L_z^2 commutes with the operator defining the PDE (clearly so, because the PDE operator does not contain φ). In other words, because L_z^2 and the PDE operator commute, they will have simultaneous eigenfunctions, and the overall solutions of the PDE can be labeled to identify the L_z^2 eigenfunction that was chosen.

Looking now at the situation in spherical polar coordinates, we note that if k^2 is independent of the angles, i.e., $k^2 = k^2(r)$, then our PDE always has the same angular solutions $\Theta_{lm}(\theta)\Phi_m(\varphi)$. Looking further at the angular terms of our PDE, we can identify them as the operator L^2, and we see that the angular solutions we have found are eigenfunctions of this operator. When the PDE operator is independent of the angles, it will commute with L^2 and the solutions to the PDE can be labeled accordingly. These symmetry features are very important and are discussed in great detail in Chapter 16.

Exercises

9.4.1 By letting the operator $\nabla^2 + k^2$ act on the general form $a_1\psi_1(x, y, z) + a_2\psi_2(x, y, z)$, show that it is linear, i.e., that $(\nabla^2 + k^2)(a_1\psi_1 + a_2\psi_2) = a_1(\nabla^2 + k^2)\psi_1 + a_2(\nabla^2 + k^2)\psi_2$.

9.4.2 Show that the Helmholtz equation,

$$\nabla^2\psi + k^2\psi = 0,$$

is still separable in circular cylindrical coordinates if k^2 is generalized to $k^2 + f(\rho) + (1/\rho^2)g(\varphi) + h(z)$.

9.4.3 Separate variables in the Helmholtz equation in spherical polar coordinates, splitting off the radial dependence **first**. Show that your separated equations have the same form as Eqs. (9.74), (9.77), and (9.78).

9.4.4 Verify that

$$\nabla^2\psi(r,\theta,\varphi) + \left[k^2 + f(r) + \frac{1}{r^2}g(\theta) + \frac{1}{r^2\sin^2\theta}h(\varphi)\right]\psi(r,\theta,\varphi) = 0$$

is separable (in spherical polar coordinates). The functions f, g, and h are functions only of the variables indicated; k^2 is a constant.

9.4.5 An atomic (quantum mechanical) particle is confined inside a rectangular box of sides a, b, and c. The particle is described by a wave function ψ that satisfies the Schrödinger wave equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi.$$

The wave function is required to vanish at each surface of the box (but not to be identically zero). This condition imposes constraints on the separation constants and therefore on the energy E. What is the smallest value of E for which such a solution can be obtained?

ANS. $E = \dfrac{\pi^2\hbar^2}{2m}\left(\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}\right).$

9.4.6 The quantum mechanical angular momentum operator is given by $\mathbf{L} = -i(\mathbf{r} \times \nabla)$. Show that

$$\mathbf{L} \cdot \mathbf{L}\psi = l(l+1)\psi$$

leads to the associated Legendre equation.
Hint. Section 8.3 and Exercise 8.3.1 may be helpful.

9.4.7 The 1-D Schrödinger wave equation for a particle in a potential field $V = \frac{1}{2}kx^2$ is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\psi = E\psi(x).$$

(a) Defining

$$a = \left(\frac{mk}{\hbar^2}\right)^{1/4}, \quad \lambda = \frac{2E}{\hbar}\left(\frac{m}{k}\right)^{1/2},$$

and setting $\xi = ax$, show that

$$\frac{d^2\psi(\xi)}{d\xi^2} + (\lambda - \xi^2)\psi(\xi) = 0.$$

(b) Substituting

$$\psi(\xi) = y(\xi)e^{-\xi^2/2},$$

show that $y(\xi)$ satisfies the Hermite differential equation.

9.5 LAPLACE AND POISSON EQUATIONS

The Laplace equation can be considered the prototypical elliptic PDE. At this point we supplement the discussion motivated by the method of separation of variables with some additional observations. The importance of Laplace's equation for electrostatics has stimulated the development of a great variety of methods for its solution in the presence of boundary conditions ranging from simple and symmetrical to complicated and convoluted. Techniques for present-day engineering problems tend to rely heavily on computational methods. The thrust of this section, however, will be on general properties of the Laplace equation and its solutions.

The basic properties of the Laplace equation are independent of the coordinate system in which it is expressed; we assume for the moment that we will use Cartesian coordinates. Then, because the PDE sets the sum of the second derivatives, $\partial^2\psi/\partial x_i^2$, to zero, it is obvious that if any of the second derivatives has a positive sign, at least one of the others must be negative. This point is illustrated in Example 9.4.1, where the x and y dependence of a solution to the Laplace equation was sinusoidal, and as a result, the z dependence was exponential (corresponding to different signs for the second derivative). Since the second derivative is a measure of curvature, we conclude that if ψ has positive curvature in any coordinate direction, it must have negative curvature in some other coordinate direction. That observation, in turn, means that all the **stationary points** of ψ (points where its first derivatives in all directions vanish) must be **saddle points**, not maxima or minima.