In the name of God

Group Theory
(November, 16, 2013)

1. Let $G = \langle g \rangle$ be a cyclic group and $H \leq G$. Prove that H is cyclic.

2. State and prove the Lagrange Theorem.

3. Let $H_1 < H_2 < \cdots$ be a chain of subgroups of a group G and $H = \bigcup_{n=1}^{\infty} H_n$. Show that
(a) H is a subgroup of G.
(b) H is not finitely generated.
(c) if H_n, $n = 1, 2, \ldots$, is a simple group, then H is a simple group.

4. (for Ph. D. students) Let N be a normal subgroup of a finite group G such that $(|N|, |G/N|) = 1$. Show that N is a characteristic subgroup of G.

5. (for Ph. D. students) Show that $\mathbb{Q}$ has no maximal subgroup.