In the name of God

Linear Groups

(May, 01, 2013)

1. Prove that $\mathrm{Aut}\left(\frac{\mathrm{GF(q^m)}}{\mathrm{GF(q)}}\right)$ is a cyclic group of order m.

2. Show that the number of one-dimensional subspaces of $V_n(q)$ is equal to $\dfrac{q^n-1}{q-1}$, which is equal to the number of hyperplanes of $V_n(q)$.

3. Let T be a transvection with hyperplane H. Show that
   (I) there exists a linear functional μ on $V_n(F)$ such that $H = \ker\mu$.
   (II) there exists a non-zero vector $a \in H$ such that $T(v) = v - \mu(v)a$, for all $v \in V_n(F)$.

4. State and prove the Iwasawa's Theorem.

5. Prove that if $n \geq 3$, then $\mathrm{GL}_n(F)' = \mathrm{SL}_n(F)' = \mathrm{SL}_n(F)$.

6. Show that if $(n, q-1) = 1$, then

   $$\mathrm{GL}_n(q) \cong \mathbb{Z}_{q-1} \times \mathrm{SL}_n(q).$$

   Also show that if $(n, q-1) > 1$, then this result may not be true.

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1. Prove that $|\mathrm{SP_{2n}(q)}| = \mathrm{q^{n^2}} \prod_{\mathrm{i=1}}^{\mathrm{n}} (\mathrm{q^{2i}} - 1)$.

2. Let $|\mathrm{SP_{2n}(F)}|$ is the group of isometries of (V, f), where (V, f) is a non-degenerate symplectic space on a field F. Let G be the subgroup of $\mathrm{SP_{2n}(F)}$ generated by all symplectic transvections. Show that G acts transitively on non-zero vectors of V. Also show that G acts transitively on hyperbolic pairs.

3. Let (V, f) be a non-degenerate orthogonal (or Hermitian with the field automorphism τ) space on a field F. In the orthogonal case suppose that $\mathrm{Char}(F) \neq 2$. Show that
   (a) There exists a non-zero vector $v \in V$ such that $f(v, v) \neq 0$.
   (b) V has an orthogonal basis.
   (c) The determinant fucntion is an epimorphism from $\mathrm{GU}(V, f)$ on to the multiplicative subgroup $\{a \in F^{\times} \mid aa^{\tau} = 1\}$ of $F^{\times}$.

4. Prove that $\mathrm{SU}_2(q^2) \cong \mathrm{SL}_2(q)$.

5. Let (V, f) be a non-degenerate orthogonal space on a field F, where $\mathrm{Char}(F) \neq 2$, of dimension 2. Show that
   (a) If V has a non-zero isotropic vector, then V is a hyperbolic plane.
   (b) If V has no non-zero isotropic vector, then V has a basis $\{v_1, v_2\}$ such that $f(v_1, v_1) = 1$ and $f(v_2, v_2) = -k$, where $k \in F$ is non-square.

6. Show that if q is odd, then $\mathrm{SO}_2^+(q) \cong \mathbb{Z}_{q+1}$.