Exercises

7.6.1 You know that the three unit vectors $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$ are mutually perpendicular (orthogonal). Show that $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$ are linearly independent. Specifically, show that no relation of the form of Eq. (7.54) exists for $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$.

7.6.2 The criterion for the linear **independence** of three vectors **A, B**, and **C** is that the equation

$$a\mathbf{A} + b\mathbf{B} + c\mathbf{C} = 0,$$

analogous to Eq. (7.54), has no solution other than the trivial $a = b = c = 0$. Using components $\mathbf{A} = (A_1, A_2, A_3)$, and so on, set up the determinant criterion for the existence or nonexistence of a nontrivial solution for the coefficients a, b, and c. Show that your criterion is equivalent to the scalar triple product $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} \neq 0$.

7.6.3 Using the Wronskian determinant, show that the set of functions

$$\left\{1, \frac{x^n}{n!}(n = 1, 2, \ldots, N)\right\}$$

is linearly independent.

7.6.4 If the Wronskian of two functions y_1 and y_2 is identically zero, show by direct integration that

$$y_1 = cy_2,$$

that is, that y_1 and y_2 are linearly dependent. Assume the functions have continuous derivatives and that at least one of the functions does not vanish in the interval under consideration.

7.6.5 The Wronskian of two functions is found to be zero at $x_0 - \varepsilon \leq x \leq x_0 + \varepsilon$ for arbitrarily small $\varepsilon > 0$. Show that this Wronskian vanishes for all x and that the functions are linearly dependent.

7.6.6 The three functions $\sin x$, e^x, and e^{-x} are linearly independent. No one function can be written as a linear combination of the other two. Show that the Wronskian of $\sin x$, e^x, and e^{-x} vanishes but only at isolated points.

ANS. $W = 4\sin x$,
$W = 0$ for $x = \pm n\pi$, $\quad n = 0, 1, 2, \ldots$.

7.6.7 Consider two functions $\varphi_1 = x$ and $\varphi_2 = |x|$. Since $\varphi_1' = 1$ and $\varphi_2' = x/|x|$, $W(\varphi_1, \varphi_2) = 0$ for any interval, including $[-1, +1]$. Does the vanishing of the Wronskian over $[-1, +1]$ prove that φ_1 and φ_2 are linearly dependent? Clearly, they are not. What is wrong?

7.6.8 Explain that **linear independence** does not mean the absence of any dependence. Illustrate your argument with $\cosh x$ and e^x.

7.6.9 Legendre's differential equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

has a regular solution $P_n(x)$ and an irregular solution $Q_n(x)$. Show that the Wronskian of P_n and Q_n is given by

$$P_n(x)Q'_n(x) - P'_n(x)Q_n(x) = \frac{A_n}{1-x^2},$$

with A_n **independent** of x.

7.6.10 Show, by means of the Wronskian, that a linear, second-order, homogeneous ODE of the form

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$

cannot have three independent solutions.

Hint. Assume a third solution and show that the Wronskian vanishes for all x.

7.6.11 Show the following when the linear second-order differential equation $py'' + qy' + ry = 0$ is expressed in self-adjoint form:

(a) The Wronskian is equal to a constant divided by p:

$$W(x) = \frac{C}{p(x)}.$$

(b) A second solution $y_2(x)$ is obtained from a first solution $y_1(x)$ as

$$y_2(x) = Cy_1(x)\int^x \frac{dt}{p(t)[y_1(t)]^2}.$$

7.6.12 Transform our linear, second-order ODE

$$y'' + P(x)y' + Q(x)y = 0$$

by the substitution

$$y = z\exp\left[-\frac{1}{2}\int^x P(t)dt\right]$$

and show that the resulting differential equation for z is

$$z'' + q(x)z = 0,$$

where

$$q(x) = Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}P^2(x).$$

Note. This substitution can be derived by the technique of Exercise 7.6.25.

7.6.13 Use the result of Exercise 7.6.12 to show that the replacement of $\varphi(r)$ by $r\varphi(r)$ may be expected to eliminate the first derivative from the Laplacian in spherical polar coordinates. See also Exercise 3.10.34.

7.6.14 By direct differentiation and substitution show that

$$y_2(x) = y_1(x) \int^x \frac{\exp[-\int^s P(t)dt]}{[y_1(s)]^2} ds$$

satisfies, like $y_1(x)$, the ODE

$$y_2''(x) + P(x)y_2'(x) + Q(x)y_2(x) = 0.$$

Note. The Leibniz formula for the derivative of an integral is

$$\frac{d}{d\alpha} \int_{g(\alpha)}^{h(\alpha)} f(x,\alpha)dx = \int_{g(\alpha)}^{h(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha} dx + f[h(\alpha),\alpha] \frac{dh(\alpha)}{d\alpha} - f[g(\alpha),\alpha] \frac{dg(\alpha)}{d\alpha}.$$

7.6.15 In the equation

$$y_2(x) = y_1(x) \int^x \frac{\exp[-\int^s P(t)dt]}{[y_1(s)]^2} ds,$$

$y_1(x)$ satisfies

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0.$$

The function $y_2(x)$ is a linearly **independent** second solution of the same equation. Show that the inclusion of lower limits on the two integrals leads to nothing new, that is, that it generates only an overall constant factor and a constant multiple of the known solution $y_1(x)$.

7.6.16 Given that one solution of

$$R'' + \frac{1}{r}R' - \frac{m^2}{r^2}R = 0$$

is $R = r^m$, show that Eq. (7.67) predicts a second solution, $R = r^{-m}$.

7.6.17 Using

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

as a solution of the linear oscillator equation, follow the analysis that proceeds through Eq. (7.81) and show that in that equation $c_n = 0$, so that in this case the second solution does not contain a logarithmic term.

7.6.18 Show that when n is **not** an integer in Bessel's ODE, Eq. (7.40), the second solution of Bessel's equation, obtained from Eq. (7.67), does **not** contain a logarithmic term.

7.6.19 (a) One solution of Hermite's differential equation

$$y'' - 2xy' + 2\alpha y = 0$$

for $\alpha = 0$ is $y_1(x) = 1$. Find a second solution, $y_2(x)$, using Eq. (7.67). Show that your second solution is equivalent to y_{odd} (Exercise 8.3.3).

(b) Find a second solution for $\alpha = 1$, where $y_1(x) = x$, using Eq. (7.67). Show that your second solution is equivalent to y_{even} (Exercise 8.3.3).

7.6.20 One solution of Laguerre's differential equation

$$xy'' + (1 - x)y' + ny = 0$$

for $n = 0$ is $y_1(x) = 1$. Using Eq. (7.67), develop a second, linearly independent solution. Exhibit the logarithmic term explicitly.

7.6.21 For Laguerre's equation with $n = 0$,

$$y_2(x) = \int^x \frac{e^s}{s} ds.$$

(a) Write $y_2(x)$ as a logarithm plus a power series.

(b) Verify that the integral form of $y_2(x)$, previously given, is a solution of Laguerre's equation $(n = 0)$ by direct differentiation of the integral and substitution into the differential equation.

(c) Verify that the series form of $y_2(x)$, part (a), is a solution by differentiating the series and substituting back into Laguerre's equation.

7.6.22 One solution of the Chebyshev equation

$$(1 - x^2)y'' - xy' + n^2 y = 0$$

for $n = 0$ is $y_1 = 1$.

(a) Using Eq. (7.67), develop a second, linearly independent solution.

(b) Find a second solution by direct integration of the Chebyshev equation.

Hint. Let $v = y'$ and integrate. Compare your result with the second solution given in Section 18.4.

ANS. (a) $y_2 = \sin^{-1} x$.
(b) The second solution, $V_n(x)$, is not defined for $n = 0$.

7.6.23 One solution of the Chebyshev equation

$$(1 - x^2)y'' - xy' + n^2 y = 0$$

for $n = 1$ is $y_1(x) = x$. Set up the Wronskian double integral solution and derive a second solution, $y_2(x)$.

ANS. $y_2 = -(1 - x^2)^{1/2}$.

7.6.24 The radial Schrödinger wave equation for a spherically symmetric potential can be written in the form

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+l(l+1)\frac{\hbar^2}{2mr^2}+V(r)\right]y(r)=Ey(r).$$

The potential energy $V(r)$ may be expanded about the origin as

$$V(r)=\frac{b_{-1}}{r}+b_0+b_1r+\cdots.$$

(a) Show that there is one (regular) solution $y_1(r)$ starting with r^{l+1}.

(b) From Eq. (7.69) show that the irregular solution $y_2(r)$ diverges at the origin as r^{-l}.

7.6.25 Show that if a second solution, y_2, is assumed to be related to the first solution, y_1, according to $y_2(x)=y_1(x)f(x)$, substitution back into the original equation

$$y_2''+P(x)y_2'+Q(x)y_2=0$$

leads to

$$f(x)=\int^x\frac{\exp[-\int^s P(t)dt]}{[y_1(s)]^2}ds,$$

in agreement with Eq. (7.67).

7.6.26 (a) Show that

$$y''+\frac{1-\alpha^2}{4x^2}y=0$$

has two solutions:

$$y_1(x)=a_0x^{(1+\alpha)/2},$$

$$y_2(x)=a_0x^{(1-\alpha)/2}.$$

(b) For $\alpha=0$ the two linearly independent solutions of part (a) reduce to the single solution $y_{1'}=a_0x^{1/2}$. Using Eq. (7.68) derive a second solution,

$$y_{2'}(x)=a_0x^{1/2}\ln x.$$

Verify that $y_{2'}$ is indeed a solution.

(c) Show that the second solution from part (b) may be obtained as a limiting case from the two solutions of part (a):

$$y_{2'}(x)=\lim_{\alpha\to 0}\left(\frac{y_1-y_2}{\alpha}\right).$$