with $\dot{y} = dy/dt$, etc., constant $A, B, C, D, K$, and $t^2 + At + B = (t - t_1)(t - t_2)$, $t_1 \neq t_2$, can be reduced to the hypergeometric equation with independent variable

$$x = \frac{t - t_1}{t_2 - t_1}$$

and parameters related by $Ct_1 + D = -c(t_2 - t_1)$, $C = a + b + 1$, $K = ab$. From this you see that (15) is a "normalized form" of the more general (18) and that various cases of (18) can thus be solved in terms of hypergeometric functions.

**15–20 HYPERGEOMETRIC ODE**

Find a general solution in terms of hypergeometric functions.

**15.** $2x(1 - x)y'' - (1 + 6x)y' - 2y = 0$

**16.** $x(1 - x)y'' + (\frac{1}{2} + 2x)y' - 2y = 0$

**17.** $4x(1 - x)y'' + y' + 8y = 0$

**18.** $4(t^2 - 3t + 2)\ddot{y} - 2\dot{y} + y = 0$

**19.** $2(t^2 - 5t + 6)\ddot{y} + (2t - 3)\dot{y} - 8y = 0$

**20.** $3t(1 + t)\ddot{y} + t\dot{y} - y = 0$

## 5.4 Bessel's Equation. Bessel Functions $J_\nu(x)$

One of the most important ODEs in applied mathematics in **Bessel's equation**,[6]

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{1}$$

where the parameter $\nu$ (nu) is a given real number which is positive or zero. Bessel's equation often appears if a problem shows cylindrical symmetry, for example, as the membranes in Sec.12.9. The equation satisfies the assumptions of Theorem 1. To see this, divide (1) by $x^2$ to get the standard form $y'' + y'/x + (1 - \nu^2/x^2)y = 0$. Hence, according to the Frobenius theory, it has a solution of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad (a_0 \neq 0). \tag{2}$$

Substituting (2) and its first and second derivatives into Bessel's equation, we obtain

$$\sum_{m=0}^{\infty} (m + r)(m + r - 1)a_m x^{m+r} + \sum_{m=0}^{\infty} (m + r)a_m x^{m+r}$$
$$+ \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

We equate the sum of the coefficients of $x^{s+r}$ to zero. Note that this power $x^{s+r}$ corresponds to $m = s$ in the first, second, and fourth series, and to $m = s - 2$ in the third series. Hence for $s = 0$ and $s = 1$, the third series does not contribute since $m \geqq 0$.

[6]FRIEDRICH WILHELM BESSEL (1784–1846), German astronomer and mathematician, studied astronomy on his own in his spare time as an apprentice of a trade company and finally became director of the new Königsberg Observatory.

Formulas on Bessel functions are contained in Ref. [GenRef10] and the standard treatise [A13].

For $s = 2, 3, \cdots$ all four series contribute, so that we get a general formula for all these $s$. We find

$$\text{(3)} \quad \begin{aligned} &\text{(a)} \quad r(r-1)a_0 + ra_0 - \nu^2 a_0 = 0 && (s = 0) \\ &\text{(b)} \quad (r+1)ra_1 + (r+1)a_1 - \nu^2 a_1 = 0 && (s = 1) \\ &\text{(c)} \quad (s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - \nu^2 a_s = 0 && (s = 2, 3, \cdots). \end{aligned}$$

From (3a) we obtain the **indicial equation** by dropping $a_0$,

$$\textbf{(4)} \qquad (r + \nu)(r - \nu) = 0.$$

The roots are $r_1 = \nu\ (\geqq 0)$ and $r_2 = -\nu$.

**Coefficient Recursion for $r = r_1 = \nu$.** For $r = \nu$, Eq. (3b) reduces to $(2\nu + 1)a_1 = 0$. Hence $a_1 = 0$ since $\nu \geqq 0$. Substituting $r = \nu$ in (3c) and combining the three terms containing $a_s$ gives simply

$$\text{(5)} \qquad (s + 2\nu)sa_s + a_{s-2} = 0.$$

Since $a_1 = 0$ and $\nu \geqq 0$, it follows from (5) that $a_3 = 0, a_5 = 0, \cdots$. Hence we have to deal only with *even-numbered* coefficients $a_s$ with $s = 2m$. For $s = 2m$, Eq. (5) becomes

$$(2m + 2\nu)2ma_{2m} + a_{2m-2} = 0.$$

Solving for $a_{2m}$ gives the recursion formula

$$\textbf{(6)} \qquad a_{2m} = -\frac{1}{2^2 m(\nu + m)}\, a_{2m-2}, \qquad m = 1, 2, \cdots.$$

From (6) we can now determine $a_2, a_4, \cdots$ successively. This gives

$$a_2 = -\frac{a_0}{2^2(\nu + 1)}$$

$$a_4 = -\frac{a_2}{2^2 2(\nu + 2)} = \frac{a_0}{2^4 2!\,(\nu + 1)(\nu + 2)}$$

and so on, and in general

$$\textbf{(7)} \qquad a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(\nu + 1)(\nu + 2)\cdots(\nu + m)}, \qquad m = 1, 2, \cdots.$$

## Bessel Functions $J_n(x)$ for Integer $\nu = n$

***Integer values of $\nu$ are denoted by $n$.*** This is standard. For $\nu = n$ the relation (7) becomes

$$\text{(8)} \qquad a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(n + 1)(n + 2)\cdots(n + m)}, \qquad m = 1, 2, \cdots.$$

$a_0$ is still arbitrary, so that the series (2) with these coefficients would contain this arbitrary factor $a_0$. This would be a highly impractical situation for developing formulas or computing values of this new function. Accordingly, we have to make a choice. The choice $a_0 = 1$ would be possible. A simpler series (2) could be obtained if we could absorb the growing product $(n + 1)(n + 2) \cdots (n + m)$ into a factorial function $(n + m)!$ What should be our choice? Our choice should be

$$a_0 = \frac{1}{2^n n!} \tag{9}$$

because then $n!\,(n + 1) \cdots (n + m) = (n + m)!$ in (8), so that (8) simply becomes

$$a_{2m} = \frac{(-1)^m}{2^{2m+n} m!\,(n + m)!}, \qquad m = 1, 2, \cdots. \tag{10}$$

By inserting these coefficients into (2) and remembering that $c_1 = 0$, $c_3 = 0, \cdots$ we obtain a particular solution of Bessel's equation that is denoted by $J_n(x)$:

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n + m)!} \qquad (n \geqq 0). \tag{11}$$

$J_n(x)$ is called the **Bessel function of the first kind** *of order n.* The series (11) converges for all $x$, as the ratio test shows. Hence $J_n(x)$ is defined for all $x$. The series converges very rapidly because of the factorials in the denominator.

### EXAMPLE 1 Bessel Functions $J_0(x)$ and $J_1(x)$

For $n = 0$ we obtain from (11) the **Bessel function of order 0**

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + - \cdots \tag{12}$$

which looks similar to a cosine (Fig. 110). For $n = 1$ we obtain the **Bessel function of order 1**

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!\,(m + 1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!\,2!} + \frac{x^5}{2^5 2!\,3!} - \frac{x^7}{2^7 3!\,4!} + - \cdots, \tag{13}$$

which looks similar to a sine (Fig. 110). But the zeros of these functions are not completely regularly spaced (see also Table A1 in App. 5) and the height of the "waves" decreases with increasing $x$. Heuristically, $n^2/x^2$ in (1) in standard form [(1) divided by $x^2$] is zero (if $n = 0$) or small in absolute value for large $x$, and so is $y'/x$, so that then Bessel's equation comes close to $y'' + y = 0$, the equation of $\cos x$ and $\sin x$; also $y'/x$ acts as a "damping term," in part responsible for the decrease in height. One can show that for large $x$,

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{14}$$

where $\sim$ is read "**asymptotically equal**" and means that *for fixed n* the quotient of the two sides approaches 1 as $x \to \infty$.

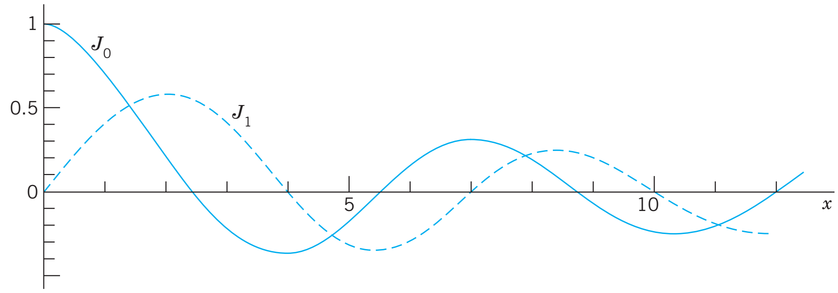


**Fig. 110.** Bessel functions of the first kind $J_0$ and $J_1$

Formula (14) is surprisingly accurate even for smaller $x\,(>0)$. For instance, it will give you good starting values in a computer program for the basic task of computing zeros. For example, for the first three zeros of $J_0$ you obtain the values 2.356 (2.405 exact to 3 decimals, error 0.049), 5.498 (5.520, error 0.022), 8.639 (8.654, error 0.015), etc. ■

## Bessel Functions $J_\nu(x)$ for any $\nu \geqq 0$. Gamma Function

We now proceed from integer $\nu = n$ to any $\nu \geqq 0$. We had $a_0 = 1/(2^n n!)$ in (9). So we have to extend the factorial function $n!$ to any $\nu \geqq 0$. For this we choose

$$a_0 = \frac{1}{2^\nu \Gamma(\nu + 1)} \tag{15}$$

with the **gamma function** $\Gamma(\nu + 1)$ defined by

$$\Gamma(\nu + 1) = \int_0^\infty e^{-t} t^\nu \, dt \qquad (\nu > -1). \tag{16}$$

(**CAUTION!** Note the convention $\nu + 1$ on the left but $\nu$ in the integral.) Integration by parts gives

$$\Gamma(\nu + 1) = -e^{-t} t^\nu \Big|_0^\infty + \nu \int_0^\infty e^{-t} t^{\nu - 1} \, dt = 0 + \nu\Gamma(\nu).$$

This is the basic functional relation of the gamma function

$$\Gamma(\nu + 1) = \nu\Gamma(\nu). \tag{17}$$

Now from (16) with $\nu = 0$ and then by (17) we obtain

$$\Gamma(1) = \int_0^\infty e^{-t} \, dt = -e^{-t} \Big|_0^\infty = 0 - (-1) = 1$$

and then $\Gamma(2) = 1 \cdot \Gamma(1) = 1!$, $\Gamma(3) = 2\Gamma(1) = 2!$ and in general

$$\Gamma(n + 1) = n! \qquad (n = 0, 1, \cdots). \tag{18}$$

Hence ***the gamma function generalizes the factorial function*** **to arbitrary positive** $\nu$. Thus (15) with $\nu = n$ agrees with (9).

Furthermore, from (7) with $a_0$ given by (15) we first have

$$a_{2m} = \frac{(-1)^m}{2^{2m} m!\,(\nu+1)(\nu+2)\cdots(\nu+m)2^\nu\Gamma(\nu+1)}.$$

Now (17) gives $(\nu+1)\Gamma(\nu+1) = \Gamma(\nu+2)$, $(\nu+2)\Gamma(\nu+2) = \Gamma(\nu+3)$ and so on, so that

$$(\nu+1)(\nu+2)\cdots(\nu+m)\Gamma(\nu+1) = \Gamma(\nu+m+1).$$

Hence because of our (standard!) choice (15) of $a_0$ the coefficients (7) are simply

$$a_{2m} = \frac{(-1)^m}{2^{2m+\nu} m!\,\Gamma(\nu+m+1)}. \tag{19}$$

With these coefficients and $r = r_1 = \nu$ we get from (2) a particular solution of (1), denoted by $J_\nu(x)$ and given by

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\,\Gamma(\nu+m+1)}. \tag{20}$$

$J_\nu(x)$ is called the **Bessel function of the first kind of order $\nu$**. The series (20) converges for all $x$, as one can verify by the ratio test.

## Discovery of Properties from Series

Bessel functions are a model case for showing how to discover properties and relations of functions from series by which they are *defined*. Bessel functions satisfy an incredibly large number of relationships—look at Ref. [A13] in App. 1; also, find out what your CAS knows. In Theorem 3 we shall discuss four formulas that are backbones in applications and theory.

**THEOREM 1**

**Derivatives, Recursions**

*The derivative of $J_\nu(x)$ with respect to $x$ can be expressed by $J_{\nu-1}(x)$ or $J_{\nu+1}(x)$ by the formulas*

(21)

$$\textbf{(a)}\quad [x^\nu J_\nu(x)]' = x^\nu J_{\nu-1}(x)$$

$$\textbf{(b)}\quad [x^{-\nu} J_\nu(x)]' = -x^{-\nu} J_{\nu+1}(x).$$

*Furthermore, $J_\nu(x)$ and its derivative satisfy the recurrence relations*

(21)

$$\textbf{(c)}\quad J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$$

$$\textbf{(d)}\quad J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x).$$

**PROOF** **(a)** We multiply (20) by $x^{\nu}$ and take $x^{2\nu}$ under the summation sign. Then we have

$$x^{\nu}J_{\nu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2\nu}}{2^{2m+\nu} m!\, \Gamma(\nu + m + 1)}.$$

We now differentiate this, cancel a factor 2, pull $x^{2\nu-1}$ out, and use the functional relationship $\Gamma(\nu + m + 1) = (\nu + m)\Gamma(\nu + m)$ [see (17)]. Then (20) with $\nu - 1$ instead of $\nu$ shows that we obtain the right side of (21a). Indeed,

$$(x^{\nu}J_{\nu})' = \sum_{m=0}^{\infty} \frac{(-1)^m 2(m + \nu)x^{2m+2\nu-1}}{2^{2m+\nu} m!\, \Gamma(\nu + m + 1)} = x^{\nu}x^{\nu-1} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu-1} m!\, \Gamma(\nu + m)}.$$

**(b)** Similarly, we multiply (20) by $x^{-\nu}$, so that $x^{\nu}$ in (20) cancels. Then we differentiate, cancel $2m$, and use $m! = m(m - 1)!$. This gives, with $m = s + 1$,

$$(x^{-\nu}J_{\nu})' = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m+\nu-1}(m - 1)!\, \Gamma(\nu + m + 1)} = \sum_{s=0}^{\infty} \frac{(-1)^{s+1} x^{2s+1}}{2^{2s+\nu+1} s!\, \Gamma(\nu + s + 2)}.$$

Equation (20) with $\nu + 1$ instead of $\nu$ and $s$ instead of $m$ shows that the expression on the right is $-x^{-\nu}J_{\nu+1}(x)$. This proves (21b).

**(c), (d)** We perform the differentiation in (21a). Then we do the same in (21b) and multiply the result on both sides by $x^{2\nu}$. This gives

$$\text{(a*)} \qquad \nu x^{\nu-1}J_{\nu} + x^{\nu}J_{\nu}' = x^{\nu}J_{\nu-1}$$

$$\text{(b*)} \qquad -\nu x^{\nu-1}J_{\nu} + x^{\nu}J_{\nu}' = -x^{\nu}J_{\nu+1}.$$

Substracting (b*) from (a*) and dividing the result by $x^{\nu}$ gives (21c). Adding (a*) and (b*) and dividing the result by $x^{\nu}$ gives (21d). ■

**EXAMPLE 2** **Application of Theorem 1 in Evaluation and Integration**

Formula (21c) can be used recursively in the form

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x) - J_{\nu-1}(x)$$

for calculating Bessel functions of higher order from those of lower order. For instance, $J_2(x) = 2J_1(x)/x - J_0(x)$, so that $J_2$ can be obtained from tables of $J_0$ and $J_1$ (in App. 5 or, more accurately, in Ref. [GenRef1] in App. 1).

To illustrate how Theorem 1 helps in integration, we use (21b) with $\nu = 3$ integrated on both sides. This evaluates, for instance, the integral

$$I = \int_1^2 x^{-3}J_4(x)\, dx = -x^{-3}J_3(x)\Big|_1^2 = -\frac{1}{8} J_3(2) + J_3(1).$$

A table of $J_3$ (on p. 398 of Ref. [GenRef1]) or your CAS will give you

$$-\tfrac{1}{8} \cdot 0.128943 + 0.019563 = 0.003445.$$

Your CAS (or a human computer in precomputer times) obtains $J_3$ from (21), first using (21c) with $\nu = 2$, that is, $J_3 = 4x^{-1}J_2 - J_1$, then (21c) with $\nu = 1$, that is, $J_2 = 2x^{-1}J_1 - J_0$. Together,

$$I = x^{-3}(4x^{-1}(2x^{-1}J_1 - J_0) - J_1)\Big|_1^2$$
$$= -\tfrac{1}{8}[2J_1(2) - 2J_0(2) - J_1(2)] + [8J_1(1) - 4J_0(1) - J_1(1)]$$
$$= -\tfrac{1}{8}J_1(2) + \tfrac{1}{4}J_0(2) + 7J_1(1) - 4J_0(1).$$

This is what you get, for instance, with Maple if you type int($\cdots$). And if you type evalf(int($\cdots$)), you obtain 0.003445448, in agreement with the result near the beginning of the example. ■

## Bessel Functions $J_\nu$ with Half-Integer $\nu$ Are Elementary

We discover this remarkable fact as another property obtained from the series (20) and confirm it in the problem set by using Bessel's ODE.

**EXAMPLE 3** **Elementary Bessel Functions $J_\nu$ with $\nu = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots$. The Value $\Gamma(\frac{1}{2})$**

We first prove (Fig. 111)

$$\text{(22)}\qquad \text{(a)}\quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \qquad \text{(b)}\quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$$

The series (20) with $\nu = \frac{1}{2}$ is

$$J_{1/2}(x) = \sqrt{x}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+1/2}m!\,\Gamma(m+\frac{3}{2})} = \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2^{2m+1}m!\,\Gamma(m+\frac{3}{2})}.$$

The denominator can be written as a product $AB$, where (use (16) in $B$)

$$A = 2^m m! = 2m(2m-2)(2m-4)\cdots 4\cdot 2,$$
$$B = 2^{m+1}\Gamma(m+\tfrac{3}{2}) = 2^{m+1}(m+\tfrac{1}{2})(m-\tfrac{1}{2})\cdots\tfrac{3}{2}\cdot\tfrac{1}{2}\Gamma(\tfrac{1}{2})$$
$$= (2m+1)(2m-1)\cdots 3\cdot 1\cdot\sqrt{\pi};$$

here we used (proof below)

$$\text{(23)}\qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

The product of the right sides of $A$ and $B$ can be written

$$AB = (2m+1)2m(2m-1)\cdots 3\cdot 2\cdot 1\sqrt{\pi} = (2m+1)!\sqrt{\pi}.$$

Hence

$$J_{1/2}(x) - \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!} = \sqrt{\frac{2}{\pi x}}\sin x.$$

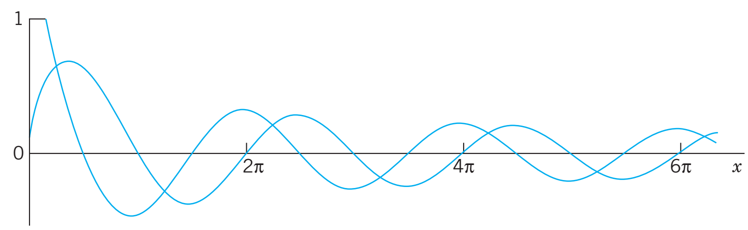


**Fig. 111.** Bessel functions $J_{1/2}$ and $J_{-1/2}$

This proves (22a). Differentiation and the use of (21a) with $\nu = \frac{1}{2}$ now gives

$$[\sqrt{x} J_{1/2}(x)]' = \sqrt{\frac{2}{\pi}} \cos x = x^{1/2} J_{-1/2}(x).$$

This proves (22b). From (22) follow further formulas successively by (21c), used as in Example 2.

We finally prove $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ by a standard trick worth remembering. In (15) we set $t = u^2$. Then $dt = 2u\,du$ and

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-1/2}\, dt = 2\int_0^\infty e^{-u^2}\, du.$$

We square on both sides, write $v$ instead of $u$ in the second integral, and then write the product of the integrals as a double integral:

$$\Gamma\left(\frac{1}{2}\right)^2 = 4\int_0^\infty e^{-u^2}\, du \int_0^\infty e^{-v^2}\, dv = 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)}\, du\, dv.$$

We now use polar coordinates $r, \theta$ by setting $u = r\cos\theta$, $v = r\sin\theta$. Then the element of area is $du\,dv = r\,dr\,d\theta$ and we have to integrate over $r$ from 0 to $\infty$ and over $\theta$ from 0 to $\pi/2$ (that is, over the first quadrant of the $uv$-plane):

$$\Gamma\left(\frac{1}{2}\right)^2 = 4\int_0^{\pi/2} \int_0^\infty e^{-r^2} r\, dr\, d\theta = 4 \cdot \frac{\pi}{2} \int_0^\infty e^{-r^2} r\, dr = 2\left(-\frac{1}{2}\right) e^{-r^2} \Big|_0^\infty = \pi.$$

By taking the square root on both sides we obtain (23). ■

## General Solution. Linear Dependence

For a general solution of Bessel's equation (1) in addition to $J_\nu$ we need a second linearly independent solution. For $\nu$ not an integer this is easy. Replacing $\nu$ by $-\nu$ in (20), we have

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m!\, \Gamma(m - \nu + 1)}. \tag{24}$$

Since Bessel's equation involves $\nu^2$, the functions $J_\nu$ and $J_{-\nu}$ are solutions of the equation for the same $\nu$. If $\nu$ is not an integer, they are linearly independent, because the first terms in (20) and in (24) are finite nonzero multiples of $x^\nu$ and $x^{-\nu}$. Thus, if $\nu$ is not an integer, a general solution of Bessel's equation for all $x \neq 0$ is

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

This cannot be the general solution for an integer $\nu = n$ because, in that case, we have linear dependence. It can be seen that the first terms in (20) and (24) are finite nonzero multiples of $x^\nu$ and $x^{-\nu}$, respectively. This means that, for any integer $\nu = n$, we have linear dependence because

$$J_{-n}(x) = (-1)^n J_n(x) \qquad (n = 1, 2, \cdots). \tag{25}$$

**PROOF** To prove (25), we use (24) and let $\nu$ approach a positive integer $n$. Then the gamma function in the coefficients of the first $n$ terms becomes infinite (see Fig. 553 in App. A3.1), the coefficients become zero, and the summation starts with $m = n$. Since in this case $\Gamma(m - n + 1) = (m - n)!$ by (18), we obtain

$$(26) \qquad J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m!\,(m-n)!} = \sum_{s=0}^{\infty} \frac{(-1)^{n+s} x^{2s+n}}{2^{2s+n}(n+s)!\,s!} \qquad (m = n + s).$$

The last series represents $(-1)^n J_n(x)$, as you can see from (11) with $m$ replaced by $s$. This completes the proof. ■

The difficulty caused by (25) will be overcome in the next section by introducing further Bessel functions, called *of the second kind* and denoted by $Y_\nu$.

## PROBLEM SET 5.4

1. **Convergence.** Show that the series (11) converges for all $x$. Why is the convergence very rapid?

### 2–10 ODEs REDUCIBLE TO BESSEL'S ODE

This is just a sample of such ODEs; some more follow in the next problem set. Find a general solution in terms of $J_\nu$ and $J_{-\nu}$ or indicate when this is not possible. Use the indicated substitutions. Show the details of your work.

2. $x^2y'' + xy' + (x^2 - \frac{4}{49})y = 0$
3. $xy'' + y' + \frac{1}{4}y = 0 \quad (\sqrt{x} = z)$
4. $y'' + (e^{-2x} - \frac{1}{9})y = 0 \quad (e^{-x} = z)$
5. **Two-parameter ODE**
   $x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0 \quad (\lambda x = z)$
6. $x^2y'' + \frac{1}{4}(x + \frac{3}{4})y = 0 \quad (y = u\sqrt{x}, \sqrt{x} = z)$
7. $x^2y'' + xy' + \frac{1}{4}(x^2 - 1)y = 0 \quad (x = 2z)$
8. $(2x + 1)^2y'' + 2(2x + 1)y' + 16x(x + 1)y = 0$
   $(2x + 1 = z)$
9. $xy'' + (2\nu + 1)y' + xy = 0 \quad (y = x^{-\nu}u)$
10. $x^2y'' + (1 - 2\nu)xy' + \nu^2(x^{2\nu} + 1 - \nu^2)y = 0$
    $(y = x^\nu u, x^\nu = z)$
11. **CAS EXPERIMENT. Change of Coefficient.** Find and graph (on common axes) the solutions of

$$y'' + kx^{-1}y' + y = 0,\ y(0) = 1,\ y'(0) = 0,$$

for $k = 0, 1, 2, \cdots, 10$ (or as far as you get useful graphs). For what $k$ do you get elementary functions? Why? Try for noninteger $k$, particularly between 0 and 2, to see the continuous change of the curve. Describe the change of the location of the zeros and of the extrema as $k$ increases from 0. Can you interpret the ODE as a model in mechanics, thereby explaining your observations?

12. **CAS EXPERIMENT. Bessel Functions for Large $x$.**

**(a)** Graph $J_n(x)$ for $n = 0, \cdots, 5$ on common axes.

**(b)** Experiment with (14) for integer $n$. Using graphs, find out from which $x = x_n$ on the curves of (11) and (14) practically coincide. How does $x_n$ change with $n$?

**(c)** What happens in (b) if $n = \pm\frac{1}{2}$? (Our usual notation in this case would be $\nu$.)

**(d)** How does the error of (14) behave as a function of $x$ for fixed $n$? [Error = exact value minus approximation (14).]

**(e)** Show from the graphs that $J_0(x)$ has extrema where $J_1(x) = 0$. Which formula proves this? Find further relations between zeros and extrema.

### 13–15 ZEROS

of Bessel functions play a key role in modeling (e.g. of vibrations; see Sec. 12.9).

13. **Interlacing of zeros.** Using (21) and Rolle's theorem, show that between any two consecutive positive zeros of $J_n(x)$ there is precisely one zero of $J_{n+1}(x)$.
14. **Zeros.** Compute the first four positive zeros of $J_0(x)$ and $J_1(x)$ from (14). Determine the error and comment.
15. **Interlacing of zeros.** Using (21) and Rolle's theorem, show that between any two consecutive zeros of $J_0(x)$ there is precisely one zero of $J_1(x)$.

### 16–18 HALF-INTEGER PARAMETER: APPROACH BY THE ODE

16. **Elimination of first derivative.** Show that $y = uv$ with $v(x) = \exp(-\frac{1}{2}\int p(x)\,dx)$ gives from the ODE $y'' + p(x)y' + q(x)y = 0$ the ODE

$$u'' + \left[q(x) - \tfrac{1}{4}p(x)^2 - \tfrac{1}{2}p'(x)\right]u = 0,$$

not containing the first derivative of $u$.

**17. Bessel's equation.** Show that for (1) the substitution in Prob. 16 is $y = ux^{-1/2}$ and gives

$$x^2u'' + (x^2 + \tfrac{1}{4} - \nu^2)u = 0. \tag{27}$$

**18. Elementary Bessel functions.** Derive (22) in Example 3 from (27).

### 19–25 APPLICATION OF (21): DERIVATIVES, INTEGRALS

Use the powerful formulas (21) to do Probs. 19–25. Show the details of your work.

**19. Derivatives.** Show that $J_0'(x) = -J_1(x)$, $J_1'(x) = J_0(x) - J_1(x)/x$, $J_2'(x) = \frac{1}{2}[J_1(x) - J_3(x)]$.

**20. Bessel's equation.** Derive (1) from (21).

**21. Basic integral formula.** Show that

$$\int x^\nu J_{\nu-1}(x)\,dx = x^\nu J_\nu(x) + c.$$

**22. Basic integral formulas.** Show that

$$\int x^{-\nu} J_{\nu+1}(x)\,dx = -x^{-\nu} J_\nu(x) + c,$$

$$\int J_{\nu+1}(x)\,dx = \int J_{\nu-1}(x)\,dx - 2J_\nu(x).$$

**23. Integration.** Show that $\int x^2 J_0(x)\,dx = x^2 J_1(x) + xJ_0(x) - \int J_0(x)\,dx$. (The last integral is nonelementary; tables exist, e.g., in Ref. [A13] in App. 1.)

**24. Integration.** Evaluate $\int x^{-1} J_4(x)\,dx$.

**25. Integration.** Evaluate $\int J_5(x)\,dx$.

# 5.5 Bessel Functions $Y_\nu(x)$. General Solution

To obtain a general solution of Bessel's equation (1), Sec. 5.4, for any $\nu$, we now introduce **Bessel functions of the second kind** $Y_\nu(x)$, beginning with the case $\nu = n = 0$.

When $n = 0$, Bessel's equation can be written (divide by $x$)

$$xy'' + y' + xy = 0. \tag{1}$$

Then the indicial equation (4) in Sec. 5.4 has a double root $r = 0$. This is Case 2 in Sec. 5.3. In this case we first have only one solution, $J_0(x)$. From (8) in Sec. 5.3 we see that the desired second solution must be of the form

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} A_m x^m. \tag{2}$$

We substitute $y_2$ and its derivatives

$$y_2' = J_0' \ln x + \frac{J_0}{x} + \sum_{m=1}^{\infty} mA_m x^{m-1}$$

$$y_2'' = J_0'' \ln x + \frac{2J_0'}{x} - \frac{J_0}{x^2} + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-2}$$

into (1). Then the sum of the three logarithmic terms $xJ_0'' \ln x$, $J_0' \ln x$, and $xJ_0 \ln x$ is zero because $J_0$ is a solution of (1). The terms $-J_0/x$ and $J_0/x$ (from $xy''$ and $y'$) cancel. Hence we are left with

$$2J_0' + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-1} + \sum_{m=1}^{\infty} mA_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0.$$

Addition of the first and second series gives $\Sigma m^2 A_m x^{m-1}$. The power series of $J_0'(x)$ is obtained from (12) in Sec. 5.4 and the use of $m!/m = (m-1)!$ in the form

$$J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m 2m x^{2m-1}}{2^{2m}(m!)^2} = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m!\,(m-1)!}.$$

Together with $\Sigma m^2 A_m x^{m-1}$ and $\Sigma A_m x^{m+1}$ this gives

$$\text{(3*)} \qquad \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2} m!\,(m-1)!} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0.$$

First, we show that the $A_m$ with odd subscripts are all zero. The power $x^0$ occurs only in the second series, with coefficient $A_1$. Hence $A_1 = 0$. Next, we consider the even powers $x^{2s}$. The first series contains none. In the second series, $m - 1 = 2s$ gives the term $(2s+1)^2 A_{2s+1} x^{2s}$. In the third series, $m + 1 = 2s$. Hence by equating the sum of the coefficients of $x^{2s}$ to zero we have

$$(2s+1)^2 A_{2s+1} + A_{2s-1} = 0, \qquad s = 1, 2, \cdots.$$

Since $A_1 = 0$, we thus obtain $A_3 = 0, A_5 = 0, \cdots$, successively.

We now equate the sum of the coefficients of $x^{2s+1}$ to zero. For $s = 0$ this gives

$$-1 + 4A_2 = 0, \qquad \text{thus} \qquad A_2 = \tfrac{1}{4}.$$

For the other values of $s$ we have in the first series in (3*) $2m - 1 = 2s + 1$, hence $m = s + 1$, in the second $m - 1 = 2s + 1$, and in the third $m + 1 = 2s + 1$. We thus obtain

$$\frac{(-1)^{s+1}}{2^{2s}(s+1)!\,s!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0.$$

For $s = 1$ this yields

$$\tfrac{1}{8} + 16A_4 + A_2 = 0, \qquad \text{thus} \qquad A_4 = -\tfrac{3}{128}$$

and in general

$$\text{(3)} \qquad A_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right), \qquad m = 1, 2, \cdots.$$

Using the short notations

$$\text{(4)} \qquad h_1 = 1 \qquad h_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m} \qquad m = 2, 3, \cdots$$

and inserting (4) and $A_1 = A_3 = \cdots = 0$ into (2), we obtain the result

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m}$$

$$\text{(5)} \qquad = J_0(x)\ln x + \frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13{,}824}x^6 - + \cdots.$$

Since $J_0$ and $y_2$ are linearly independent functions, they form a basis of (1) for $x > 0$. Of course, another basis is obtained if we replace $y_2$ by an independent particular solution of the form $a(y_2 + bJ_0)$, where $a\ (\neq 0)$ and $b$ are constants. It is customary to choose $a = 2/\pi$ and $b = \gamma - \ln 2$, where the number $\gamma = 0.57721566490 \cdots$ is the so-called **Euler constant**, which is defined as the limit of

$$1 + \frac{1}{2} + \cdots + \frac{1}{s} - \ln s$$

as $s$ approaches infinity. The standard particular solution thus obtained is called the **Bessel function of the second kind** *of order zero* (Fig. 112) or **Neumann's function** *of order zero* and is denoted by $Y_0(x)$. Thus [see (4)]

$$\text{(6)} \qquad Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2} + \gamma\right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2} x^{2m}\right].$$

For small $x > 0$ the function $Y_0(x)$ behaves about like $\ln x$ (see Fig. 112, why?), and $Y_0(x) \to -\infty$ as $x \to 0$.

## Bessel Functions of the Second Kind $Y_n(x)$

For $\nu = n = 1, 2, \cdots$ a second solution can be obtained by manipulations similar to those for $n = 0$, starting from (10), Sec. 5.4. It turns out that in these cases the solution also contains a logarithmic term.

The situation is not yet completely satisfactory, because the second solution is defined differently, depending on whether the order $\nu$ is an integer or not. To provide uniformity of formalism, it is desirable to adopt a form of the second solution that is valid for all values of the order. For this reason we introduce a standard second solution $Y_\nu(x)$ defined for all $\nu$ by the formula

$$\text{(7)} \qquad \begin{aligned} &\text{(a)} \quad Y_\nu(x) = \frac{1}{\sin \nu\pi}\left[J_\nu(x)\cos \nu\pi - J_{-\nu}(x)\right] \\ &\text{(b)} \quad Y_n(x) = \lim_{\nu \to n} Y_\nu(x). \end{aligned}$$

This function is called the **Bessel function of the second kind** *of order* $\nu$ or **Neumann's function**[7] *of order* $\nu$. Figure 112 shows $Y_0(x)$ and $Y_1(x)$.

Let us show that $J_\nu$ and $Y_\nu$ are indeed linearly independent for all $\nu$ (and $x > 0$).

For noninteger order $\nu$, the function $Y_\nu(x)$ is evidently a solution of Bessel's equation because $J_\nu(x)$ and $J_{-\nu}(x)$ are solutions of that equation. Since for those $\nu$ the solutions $J_\nu$ and $J_{-\nu}$ are linearly independent and $Y_\nu$ involves $J_{-\nu}$, the functions $J_\nu$ and $Y_\nu$ are

[7] CARL NEUMANN (1832–1925), German mathematician and physicist. His work on potential theory using integer equation methods inspired VITO VOLTERRA (1800–1940) of Rome, ERIK IVAR FREDHOLM (1866–1927) of Stockholm, and DAVID HILBERT (1962–1943) of Göttingen (see the footnote in Sec. 7.9) to develop the field of integral equations. For details see Birkhoff, G. and E. Kreyszig, The Establishment of Functional Analysis, *Historia Mathematica* 11 (1984), pp. 258–321.

The solutions $Y_\nu(x)$ are sometimes denoted by $N_\nu(x)$; in Ref. [A13] they are called **Weber's functions**; Euler's constant in (6) is often denoted by $C$ or $\ln \gamma$.

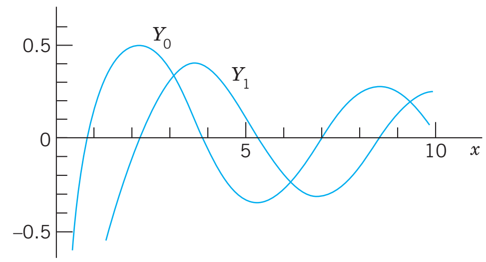


**Fig. 112.** Bessel functions of the second kind $Y_0$ and $Y_1$. (For a small table, see App. 5.)

linearly independent. Furthermore, it can be shown that the limit in (7b) exists and $Y_n$ is a solution of Bessel's equation for integer order; see Ref. [A13] in App. 1. We shall see that the series development of $Y_n(x)$ contains a logarithmic term. Hence $J_n(x)$ and $Y_n(x)$ are linearly independent solutions of Bessel's equation. The series development of $Y_n(x)$ can be obtained if we insert the series (20) in Sec. 5.4 and (2) in this section for $J_\nu(x)$ and $J_{-\nu}(x)$ into (7a) and then let $\nu$ approach $n$; for details see Ref. [A13]. The result is

$$Y_n(x) = \frac{2}{\pi} J_n(x)\left(\ln\frac{x}{2} + \gamma\right) + \frac{x^n}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}(h_m + h_{m+n})}{2^{2m+n}m!\,(m+n)!}x^{2m} \tag{8}$$
$$- \frac{x^{-n}}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{2^{2m-n}m!}x^{2m}$$

where $x > 0$, $n = 0, 1, \cdots$, and [as in (4)] $h_0 = 0, \quad h_1 = 1,$

$$h_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}, \qquad h_{m+n} = 1 + \frac{1}{2} + \cdots + \frac{1}{m+n}.$$

For $n = 0$ the last sum in (8) is to be replaced by 0 [giving agreement with (6)].

Furthermore, it can be shown that

$$Y_{-n}(x) = (-1)^n Y_n(x).$$

Our main result may now be formulated as follows.

**THEOREM 1**

**General Solution of Bessel's Equation**

*A general solution of Bessel's equation for all values of $\nu$ (and $x > 0$) is*

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x). \tag{9}$$

We finally mention that there is a practical need for solutions of Bessel's equation that are complex for real values of $x$. For this purpose the solutions

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + iY_\nu(x) \\ H_\nu^{(2)}(x) &= J_\nu(x) - iY_\nu(x) \end{aligned} \tag{10}$$

are frequently used. These linearly independent functions are called **Bessel functions of the third kind** *of order* $\nu$ or *first and second* **Hankel functions**[8] *of order* $\nu$.

This finishes our discussion on Bessel functions, except for their "orthogonality," which we explain in Sec. 11.6. Applications to vibrations follow in Sec. 12.10.

## PROBLEM SET 5.5

### 1–9 FURTHER ODE's REDUCIBLE TO BESSEL'S ODE

Find a general solution in terms of $J_\nu$ and $Y_\nu$. Indicate whether you could also use $J_{-\nu}$ instead of $Y_\nu$. Use the indicated substitution. Show the details of your work.

1. $x^2y'' + xy' + (x^2 - 16)\,y = 0$
2. $xy'' + 5y' + xy = 0 \quad (y = u/x^2)$
3. $9x^2y'' + 9xy' + (36x^4 - 16)y = 0 \quad (x^2 = z)$
4. $y'' + xy = 0 \quad (y = u\sqrt{x}, \frac{2}{3}x^{3/2} = z)$
5. $4xy'' + 4y' + y = 0 \quad (\sqrt{x} = z)$
6. $xy'' + y' + 36y = 0 \quad (12\sqrt{x} = z)$
7. $y'' + k^2x^2y = 0 \quad (y = u\sqrt{x}, \frac{1}{2}kx^2 = z)$
8. $y'' + k^2x^4y = 0 \quad (y = u\sqrt{x}, \frac{1}{3}kx^3 = z)$
9. $xy'' - 5y' + xy = 0 \quad (y = x^3u)$
10. **CAS EXPERIMENT. Bessel Functions for Large $x$.** It can be shown that for large $x$,

$$(11) \qquad Y_n(x) \sim \sqrt{2/(\pi x)}\,\sin\,(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)$$

with $\sim$ defined as in (14) of Sec. 5.4.

**(a)** Graph $Y_n(x)$ for $n = 0, \cdots, 5$ on common axes. Are there relations between zeros of one function and extrema of another? For what functions?

**(b)** Find out from graphs from which $x = x_n$ on the curves of (8) and (11) (both obtained from your CAS) practically coincide. How does $x_n$ change with $n$?

**(c)** Calculate the first ten zeros $x_m$, $m = 1, \cdots, 10$, of $Y_0(x)$ from your CAS and from (11). How does the error behave as $m$ increases?

**(d)** Do (c) for $Y_1(x)$ and $Y_2(x)$. How do the errors compare to those in (c)?

### 11–15 HANKEL AND MODIFIED BESSEL FUNCTIONS

11. **Hankel functions.** Show that the Hankel functions (10) form a basis of solutions of Bessel's equation for any $\nu$.
12. **Modified Bessel functions of the first kind of order** $\nu$ are defined by $I_\nu(x) = i^{-\nu}J_\nu(ix)$, $i = \sqrt{-1}$. Show that $I_\nu$ satisfies the ODE

$$(12) \qquad x^2y'' + xy' - (x^2 + \nu^2)y = 0.$$

13. **Modified Bessel functions.** Show that $I_\nu(x)$ has the representation

$$(13) \qquad I_\nu(x) = \sum_{m=0}^{\infty} \frac{x^{2m+\nu}}{2^{2m+\nu}m!\,\Gamma(m + \nu + 1)}.$$

14. **Reality of $I_\nu$.** Show that $I_\nu(x)$ is real for all real $x$ (and real $\nu$), $I_\nu(x) \neq 0$ for all real $x \neq 0$, and $I_{-n}(x) = I_n(x)$, where $n$ is any integer.
15. **Modified Bessel functions of the third kind** (sometimes called *of the second kind*) are defined by the formula (14) below. Show that they satisfy the ODE (12).

$$(14) \qquad K_\nu(x) = \frac{\pi}{2\sin\nu\pi}\,[I_{-\nu}(x) - I_\nu(x)].$$

## CHAPTER 5 REVIEW QUESTIONS AND PROBLEMS

1. Why are we looking for power series solutions of ODEs?
2. What is the difference between the two methods in this chapter? Why do we need two methods?
3. What is the indicial equation? Why is it needed?
4. List the three cases of the Frobenius method, and give examples of your own.
5. Write down the most important ODEs in this chapter from memory.
6. Can a power series solution reduce to a polynomial? When? Why is this important?
7. What is the hypergeometric equation? Where does the name come from?
8. List some properties of the Legendre polynomials.
9. Why did we introduce two kinds of Bessel functions?
10. Can a Bessel function reduce to an elementary function? When?

[8]HERMANN HANKEL (1839–1873), German mathematician.

**11–20** **POWER SERIES METHOD OR FROBENIUS METHOD**

Find a basis of solutions. Try to identify the series as expansions of known functions. Show the details of your work.

**11.** $y'' + 4y = 0$

**12.** $xy'' + (1 - 2x)y' + (x - 1)y = 0$

**13.** $(x - 1)^2 y'' - (x - 1)y' - 35y = 0$

**14.** $16(x + 1)^2 y'' + 3y = 0$

**15.** $x^2 y'' + xy' + (x^2 - 5)y = 0$

**16.** $x^2 y'' + 2x^3 y' + (x^2 - 2)y = 0$

**17.** $xy'' - (x + 1)y' + y = 0$

**18.** $xy'' + 3y' + 4x^3 y = 0$

**19.** $y'' + \frac{1}{4x} y = 0$

**20.** $xy'' + y' - xy = 0$

## SUMMARY OF CHAPTER 5

# Series Solution of ODEs. Special Functions

The **power series method** gives solutions of linear ODEs

$$(1) \qquad y'' + p(x)y' + q(x)y = 0$$

with ***variable coefficients*** $p$ and $q$ in the form of a power series (with any center $x_0$, e.g., $x_0 = 0$)

$$(2) \qquad y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots.$$

Such a solution is obtained by substituting (2) and its derivatives into (1). This gives a ***recurrence formula*** for the coefficients. You may program this formula (or even obtain and graph the whole solution) on your CAS.

If $p$ and $q$ are **analytic** at $x_0$ (that is, representable by a power series in powers of $x - x_0$ with positive radius of convergence; Sec. 5.1), then (1) has solutions of this form (2). The same holds if $\tilde{h}$, $\tilde{p}$, $\tilde{q}$ in

$$\tilde{h}(x)y'' + \tilde{p}(x)y' + \tilde{q}(x)y = 0$$

are analytic at $x_0$ and $\tilde{h}(x_0) \neq 0$, so that we can divide by $\tilde{h}$ and obtain the standard form (1). ***Legendre's equation*** is solved by the power series method in Sec. 5.2.

The ***Frobenius method*** (Sec. 5.3) extends the power series method to ODEs

$$(3) \qquad y'' + \frac{a(x)}{x - x_0} y' + \frac{b(x)}{(x - x_0)^2} y = 0$$

whose coefficients are **singular** (i.e., not analytic) at $x_0$, but are "not too bad," namely, such that $a$ and $b$ are analytic at $x_0$. Then (3) has at least one solution of the form

$$(4) \quad y(x) = (x - x_0)^r \sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0(x - x_0)^r + a_1(x - x_0)^{r+1} + \cdots$$