In the name of God

Group Theory
(November, 27, 2006)

1. (a) Show by an example that the product of two subnormal subgroup of a group need not be a subgroup.
   (b) If H sn G and $K \trianglelefteq G$ then HK sn G.

2. Suppose that G is nilpotent. Then for any central series of G, say
$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G,$$
$$\Gamma_{r-i+1}(G) \leq G_i \leq Z_i(G) \text{ for each } i = 0, 1, \ldots, r.$$
   Furthermore, the least integer c such that $\Gamma_{c+1}(G) = 1$ is equal to the least integer c such that $Z_c(G) = G$.

3. Let p and q be primes such that $p > q$. If $p \not\equiv 1 \bmod q$ the $v(pq) = 1$, while if $p \equiv 1 \bmod q$ the $v(pq) = 2$.

4. (a) What is the wreath product of two groups? Describe its fundamental properties.
   (b) Let G be any soluble group, say of derived length n. Then $G \wr C_2$ is soluble of derived length $n + 1$, where $\wr$ denotes the natural wreath product.

5. Let G be a finite group.
   (a) If $K \trianglelefteq G$ then $\mathrm{Fitt}(K) \leq \mathrm{Fitt}(G)$.
   (b) Show by an example that $\mathrm{Fitt}(G)$ need not contain $\mathrm{Fitt}(H)$ for every subgroup H of G.

In the name of God

Group Theory
(December, 02, 2013)

1. Let G be a group and X be a set. Show that there exists a homomorphism $G \longrightarrow \mathrm{Sym}(X)$ if and only if there exists a function
$$\begin{aligned} X \times G &\longrightarrow X \\ (x, g) &\mapsto xg \end{aligned}$$
such that $x1 = x$ and $x(gh) = (xg)h$, for all $x \in X$ and $g, h \in G$.

2. Show that if G is a finite group of order p^2q^2, where p and q are prime numbers, then G is not simple.

3. Let H be a normal subgroup of a finite group G, such that $(|H|, |G : H|) = 1$. Prove that H has a complement in G.

4. Let G be a finite primitive permutation group on a set X and $1 \neq N \trianglelefteq G$. Then N acts transitively on X. Moreover if N is regular on X, then N is a minimal normal subgroup of G.

5. Suppose that G is a Frobenius group on a set X with kernel K. Show that
(a) $K = \{g \in G \mid \mathrm{Fix}(g) = \emptyset\} \cup \{1\}$, where $\mathrm{Fix}(g) = \{x \in X \mid xg = x\}$.
(b) For all $1 \neq u \in K$, $C_G(u) \subseteq K$; and for all $1 \neq g \in G_x$, $C_G(g) \subseteq G_x$.
(c) $Z(G) = 1$

6. (Ph. D. students) A regular permutation group of finite degree is primitive if and only if it has prime order.

7. (Ph. D. students) Show that every non-abelian group of order 8 is isomorphic to D_8 or Q_8.

In the name of God

Group Theory
(November, 16, 2013)

1. Let $G = \langle g \rangle$ be a cyclic group and $H \leq G$. Prove that H is cyclic.
2. State and prove the Lagrange Theorem.
3. Let $H_1 < H_2 < \cdots$ be a chain of subgroups of a group G and $H = \bigcup_{n=1}^{\infty} H_n$. Show that
   (a) H is a subgroup of G.
   (b) H is not finitely generated.
   (c) if H_n, $n = 1, 2, \ldots$, is a simple group, then H is a simple group.
4. (for Ph. D. students) Let N be a normal subgroup of a finite group G such that $(|N|, |G/N|) = 1$. Show that N is a characteristic subgroup of G.
5. (for Ph. D. students) Show that $\mathbb{Q}$ has no maximal subgroup.

In the name of God

Group Theory
(January, 08, 2013)

1. Let G be a finite group of order $2m$, where $m > 1$ is odd. Then G has an normal subgroup of order m.
2. Show that every group of order p^2q^2, where p and q are primes, is not simple.

In the name of God

Group Theory
(January, 08, 2013)

Every question has 15 scores

1. Give the exact definition of the following concepts: Free group, Free abelian group, Wreath product, Holomorph, Solvable group,

2. Let $G \neq 1$ be a finite group. If G is characteristically simple, then G is a direct product of isomorphic simple groups.

3. Let H be a subgroup of an abelian group G. If G/H is free abelian, then there exists a subgroup K of G such that $G = H \oplus K$.

4. Let G be a finitely generated abelian group. If G is torsuion free, then G is a free abelian group with finite rank.

5. Let H be a minimal normal subgroup of a solvable group G. Then either H is an elementary abelian p group, for some prime p or is a direct product of copies of $\mathbb{Q}$, the additive group of rational numbers.

6. If G is a nilpotent group then every subgroup of G is a subnormal subgroup. Show that if G is finite, then the converse is also true.

7. Let M and M be normal nilpotent subgroup of a group. Then MN is normal and nilpotent.

In the name of God

Group Theory
(November, 18, 2012)

1. Let p be a prime. If H is a p-subgroup of a finite group G, then
$$|G : H| \equiv |N_G(H) : H| \pmod{p}.$$
Moreover if $p \mid |G : H|$, then $H < N_G(H)$.

2. If G is a finite simple group of order 60, then $G \cong A_5$.

3. Let H be an abelian normal subgroup of a finite group G such that $(|H|, |G : H|) = 1$. Then H has a complement in G.

4. Let G be a primitive permutation group on a set X and $1 \neq N \trianglelefteq G$. Then N is transitive on X. Moreover If N is regular on X, then N is a minimal normal subgroup of G.

5. Let G be a finite Frobenius group with Frobenius kernel K and Frobenius complement H. Show that $|K| = |G : H|$ and $|G : H| \equiv 1 \pmod{|H|}$

6. Let G be a finite group of order $2p$, where p is a prime. Prove that either $G \cong \mathbb{Z}_{2p}$ or $G \cong D_{2p}$.

7. (Ph. D. students) Let G be a Frobenius group on a set X with Frobenius kernel K. Show that for all $1 \neq u \in K$, $C_G(u) \subseteq K$ and for all $1 \neq g \in K$, $C_G(g) \subseteq G_x$.

In the name of God

Group Theory
(June, 20, 2012)

Answer to six questions only

1. Let $G = \langle g_1, \ldots, g_n \rangle$ be a finitely generated abelian torsion free group. Show that is free abelian of finite rank.

2. Prove that in a polycyclic group G the number of infinite factors in a cyclic series is independent of the series and hence is an invariant of G.

3. Let G be a finite group. Then G is nilpotent if and only if $G' \leq \Phi(G)$.

4. Let G be a supersolvable group. Prove that $F(G)$ is nilpotent and $G/F(G)$ is a finite abelian group.

5. Show that the additive group of rational number $\mathbb{Q}$ is not free abelian.

6. Prove that a finite group G is nilpotent if and only if elements of co-prime order commute.

7. Let G be a group. Let H be a proper subgroup and A be a normal abelian subgroup of G, such that $G = HA$. Show that H is a maximal subgroup of G if and only if $A/H \cap A$ is a minimal normal subgroup of $G/H \cap A$.

In the name of God

Group Theory
(April, 30, 2012)

Answer to six questions only

1. Let G be a finite group of order $2m$, where $m > 1$ is odd. Then G has an normal subgroup of order m.

2. Let G be a group of order 385. Then the Sylow 7-subgroup of G is contained in the center of G and the Sylow 11-subgroup of G is normal.

3. Let H be an abelian normal subgroup of a finite group G such that $(|H|, |G : H|) = 1$. Then H has a complement in G and all complements are conjugate.

4. Let G be a transitive permutation group on a set X and let $x \in X$. Then G is primitive if and only if G_x is a maximal subgroup of G.

5. Let $G = G_1 \times \cdots \times G_n$, where G_i is non-abelian simple. Then $G_1, \ldots, G_n$ are the only minimal normal subgroups of G; and every normal subgroup is a direct product of some G_i.

6. Let H be a group acting on a set X, and let G be any group. Describe the (restricted and unrestricted) wreath product of G by H.

7. Let G be a finite Frobenius group on X with kernel K and complement H. Prove, **in details**, that

$$|X| = |K| = |G : H| \equiv 1 \pmod{|H|},$$

in particular $G = KH$.

In the name of God

Group Theory
(July, 01, 2011)

1. Let G be a transitive permutation group on a set X and let $x \in X$. Then G is primitive if and only if G_x is a maximal subgroup of G.

2. (a) If G is a primitive permutation group on a set X, then either G has prime order or, for each pair of distinct elements x and y in X, $G = \langle G_x, G_y \rangle$.

   (b) Let G be a primitive permutation group on a set X. If G_z, is an abelian group for some $z \in X$, then $G_x \cap G_y = 1$, for all $x, y \in X$.

3. Suppose that $G = Dr_{i=1}^{n} G_i$, where, for each $i = 1, \ldots, n$, G_i is a simple non-abelian normal subgroup of G. Then $G_1, \ldots, G_n$ are the only minimal normal subgroups of G and every non-trivial normal subgroup of G is a direct product of some of $G_1, \ldots, G_n$.

4. Show that
   (a) $Hol(C_2 \times C_2) \cong S_4$.
   (b) $C_2 \wr C_2 \cong D_8$.

5. An abelian group G is divisible if and only if it is a direct sum of isomorphic copies of $\mathbb{Q}$ and of quasicyclic groups.

6. Let G be a soluble group. A minimal normal subgroup of G is either an elementary abelian p-group or else a direct product of copies of the additive group of rational numbers.

7. Let G be a finite group. Then G is nilpotent if and only if every subgroup is subnormal.

8. If the center of a group G is torsion-free, each upper central factor is torsion-free.

In the name of God

Group Theory
(January, 07, 2009)

1. Let G be a cyclic p-group of order $p^e > 1$ and $A := \mathrm{Aut}(G)$. Then $A = S \times T$, where S is a group of order $p^e - 1$ and T is a cyclic group of order $p - 1$.

2. Let K be an abelain normal subgroup of a finite group G such that $(|K|, |G : K|) = 1$. Then K has a complement in G, and all complements of K are conjugate in G.

3. Let G be a Frobenius group, with Frobenius complement H. If $|H|$ is even, then the Frobenius kernel is a normal subgroup.

4. (I) Let H be a subgroup of a group G. Prove that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.
(II) Let G be nilpotent and N a maximal Abelian normal subgroup of G. Prove that
(a) $C_G(N) = N$.
(b) If N is cyclic, then G' is cyclic.

5. Let G be a finite group, $C := C_G(F(G))$. Then
$$O_p(C/C \cap F(G)) = 1,$$
for every prime p

6. Let G be a π-separable finite group and $O_{\pi'}(G) = 1$. Then
$$C_G(O_\pi(G)) \leq O_\pi(G).$$

In the name of God

Group Theory
(December, 11, 2009)

1. Let $G = G_1 \times \cdots \times G_n$ and N be a normal subgroup of G.
   (a) If N is perfect, then $N = (N \cap G_1) \times \cdots \times (N \cap G_n)$.
   (b) If $G_1, \ldots, G_n$ are non-abelina simple groups, then there exists a subset $J := \{j_1, \ldots, j_m\} \subseteq \{1, \ldots, n\}$ such that
$$N = G_{j_1} \times \cdots \times G_{j_m} \quad \text{and} \quad G_k \cap N = 1 \quad \text{for} \quad k \notin J.$$
2. Let $\mathcal{M}$ be a finite set of minimal normal subgroup of G, and let $M = \prod_{N \in \mathcal{M}} N$. Let U be a normal subgroup of G. Then there exist $N_1, \ldots, N_k \in \mathcal{M}$ such that
$$UM = U \times N_1 \times \cdots \times N_k.$$
3. Let G be a finite abelian group and U a cyclic subgroup of maximal order n G. Then there exists a complement V of U in G.
4. The automorphism of a group order p, a prime, is cyclic.
5. Let P be a p-subgroup of G and p be a divisor of $|G : P|$. Then $P < N_G(P)$.
6. Let G be not 3-closed and $|G| = 12$. Then G is 2-closed.

In the name of God

Group Theory
(January, 20, 2007)

1. Let H act on K, say with action φ, and let $J = \text{Im } \varphi \leq \text{Aut } K$. If the action is faithful then the group $H \ltimes_{\varphi} K$ is isomorphic to the relative holomorph JK of K.

2. Let G be a finite group such that all Sylow subgroup of G are cyclic. Then G is Soluble. Moreover, G/G' and $G\,'$ are both cyclic, G splits over G', and G' is a Hall subgroup of G.

3. Let G be a finite group and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(Q)/C_G(Q)$ is a p-subgroup for every subgroup Q of P.

4. Let G be a finite group. Then G is p-nilpotent if and only if every chief factor of G of order divisible by p is central. Conclude that G is nilpotent if and inly if G is p-nilpotent for every prime p.

5. State and prove the Burnside's basis theorem.

6. Suppose that A is an abelian minimal normal subgroup of a finite group G. Then either $A \leq \Phi(G)$ or G splits over A.