Example 8.2.1 LAGUERRE FUNCTIONS

Consider the eigenvalue problem $\mathcal{L}\psi = \lambda\psi$, with

$$\mathcal{L} = x\frac{d^2}{dx^2} + (1-x)\frac{d}{dx}, \tag{8.21}$$

subject to (a) ψ nonsingular on $0 \le x < \infty$, and (b) $\lim_{x\to\infty}\psi(x) = 0$. Condition (a) is simply a requirement that we use the solution of the differential equation that is regular at $x = 0$; and condition (b) is a typical Dirichlet boundary condition.

The operator $\mathcal{L}$ is not self-adjoint, with $p_0 = x$ and $p_1 = 1 - x$. But we can form

$$w(x) = \frac{1}{x}\exp\left(\int \frac{1-x}{x}dx\right) = \frac{1}{x}e^{\ln x - x} = e^{-x}. \tag{8.22}$$

The boundary terms, for arbitrary eigenfunctions u and v, are of the form

$$\left[xe^{-x}\left(v^*u' - (v^*)'u\right)\right]_0^\infty;$$

their contributions at $x = \infty$ vanish because u and v go to zero; the common factor x causes the $x = 0$ contribution to vanish also. We therefore have a self-adjoint problem, with u and v of different eigenvalues orthogonal under the definition

$$\langle v|u\rangle = \int_0^\infty v^*(x)u(x)e^{-x}dx.$$

The eigenvalue equation of this example is that whose solutions are the Laguerre polynomials; what we have shown here is that they are orthogonal on $(0,\infty)$ with weight e^{-x}. ■

Exercises

8.2.1 Show that Laguerre's ODE, Table 7.1, may be put into self-adjoint form by multiplying by e^{-x} and that $w(x) = e^{-x}$ is the weighting function.

8.2.2 Show that the Hermite ODE, Table 7.1, may be put into self-adjoint form by multiplying by e^{-x^2} and that this gives $w(x) = e^{-x^2}$ as the appropriate weighting function.

8.2.3 Show that the Chebyshev ODE, Table 7.1, may be put into self-adjoint form by multiplying by $(1 - x^2)^{-1/2}$ and that this gives $w(x) = (1 - x^2)^{-1/2}$ as the appropriate weighting function.

8.2.4 The Legendre, Chebyshev, Hermite, and Laguerre equations, given in Table 7.1, have solutions that are polynomials. Show that ranges of integration that guarantee that the Hermitian operator boundary conditions will be satisfied are

(a) Legendre $[-1, 1]$, (b) Chebyshev $[-1, 1]$,
(c) Hermite $(-\infty, \infty)$, (d) Laguerre $[0, \infty)$.

8.2.5 The functions $u_1(x)$ and $u_2(x)$ are eigenfunctions of the same Hermitian operator but for distinct eigenvalues λ_1 and λ_2. Prove that $u_1(x)$ and $u_2(x)$ are linearly independent.

8.2.6 Given that

$$P_1(x) = x \quad \text{and} \quad Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

are solutions of Legendre's differential equation (Table 7.1) corresponding to different eigenvalues:

(a) Evaluate their orthogonality integral

$$\int_{-1}^{1} \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)dx.$$

(b) Explain why these two functions are not orthogonal, that is, why the proof of orthogonality does not apply.

8.2.7 $T_0(x) = 1$ and $V_1(x) = (1-x^2)^{1/2}$ are solutions of the Chebyshev differential equation corresponding to different eigenvalues. Explain, in terms of the boundary conditions, why these two functions are not orthogonal on the range $(-1, 1)$ with the weighting function found in Exercise 8.2.3.

8.2.8 A set of functions $u_n(x)$ satisfies the Sturm-Liouville equation

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}u_n(x)\right] + \lambda_n w(x) u_n(x) = 0.$$

The functions $u_m(x)$ and $u_n(x)$ satisfy boundary conditions that lead to orthogonality. The corresponding eigenvalues λ_m and λ_n are distinct. Prove that for appropriate boundary conditions, $u'_m(x)$ and $u'_n(x)$ are orthogonal with $p(x)$ as a weighting function.

8.2.9 Linear operator A has n distinct eigenvalues and n corresponding eigenfunctions: $A\psi_i = \lambda_i\psi_i$. Show that the n eigenfunctions are linearly independent. Do not assume A to be Hermitian.
Hint. Assume linear dependence, i.e., that $\psi_n = \sum_{i=1}^{n-1} a_i\psi_i$. Use this relation and the operator-eigenfunction equation first in one order and then in the reverse order. Show that a contradiction results.

8.2.10 The ultraspherical polynomials $C_n^{(\alpha)}(x)$ are solutions of the differential equation

$$\left\{(1-x^2)\frac{d^2}{dx^2} - (2\alpha+1)x\frac{d}{dx} + n(n+2\alpha)\right\} C_n^{(\alpha)}(x) = 0.$$

(a) Transform this differential equation into self-adjoint form.

(b) Find an interval of integration and weighting factor that make $C_n^{(\alpha)}(x)$ of the same α but different n orthogonal.

Note. Assume that your solutions are polynomials.