t)\right)$ *is non-singular (with probability one). Then if* $\{N_t\}_{t\geq 0}$ *is any one-dimensional* $\left(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}\right)$*-martingale there exists an* n*-dimensional* $\{\mathcal{F}_t\}_{t\geq 0}$*-previsible process* $\{\phi_t\}_{t\geq 0} = \{\phi_t^1, \ldots, \phi_t^n\}_{t\geq 0}$ *such that*

$$N_t = N_0 + \sum_{j=1}^{n}\int_0^t \phi_s^j dM_s^j.$$

A proof of this result is beyond our scope here. It can be found, for example, in Protter (1990). Notice that the non-singularity of the matrix σ reflects our remark about non-vanishing quadratic variation after the proof of Theorem 4.6.2.

We are now in a position to verify that our guess was correct: the value of a claim in the multifactor world is its discounted expected value under the martingale measure $\mathbb{Q}$.

Let $C_T \in \mathcal{F}_T$ be a claim at time T and let $\mathbb{Q}$ be the martingale measure obtained above. We write

$$M_t = \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} C_T \middle| \mathcal{F}_t \right].$$

Since, by assumption, the matrix $\sigma = \left(\sigma_{ij}\right)$ is invertible, the n-factor Martingale Representation Theorem tells us that there is an $\{\mathcal{F}_t\}_{t\geq 0}$-previsible process $\{\phi_t^1, \ldots, \phi_t^n\}_{t\geq 0}$ such that

$$M_t = M_0 + \sum_{j=1}^{n} \int_0^t \phi_s^j \, d\tilde{S}_s^j.$$

Our hedging strategy will be to hold ϕ_t^i units of the ith stock at time t for each $i = 1, \ldots, n$, and to hold ψ_t units of bond where

$$\psi_t = M_t - \sum_{j=1}^{n} \phi_t^j \tilde{S}_t^j.$$

The value of the portfolio is then $V_t = B_t M_t$, which at time T is exactly the value of the claim, and the portfolio is self-financing in that

$$dV_t = \sum_{j=1}^{n} \phi_t^j \, dS_t^j + \psi_t dB_t.$$

In the absence of arbitrage the value of the derivative at time t is

$$V_t = B_t \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} C_T \middle| \mathcal{F}_t \right] = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[C_T | \mathcal{F}_t \right]$$

as predicted.

Remark: The multifactor market that we have constructed is complete and arbitrage-free. We have simplified the exposition by insisting that the number of sources of noise in our market is exactly matched by the number of risky tradable assets that we are modelling. More generally, we could model k risky assets driven by d sources of noise. Existence of a martingale measure corresponds to existence of a solution to (7.8). It is *uniqueness* of the martingale measure that provides us with the Martingale Representation Theorem and hence the ability to replicate any claim. For a complete arbitrage-free market we then require that $d \leq k$ and that σ has full rank. That is, the number of independent sources of randomness should exactly match the number of 'independent' risky assets trading in our market. □

The multi-dimensional Black–Scholes equation

In Exercise 7 you are asked to use a delta-hedging argument to obtain this price as the solution to the multidimensional Black–Scholes equation. This partial differential equation can also be obtained directly from the expectation price and a multidimensional version of the Feynman–Kac stochastic representation. We quote the appropriate version of this useful result here.

Theorem 7.2.6 (Multidimensional Feynman–Kac stochastic representation) *Let $\sigma(t,x) = (\sigma_{ij}(t,x))$ be a real symmetric $n \times n$ matrix, $\Phi : \mathbb{R}^n \to \mathbb{R}$ and $\mu_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, n$, be real-valued functions and r be a constant. We suppose that the function $F(t,x)$, defined for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, solves the boundary value problem*

$$\frac{\partial F}{\partial t}(t,x) + \sum_{i=1}^{n} \mu_i(t,x)\frac{\partial F}{\partial x_i}(t,x) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(t,x)\frac{\partial^2 F}{\partial x_i \partial x_j}(t,x) - rF(t,x) = 0,$$

$$F(T,x) = \Phi(x),$$

where $C_{ij}(t,x) = \sum_{k=1}^{n} \sigma_{ik}(t,x)\sigma_{jk}(t,x)$.

Assume further that for each $i = 1, \dots, n$, the process $\{X_t^i\}_{t\geq 0}$ solves the stochastic differential equation

$$dX_t^i = \mu_i(t, X_t)dt + \sum_{j=1}^{n} \sigma_{ij}(t, X_t)dW_t^j$$

where $X_t = \{X_t^1, \dots, X_t^n\}$. Finally, suppose that

$$\int_0^T \mathbb{E}\left[\sum_{j=1}^{n}\left(\sigma_{ij}(s,X_s)\frac{\partial F}{\partial x_i}(s,X_s)\right)^2\right] ds < \infty, \quad i = 1, \dots, n.$$

Then

$$F(t,x) = e^{-r(T-t)}\mathbb{E}\left[\Phi(X_T) \mid X_t = x\right].$$

Corollary 7.2.7 *Let $S_t = \{S_t^1, \dots, S_t^n\}$ be as above and $C_T = \Phi(S_T)$ be a claim at time T. Then the price of the claim at time $t < T$,*

$$V_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_T) \mid \mathcal{F}_t\right] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_T) \mid S_t = x\right] \triangleq F(t,x)$$

satisfies

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(t,x)x_i x_j\frac{\partial^2 F}{\partial x_i \partial x_j}(t,x) + r\sum_{i=1}^{n} x_i\frac{\partial F}{\partial x_i}(t,x) - rF(t,x) = 0,$$

$$F(T,x) = \Phi(x).$$

Proof: The process $\{S_t\}_{t\geq 0}$ is governed by

$$dS_t^i = rS_t^i dt + \sum_{j=1}^{n} \sigma_{ij}(t,S_t)S_t^j dX_t^j,$$

where $\{X_t^j\}_{t\geq 0}$, $j = 1, \dots, n$, are $\mathbb{Q}$-Brownian motions, so the result follows from an application of Theorem 7.2.6. □

Numeraires

The more assets there are in our market, the more freedom we have in choosing our 'numeraire' or 'reference asset'. Usually it is chosen to be a cash bond, but in fact it could be any of the tradable assets available. In the context of foreign exchange we checked that we could use as reference the riskless bond in either currency and always obtain the same value for a claim. Here we consider two numeraires in the *same* market, but they may have non-zero volatility.

Suppose that our market consists of $n+2$ tradable assets whose prices we denote by $\{B_t^1, B_t^2, S_t^1, \ldots, S_t^n\}_{t\geq 0}$. We compare the prices obtained for a derivative by two traders, one of whom chooses $\{B_t^1\}_{t\geq 0}$ as numeraire and the other of whom chooses $\{B_t^2\}_{t\geq 0}$. We always assume our multidimensional geometric Brownian motion model for the evolution of prices, but now neither of the processes $\{B_t^i\}_{t\geq 0}$ necessarily has finite variation.

If we choose $\{B_t^1\}_{t\geq 0}$ as numeraire then we first find an equivalent measure, $\mathbb{Q}^1$, under which the asset prices discounted by $\{B_t^1\}_{t\geq 0}$, that is

$$\left\{\frac{B_t^2}{B_t^1}, \frac{S_t^1}{B_t^1}, \ldots, \frac{S_t^n}{B_t^1}\right\}_{t\geq 0},$$

are all $\mathbb{Q}^1$-martingales. The value that we obtain for a derivative with payoff C_T at time T is then

$$V_t^1 = B_t^1 \mathbb{E}^{\mathbb{Q}^1}\left[\left.\frac{C_T}{B_T^1}\right|\mathcal{F}_t\right]$$

(see Exercise 7).

If instead we had chosen $\{B_t^2\}_{t\geq 0}$ as our numeraire then the price would have been

$$V_t^2 = B_t^2 \mathbb{E}^{\mathbb{Q}^2}\left[\left.\frac{C_T}{B_T^2}\right|\mathcal{F}_t\right]$$

where $\mathbb{Q}^2$ is an equivalent probability measure under which

$$\left\{\frac{B_t^1}{B_t^2}, \frac{S_t^1}{B_t^2}, \ldots, \frac{S_t^n}{B_t^2}\right\}_{t\geq 0}$$

are all martingales. We have not proved that such a measure $\mathbb{Q}^2$ is unique, but if a claim can be replicated we obtain the same price for any measure $\mathbb{Q}^2$ with this property.

Suppose that we choose $\mathbb{Q}^2$ so that its Radon–Nikodym derivative with respect to $\mathbb{Q}^1$ is given by

$$\left.\frac{d\mathbb{Q}^2}{d\mathbb{Q}^1}\right|_{\mathcal{F}_t} = \frac{B_t^2}{B_t^1}.$$

Notice that since $\mathbb{Q}^1$ is a martingale measure for an investor choosing $\{B_t^1\}_{t\geq 0}$ as numeraire, we know that $\{B_t^2/B_t^1\}_{t\geq 0}$ is a $\mathbb{Q}^1$-martingale. Recall that if

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \zeta_t, \quad \text{for all } t > 0,$$

then, for $0 \leq s \leq t$,

$$\mathbb{E}^{\mathbb{Q}}\left[X_t \middle| \mathcal{F}_s\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{\zeta_t}{\zeta_s} X_t \middle| \mathcal{F}_s\right].$$

We first apply this to check that $\{S_t^i/B_t^2\}_{t\geq 0}$ is a $\mathbb{Q}^2$-martingale for each $i = 1, \ldots, n$.

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^2}\left[\frac{S_t^i}{B_t^2}\middle| \mathcal{F}_s\right] &= \mathbb{E}^{\mathbb{Q}^1}\left[\frac{B_t^2}{B_t^1}\frac{B_s^1}{B_s^2}\frac{S_t^i}{B_t^2}\middle| \mathcal{F}_s\right] \\ &= \mathbb{E}^{\mathbb{Q}^1}\left[\frac{B_s^1}{B_s^2}\frac{S_t^i}{B_t^1}\middle| \mathcal{F}_s\right] \\ &= \frac{B_s^1}{B_s^2}\frac{S_s^i}{B_s^1} = \frac{S_s^i}{B_s^2}, \end{aligned}$$

where the last line follows since B_s^1 and B_s^2 are $\mathcal{F}_s$-measurable and $\{S_t^i/B_t^1\}_{t\geq 0}$ is a $\mathbb{Q}^1$-martingale. In other words, $\{S_t^i/B_t^2\}_{t\geq 0}$ is a $\mathbb{Q}^2$-martingale as required. That $\{B_t^1/B_t^2\}_{t\geq 0}$ is a $\mathbb{Q}^2$-martingale follows in the same way.

The price for our derivative given that we chose $\{B_t^2\}_{t\geq 0}$ as numeraire is then

$$\begin{aligned} V_t^2 &= \mathbb{E}^{\mathbb{Q}^2}\left[\frac{B_t^2}{B_T^2} C_T \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}^1}\left[\frac{B_T^2}{B_T^1}\frac{B_t^1}{B_t^2}\frac{B_t^2}{B_T^2} C_T \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}^1}\left[\frac{B_t^1}{B_T^1} C_T \middle| \mathcal{F}_t\right] = V_t^1. \end{aligned}$$

In other words, the choice of numeraire is unimportant – we always arrive at the same price.

Quantos

We now apply our multifactor technology in an example. We are going to price a *quanto forward contract*.

Definition 7.2.8 *A financial asset is called a* quanto product *if it is denominated in a currency other than that in which it is traded.*

A quanto forward contract is also known as a *guaranteed exchange rate forward*. It is most easily explained through an example.

Example 7.2.9 *BP, a UK company, has a Sterling denominated stock price that we denote by $\{S_t\}_{t\geq 0}$. For a dollar investor, a quanto forward contract on BP stock with maturity T has payoff $(S_T - K)$ converted into dollars according to some prearranged exchange rate. That is the payout will be $\$E(S_T - K)$ for some preagreed E, where S_T is the* Sterling *asset price at time T.*

As in our foreign exchange market of §5.3 we shall assume that there is a riskless cash bond in each of the dollar and Sterling markets, but now we have two random processes to model, the stock price, $\{S_t\}_{t\geq 0}$ and the exchange rate, that is the value of one pound in dollars which we denote by $\{E_t\}_{t\geq 0}$. This will then require a *two-factor* model.

Black–Scholes quanto model: We write $\{B_t\}_{t\geq 0}$ for the dollar cash bond and $\{D_t\}_{t\geq 0}$ for its Sterling counterpart. Writing E_t for the dollar worth of one pound at time t and S_t for the Sterling asset price at time t, our model is

$$\begin{aligned} \text{Dollar bond} \quad & B_t = e^{rt}, \\ \text{Sterling bond} \quad & D_t = e^{ut}, \\ \text{Sterling asset price} \quad & S_t = S_0 \exp\left(\nu t + \sigma_1 W_t^1\right), \\ \text{Exchange rate} \quad & E_t = E_0 \exp\left(\lambda t + \rho\sigma_2 W_t^1 + \sqrt{1-\rho^2}\sigma_2 W_t^2\right), \end{aligned}$$

where $\{W_t^1\}_{t\geq 0}$ and $\{W_t^2\}_{t\geq 0}$ are independent $\mathbb{P}$-Brownian motions and r, u, ν, λ, σ_1, σ_2 and ρ are constants.

In this model the volatilities of $\{S_t\}_{t\geq 0}$ and $\{E_t\}_{t\geq 0}$ are σ_1 and σ_2 respectively and $\{W_t^1, \rho W_t^1 + \sqrt{1-\rho^2} W_t^2\}_{t\geq 0}$ is a pair of correlated Brownian motions with correlation coefficient ρ. There is no extra generality in replacing the expressions for S_t and E_t by

$$S_t = S_0 \exp\left(\nu t + \sigma_{11}\tilde{W}_t^1 + \sigma_{12}\tilde{W}_t^2\right),$$
$$E_t = E_0 \exp\left(\lambda t + \sigma_{21}\tilde{W}_t^1 + \sigma_{22}\tilde{W}_t^2\right),$$

for independent Brownian motions $\{\tilde{W}_t^1, \tilde{W}_t^2\}_{t\geq 0}$.

Pricing a quanto forward contract

What is the value of K that makes the value at time zero of the quanto forward contract zero?

As in our discussion of foreign exchange, the first step is to reformulate the problem in terms of the dollar tradables. We now have three dollar tradables: the dollar worth of the Sterling bond, $E_t D_t$; the dollar worth of the stock, $E_t S_t$; and the dollar cash bond, B_t. Choosing the dollar cash bond as numeraire, we first find the stochastic differential equations governing the discounted values of the other two dollar tradables. We write $Y_t = B_t^{-1} E_t D_t$ and $Z_t = B_t^{-1} E_t S_t$. Since

$$dE_t = \left(\lambda + \frac{1}{2}\sigma_2^2\right) E_t dt + \rho\sigma_2 E_t dW_t^1 + \sqrt{1-\rho^2}\sigma_2 E_t dW_t^2,$$

application of our multifactor integration by parts formula gives

$$d(E_t D_t) = u E_t D_t dt + \left(\lambda + \frac{1}{2}\sigma_2^2\right) E_t D_t dt + \rho\sigma_2 E_t D_t dW_t^1 + \sqrt{1-\rho^2}\,\sigma_2 E_t D_t dW_t^2$$

and

$$dY_t = \left(\lambda + \frac{1}{2}\sigma_2^2 + u - r\right) Y_t dt + Y_t \left(\rho\sigma_2 dW_t^1 + \sqrt{1-\rho^2}\sigma_2 dW_t^2\right).$$

Similarly, since

$$dS_t = \left(\nu + \frac{1}{2}\sigma_1^2\right) S_t dt + \sigma_1 S_t dW_t^1,$$

$$\begin{aligned} d(E_t S_t) &= \left(\nu + \frac{1}{2}\sigma_1^2\right) E_t S_t dt + \sigma_1 E_t S_t dW_t^1 \\ &\quad + \left(\lambda + \frac{1}{2}\sigma_2^2\right) S_t E_t dt + \rho\sigma_2 S_t E_t dW_t^1 \\ &\quad + \sqrt{1-\rho^2}\sigma_2 S_t E_t dW_t^2 + \rho\sigma_1\sigma_2 S_t E_t dt \end{aligned}$$

and so

$$\begin{aligned} dZ_t &= \left(\nu + \frac{1}{2}\sigma_1^2 + \lambda + \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 - r\right) Z_t dt \\ &\quad + (\sigma_1 + \rho\sigma_2) Z_t dW_t^1 + \sqrt{1-\rho^2}\sigma_2 Z_t dW_t^2. \end{aligned}$$

Now we seek a change of measure to make these two processes martingales. Our calculations after the proof of Theorem 7.2.4 reduce this to finding θ_1, θ_2 such that

$$\lambda + \frac{1}{2}\sigma_2^2 + u - r - \theta_1\rho\sigma_2 - \theta_2\sqrt{1-\rho^2}\sigma_2 = 0$$

and

$$\nu + \frac{1}{2}\sigma_1^2 + \lambda + \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 - r - \theta_1(\sigma_1 + \rho\sigma_2) - \theta_2\sqrt{1-\rho^2}\sigma_2 = 0.$$

Solving this pair of simultaneous equations gives

$$\theta_1 = \frac{\nu + \frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 - u}{\sigma_1}$$

and

$$\theta_2 = \frac{\lambda + \frac{1}{2}\sigma_2^2 + u - r - \rho\sigma_2\theta_1}{\sqrt{1-\rho^2}\sigma_2}.$$

Under the martingale measure, $\mathbb{Q}$, $\{X_t^1\}_{t\geq 0}$ and $\{X_t^2\}_{t\geq 0}$ defined by $X_t^1 = W_t^1 + \theta_1 t$ and $X_t^2 = W_t^2 + \theta_2 t$ are independent Brownian motions. We have

$$S_t = S_0 \exp\left(\left(u - \rho\sigma_1\sigma_2 - \frac{1}{2}\sigma_1^2\right)t + \sigma_1 X_t^1\right).$$

In particular,

$$S_T = \exp(-\rho\sigma_1\sigma_2 T) S_0 e^{uT} \exp\left(\sigma_1 X_T^1 - \frac{1}{2}\sigma_1^2 T\right)$$

and we are finally in a position to price the forward. Since $\{X_t^1\}_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion,

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\sigma_1 X_T^1 - \frac{1}{2}\sigma_1^2 T\right)\right] = 1,$$

so

$$\begin{aligned} V_0 &= e^{-rT} E\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)\right] \\ &= e^{-rT} E\left(\exp\left(-\rho\sigma_1\sigma_2 T\right) S_0 e^{uT} - K\right). \end{aligned}$$

Writing $F = S_0 e^{uT}$ for the forward price in the Sterling market and setting $V_0 = 0$ we see that we should take

$$K = F \exp\left(-\rho\sigma_1\sigma_2 T\right).$$

Remark: The exchange rate is given by

$$E_t = E_0 \exp\left(\left(r - u - \frac{1}{2}\sigma_2^2\right)t + \rho\sigma_2 X_t^1 + \sqrt{1-\rho^2}\sigma_2 X_t^2\right).$$

It is reassuring to observe that $\rho X_t^1 + \sqrt{1-\rho^2} X_t^2$ is a $\mathbb{Q}$-Brownian motion with variance one so that this expression for $\{E_t\}_{t\geq 0}$ is precisely that obtained in §5.3. Notice also that the discounted stock price process $e^{-rt}S_t$ is *not* a martingale; there is an extra term, reflecting the fact that the *Sterling* price is *not* a dollar tradable. □

7.3 Asset prices with jumps

The Black–Scholes framework is highly flexible. The critical assumptions are continuous time trading and that the dynamics of the asset price are continuous. Indeed, provided this second condition is satisfied, the Black–Scholes price can be justified as an asymptotic approximation to the arbitrage price under discrete trading, as the trading interval goes to zero. But are asset prices continuous?

So far, we have always assumed that any contracts written will be honoured. In particular, if a government or company issues a bond, we have ignored the possibility that they might default on that contract at maturity. But defaults do happen. This has been dramatically illustrated in recent years by credit crises in Asia, Latin America and Russia. If a company A holds a substantial quantity of company B's debt securities, then a default by B might be expected to have the knock-on effect of a sudden drop in company A's share price. How can we incorporate these market 'shocks' into our model?

A Poisson process of jumps

By their very nature, defaults are unpredictable. If we assume that we have absolutely no information to help us predict the default times or other market shocks, then we should model them by a Poisson random variable. That is the time between shocks is exponentially distributed and the number of shocks by time t, denoted by N_t, is a Poisson random variable with parameter λt for some $\lambda > 0$. Between shocks we assume that an asset price follows our familiar geometric Brownian motion model.

A typical model for the evolution of the price of a risky asset with jumps is

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t - \delta dN_t, \tag{7.9}$$

where $\{W_t\}_{t\geq 0}$ and $\{N_t\}_{t\geq 0}$ are independent. To make sense of equation (7.9) we write it in integrated form, but then we must define the stochastic integral with respect to $\{N_t\}_{t\geq 0}$. Writing τ_i for the time of the ith jump of the Poisson process, we define

$$\int_0^t f(u, S_u)dN_u = \sum_{i=1}^{N_t} f\left(\tau(i)-, S_{\tau(i)-}\right).$$

For the model (7.9), if there is a shock, then the asset price is decreased by a factor of $(1-\delta)$. This observation tells us that the solution to (7.9) is

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)(1-\delta)^{N_t}.$$

To deal with more general models we must extend our theory of stochastic calculus to incorporate processes with jumps. As usual, the first step is to find an (extended) Itô formula.

Assumption: We assume that asset price processes are càdlàg, that is they are right continuous with left limits.

Theorem 7.3.1 (Itô's formula with jumps) *Suppose*

$$dY_t = \mu_t dt + \sigma_t dW_t + \nu_t dN_t$$

where, under $\mathbb{P}$, $\{W_t\}_{t\geq 0}$ *is a standard Brownian motion and* $\{N_t\}_{t\geq 0}$ *is a Poisson process with intensity* λ. *If* f *is a twice continuously differentiable function on* $\mathbb{R}$ *then*

$$\begin{aligned} f(Y_t) &= f(Y_0) + \int_0^t f'(Y_{s-})dY_s + \frac{1}{2}\int_0^t f''(Y_{s-})\sigma_s^2 ds \\ &\quad - \sum_{i=1}^{N_t} f'(T_{\tau_i-})\left(Y_{\tau_i} - Y_{\tau_i-}\right) + \sum_{i=1}^{N_t}\left(f(Y_{\tau_i}) - f(Y_{\tau_i-})\right), \end{aligned} \tag{7.10}$$

where $\{\tau_i\}$ *are the times of the jumps of the Poisson process.*

We don't prove this here, but heuristically it is not difficult to see that this should be the correct result. The first three terms are exactly what we'd expect if the process $\{Y_t\}_{t\geq 0}$ were continuous, but now, because of the discontinuities, we must distinguish Y_{s-} from Y_s. In between jumps of $\{N_t\}_{t\geq 0}$, precisely this equation should apply, but we must compensate for changes at jump times. In the first three terms we have included a term of the form $\sum_{i=1}^{N_t} f'(Y_{\tau_i-})\left(Y_{\tau_i} - Y_{\tau_i-}\right)$ and the first sum in equation (7.10) corrects for this. Since N_t is finite, we do not have to correct the term involving f''. Now we add in the *actual* contribution from the jump times and this is the second sum.

Compensation

As usual a key rôle will be played by martingales. Evidently a Poisson process, $\{N_t\}_{t\geq 0}$ with intensity λ under $\mathbb{P}$ is not a $\mathbb{P}$-martingale – it is monotone increasing. But we can write it as a martingale plus a drift. In Exercise 13 it is shown that the process $\{M_t\}_{t\geq 0}$ defined by $M_t = N_t - \lambda t$ *is* a $\mathbb{P}$-martingale.

More generally we can consider time-inhomogeneous Poisson processes. For such processes the intensity $\{\lambda_t\}_{t\geq 0}$ is a function of time. The probability of a jump in the time interval $[t, t + \delta t)$ is $\lambda_t \delta t + o(\delta t)$. Thus, for example, the probability that there is no jump in the interval $[s, t]$ is $\exp\left(-\int_s^t \lambda_u du\right)$. The corresponding *Poisson martingale* is $M_t = N_t - \int_0^t \lambda_s ds$. The process $\{\Lambda_t\}_{t\geq 0}$ given by $\Lambda_t = \int_0^t \lambda_s ds$ is the *compensator* of $\{N_t\}_{t\geq 0}$.

In Exercise 14 it is shown that just as integration with respect to Brownian martingales gives rise to (local) martingales, so integration with respect to Poisson martingales gives rise to martingales.

Poisson exponential martingales

Example 7.3.2 *Let $\{N_t\}_{t\geq 0}$ be a Poisson process with intensity $\{\lambda_t\}_{t\geq 0}$ under $\mathbb{P}$ where for each $t > 0$, $\int_0^t \lambda_s ds < \infty$. For a given bounded deterministic function $\{\alpha_t\}_{t\geq 0}$, let*

$$L_t = \exp\left(\int_0^t \alpha_s dM_s + \int_0^t \left(1 + \alpha_s - e^{\alpha_s}\right)\lambda_s ds\right) \qquad (7.11)$$

where $dM_s = dN_s - \lambda_s ds$. Find the stochastic differential equation satisfied by $\{L_t\}_{t\geq 0}$ and deduce that $\{L_t\}_{t\geq 0}$ is a $\mathbb{P}$-martingale.

Solution: First write

$$Z_t = \int_0^t \alpha_s dM_s + \int_0^t \left(1 + \alpha_s - e^{\alpha_s}\right)\lambda_s ds$$

so that $L_t = e^{Z_t}$. Then

$$dZ_t = \alpha_t dN_t - \alpha_t \lambda_t dt + \left(1 + \alpha_t - e^{\alpha_t}\right)\lambda_t dt$$

and by our generalised Itô formula

$$dL_t = L_{t-} dZ_t + \left(-e^{Z_{t-}}\alpha_t + e^{Z_{t-}+\alpha_t} - e^{Z_{t-}}\right) dN_t,$$

where we have used the fact that if a jump in $\{Z_t\}_{t\geq 0}$ takes place at time t, then that jump is of size α_{t-}. Substituting and rearranging give

$$\begin{aligned} dL_t &= L_{t-}\alpha_t dM_t + L_{t-}\left(1 + \alpha_t - e^{\alpha_t}\right)\lambda_t dt - L_{t-}\left(1 + \alpha_t - e^{\alpha_t}\right) dN_t \\ &= L_{t-}\left(e^{\alpha_t} - 1\right) dM_t. \end{aligned}$$

By Exercise 14, $\{L_t\}_{t\geq 0}$ is a $\mathbb{P}$-martingale. □

Definition 7.3.3 *Processes of the form of $\{L_t\}_{t\geq 0}$ defined by (7.11) will be called* Poisson exponential martingales.

Our Poisson exponential martingales and Brownian exponential martingales are examples of Doléans–Dade exponentials.

Definition 7.3.4 *For a semimartingale $\{X_t\}_{t\geq 0}$ with $X_0 = 0$, the* Doléans–Dade exponential *of $\{X_t\}_{t\geq 0}$ is the unique semimartingale solution $\{Z_t\}_{t\geq 0}$ to*

$$Z_t = 1 + \int_0^t Z_{s-} dX_s.$$

Change of measure

In the same way as we used Brownian exponential martingales to change measure and thus 'transform drift' in the continuous world, so we shall combine Brownian and Poisson exponential martingales in our discontinuous asset pricing models. A change of drift for a Poisson martingale will correspond to a change of intensity for the Poisson process $\{N_t\}_{t\geq 0}$. More precisely, we have the following version of the Girsanov Theorem.

Theorem 7.3.5 (Girsanov Theorem for asset prices with jumps) *Let $\{W_t\}_{t\geq 0}$ be a standard $\mathbb{P}$-Brownian motion and $\{N_t\}_{t\geq 0}$ a (possibly time-inhomogeneous) Poisson process with intensity $\{\lambda_t\}_{t\geq 0}$ under $\mathbb{P}$. That is*

$$M_t = N_t - \int_0^t \lambda_u du$$

is a $\mathbb{P}$-martingale. We write $\mathcal{F}_t$ for the σ-field generated by $\mathcal{F}_t^W \cup \mathcal{F}_t^N$. Suppose that $\{\theta_t\}_{t\geq 0}$ and $\{\phi_t\}_{t\geq 0}$ are $\{\mathcal{F}_t\}_{t\geq 0}$-previsible processes with ϕ_t positive for each t, such that

$$\int_0^t \|\theta_s\|^2 ds < \infty \quad \textit{and} \quad \int_0^t \phi_s \lambda_s ds < \infty.$$

Then under the measure $\mathbb{Q}$ whose Radon–Nikodym derivative with respect to $\mathbb{P}$ is given by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t$$

where $L_0 = 1$ and

$$\frac{dL_t}{L_{t-}} = \theta_t dW_t - (1 - \phi_t)\, dM_t,$$

the process $\{X_t\}_{t\geq 0}$ defined by $X_t = W_t - \int_0^t \theta_s ds$ is a Brownian motion and $\{N_t\}_{t\geq 0}$ has intensity $\{\phi_t \lambda_t\}_{t\geq 0}$.

In Exercise 16 it is shown that $\{L_t\}_{t\geq 0}$ is actually the product of a Brownian exponential martingale and a Poisson exponential martingale.

The proof of Theorem 7.3.5 is once again beyond our scope, but to check that the processes $\{X_t\}_{t\geq 0}$ and $\left\{N_t - \int_0^t \phi_s \lambda_s ds\right\}_{t\geq 0}$ are both local martingales under $\mathbb{Q}$ is an exercise based on the Itô formula.

Heuristics: An informal justification of the result is based on the extended multiplication table:

$\times$	dW_t	dN_t	dt
dW_t	dt	0	0
dN_t	0	dN_t	0
dt	0	0	0

Thus, for example,

$$\begin{aligned} d\left(L_t\big(N_t-\int_0^t\phi_s\lambda_s ds\big)\right) &= \left(N_t-\int_0^t\phi_s\lambda_s ds\right)dL_t+L_t\left(dN_t-\phi_t\lambda_t dt\right) \\ &\quad -L_t(1-\phi_t)(dN_t)^2 \\ &= \left(N_t-\int_0^t\phi_s\lambda_s ds\right)dL_t+L_t\left(dM_t+\lambda_t dt\right) \\ &\quad -L_t\phi_t\lambda_t dt-L_t(1-\phi_t)\left(dM_t+\lambda_t dt\right) \\ &= \left(N_t-\int_0^t\phi_s\lambda_s ds\right)dL_t+L_t\phi_t dM_t. \end{aligned}$$

Since $\{M_t\}_{t\geq 0}$ and $\{L_t\}_{t\geq 0}$ are $\mathbb{P}$-martingales, subject to appropriate boundedness assumptions, $\left\{L_t\left(N_t-\int_0^t\phi_s\lambda_s ds\right)\right\}_{t\geq 0}$ should be a $\mathbb{P}$-martingale and consequently $\left\{\left(N_t-\int_0^t\phi_s\lambda_s ds\right)\right\}_{t\geq 0}$ should be a $\mathbb{Q}$-martingale. □

Our instinct is to use the extended Girsanov Theorem to find an equivalent probability measure under which the discounted asset price is a martingale.

Suppose then that

$$\frac{dS_t}{S_t}=\mu dt+\sigma dW_t-\delta dN_t.$$

Evidently the discounted asset price satisfies

$$\frac{d\tilde{S}_t}{\tilde{S}_t}=(\mu-r)\,dt+\sigma dW_t-\delta dN_t.$$

But now we see that there are *many* choices of $\{\theta_t\}_{t\geq 0}$ and $\{\phi_t\}_{t\geq 0}$ in Theorem 7.3.5 that lead to a martingale measure. The difficulty of course is that our market is not *complete*, so that although for any replicable claim we can use any of the martingale measures and arrive at the same answer, there are claims that cannot be hedged. There are two independent sources of risk, the Brownian motion and the Poisson point process, and so if we are to be able to hedge arbitrary claims $C_T\in\mathcal{F}_T$, we need two tradable risky assets subject to the same two noises.

Market price of risk

So if there *are* enough assets available to hedge claims, can we find a measure under which once discounted they are *all* martingales? Remember that otherwise there will be arbitrage opportunities in our market.

If the asset price has no jumps, we can write

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt+\sigma dW_t \\ &= (r+\gamma\sigma)\,dt+\sigma dW_t, \end{aligned}$$

where $\gamma=(\mu-r)/\sigma$ is the market price of risk. We saw in Chapter 5 that in the absence of arbitrage (so when there *is* an equivalent martingale measure for our market), γ will be the same for *all* assets driven by $\{W_t\}_{t\geq 0}$.

If the asset price has jumps, then investors will expect to be compensated for the additional risk associated with the possibility of downward jumps, even if we have 'compensated' the jumps (replaced dN_t by dM_t) so that their mean is zero. The price of such an asset is governed by

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu dt + \sigma dW_t + \nu dM_t \\ &= (r + \gamma\sigma + \eta\lambda\nu)\, dt + \sigma dW_t + \nu dM_t\end{aligned}$$

where ν measures the sensitivity of the asset price to the market shock and η is the excess rate of return per unit of jump risk. Again if there is to be a martingale measure under which *all* the discounted asset prices are martingales, then σ and η should be the same for all assets whose prices are driven by $\{W_t\}_{t\geq 0}$ and $\{N_t\}_{t\geq 0}$. The martingale measure, $\mathbb{Q}$, will then be the measure $\mathbb{Q}$ of Theorem 7.3.5 under which

$$W_t + \int_0^t \frac{\mu - r}{\sigma} ds \quad \text{and} \quad M_t - \int_0^t \eta\lambda ds$$

are martingales. That is we take $\theta = \gamma$ and $\phi = -\eta$.

Multiple noises

The same ideas can be extended to assets driven by larger numbers of independent noises. For example, we might have n assets with dynamics

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \sum_{\alpha=1}^{n} \sigma_{i\alpha} dW_t^\alpha + \sum_{\beta=1}^{m} \nu_{i\beta} dM_t^\beta$$

where, under $\mathbb{P}$, $\{W_t^\alpha\}_{t\geq 0}$, $\alpha = 1, \ldots, n$, are independent Brownian motions and $\{M_t^\beta\}_{t\geq 0}$, $\beta = 1, \ldots, m$, are independent Poisson martingales.

There will be an equivalent martingale measure under which *all* the discounted asset prices are martingales if we can associate a unique market price of risk with each source of noise. In this case we can write

$$\mu_i = r + \sum_{\alpha=1}^{n} \gamma_\alpha \sigma_{i\alpha} + \sum_{\beta=1}^{m} \eta_\beta \lambda_\beta \nu_{i\beta}.$$

All discounted asset prices will be martingales under the measure $\mathbb{Q}$ for which

$$\tilde{W}_t^\alpha = W_t^\alpha + \gamma_\alpha t$$

is a martingale for each α and

$$\tilde{M}_t^\beta = M_t^\beta + \eta_\beta \lambda_\beta t$$

is a martingale for each β.

As always it is *replication* that drives the theory. Note that in order to be able to hedge arbitrary $C_T \in \mathcal{F}_T$ we'll require $n + m$ 'independent' tradable risky assets driven by these sources of noise. With fewer assets at our disposal there will be claims C_T that we cannot hedge.

All this is little changed if we take the coefficients μ, σ, λ to be adapted to the filtration generated by $\{W_t^i\}_{t\geq 0}$, $i = 1, \ldots, n$; see Exercise 15. Since we are not introducing any extra sources of noise, the same number of assets will be needed for market completeness. These ideas form the basis of Jarrow–Madan theory.

7.4 Model error

Even in the absence of jumps (or between jumps) we have given only a very vague justification for the Samuelson model

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{7.12}$$

Moreover, although we have shown that under this model the pricing and hedging of derivatives are dictated by the single parameter σ, we have said nothing about how actually to estimate this number from market data. So what is market practice?

Implied volatility

Vanilla options are generally traded on exchanges, so if a trader wants to know the price of, say, a European call option, then she can read it from her trading screen. However, for an over-the-counter derivative, the price is not quoted on an exchange and so one needs a pricing model. The normal practice is to build a Black–Scholes model and then *calibrate* it to the market – that is estimate σ from the market. But it is *not* usual to estimate σ directly from data for the stock price. Instead one uses the quoted price for exchange-traded options written on the same stock. The procedure is simple: for given strike price and maturity, we can think of the Black–Scholes pricing formula for a European option as a mapping from volatility, σ, to price V. In Exercise 17, it is shown that for vanilla options this mapping is strictly monotone and so can be inverted to infer σ from the price. In other words, given the option price one can recover the corresponding value of σ in the Black–Scholes formula. This number is the so-called *implied volatility*.

If the markets really did follow our Black–Scholes model, then this procedure would give the same value of σ, irrespective of the strike price and maturity of the exchange-traded option chosen. Sadly, this is far from what we observe in reality: not only is there dependence on the strike price for a fixed maturity, giving rise to the famous volatility smile, but also implied volatility tends to increase with time to maturity (Figure 7.1). Market practice is to choose as volatility parameter for pricing an over-the-counter option the implied volatility obtained from 'comparable' exchange-traded options.

Hedging error

This procedure can be expected to lead to a *consistent* price for exchange-traded and over-the-counter options and model error is not a serious problem. The difficulties arise in *hedging*. Even for exchange-traded options a model is required to determine the replicating portfolio. We follow Davis (2001).

Suppose that the true stock price process follows

$$dS_t = \alpha_t S_t dS_t + \beta_t S_t dW_t$$

where $\{\alpha_t\}_{t\geq 0}$ and $\{\beta_t\}_{t\geq 0}$ are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes, but we price *and hedge* an option with payoff $\Phi(S_T)$ at time T as though $\{S_t\}_{t\geq 0}$ followed equation (7.12) for some parameter σ.

Our estimate for the value of the option at time $t < T$ will be $V(t, S_t)$ where

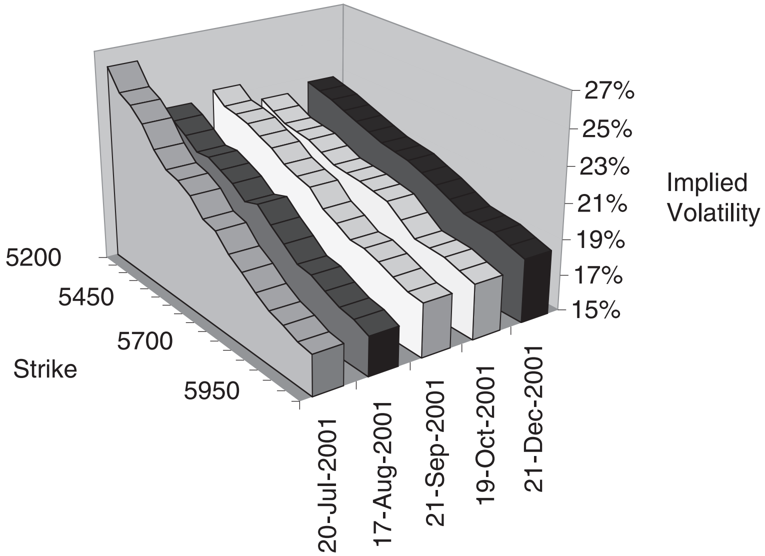

Figure 7.1 Implied volatility as a function of strike price and maturity for European call options based on the FTSE stock index.

$V(t, x)$ satisfies the Black–Scholes partial differential equation

$$\frac{\partial V}{\partial t}(t, x) + rx\frac{\partial V}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}(t, x) - rV(t, x) = 0,$$
$$V(T, x) = \Phi(x).$$

Our hedging portfolio consists at time t of $\phi_t = \frac{\partial V}{\partial x}(t, S_t)$ units of stock and cash bonds with total value $\psi_t e^{rt} \triangleq V(t, S_t) - \phi_t S_t$.

Our first worry is that because of model misspecification, the portfolio is not self-financing. So what is the cost of following such a strategy? Since the cost of purchasing the 'hedging' portfolio at time t is $V(t, S_t)$, the incremental cost of the strategy over an infinitesimal time interval $[t, t + \delta t)$ is

$$\frac{\partial V}{\partial x}(t, S_t)\big(S_{t+\delta t} - S_t\big) + \left(V(t, S_t) - \frac{\partial V}{\partial x}(t, S_t)S_t\right)(e^{r\delta t} - 1)$$
$$- V(t + \delta t, S_{t+\delta t}) + V(t, S_t).$$

In other words, writing Z_t for our net position at time t, we have

$$dZ_t = \frac{\partial V}{\partial x}(t, S_t)dS_t + \left(V(t, S_t) - \frac{\partial V}{\partial x}(t, S_t)S_t\right)rdt - dV(t, S_t).$$

Since $V(t, x)$ solves the Black–Scholes partial differential equation, applying Itô's formula gives

$$\begin{aligned} dZ_t &= \frac{\partial V}{\partial x}(t, S_t)dS_t + \left(V(t, S_t) - \frac{\partial V}{\partial x}(t, S_t)S_t\right) r dt \\ &\quad - \frac{\partial V}{\partial t}(t, S_t)dt - \frac{\partial V}{\partial x}(t, S_t)dS_t - \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t, S_t)\beta_t^2 S_t^2 dt \\ &= \frac{1}{2}S_t^2 \frac{\partial^2 V}{\partial x^2}\left(\sigma^2 - \beta_t^2\right) dt. \end{aligned}$$

Irrespective of the model, $V(T, S_T) = \Phi(S_T)$ precisely matches the claim against us at time T, so our net position at time T (having honoured the claim $\Phi(S_T)$ against us) is

$$Z_T = \int_0^T \frac{1}{2}S_t^2 \frac{\partial^2 V}{\partial x^2}(t, S_t)\left(\sigma^2 - \beta_t^2\right) dt.$$

For European call and put options $\frac{\partial^2 V}{\partial x^2} > 0$ (see Exercise 18) and so if $\sigma^2 > \beta_t^2$ for all $t \in [0, T]$ our hedging strategy makes a profit. This means that regardless of the price dynamics, we make a profit if the parameter σ in our Black–Scholes model dominates the true diffusion coefficient β. This is key to successful hedging. Our calculation won't work if the price process has jumps, although by choosing σ large enough one can still arrange for Z_T to have positive expectation.

The choice of σ is still a tricky matter. If we are too cautious no one will buy the option, too optimistic and we are exposed to the risk associated with changes in volatility and we should try to hedge that risk. Such hedging is known as *vega hedging*, the Greek *vega* of an option being the sensitivity of its Black–Scholes price to changes in σ. The idea is the same as that of delta hedging (Exercise 5 of Chapter 5). For example, if we buy an over-the-counter option for which $\frac{\partial V}{\partial \sigma} = v$, then we also sell a number v/v' of a comparable exchange traded option whose value is V' and for which $\frac{\partial V'}{\partial \sigma} = v'$. The resulting portfolio is said to be *vega-neutral*.

Stochastic volatility and implied volatility

Since we cannot observe the volatility directly, it is natural to try to model it as a random process. A huge amount of effort has gone into developing so-called *stochastic volatility models*. Fat-tailed returns distributions observed in data can be modelled in this framework and sometimes 'jumps' in the asset price can be best modelled by jumps in the volatility. For example if jumps occur according to a Poisson process with constant rate and at the time, τ, of a jump, $S_\tau/S_{\tau-}$ has a lognormal distribution, then the distribution of S_t will be lognormal but with variance parameter given by a multiple of a Poisson random variable (Exercise 19). Stochastic volatility can also be used to model the 'smile' in the implied volatility curve and we end this chapter by finding the correspondence between the choice of a stochastic volatility model and of an implied volatility model. Once again we follow Davis (2001). A typical stochastic volatility model takes the form

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dW_t^1, \\ d\sigma_t &= a(S_t, \sigma_t)dt + b(S_t, \sigma_t)\left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right), \end{aligned}$$

where $\{W_t^1\}_{t\geq 0}$, $\{W_t^2\}_{t\geq 0}$ are independent $\mathbb{P}$-Brownian motions, ρ is a constant in (0, 1) and the coefficients $a(x, \sigma)$ and $b(x, \sigma)$ define the volatility model.

As usual we'd like to find a martingale measure. If $\mathbb{Q}$ is equivalent to $\mathbb{P}$, then its Radon–Nikodym derivative with respect to $\mathbb{P}$ takes the form

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\int_0^t \hat{\theta}_s dW_s^1 - \frac{1}{2}\int_0^t \hat{\theta}_s^2 ds - \int_0^t \theta_s dW_s^2 - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$$

for some integrands $\{\hat{\theta}_t\}_{t\geq 0}$ and $\{\theta_t\}_{t\geq 0}$. In order for the discounted asset price $\{\tilde{S}_t\}_{t\geq 0}$ to be a $\mathbb{Q}$-martingale, we choose

$$\hat{\theta}_t = \frac{\mu - r}{\sigma_t}.$$

The choice of $\{\theta_t\}_{t\geq 0}$ however is arbitrary as $\{\sigma_t\}_{t\geq 0}$ is not a tradable and so no arbitrage argument can be brought to bear to dictate its drift. Under $\mathbb{Q}$,

$$X_t^1 = W_t^1 + \int_0^t \hat{\theta}_s ds$$

and

$$X_t^2 = W_t^2 + \int_0^t \theta_s ds$$

are independent Brownian motions. The dynamics of $\{S_t\}_{t\geq 0}$ and $\{\sigma_t\}_{t\geq 0}$ are then most conveniently written as

$$dS_t = rS_t dt + \sigma_t S_t dX_t^1$$

and

$$d\sigma_t = \tilde{a}(S_t, \sigma_t)dt + b(S_t, \sigma_t)\left(\rho dX_t^1 + \sqrt{1-\rho^2}dX_t^2\right)$$

where

$$\tilde{a}(S_t, \sigma_t) = a(S_t, \sigma_t) - b(S_t, \sigma_t)\left(\rho\hat{\theta}_t + \sqrt{1-\rho^2}\theta_t\right).$$

We now *introduce* a second tradable asset. Suppose that we have an option written on $\{S_t\}_{t\geq 0}$ whose exercise value at time T is $\Phi(S_T)$. We *define* its value at times $t < T$ to be the discounted value of $\Phi(S_T)$ under the measure $\mathbb{Q}$. That is

$$V(t, S_t, \sigma_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\Phi(S_T)\middle|\mathcal{F}_t\right].$$

Our multidimensional Feynman–Kac Stochastic Representation Theorem (combined with the usual product rule) tells us that the function $V(t, x, \sigma)$ solves the partial differential equation

$$\frac{\partial V}{\partial t}(t,x,\sigma) + rx\frac{\partial V}{\partial x}(t,x,\sigma) + \tilde{a}(t,x,\sigma)\frac{\partial V}{\partial \sigma}(t,x,\sigma) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 V}{\partial x^2}(t,x,\sigma)$$
$$+\frac{1}{2}b(t,x,\sigma)^2\frac{\partial^2 V}{\partial \sigma^2}(t,x,\sigma) + \rho\sigma x b(t,x,\sigma)\frac{\partial^2 V}{\partial x\partial\sigma}(t,x,\sigma) - rV(t,x,\sigma) = 0.$$

Writing $Y_t = V(t, S_t, \sigma_t)$ and suppressing the dependence of V, $\tilde{a}$ and b on (t, S_t, σ_t) in our notation, an application of Itô's formula tells us that

$$\begin{aligned} dY_t &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dS_t + \frac{\partial V}{\partial \sigma}d\sigma_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\sigma_t^2 S_t^2 dt \\ &\quad + \frac{\partial^2 V}{\partial x \partial \sigma}\rho b \sigma_t S_t dt + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2} b^2 dt \\ &= \left(rV - rS_t \frac{\partial V}{\partial x} - \tilde{a}\frac{\partial V}{\partial \sigma} - \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 V}{\partial x^2} - \frac{1}{2}b^2 \frac{\partial^2 V}{\partial \sigma^2} - \rho \sigma_t S_t b \frac{\partial^2 V}{\partial x \partial \sigma} \right) dt \\ &\quad + rS_t \frac{\partial V}{\partial x}dt + \sigma_t S_t \frac{\partial V}{\partial x}dX_t^1 + \tilde{a}\frac{\partial V}{\partial \sigma}dt + b\rho \frac{\partial V}{\partial \sigma}dX_t^1 + b\sqrt{1-\rho^2}\frac{\partial V}{\partial \sigma}dX_t^2 \\ &\quad + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 V}{\partial x^2}dt + \rho b \sigma_t S_t \frac{\partial^2 V}{\partial x \partial \sigma}dt + \frac{1}{2}b^2 \frac{\partial^2 V}{\partial \sigma^2}dt \\ &= rY_t dt + \sigma_t S_t \frac{\partial V}{\partial x}dX_t^1 + b\rho \frac{\partial V}{\partial \sigma}dX_t^1 + b\sqrt{1-\rho^2}\frac{\partial V}{\partial \sigma}dX_t^2. \end{aligned}$$

If the mapping $\sigma \mapsto y = V(t, x, \sigma)$ is invertible so that $\sigma = D(t, x, y)$ for some nice function D, then

$$dY_t = rY_t dt + c(t, S_t, Y_t)dX_t^1 + d(t, S_t, Y_t)dX_t^2$$

for some functions c and d.

We have now created a complete market model with tradables $\{S_t\}_{t \geq 0}$ and $\{Y_t\}_{t \geq 0}$ for which $\mathbb{Q}$ is the unique martingale measure. Of course, we have actually created one such market for each choice of $\{\theta_t\}_{t \geq 0}$ and it is the choice of $\{\theta_t\}_{t \geq 0}$ that specifies the functions c and d and it is precisely these functions that tell us how to hedge.

So what model for implied volatility corresponds to this stochastic volatility model? The implied volatility, $\hat{\sigma}(t)$, will be such that Y_t is the Black–Scholes price evaluated at (t, S_t) if the volatility in equation (7.12) is taken to be $\hat{\sigma}(t)$. In this way each choice of $\{\theta_t\}_{t \geq 0}$, or equivalently model for $\{Y_t\}_{t \geq 0}$, provides a model for the implied volatility.

There is a huge literature on stochastic volatility. A good starting point is Fouque, Papanicolau and Sircar (2000).

Exercises

1 Check that the replicating portfolio defined in §7.1 is self-financing.

2 Suppose that $\{W_t^1\}_{t \geq 0}$ and $\{W_t^2\}_{t \geq 0}$ are independent Brownian motions under $\mathbb{P}$ and let ρ be a constant with $0 < \rho < 1$. Find coefficients $\{\alpha_{ij}\}_{i,j=1,2}$ such that

$$\tilde{W}_t^1 = \alpha_{11} W_t^1 + \alpha_{12} W_t^2$$

and

$$\tilde{W}_t^2 = \alpha_{21} W_t^1 + \alpha_{22} W_t^2$$

define two standard Brownian motions under $\mathbb{P}$ with $\mathbb{E}\left[\tilde{W}_t^1 \tilde{W}_t^2\right] = \rho t$. Is your solution unique?

3 Suppose that $F(t, x)$ solves the time-inhomogeneous Black–Scholes partial differential equation

$$\frac{\partial F}{\partial t}(t, x) + \frac{1}{2}\sigma^2(t)x^2\frac{\partial^2 F}{\partial x^2}(t, x) + r(t)x\frac{\partial F}{\partial x}(t, x) - r(t)F(t, x) = 0, \tag{7.13}$$

subject to the boundary conditions appropriate to pricing a European call option. Substitute

$$y = xe^{\alpha(t)}, \quad v = Fe^{\beta(t)}, \quad \tau = \gamma(t)$$

and choose $\alpha(t)$ and $\beta(t)$ to eliminate the coefficients of v and $\frac{\partial v}{\partial y}$ in the resulting equation and $\gamma(t)$ to remove the remaining time dependence so that the equation becomes

$$\frac{\partial v}{\partial \tau}(\tau, y) = \frac{1}{2}y^2\frac{\partial^2 v}{\partial y^2}(\tau, y).$$

Notice that the coefficients in this equation are independent of time and there is no reference to r or σ. Deduce that the solution to equation (7.13) can be obtained by making appropriate substitutions in the classical Black–Scholes formula.

4 Let $\{W_t^i\}_{t\geq 0}$, $i = 1, \dots, n$, be independent Brownian motions. Show that $\{R_t\}_{t\geq 0}$ defined by

$$R_t = \sqrt{\sum_{i=1}^{n}(W_t^i)^2}$$

satisfies a stochastic differential equation. The process $\{R_t\}_{t\geq 0}$ is the radial part of Brownian motion in $\mathbb{R}^n$ and is known as the *n-dimensional Bessel process*.

5 Recall that we define two-dimensional Brownian motion, $\{X_t\}_{t\geq 0}$, by $X_t = (W_t^1, W_t^2)$, where $\{W_t^1\}_{t\geq 0}$ and $\{W_t^2\}_{t\geq 0}$ are independent (one-dimensional) standard Brownian motions. Find the Kolmogorov backward equation for $\{X_t\}_{t\geq 0}$.
Repeat your calculation if $\{W_t^1\}_{t\geq 0}$ and $\{W_t^2\}_{t\geq 0}$ are replaced by *correlated* Brownian motions, $\{\tilde{W}_t^1\}_{t\geq 0}$ and $\{\tilde{W}_t^1\}_{t\geq 0}$ with $\mathbb{E}\left[d\tilde{W}_t^1 d\tilde{W}_t^2\right] = \rho dt$ for some $-1 < \rho < 1$.

6 Use a delta-hedging argument to obtain the result of Corollary 7.2.7.

7 Repeat the Black–Scholes analysis of §7.2 in the case when the chosen numeraire, $\{B_t\}_{t\geq 0}$, has non-zero volatility and check that the fair price of a derivative with payoff C_T at time T is once again

$$V_t = B_t\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{C_T}{B_T}\right|\mathcal{F}_t\right]$$

for a suitable choice of $\mathbb{Q}$ (which you should specify).

8 Two traders, operating in the same complete arbitrage-free Black–Scholes market of §7.2, sell identical options, but make different choices of numeraire. How will their hedging strategies differ?

9 Find a portfolio that replicates the quanto forward contract of Example 7.2.9.

10 A *quanto digital contract* written on the BP stock of Example 7.2.9 pays \$1 at time T if the BP Sterling stock price, S_T, is larger than K. Assuming the Black–Scholes quanto model of §7.2, find the time zero price of such an option and the replicating portfolio.

11 A *quanto call option* written on the BP stock of Example 7.2.9 is worth $E(S_T - K)_+$ *dollars* at time T, where S_T is the *Sterling* stock price. Assuming the Black–Scholes quanto model of §7.2, find the time zero price of the option and the replicating portfolio.

12 *Asian options* Suppose that our market, consisting of a riskless cash bond, $\{B_t\}_{t\geq 0}$, and a single risky asset with price $\{S_t\}_{t\geq 0}$, is governed by

$$dB_t = rB_t dt, \quad B_0 = 1$$

and

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion.
An option is written with payoff $C_T = \Phi(S_T, Z_T)$ at time T where

$$Z_t = \int_0^t g(u, S_u)du$$

for some (deterministic) real-valued function g on $\mathbb{R}_+ \times \mathbb{R}$.
From our general theory we know that the value of such an option at time t satisfies

$$V_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_T, Z_T)|\mathcal{F}_t\right]$$

where $\mathbb{Q}$ is the measure under which $\{S_t/B_t\}_{t\geq 0}$ is a martingale.
Show that $V_t = F(t, S_t, Z_t)$ where the real-valued function $F(t, x, z)$ on $\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}$ solves

$$\frac{\partial F}{\partial t} + rx\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 F}{\partial x^2} + g\frac{\partial F}{\partial z} - rF = 0,$$
$$F(T, x, z) = \Phi(x, z).$$

Show further that the claim C_T can be hedged by a self-financing portfolio consisting at time t of

$$\phi_t = \frac{\partial F}{\partial x}(t, S_t, Z_t)$$

units of stock and

$$\psi_t = e^{-rt}\left(F(t, S_t, Z_t) - S_t\frac{\partial F}{\partial x}(t, S_t, Z_t)\right)$$

cash bonds.

13 Suppose that $\{N_t\}_{t\geq 0}$ is a Poisson process whose intensity under $\mathbb{P}$ is $\{\lambda_t\}_{t\geq 0}$. Show that $\{M_t\}_{t\geq 0}$ defined by

$$M_t = N_t - \int_0^t \lambda_s ds$$

is a $\mathbb{P}$-martingale with respect to the σ-field generated by $\{N_t\}_{t\geq 0}$.

14 Suppose that $\{N_t\}_{t\geq 0}$ is a Poisson process under $\mathbb{P}$ with intensity $\{\lambda_t\}_{t\geq 0}$ and $\{M_t\}_{t\geq 0}$ is the corresponding Poisson martingale. Check that for an $\{\mathcal{F}_t^M\}_{t\geq 0}$-predictable process $\{f_t\}_{t\geq 0}$,

$$\int_0^t f_s dM_s$$

is a $\mathbb{P}$-martingale.

15 Show that our analysis of §7.3 is still valid if we allow the coefficients in the stochastic differential equations driving the asset prices to be $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes, provided we make some boundedness assumptions that you should specify.

16 Show that the process $\{L_t\}_{t\geq 0}$ in Theorem 7.3.5 is the product of a Poisson exponential martingale and a Brownian exponential martingale and hence prove that it is a martingale.

17 Show that in the classical Black–Scholes model the *vega* for a European call (or put) option is strictly positive. Deduce that for vanilla options we can infer the volatility parameter of the Black–Scholes model from the price.

18 Suppose that $V(t, x)$ is the Black–Scholes price of a European call (or put) option at time t given that the stock price at time t is x. Prove that $\frac{\partial^2 V}{\partial x^2} \geq 0$.

19 Suppose that an asset price $\{S_t\}_{t\geq 0}$ follows a geometric Brownian motion with jumps occurring according to a Poisson process with constant intensity λ. At the time, τ, of each jump, independently, $S_\tau / S_{\tau-}$ has a lognormal distribution. Show that, for each fixed t, S_t has a lognormal distribution with the variance parameter σ^2 given by a multiple of a Poisson random variable.

Bibliography

Background reading:

- *Probability, an Introduction,* Geoffrey Grimmett and Dominic Welsh, Oxford University Press (1986).
- *Options, Futures and Other Derivative Securities,* John Hull, Prentice-Hall (Second edition 1993).

Grimmett & Welsh contains all the concepts that we assume from probability theory. Hull is popular with practitioners. It explains the operation of markets in some detail before turning to modelling.

Supplementary textbooks:

- *Arbitrage Theory in Continuous Time,* Tomas Björk, Oxford University Press (1998).
- *Dynamic Asset Pricing Theory,* Darrell Duffie, Princeton University Press (1992).
- *Introduction to Stochastic Calculus Applied to Finance,* Damien Lamberton and Bernard Lapeyre, translated by Nicolas Rabeau and François Mantion, Chapman and Hall (1996).
- *The Mathematics of Financial Derivatives,* Paul Wilmott, Sam Howison and Jeff Dewynne, Cambridge University Press (1995).

These all represent useful supplementary reading. The first three employ a variety of techniques while Wilmott, Howison & Dewynne is devoted exclusively to the partial differential equations approach.

Further topics in financial mathematics:

- *Financial Calculus: an Introduction to Derivatives Pricing,* Martin Baxter and Andrew Rennie, Cambridge University Press (1996).
- *Derivatives in Financial Markets with Stochastic Volatility,* Jean-Pierre Fouque, George Papanicolau and Ronnie Sircar, Cambridge University Press (2000).
- *Continuous Time Finance,* Robert Merton, Blackwell (1990).
- *Martingale Methods in Financial Modelling,* Marek Musiela and Marek Rutkowski, Springer-Verlag (1998).

Although aimed at practitioners rather than university courses, Chapter 5 of Baxter & Rennie provides a good starting point for the study of interest rates. Fouque, Papanicolau & Sircar is a highly accessible text that would provide an excellent basis for a special topic in a *second* course in financial mathematics. Merton is a synthesis of the remarkable research contributions of its Nobel-prize-winning author. Musiela & Rutkowski provides an encyclopaedic reference.

Brownian motion, martingales and stochastic calculus:

- *Introduction to Stochastic Integration,* Kai Lai Chung and Ruth Williams, Birkhäuser (Second edition 1990).
- *Stochastic Differential Equations and Diffusion Processes,* Nobuyuki Ikeda and Shinzo Watanabe, North-Holland (Second edition 1989).
- *Brownian Motion and Stochastic Calculus,* Ioannis Karatzas and Steven Shreve, Springer-Verlag (Second edition 1991).
- *Probability with Martingales,* David Williams, Cambridge University Press (1991).

Williams is an excellent introduction to discrete parameter martingales and much more (integration, conditional expectation, measure, …). The others all deal with the continuous world. Chung & Williams is short enough that it can simply be read cover to cover.
A further useful reference is *Handbook of Brownian Motion: Facts and Formulae,* Andrei Borodin and Paavo Salminen, Birkhäuser (1996).

Additional references from the text:

- Louis Bachelier, La théorie de la speculation. *Ann Sci Ecole Norm Sup* **17** (1900), 21–86. English translation in *The Random Character of Stock Prices*, Paul Cootner (ed), MIT Press (1964), reprinted Risk Books (2000)
- J Cox, S Ross and M Rubinstein, Option pricing, a simplified approach. *J Financial Econ* **7** (1979), 229–63.
- M Davis, Mathematics of financial markets, in *Mathematics Unlimited – 2001 and Beyond,* Bjorn Engquist and Wilfried Schmid (eds), Springer-Verlag (2001).
- D Freedman, *Brownian Motion and Diffusion,* Holden-Day (1971).
- J M Harrison and D M Kreps, Martingales and arbitrage in multiperiod securities markets. *J Econ Theory* **20** (1979), 381–408
- J M Harrison and S R Pliska, Martingales and stochastic integrals in the theory of continuous trading. *Stoch Proc Appl* **11** (1981), 215–60.
- F B Knight, *Essentials of Brownian Motion and Diffusion,* Mathematical Surveys, volume 18, American Mathematical Society (1981).
- T J Lyons, Uncertain volatility and the risk-free synthesis of derivatives. *Appl Math Finance* **2** (1995), 117–33.
- P Protter, *Stochastic Integration and Differential Equations,* Springer-Verlag (1990).
- D Revuz and M Yor, *Continuous Martingales and Brownian Motion,* Springer-Verlag (Third edition 1998).
- P A Samuelson, Proof that properly anticipated prices fluctuate randomly, *Industrial Management Review* **6**, (1965), 41–50.

Notation

Financial instruments and the Black–Scholes model

T, maturity time.
C_T, value of claim at time T.
$\{S_n\}_{n\geq 0}$, $\{S_t\}_{t\geq 0}$, value of the underlying stock.
K, the strike price in a vanilla option.
$(S_T - K)_+ = \max\{(S_T - K), 0\}$.
r, continuously compounded interest rate.
σ, volatility.
$\mathbb{P}$, a probability measure, usually the market measure.
$\mathbb{Q}$, a martingale measure equivalent to the market measure.
$\mathbb{E}^{\mathbb{Q}}$, the expectation under $\mathbb{Q}$.
$\frac{d\mathbb{Q}}{d\mathbb{P}}$ the Radon–Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$.
$\{\tilde{S}_t\}_{t\geq 0}$, the *discounted* value of the underlying stock. In general, for a process $\{Y_t\}_{t\geq 0}$, $\tilde{Y}_t = Y_t/B_t$ where $\{B_t\}_{t\geq 0}$ is the value of the riskless cash bond at time t.
$V(t, x)$, the value of a portfolio at time t if the stock price $S_t = x$. Also the Black–Scholes price of an option.

General probability

$(\Omega, \mathcal{F}, \mathbb{P})$, probability triple.
$\mathbb{P}[A|B]$, conditional probability of A given B.
Φ, standard normal distribution function.
$p(t, x, y)$, transition density of Brownian motion.
$X \stackrel{\mathcal{D}}{=} Y$, the random variables X and Y have the same distribution.
$Z \sim N(0, 1)$, the random variable Z has a standard normal distribution.
$\mathbb{E}[X; A]$, see Definition 2.3.4.

Martingales and other stochastic processes

$\{M_t\}_{t\geq 0}$, a martingale under some specified probability measure.
$\{[M]_t\}_{t\geq 0}$, the quadratic variation of $\{M_t\}_{t\geq 0}$.
$\{\mathcal{F}_n\}_{n\geq 0}$, $\{\mathcal{F}_t\}_{t\geq 0}$, filtration.

$\{\mathcal{F}_n^X\}_{n\geq 0}$ (resp. $\{\mathcal{F}_t^X\}_{t\geq 0}$), filtration generated by the process $\{X_n\}_{n\geq 0}$ (resp. $\{X_t\}_{t\geq 0}$).
$\mathbb{E}[X|\mathcal{F}]$, $\mathbb{E}\left[X_{n+1}\middle| X_n\right]$, conditional expectation; see pages 30ff.
$\{W_t\}_{t\geq 0}$, Brownian motion under a specified measure, usually the market measure.
$X^*(t)$, $X_*(t)$, maximum and minimum processes corresponding to $\{X_t\}_{t\geq 0}$.

Miscellaneous

$\stackrel{\Delta}{=}$, defined equal to.
$\delta(\pi)$, the mesh of the partition π.
$f\big|_x$, the function f evaluated at x.
θ^t (for a vector or matrix θ), the transpose of θ.
$x > 0$, $x \gg 0$ for a vector $x \in \mathbb{R}^n$, see page 11.

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