Sorting Algorithms

Topic Overview

- Issues in Sorting on Parallel Computers
- Sorting Networks
- Bubble Sort and its Variants
- Bucket and Sample Sort
- Other Sorting Algorithms

Sorting: Overview

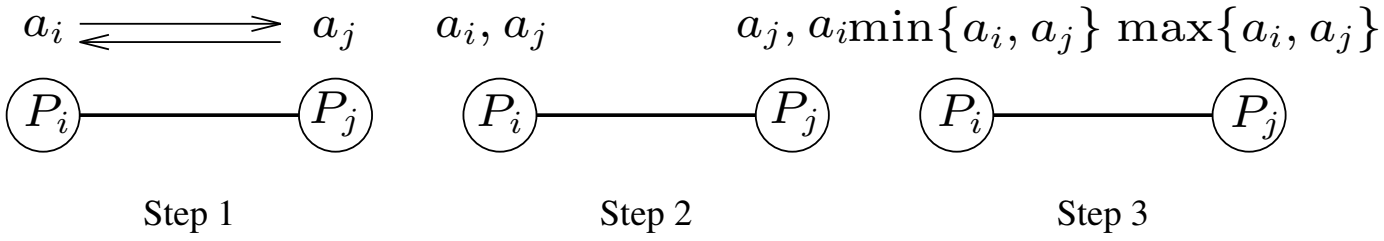
- One of the most commonly used and well-studied kernels.
- The fundamental operation of comparison-based sorting is *compare-exchange*.
- The lower bound on any comparison-based sort of n numbers is $\Theta(n \log n)$.
- We focus here on comparison-based sorting algorithms.

Sorting: Basics

What is a parallel sorted sequence? Where are the input and output lists stored?

- We assume that the input and output lists are distributed.
- The sorted list is partitioned with the property that each partitioned list is sorted and each element in processor P_i's list is less than that in P_j's list if $i < j$.

Sorting: Parallel Compare Exchange Operation



A parallel compare-exchange operation. Processes P_i and P_j send their elements to each other. Process P_i keeps $\min\{a_i, a_j\}$, and P_j keeps $\max\{a_i, a_j\}$.

Sorting: Basics

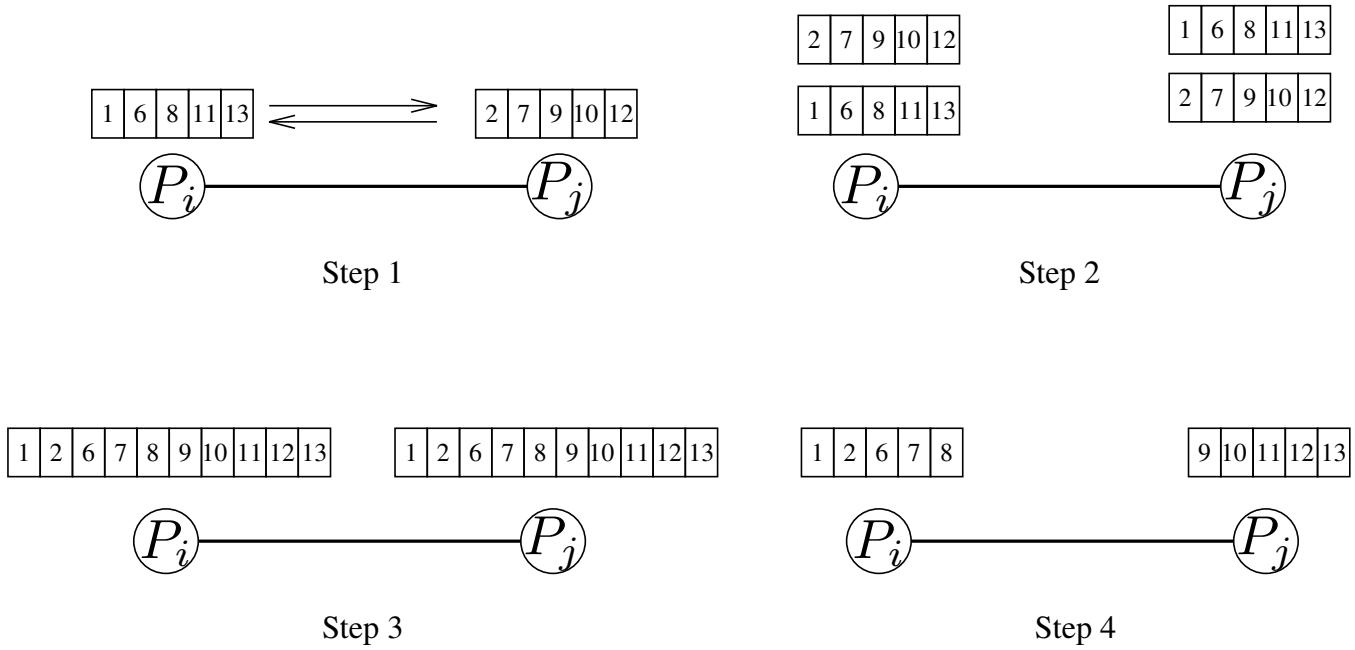
What is the parallel counterpart to a sequential comparator?

- If each processor has one element, the compare exchange operation stores the smaller element at the processor with smaller id.

- If we have more than one element per processor, we call this operation a compare split. Assume each of two processors have n/p elements.

- After the compare-split operation, the smaller n/p elements are at processor P_i and the larger n/p elements at P_j, where $i < j$.

Sorting: Parallel Compare Split Operation

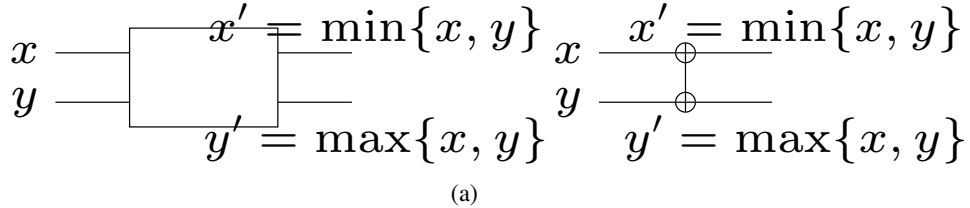


A compare-split operation. Each process sends its block of size n/p to the other process. Each process merges the received block with its own block and retains only the appropriate half of the merged block. In this example, process P_i retains the smaller elements and process P_j retains the larger elements.

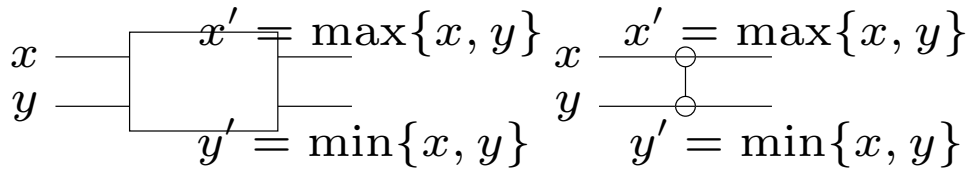
Sorting Networks

- Networks of comparators designed specifically for sorting.
- A comparator is a device with two inputs x and y and two outputs x' and y'. For an *increasing comparator*, $x' = \min\{x, y\}$ and $y' = \max\{x, y\}$; and vice-versa.
- We denote an increasing comparator by $\oplus$ and a decreasing comparator by $\ominus$.
- The speed of the network is proportional to its depth.

Sorting Networks: Comparators



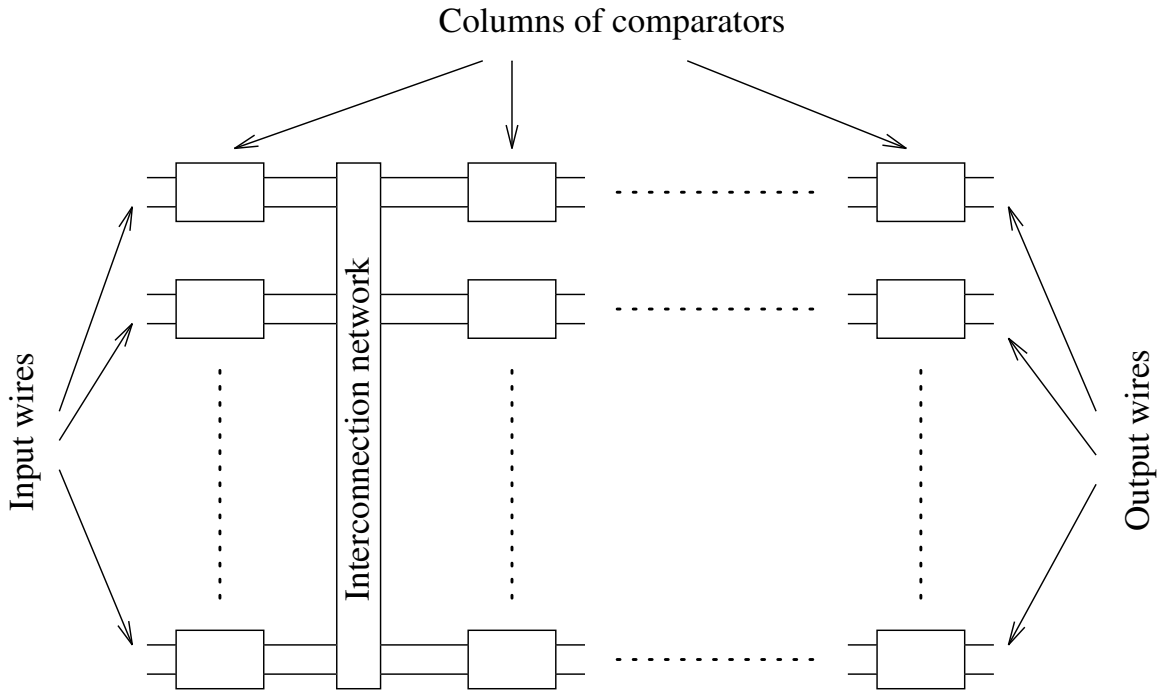
(a)



(b)

A schematic representation of comparators: (a) an increasing comparator, and (b) a decreasing comparator.

Sorting Networks



A typical sorting network. Every sorting network is made up of a series of columns, and each column contains a number of comparators connected in parallel.

Sorting Networks: Bitonic Sort

- A bitonic sorting network sorts n elements in $\Theta(\log^2 n)$ time.
- A bitonic sequence has two tones – increasing and decreasing, or vice versa. Any cyclic rotation of such networks is also considered bitonic.
- $\langle 1, 2, 4, 7, 6, 0 \rangle$ is a bitonic sequence, because it first increases and then decreases. $\langle 8, 9, 2, 1, 0, 4 \rangle$ is another bitonic sequence, because it is a cyclic shift of $\langle 0, 4, 8, 9, 2, 1 \rangle$.
- The kernel of the network is the rearrangement of a bitonic sequence into a sorted sequence.

Sorting Networks: Bitonic Sort

- Let $s = \langle a_0, a_1, \ldots, a_{n-1} \rangle$ be a bitonic sequence such that $a_0 \leq a_1 \leq \ldots \leq a_{n/2-1}$ and $a_{n/2} \geq a_{n/2+1} \geq \ldots \geq a_{n-1}$.

- Consider the following subsequences of s:

$$\begin{aligned} s_1 &= \langle \min\{a_0, a_{n/2}\}, \min\{a_1, a_{n/2+1}\}, \ldots, \min\{a_{n/2-1}, a_{n-1}\} \rangle \\ s_2 &= \langle \max\{a_0, a_{n/2}\}, \max\{a_1, a_{n/2+1}\}, \ldots, \max\{a_{n/2-1}, a_{n-1}\} \rangle \end{aligned} \quad (1)$$

- Note that s_1 and s_2 are both bitonic and each element of s_1 is less that every element in s_2.

- We can apply the procedure recursively on s_1 and s_2 to get the sorted sequence.

Sorting Networks: Bitonic Sort

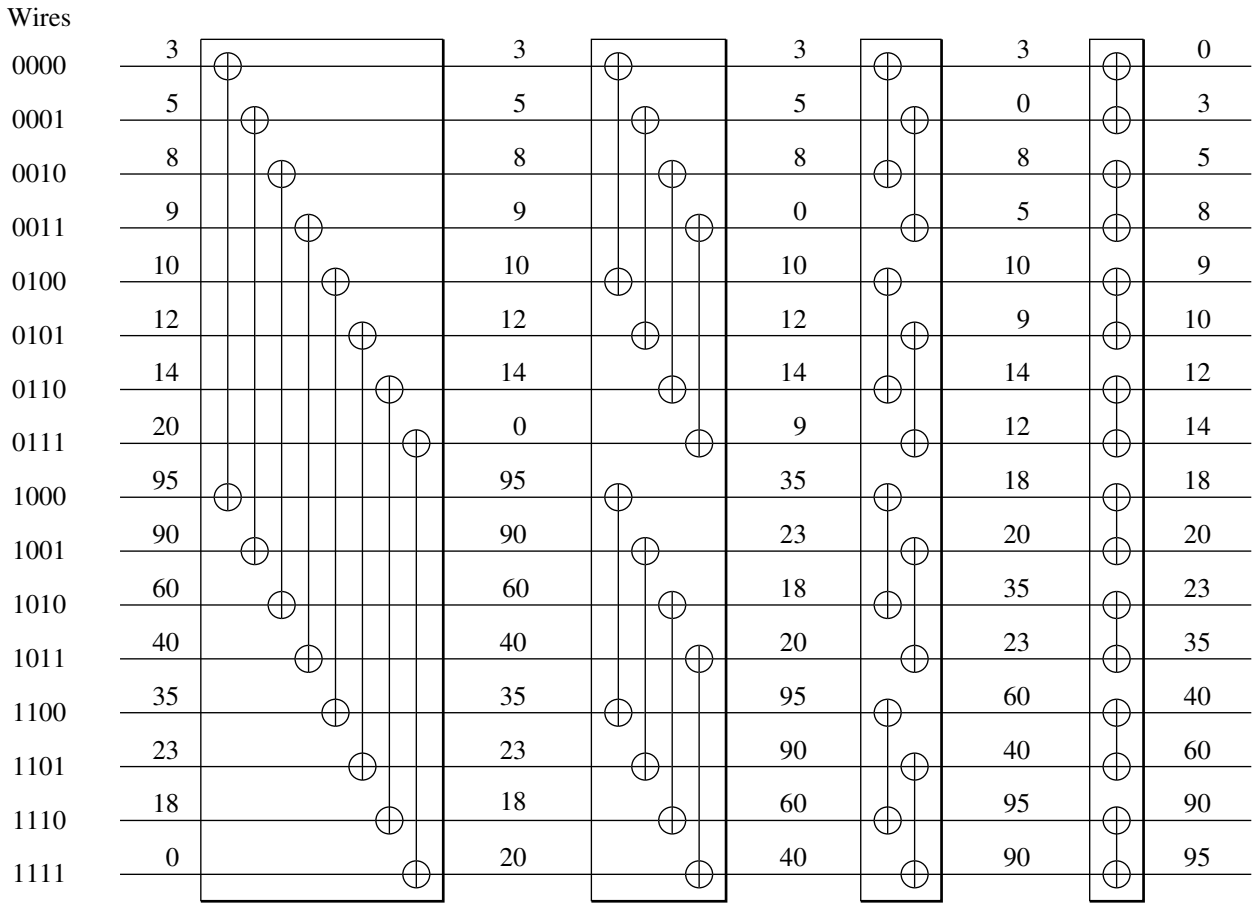
Original sequence	3	5	8	9	10	12	14	20	95	90	60	40	35	23	18	0
1st Split	3	5	8	9	10	12	14	0	95	90	60	40	35	23	18	20
2nd Split	3	5	8	0	10	12	14	9	35	23	18	20	95	90	60	40
3rd Split	3	0	8	5	10	9	14	12	18	20	35	23	60	40	95	90
4th Split	0	3	5	8	9	10	12	14	18	20	23	35	40	60	90	95

Merging a 16-element bitonic sequence through a series of $\log 16$ bitonic splits.

Sorting Networks: Bitonic Sort

- We can easily build a sorting network to implement this bitonic merge algorithm.
- Such a network is called a *bitonic merging network*.
- The network contains $\log n$ columns. Each column contains $n/2$ comparators and performs one step of the bitonic merge.
- We denote a bitonic merging network with n inputs by $\oplus$BM(n).
- Replacing the $\oplus$ comparators by $\ominus$ comparators results in a decreasing output sequence; such a network is denoted by $\ominus$BM(n).

Sorting Networks: Bitonic Sort



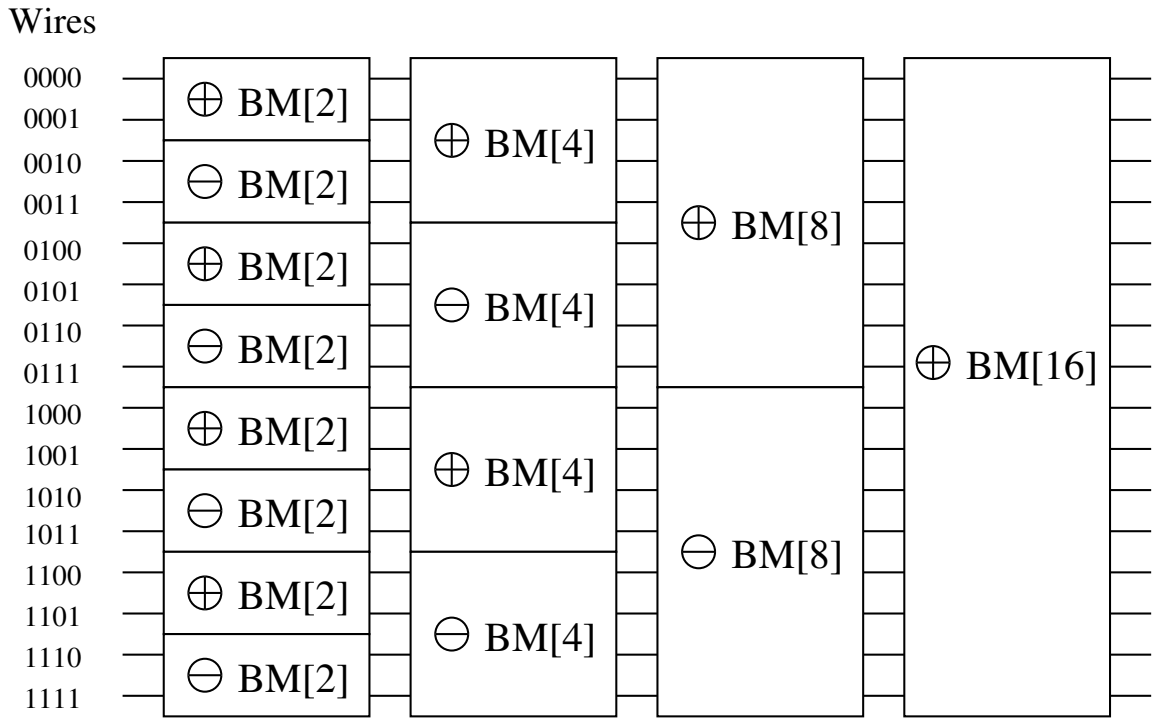
A bitonic merging network for $n = 16$. The input wires are numbered $0, 1 \ldots, n-1$, and the binary representation of these numbers is shown. Each column of comparators is drawn separately; the entire figure represents a $\oplus$BM(16) bitonic merging network. The network takes a bitonic sequence and outputs it in sorted order.

Sorting Networks: Bitonic Sort

How do we sort an unsorted sequence using a bitonic merge?

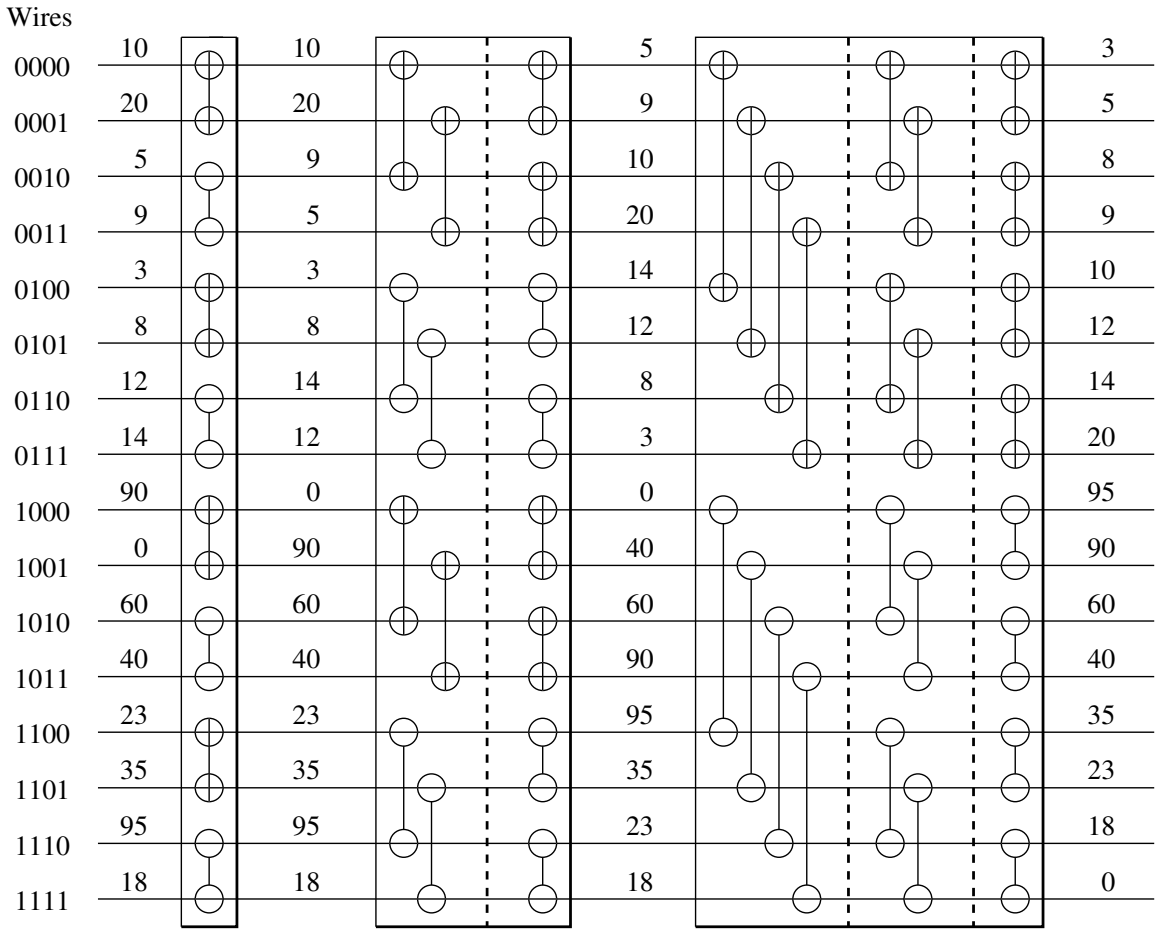
- We must first build a single bitonic sequence from the given sequence.
- A sequence of length 2 is a bitonic sequence.
- A bitonic sequence of length 4 can be built by sorting the first two elements using $\oplus$BM(2) and next two, using $\ominus$BM(2).
- This process can be repeated to generate larger bitonic sequences.

Sorting Networks: Bitonic Sort



A schematic representation of a network that converts an input sequence into a bitonic sequence. In this example, $\oplus$BM(k) and $\ominus$BM(k) denote bitonic merging networks of input size k that use $\oplus$ and $\ominus$ comparators, respectively. The last merging network ($\oplus$BM(16)) sorts the input. In this example, $n = 16$.

Sorting Networks: Bitonic Sort



The comparator network that transforms an input sequence of 16 unordered numbers into a bitonic sequence.

Sorting Networks: Bitonic Sort

- The depth of the network is $\Theta(\log^2 n)$.
- Each stage of the network contains $n/2$ comparators. A serial implementation of the network would have complexity $\Theta(n \log^2 n)$.

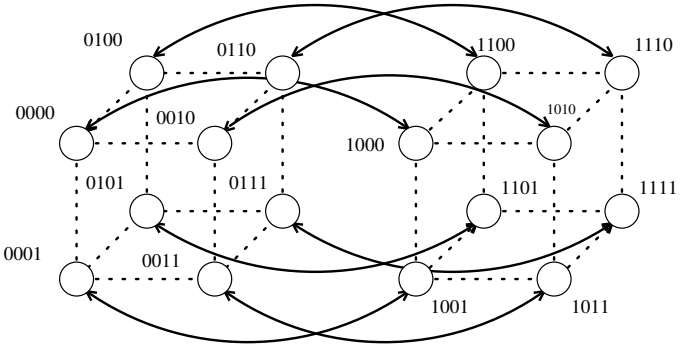
Mapping Bitonic Sort to Hypercubes

- Consider the case of one item per processor. The question becomes one of how the wires in the bitonic network should be mapped to the hypercube interconnect.

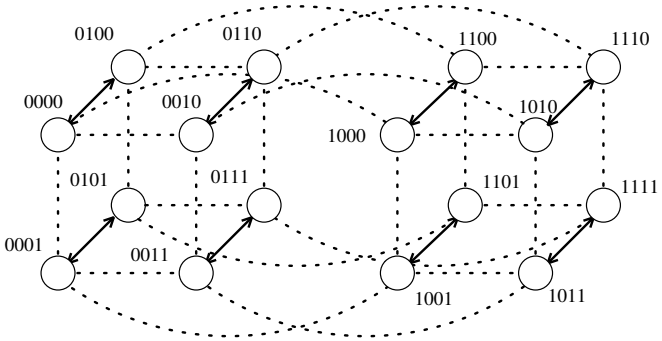
- Note from our earlier examples that the compare-exchange operation is performed between two wires only if their labels differ in exactly one bit!

- This implies a direct mapping of wires to processors. All communication is nearest neighbor!

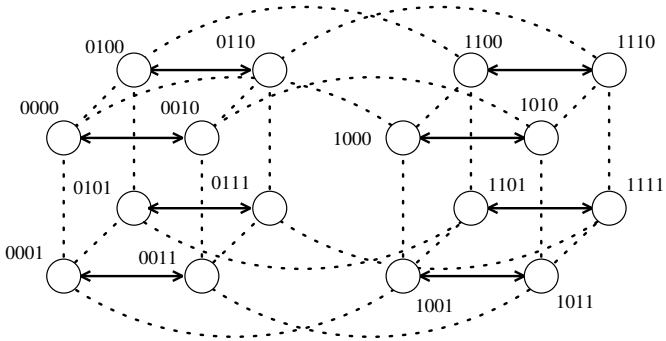
Mapping Bitonic Sort to Hypercubes



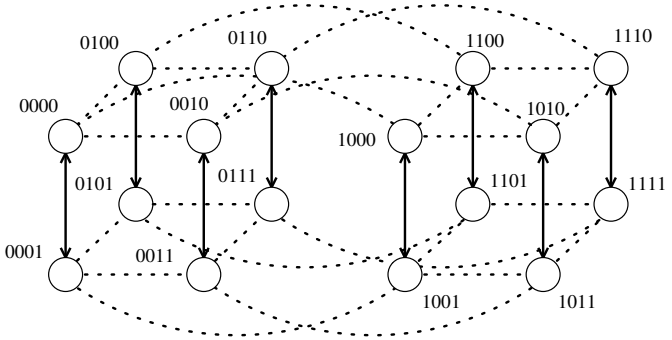
Step 1



Step 2



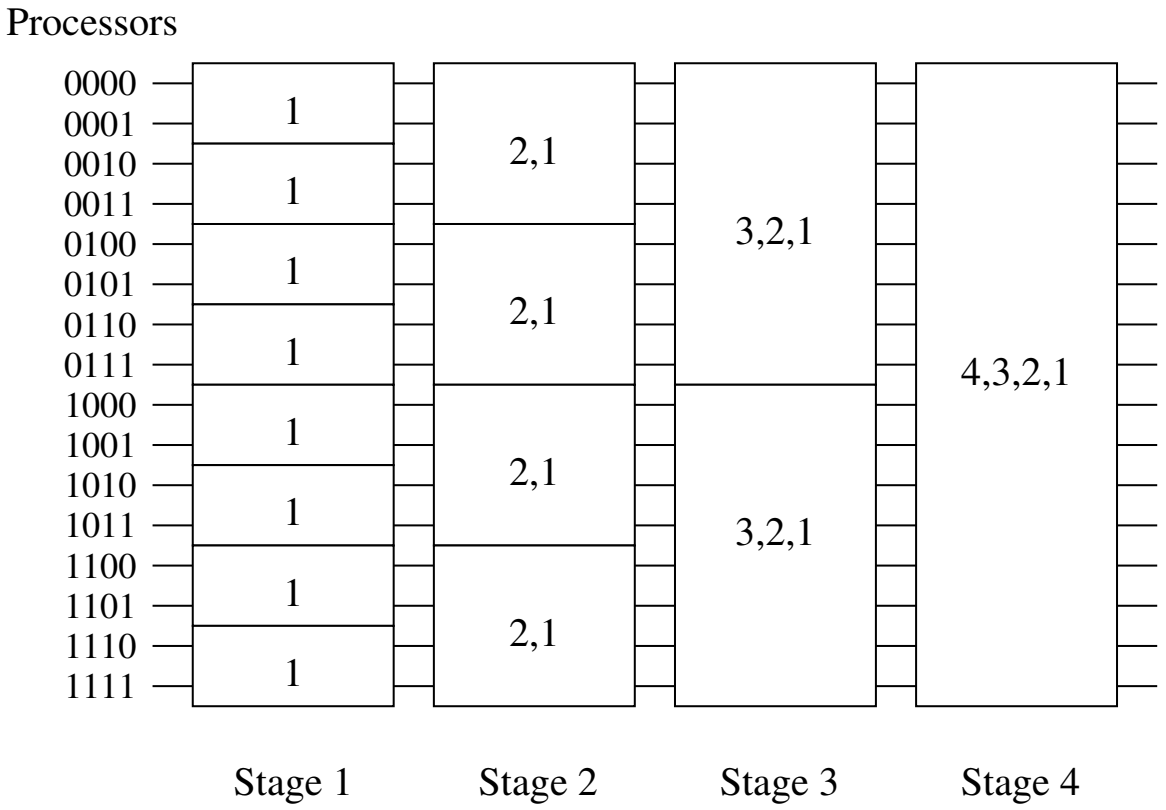
Step 3



Step 4

Communication during the last stage of bitonic sort. Each wire is mapped to a hypercube process; each connection represents a compare-exchange between processes.

Mapping Bitonic Sort to Hypercubes



Communication characteristics of bitonic sort on a hypercube. During each stage of the algorithm, processes communicate along the dimensions shown.

Mapping Bitonic Sort to Hypercubes

- During each step of the algorithm, every process performs a compare-exchange operation (single nearest neighbor communication of one word).

- Since each step takes $Theta(1)$ time, the parallel time is

$$T_P = \Theta(\log^2 n) \tag{2}$$

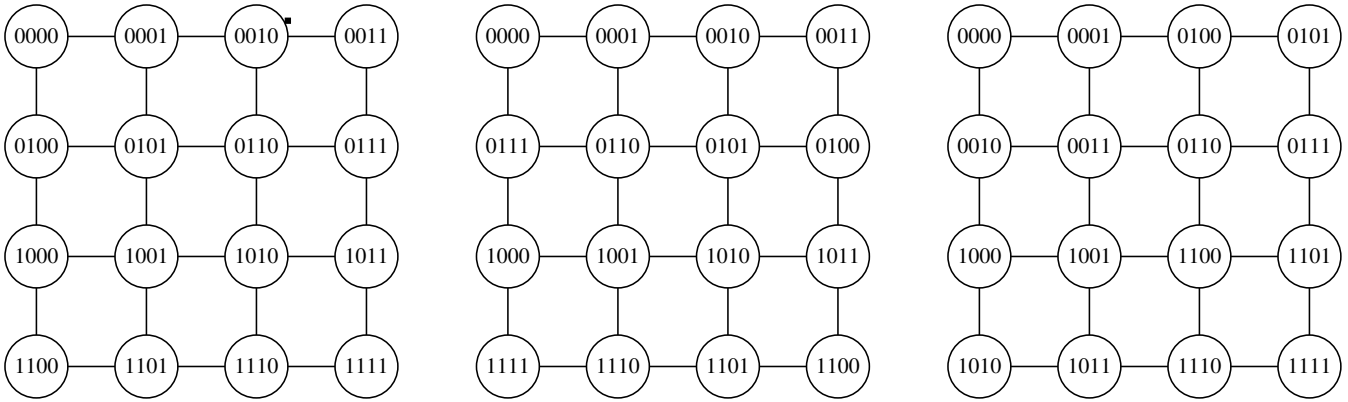
- This algorithm is cost optimal w.r.t. its serial counterpart, but not w.r.t. the best sorting algorithm.

Mapping Bitonic Sort to Meshes

- The connectivity of a mesh is lower than that of a hypercube, so we must expect some overhead in this mapping.

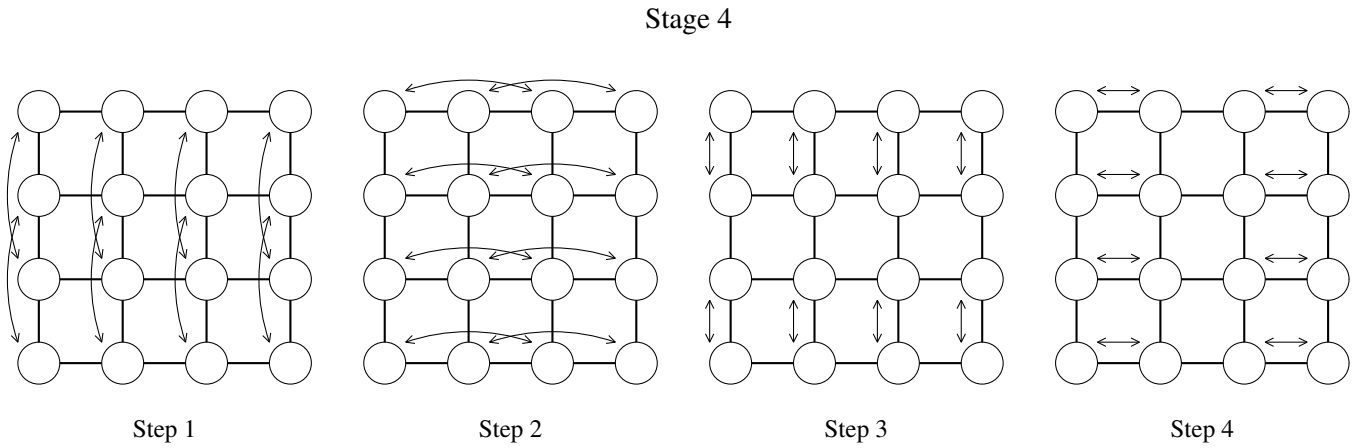
- Consider the row-major shuffled mapping of wires to processors.

Mapping Bitonic Sort to Meshes



Different ways of mapping the input wires of the bitonic sorting network to a mesh of processes: (a) row-major mapping, (b) row-major snakelike mapping, and (c) row-major shuffled mapping.

Mapping Bitonic Sort to Meshes



The last stage of the bitonic sort algorithm for $n = 16$ on a mesh, using the row-major shuffled mapping. During each step, process pairs compare-exchange their elements. Arrows indicate the pairs of processes that perform compare-exchange operations.

Block of Elements Per Processor

- Each process is assigned a block of n/p elements.
- The first step is a local sort of the local block.
- Each subsequent compare-exchange operation is replaced by a compare-split operation.
- We can effectively view the bitonic network as having $(1 + \log p)(\log p)/2$ steps.

Block of Elements Per Processor: Hypercube

- Initially the processes sort their n/p elements (using merge sort) in time $\Theta((n/p)\log(n/p))$ and then perform $\Theta(\log^2 p)$ compare-split steps.

- The parallel run time of this formulation is

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(\frac{n}{p}\log^2 p\right)}^{\text{comparisons}} + \overbrace{\Theta\left(\frac{n}{p}\log^2 p\right)}^{\text{communication}}.$$

- Comparing to an optimal sort, the algorithm can efficiently use up to $p = \Theta(2^{\sqrt{\log n}})$ processes.

- The isoefficiency function due to both communication and extra work is $\Theta(p^{\log p}\log^2 p)$.

Block of Elements Per Processor: Mesh

- The parallel runtime in this case is given by:

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(\frac{n}{p}\log^2 p\right)}^{\text{comparisons}} + \overbrace{\Theta\left(\frac{n}{\sqrt{p}}\right)}^{\text{communication}}$$

- This formulation can efficiently use up to $p = \Theta(\log^2 n)$ processes.

- The isoefficiency function is $\Theta(2^{\sqrt{p}}\sqrt{p})$.

Performance of Parallel Bitonic Sort

The performance of parallel formulations of bitonic sort for n elements on p processes.

Architecture	Maximum Number of Processes for $E = \Theta(1)$	Corresponding Parallel Run Time	Isoefficiency Function
Hypercube	$\Theta(2^{\sqrt{\log n}})$	$\Theta(n/(2^{\sqrt{\log n}}) \log n)$	$\Theta(p^{\log p} \log^2 p)$
Mesh	$\Theta(\log^2 n)$	$\Theta(n/\log n)$	$\Theta(2^{\sqrt{p}}\sqrt{p})$
Ring	$\Theta(\log n)$	$\Theta(n)$	$\Theta(2^p p)$

Bubble Sort and its Variants

The sequential bubble sort algorithm compares and exchanges adjacent elements in the sequence to be sorted:

```
1.    procedure BUBBLE_SORT(n)
2.    begin
3.        for i := n − 1 downto 1 do
4.            for j := 1 to i do
5.                compare-exchange(a_j, a_{j+1});
6.    end BUBBLE_SORT
```

Sequential bubble sort algorithm.

Bubble Sort and its Variants

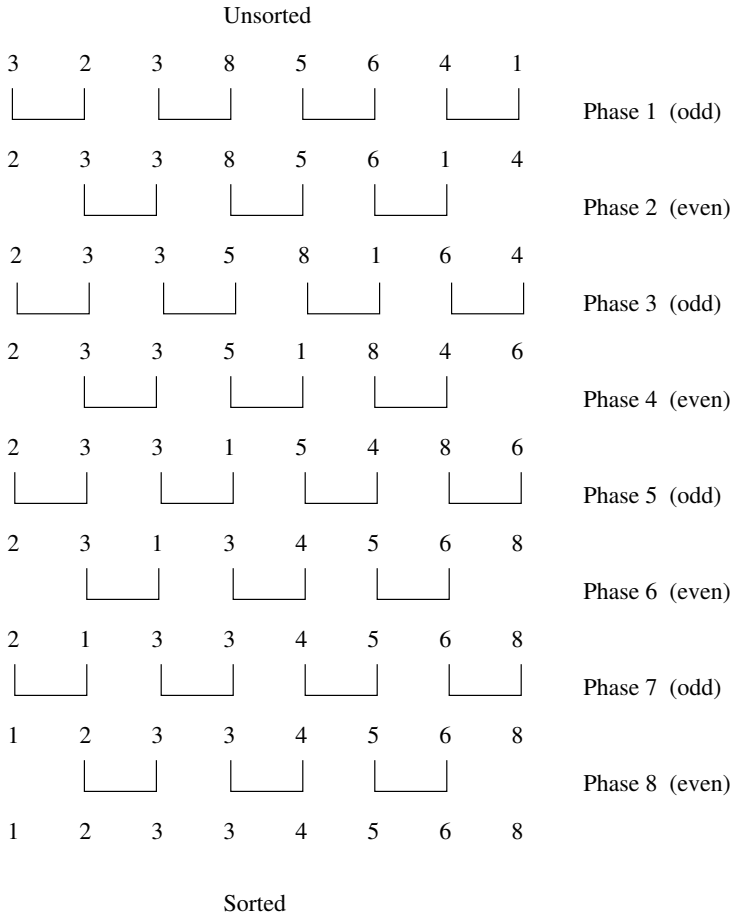
- The complexity of bubble sort is $\Theta(n^2)$.
- Bubble sort is difficult to parallelize since the algorithm has no concurrency.
- A simple variant, though, uncovers the concurrency.

Odd-Even Transposition

```
1.    procedure ODD-EVEN(n)
2.    begin
3.        for i := 1 to n do
4.        begin
5.            if i is odd then
6.                for j := 0 to n/2 − 1 do
7.                    compare-exchange(a_{2j+1}, a_{2j+2});
8.            if i is even then
9.                for j := 1 to n/2 − 1 do
10.                   compare-exchange(a_{2j}, a_{2j+1});
11.       end for
12.   end ODD-EVEN
```

Sequential odd-even transposition sort algorithm.

Odd-Even Transposition



Sorting $n = 8$ elements, using the odd-even transposition sort algorithm. During each phase, $n = 8$ elements are compared.

Odd-Even Transposition

- After n phases of odd-even exchanges, the sequence is sorted.
- Each phase of the algorithm (either odd or even) requires $\Theta(n)$ comparisons.
- Serial complexity is $\Theta(n^2)$.

Parallel Odd-Even Transposition

- Consider the one item per processor case.
- There are n iterations, in each iteration, each processor does one compare-exchange.
- The parallel run time of this formulation is $\Theta(n)$.
- This is cost optimal with respect to the base serial algorithm but not the optimal one.

Parallel Odd-Even Transposition

```
1.    procedure ODD-EVEN_PAR(n)
2.    begin
3.        id := process's label
4.        for i := 1 to n do
5.        begin
6.            if i is odd then
7.                if id is odd then
8.                    compare-exchange_min(id + 1);
9.                else
10.                   compare-exchange_max(id - 1);
11.           if i is even then
12.               if id is even then
13.                   compare-exchange_min(id + 1);
14.               else
15.                   compare-exchange_max(id - 1);
16.       end for
17.   end ODD-EVEN_PAR
```

Parallel formulation of odd-even transposition.

Parallel Odd-Even Transposition

- Consider a block of n/p elements per processor.
- The first step is a local sort.
- In each subsequent step, the compare exchange operation is replaced by the compare split operation.
- The parallel run time of the formulation is

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta(n)}^{\text{comparisons}} + \overbrace{\Theta(n).}^{\text{communication}}$$

Parallel Odd-Even Transposition

- The parallel formulation is cost-optimal for $p = O(\log n)$.
- The isoefficiency function of this parallel formulation is $\Theta(p\,2^p)$.

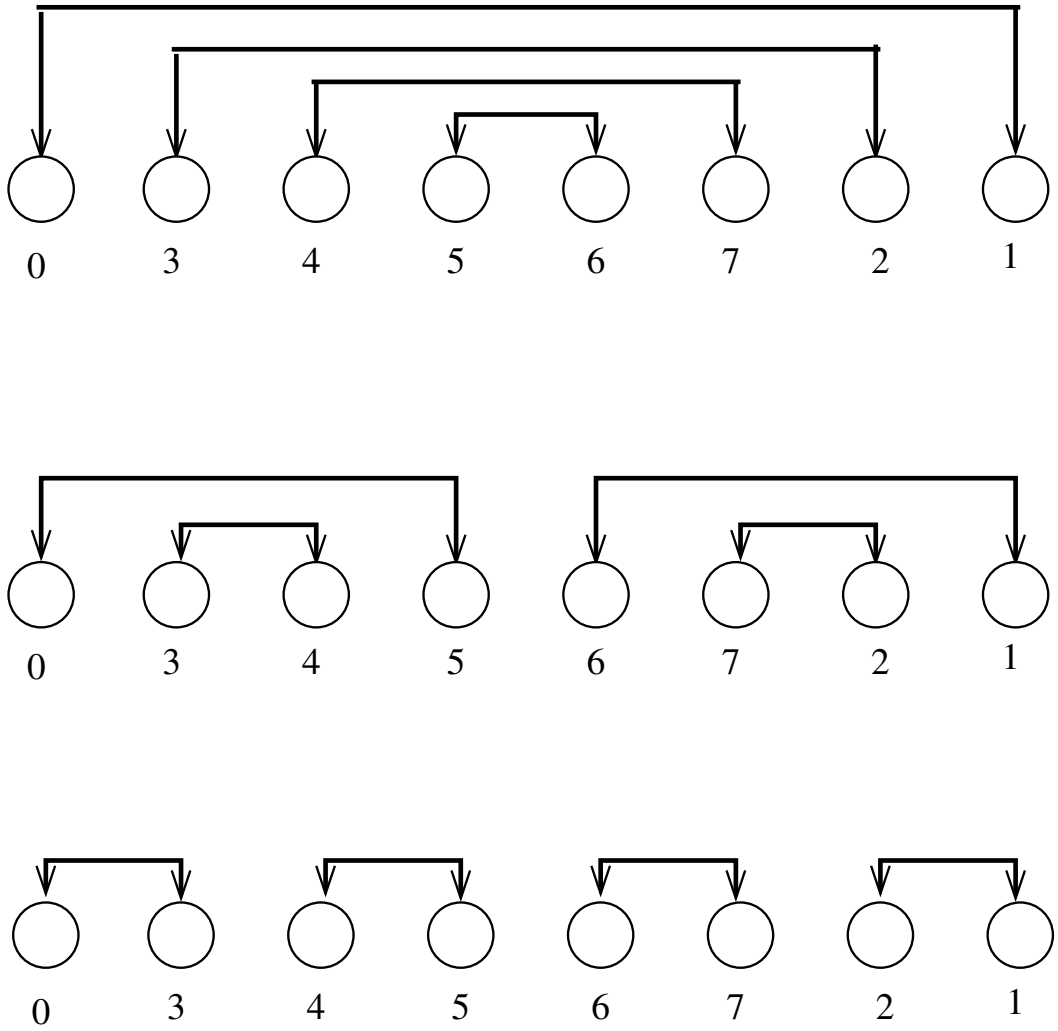
Shellsort

- Let n be the number of elements to be sorted and p be the number of processes.
- During the first phase, processes that are far away from each other in the array compare-split their elements.
- During the second phase, the algorithm switches to an odd-even transposition sort.

Parallel Shellsort

- Initially, each process sorts its block of n/p elements internally.
- Each process is now paired with its corresponding process in the reverse order of the array. That is, process P_i, where $i < p/2$, is paired with process P_{p-i-1}.
- A compare-split operation is performed.
- The processes are split into two groups of size $p/2$ each and the process repeated in each group.

Parallel Shellsort



An example of the first phase of parallel shellsort on an eight-process array.

Parallel Shellsort

- Each process performs $d = \log p$ compare-split operations.
- With $O(p)$ bisection width, the each communication can be performed in time $\Theta(n/p)$ for a total time of $\Theta((n \log p)/p)$.
- In the second phase, l odd and even phases are performed, each requiring time $\Theta(n/p)$.
- The parallel run time of the algorithm is:

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(\frac{n}{p}\log p\right)}^{\text{first phase}} + \overbrace{\Theta\left(l\frac{n}{p}\right)}^{\text{second phase}}. \quad (3)$$

Bucket and Sample Sort

- In Bucket sort, the range $[a, b]$ of input numbers is divided into m equal sized intervals, called buckets.
- Each element is placed in its appropriate bucket.
- If the numbers are uniformly divided in the range, the buckets can be expected to have roughly identical number of elements.
- Elements in the buckets are locally sorted.
- The run time of this algorithm is $\Theta(n \log(n/m))$.

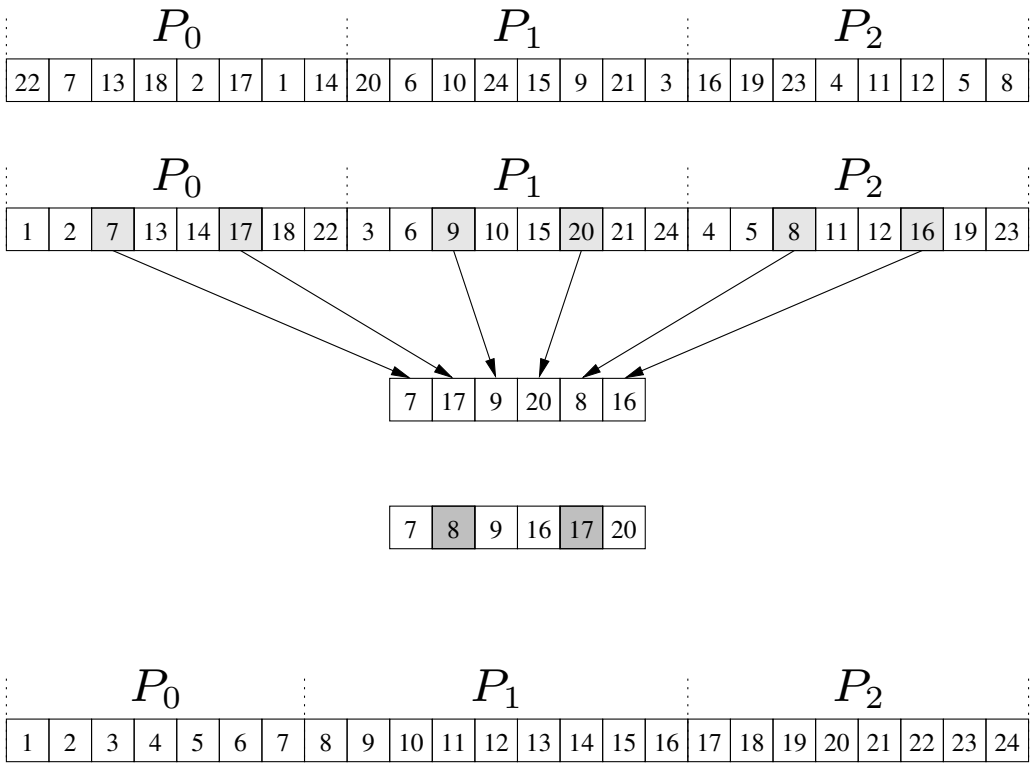
Parallel Bucket Sort

- Parallelizing bucket sort is relatively simple. We can select $m = p$.
- In this case, each processor has a range of values it is responsible for.
- Each processor runs through its local list and assigns each of its elements to the appropriate processor.
- The elements are sent to the destination processors using a single all-to-all personalized communication.
- Each processor sorts all the elements it receives.

Parallel Bucket and Sample Sort

- The critical aspect of the above algorithm is one of assigning ranges to processors. This is done by suitable splitter selection.
- The splitter selection method divides the n elements into m blocks of size n/m each, and sorts each block by using quicksort.
- From each sorted block it chooses $m - 1$ evenly spaced elements.
- The $m(m - 1)$ elements selected from all the blocks represent the sample used to determine the buckets.
- This scheme guarantees that the number of elements ending up in each bucket is less than $2n/m$.

Parallel Bucket and Sample Sort



An example of the execution of sample sort on an array with 24 elements on three processes.

Parallel Bucket and Sample Sort

- The splitter selection scheme can itself be parallelized.
- Each processor generates the $p-1$ local splitters in parallel.
- All processors share their splitters using a single all-to-all broadcast operation.
- Each processor sorts the $p(p-1)$ elements it receives and selects $p-1$ uniformly spaces splitters from them.

Parallel Bucket and Sample Sort: Analysis

- The internal sort of n/p elements requires time $\Theta((n/p)\log(n/p))$, and the selection of $p-1$ sample elements requires time $\Theta(p)$.

- The time for an all-to-all broadcast is $\Theta(p^2)$, the time to internally sort the $p(p-1)$ sample elements is $\Theta(p^2 \log p)$, and selecting $p-1$ evenly spaced splitters takes time $\Theta(p)$.

- Each process can *insert* these $p-1$ splitters in its local sorted block of size n/p by performing $p-1$ binary searches in time $\Theta(p\log(n/p))$.

- The time for reorganization of the elements is $O(n/p)$.

Parallel Bucket and Sample Sort: Analysis

- The total time is given by:

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(p^2\log p\right)}^{\text{sort sample}} + \overbrace{\Theta\left(p\log\frac{n}{p}\right)}^{\text{block partition}} + \overbrace{\Theta(n/p).}^{\text{communication}} \qquad (5)$$

- The isoefficiency of the formulation is $\Theta(p^3 \log p)$.