In the name of God

Group Theory
(January, 04, 2014)

Answer to six questions

1. Let $G = \operatorname*{Dr}_{i=1}^{n} G_i$ be a finite group, where G_i, $i = 1, \ldots, n$, is a non-abelian simple subgroup of G. Prove that $G_1, \ldots, G_n$ are the only minimal normal subgroups of G, and every non-trivial normal subgroup of G is a direct product of some of $G_1, \ldots, G_n$.

2. Let G be an abelian p-group of finite exponent and a be an element of maximum order in G. Then there exits a subgroup H of G such that $G = \langle a \rangle \oplus H$.

3. Prove that a finitely generated torsion free abelain group is a free abelain group of finite rank.

4. Let $G = HA$ be a group, where $H < G$ and A is an abelain normal subgroup of G. Then H is a maximal subgroup of G if and only if $A/H \cap A$ is a minimal normal subgroup of $G/H \cap A$.

5. Let $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ be a central series of a group G. Then $\Gamma_{n-i+1}(G) \leq G_i \leq Z_i(G)$. Conclude that $Z_c(G) = G$ if and only if $\Gamma_{c+1}(G) = 1$.

6. In a polycyclic group G the number of infinite factors in a cyclic series is independent of the series and hence is an invariant of G.

7. Show that $\mathrm{Hol}(C_2 \times C_2) \cong S_4$ and $\mathrm{Dih}(C_n) \cong D_{2n}$.

8. Let G be a finite group. Prove that the following statements are equivalent
   (i) G is nilpotent.
   (ii) For every $N \lhd G$, $Z(G/N) \neq 1$.
   (iii) For every $1 \neq N \unlhd G$, $[N, G] < N$.