In the name of God

Representation Theory of Groups 16/05/2010

Answer to four questions

(1) (a)Let G be a finite group. Show that there is one to one correspondence between representations of G and G-modules.
(b) Let V be a reducible G-module. Show that the representation which corresponds V is equivalent to $\begin{bmatrix} C(x) & 0 \\ E(x) & D(x) \end{bmatrix}$, where C and D are representations.

(2) Construct the character table of A_4.

(3) Two representations of a finite group over the complex field are equivalent if and only if they have the same character.

(4) (Reciprocity Theorem of Frobenius). Let H be a subgroup of a finite group G. If ψ and ϕ are characters of H and G, respectively, then

$$\langle \psi^G, \phi \rangle_G = \langle \psi, \phi_H \rangle_H.$$

(5) Let G be a finite group of order g. Let $\{\chi^{(1)}, \ldots, \chi^{(k)}\}$ be the set of irreducible characters of G. Then $g = \sum_{i=1}^{k} (\chi^{(i)}(1))^2$.

In the name of God

Representation Theory of Groups Quiz 2 (14/November/2011)

(1) Let $\chi^{(1)}, \ldots, \chi^{(k)}$ be the set of all irreducible characters of a finite G over $\mathbb{C}$, of degrees $f^{(1)}, \ldots, f^{(k)}$, respectively. Let ρ be the right regular representation on G. Prove, in details, that $\rho = \sum_{i=1}^{k} f^{(i)} \chi^{(i)}$.

(2) Let ϕ be a character of a finite group G over $\mathbb{C}$. Show that $\phi(x^{-1}) = \overline{\phi(x)}$, for all $x \in G$.

In the name of God

Representation Theory of Groups 17/04/2006

1. State and prove Maschke's Theorem.
2. Let G be a non-abelian group of order 27. Find $|Irr(G)|$ and $\{\chi(1) \mid \chi \in Irr(G)\}$.
3. Let χ be an irreducible character of a finite group G. Show that $\chi(1) \mid |G|$.

In the name of God

Representation Theory of Groups Quiz 1 (10/October/2011)

(1) Show that permutation representation of degree n, where $n > 1$, is always reducible. Let $M(x)$ is the permutation representation of S_3. Find an irreducible representation $D(x)$ such that $M(x) \sim \begin{bmatrix} 1 & 0 \\ E(x) & D(x) \end{bmatrix}$.

(2) State and prove Maschke's Theorem.

In the name of God

Representation Theory of Groups Quiz 1 (26/02/2006)

Let A be a semisimple finite dimensional F-algebra and let M be an irreducible A-module. Then

(a) $M(A)$ is a minimal ideal of A;

(b) if W is irreducible, then it is annihilated by $M(A)$ unless $W \cong M$;

(c) the map $x \to x_M$ is one-to-one from $M(A)$ onto $A_M \subseteq \mathrm{End}(M)$;

(d) $\mathcal{M}(A)$ is a finite set.

In the name of God

Representation Theory of Groups 28/06/2006

Note: All groups are finite and all characters are $\mathbb{C}$-characters

(1) If $\chi \in \mathrm{Irr}(\Sigma_n)$, then $\chi(g) \in \mathbb{Z}$, for all $g \in \Sigma_n$.
(10 points)

(2) Let χ be a character of G and let $Z = Z(\chi)$ and $f = \chi(1)$. Let $\mathcal{X}$ be a representation of G which affords χ. Then
(a) $Z = \{g \in G \mid \mathcal{X}(g) = \varepsilon I \text{ for some } \varepsilon \in \mathbb{C}\}$;
(b) Z is a subgroup of G;
(c) $\chi_Z = f\lambda$ for some linear character λ of Z;
(d) $Z/\ker\chi$ is cyclic;
(e) $Z/\ker\chi \subseteq \mathbf{Z}(G/\ker\chi)$.
(f) If $\chi \in \mathrm{Irr}(G)$, then $Z/\ker\chi = \mathbf{Z}(G/\ker\chi)$.
(20 points)

(3) Let G be an M-group and let $1 = f_1 < f_2 < \cdots < f_s$ be the distinct character degrees of the irreducible characters of G. Let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = f_i$. Then $G^{(i)} \leq \ker\chi$, where $G^{(i)}$ denotes the ith term of the derived series of G. Deduce that G is a soluble group.
(15 points)

(4) Let G be group and n be a positive integer. For all $g \in G$ define $\mathcal{V}_n(g) = |\{h \in G \mid h^n = g\}|$. Then $\mathcal{V}_n$ is a class function on G and $[\mathcal{V}_n, \chi] = \frac{1}{|G|}\sum_{h \in G}\chi(h^n)$, for all $\chi \in \mathrm{Irr}(G)$.
(10 points)

(5) Let N be a normal subgroup of G. Show that

$$\mathrm{Irr}(G/N) = \{\beta \in \mathrm{Irr}(G) \mid [(1_N)^G, \beta] \neq 0\}.$$

(10 points)

(6) Let H be a normal subgroup of G and let $\chi \in \mathrm{Irr}(G)$. Let θ be an irreducible constituent of χ_H and suppose $\theta = \theta_1, \theta_2, \ldots, \theta_t$ are distinct conjugates of θ in G. Then $\chi_H = e\sum_{i=1}^{t}\theta_i$, where $e = [\chi_H, \theta]$.
(10 points)

(7) Let H be a normal subgroup of G,$\theta \in \mathrm{Irr}(H)$ and $T = I_G(\theta)$. Let

$$\mathcal{A} = \{\psi \in \mathrm{Irr}(T) \mid [\psi_H, \theta] \neq 0\}, \quad \mathcal{B} = \{\chi \in \mathrm{Irr}(G) \mid [\chi_H, \theta] \neq 0\}.$$

Then
(a) If $\psi \in \mathcal{A}$, then ψ^G is irreducble;
(b) The map $\psi \mapsto psi^G$ is a bijection of $\mathcal{A}$ onto $\mathcal{B}$;
(c) If ψ^G, with $\psi \in \mathcal{A}$, then ψ is the unique irreducible constituent of χ_T which lies in $\mathcal{A}$;
(d) If $\psi^G = \chi$, with $\psi \in \mathcal{A}$, then $[\psi_H, \theta] = [\chi_H, \theta]$.
(25 points)

In the name of God

Representation Theory of Groups 28/06/2010

Note: All groups are finite and all characters are $\mathbb{C}$-characters

(1) If $\chi \in \mathrm{Irr}(\Sigma_n)$, then $\chi(g) \in \mathbb{Z}$, for all $g \in \Sigma_n$.
(10 points)

(2) Let χ be a character of G and let $Z = Z(\chi)$ and $f = \chi(1)$. Let $\mathcal{X}$ be a representation of G which affords χ. Then
(a) $Z = \{g \in G \mid \mathcal{X}(g) = \varepsilon I \text{ for some } \varepsilon \in \mathbb{C}\}$;
(b) Z is a subgroup of G;
(c) $\chi_Z = f\lambda$ for some linear character λ of Z;
(d) $Z/\ker\chi$ is cyclic;
(e) $Z/\ker\chi \subseteq \mathbf{Z}(G/\ker\chi)$.
(f) If $\chi \in \mathrm{Irr}(G)$, then $Z/\ker\chi = \mathbf{Z}(G/\ker\chi)$.
(20 points)

(3) Let G be an M-group and let $1 = f_1 < f_2 < \cdots < f_s$ be the distinct character degrees of the irreducible characters of G. Let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = f_i$. Then $G^{(i)} \leq \ker\chi$, where $G^{(i)}$ denotes the ith term of the derived series of G. Deduce that G is a soluble group.
(15 points)

(4) Let G be group and n be a positive integer. For all $g \in G$ define $\mathcal{V}_n(g) = |\{h \in G \mid h^n = g\}|$. Then $\mathcal{V}_n$ is a class function on G and $[\mathcal{V}_n, \chi] = \frac{1}{|G|}\sum_{h\in G}\chi(h^n)$, for all $\chi \in \mathrm{Irr}(G)$.
(10 points)

(5) Let N be a normal subgroup of G. Show that

$$\mathrm{Irr}(G/N) = \{\beta \in \mathrm{Irr}(G) \mid [(1_N)^G, \beta] \neq 0\}.$$

(10 points)

(6) Let H be a normal subgroup of G and let $\chi \in \mathrm{Irr}(G)$. Let θ be an irreducible constituent of χ_H and suppose $\theta = \theta_1, \theta_2, \ldots, \theta_t$ are distinct conjugates of θ in G. Then $\chi_H = e\sum_{i=1}^{t}\theta_i$, where $e = [\chi_H, \theta]$.
(10 points)

(7) Let H be a normal subgroup of G,$\theta \in \mathrm{Irr}(H)$ and $T = I_G(\theta)$. Let

$$\mathcal{A} = \{\psi \in \mathrm{Irr}(T) \mid [\psi_H, \theta] \neq 0\}, \qquad \mathcal{B} = \{\chi \in \mathrm{Irr}(G) \mid [\chi_H, \theta] \neq 0\}.$$

Then
(a) If $\psi \in \mathcal{A}$, then ψ^G is irreducble;
(b) The map $\psi \mapsto psi^G$ is a bijection of $\mathcal{A}$ onto $\mathcal{B}$;
(c) If ψ^G, with $\psi \in \mathcal{A}$, then ψ is the unique irreducible constituent of χ_T which lies in $\mathcal{A}$;
(d) If $\psi^G = \chi$, with $\psi \in \mathcal{A}$, then $[\psi_H, \theta] = [\chi_H, \theta]$.
(25 points)

In the name of God

Representation Theory of Groups, Final Exam

(1) Let G be the group of order 21 which is defined by the relations

$$a^7 = b^3 = 1, \quad b^{-1}ab = a^2.$$

Show that G has 5 conjugacy classes and construct its the character tabel.

(2) Let F be an irreducible representationof degree f and character χ. Suppose C_α is a conjugacy class of size h_α, such that $(h_\alpha, f) = 1$. Then either (i) $\chi_\alpha = f\varepsilon_0$, where ε_0 is a root of unity, or else
(ii) $\chi_\alpha = 0$.

(3) Let G be a transitive permutation group of degree n such that each permutation of G, other than the identity, leaves at most one of the objects fixed. Then those permutations which displace all the objects, together with the identity, form a normal subgroup of G of order n.

(4) Prove that, for a group of odd order g and class number k, the integer $g - k$ is divisible by 16.

(5) Let g be an element and let ψ be a character of a finite group G. Suppose that g is of order h. If for all $1 \leq r \leq h$ with $(h, r) = 1$, $\psi(g) = \psi(g^r)$, then $\psi(g)$ is rational. Moreover $\psi(g)$ is rational if and only if g is conjugate to g^r, for all $1 \leq r \leq h$ with $(h, r) = 1$.

(6) Every (complex) irreducible orthogonal representation of a finite group G is equivalent to a real orthogonal representation.

(7) (For Ph. D. students) Let F be a (complex) irreducible representation of a group G of order g, and let χ be the character of F. Then $\frac{1}{g}\sum_{y \in G} \chi(y^2) \in \{-1, 0, 1\}$.

In the name of God

Representation Theory of Groups 28/11/2011

(1) Let G be a finite group. Show that there is a one to one correspondence between representations of G and G-modules.

(2) Let V be a reducible G-module. Show that the representation which corresponds V is equivalent to

$$\begin{bmatrix} C(x) & 0 \\ E(x) & D(x) \end{bmatrix},$$

where $C(x)$ and $D(x)$ are representations and $C(x)$ is irreducible.

(2) Construct the character table of A_4.

(3) Let G be a finite group. Construct a basis for the center of $G_{\mathbb{C}}$ and conclude the dimension of the center of $\mathcal{H} = \mathrm{Hom}(G_{\mathbb{C}}, G_{\mathbb{C}})$ is equal to the number of conjugacy classes of G.

(4) Let $B(u)$ be a representation of a subgroup H of a finite group G, with degree q and character ϕ. Let $\{t_1, t_2, \ldots, t_n\}$ a right transversal of H in G. Let $B(x) = 0$, for all $x \in G \setminus H$. Show that $A(x) = [B(t_i x t_j^{-1})]_{qn \times qn}$ is a representation of G. Let $\phi^G(x)$ be the character of $A(x)$. Show that

$$\phi^G(x) = \frac{1}{|H|} \sum_{y \in G} \phi(yxy^{-1}).$$

(6) Show that in a group G of odd order no element other than 1 is conjugate to its inverse. Prove that if $1 \neq u \in G$, there exists at least one irreducible character χ such that $\chi(u)$ is not real.

(7) (FOR PhD STUDENTS ONLY) Let

$$G = \langle a, b \mid a^6 = 1,\ a^3 = (ab)^2 = b^2 \rangle.$$

Show that $G' = \{1, a^2, a^4\}$, $G/G' \cong C_4$ and $Z(G) = \{1, a^3\}$. Then construct the character table of G.

In the name of God

Representation Theory of Groups, Final Exam

Answer to only five questions

(1) Let χ be an irreducible character of degree f which takes the value χ_α for the conjugacy class C_α. Then each of the numbers $h_\alpha \chi_\alpha / f$ is an algebraic integer, where $h_\alpha = |C_\alpha|$.

(2) Let G be a transitive permutation group of degree n such that each permutation of G, other than the identity, leaves at most one of the objects fixed. Then those permutations which displace all the objects, together with the identity, form a normal subgroup of G of order n.

(3) Every (complex) irreducible orthogonal representation of a finite group G is equivalent to a real orthogonal representation.

(4) (Clifford's Theorem). Let H be a normal subgroup of G, and let χ be an irreducible character of G. Then there exists an irreducible character ξ of H such that

$$\chi_H = e \sum_{j=1}^{r} \xi_{t_j},$$

where e is a positive integer and the sum involves a complete set of conjugates of ξ.

(5) (a) Let U and V be G-modules over a field K that afford the matrix representations $A(x)$ and $B(x)$, respectively. Show that there is one-to-one correspondence between G-homomorphisms and matrices T with $TA(x) = B(x)T$.
(b) If K is algebraically closed field and $A(x)$ is irreducible, then the only matrices which commute with all the matrices $A(x)$ $(x \in G)$ are the scalar multiples of the identity matrix.

(6) Let F be a (complex) irreducible representation of a group G of order g, and let χ be the character of F. Then $\frac{1}{g} \sum_{y \in G} \chi(y^2) \in \{-1, 0, 1\}$.