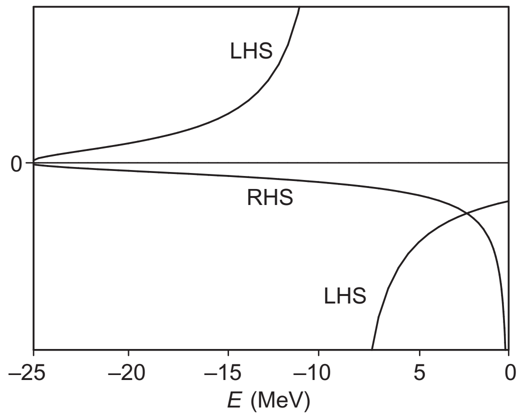

FIGURE 8.3 Left- and right-hand sides of Eq. (8.32) as a function of E for the model parameters given in the text.

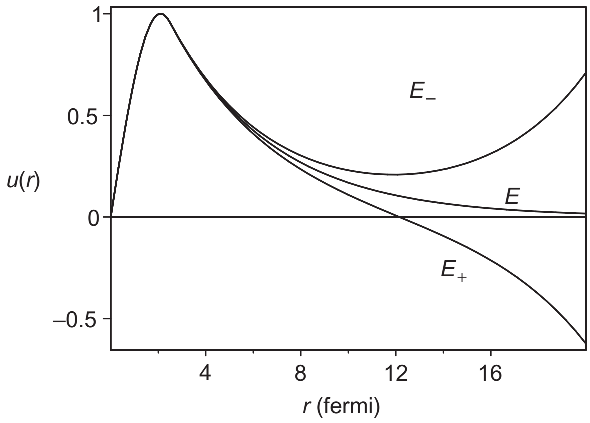

FIGURE 8.4 Wavefunctions for the deuteron problem when the energy is chosen to be less than the eigenvalue E ($E_- < E$) or greater than E ($E_+ > E$).

Exercises

8.3.1 Solve the Legendre equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

by direct series substitution.

(a) Verify that the indicial equation is

$$s(s-1) = 0.$$

(b) Using $s = 0$ and setting the coefficient $a_1 = 0$, obtain a series of even powers of x:

$$y_{\text{even}} = a_0 \left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 + \cdots \right],$$

where

$$a_{j+2} = \frac{j(j+1) - n(n+1)}{(j+1)(j+2)} a_j.$$

(c) Using $s = 1$ and noting that the coefficient a_1 must be zero, develop a series of odd powers of x:

$$y_{\text{odd}} = a_0 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 + \cdots \right],$$

where

$$a_{j+2} = \frac{(j+1)(j+2) - n(n+1)}{(j+2)(j+3)} a_j.$$

(d) Show that both solutions, y_{even} and y_{odd}, diverge for $x = \pm 1$ **if the series continue to infinity**. (Compare with Exercise 1.2.5.)

(e) Finally, show that by an appropriate choice of n, one series at a time may be converted into a polynomial, thereby avoiding the divergence catastrophe. In quantum mechanics this restriction of n to integral values corresponds to **quantization of angular momentum**.

8.3.2 Show that with the weight factor $\exp(-x^2)$ and the interval $-\infty < x < \infty$ for the scalar product, the Hermite ODE eigenvalue problem is Hermitian.

8.3.3 (a) Develop series solutions for Hermite's differential equation

$$y'' - 2xy' + 2\alpha y = 0.$$

ANS. $s(s-1) = 0$, indicial equation.

For $s = 0$,

$$a_{j+2} = 2a_j \frac{j - \alpha}{(j+1)(j+2)} \quad (j \text{ even}),$$

$$y_{\text{even}} = a_0 \left[1 + \frac{2(-\alpha)x^2}{2!} + \frac{2^2(-\alpha)(2-\alpha)x^4}{4!} + \cdots \right].$$

For $s = 1$,

$$a_{j+2} = 2a_j \frac{j + 1 - \alpha}{(j+2)(j+3)} \quad (j \text{ even}),$$

$$y_{\text{odd}} = a_1 \left[x + \frac{2(1-\alpha)x^3}{3!} + \frac{2^2(1-\alpha)(3-\alpha)x^5}{5!} + \cdots \right].$$

(b) Show that both series solutions are convergent for all x, the ratio of successive coefficients behaving, for a large index, like the corresponding ratio in the expansion of $\exp(x^2)$.

(c) Show that by appropriate choice of α, the series solutions may be cut off and converted to finite polynomials. (These polynomials, properly normalized, become the Hermite polynomials in Section 18.1.)

8.3.4 Laguerre's ODE is

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0.$$

Develop a series solution and select the parameter n to make your series a polynomial.

8.3.5 Solve the Chebyshev equation

$$(1-x^2)T_n'' - xT_n' + n^2 T_n = 0,$$

by series substitution. What restrictions are imposed on n if you demand that the series solution converge for $x = \pm 1$?

ANS. The infinite series does converge for $x = \pm 1$ and no restriction on n exists (compare with Exercise 1.2.6).

8.3.6 Solve

$$(1-x^2)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) = 0,$$

choosing the root of the indicial equation to obtain a series of **odd** powers of x. Since the series will diverge for $x = 1$, choose n to convert it into a polynomial.

8.4 Variation Method

We saw in Chapter 6 that the expectation value of a Hermitian operator H for the normalized function ψ can be written as

$$\langle H \rangle \equiv \langle \psi | H | \psi \rangle,$$

and that the expansion of this quantity in a basis consisting of the orthonormal eigenfunctions of H had the form given in Eq. (6.30):

$$\langle H \rangle = \sum_{\mu} |a_\mu|^2 \lambda_\mu,$$

where a_μ is the coefficient of the μth eigenfunction of H and λ_i is the corresponding eigenvalue. As we noted when we obtained this result, one of its consequences is that $\langle H \rangle$ is a weighted average of the eigenvalues of H, and therefore is at least as large as the smallest eigenvalue, and equal to the smallest eigenvalue only if ψ is actually an eigenfunction to which that eigenvalue corresponds.

The observations of the foregoing paragraph hold true even if we do not actually make an expansion of ψ and even if we do not actually know or have available the eigenfunctions or eigenvalues of H. The knowledge that $\langle H \rangle$ is an upper limit to the smallest eigenvalue of H is sufficient to enable us to devise a method for approximating that eigenvalue and the associated eigenfunction. This eigenfunction will be the member of the Hilbert space of our problem that yields the smallest expectation value of H, and a strategy for finding it is to search for the minimum in $\langle H \rangle$ within our Hilbert space. This is the essential idea