Exercises

7.5.1 Uniqueness theorem. The function $y(x)$ satisfies a second-order, linear, homogeneous differential equation. At $x = x_0$, $y(x) = y_0$ and $dy/dx = y_0'$. Show that $y(x)$ is unique, in that no other solution of this differential equation passes through the points (x_0, y_0) with a slope of y_0'.

Hint. Assume a second solution satisfying these conditions and compare the Taylor series expansions.

7.5.2 A series solution of Eq. (7.23) is attempted, expanding about the point $x = x_0$. If x_0 is an ordinary point, show that the indicial equation has roots $s = 0, 1$.

7.5.3 In the development of a series solution of the simple harmonic oscillator (SHO) equation, the second series coefficient a_1 was neglected except to set it equal to zero. From the coefficient of the next-to-the-lowest power of x, x^{s-1}, develop a second-indicial type equation.

(a) (SHO equation with $s = 0$). Show that a_1, may be assigned any finite value (including zero).

(b) (SHO equation with $s = 1$). Show that a_1 must be set equal to zero.

7.5.4 Analyze the series solutions of the following differential equations to see when a_1 **may** be set equal to zero without irrevocably losing anything and when a_1 **must** be set equal to zero.

(a) Legendre, (b) Chebyshev, (c) Bessel, (d) Hermite.

ANS. (a) Legendre, (b) Chebyshev, and (d) Hermite: For $s = 0$, a_1 **may** be set equal to zero; for $s = 1$, a_1 **must** be set equal to zero.
(c) Bessel: a_1 **must** be set equal to zero (except for $s = \pm n = -\frac{1}{2}$).

7.5.5 Obtain a series solution of the hypergeometric equation

$$x(x-1)y'' + [(1+a+b)x - c]y' + aby = 0.$$

Test your solution for convergence.

7.5.6 Obtain two series solutions of the confluent hypergeometric equation

$$xy'' + (c-x)y' - ay = 0.$$

Test your solutions for convergence.

7.5.7 A quantum mechanical analysis of the Stark effect (parabolic coordinates) leads to the differential equation

$$\frac{d}{d\xi}\left(\xi\frac{du}{d\xi}\right) + \left(\frac{1}{2}E\xi + \alpha - \frac{m^2}{4\xi} - \frac{1}{4}F\xi^2\right)u = 0.$$

Here α is a constant, E is the total energy, and F is a constant such that Fz is the potential energy added to the system by the introduction of an electric field.

Using the larger root of the indicial equation, develop a power-series solution about $\xi = 0$. Evaluate the first three coefficients in terms of a_o.

ANS. Indicial equation $s^2 - \frac{m^2}{4} = 0$,

$$u(\xi) = a_0 \xi^{m/2} \left\{ 1 - \frac{\alpha}{m+1}\xi + \left[\frac{\alpha^2}{2(m+1)(m+2)} - \frac{E}{4(m+2)} \right] \xi^2 + \cdots \right\}.$$

Note that the perturbation F does not appear until a_3 is included.

7.5.8 For the special case of no azimuthal dependence, the quantum mechanical analysis of the hydrogen molecular ion leads to the equation

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{du}{d\eta}\right] + \alpha u + \beta \eta^2 u = 0.$$

Develop a power-series solution for $u(\eta)$. Evaluate the first three nonvanishing coefficients in terms of a_0.

ANS. Indicial equation $s(s-1) = 0$,

$$u_{k=1} = a_0 \eta \left\{ 1 + \frac{2-\alpha}{6}\eta^2 + \left[\frac{(2-\alpha)(12-\alpha)}{120} - \frac{\beta}{20} \right] \eta^4 + \cdots \right\}.$$

7.5.9 To a good approximation, the interaction of two nucleons may be described by a mesonic potential

$$V = \frac{Ae^{-ax}}{x},$$

attractive for A negative. Show that the resultant Schrödinger wave equation

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (E - V)\psi = 0$$

has the following series solution through the first three nonvanishing coefficients:

$$\psi = a_0 \left\{ x + \frac{1}{2}A'x^2 + \frac{1}{6}\left[\frac{1}{2}A'^2 - E' - aA' \right] x^3 + \cdots \right\},$$

where the prime indicates multiplication by $2m/\hbar^2$.

7.5.10 If the parameter b^2 in Eq. (7.53) is equal to 2, Eq. (7.53) becomes

$$y'' + \frac{1}{x^2}y' - \frac{2}{x^2}y = 0.$$

From the indicial equation and the recurrence relation, **derive** a solution $y = 1 + 2x + 2x^2$. Verify that this is indeed a solution by substituting back into the differential equation.

7.5.11 The modified Bessel function $I_0(x)$ satisfies the differential equation

$$x^2 \frac{d^2}{dx^2} I_0(x) + x \frac{d}{dx} I_0(x) - x^2 I_0(x) = 0.$$

Given that the leading term in an asymptotic expansion is known to be

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}},$$

assume a series of the form

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left\{ 1 + b_1 x^{-1} + b_2 x^{-2} + \cdots \right\}.$$

Determine the coefficients b_1 and b_2.

ANS. $b_1 = \frac{1}{8}, \quad b_2 = \frac{9}{128}.$

7.5.12 The even power-series solution of Legendre's equation is given by Exercise 8.3.1. Take $a_0 = 1$ and n not an even integer, say $n = 0.5$. Calculate the partial sums of the series through x^{200}, x^{400}, x^{600}, ..., x^{2000} for $x = 0.95(0.01)1.00$. Also, write out the individual term corresponding to each of these powers.

Note. This calculation does **not** constitute proof of convergence at $x = 0.99$ or divergence at $x = 1.00$, but perhaps you can see the difference in the behavior of the sequence of partial sums for these two values of x.

7.5.13 (a) The odd power-series solution of Hermite's equation is given by Exercise 8.3.3. Take $a_0 = 1$. Evaluate this series for $\alpha = 0, x = 1, 2, 3$. Cut off your calculation after the last term calculated has dropped below the maximum term by a factor of 10^6 or more. Set an upper bound to the error made in ignoring the remaining terms in the infinite series.

(b) As a check on the calculation of part (a), show that the Hermite series $y_{\text{odd}}(\alpha = 0)$ corresponds to $\int_0^x \exp(x^2) dx$.

(c) Calculate this integral for $x = 1, 2, 3$.

7.6 OTHER SOLUTIONS

In Section 7.5 a solution of a second-order homogeneous ODE was developed by substituting in a power series. By Fuchs' theorem this is possible, provided the power series is an expansion about an ordinary point or a nonessential singularity.[5] There is no guarantee that this approach will yield the two independent solutions we expect from a linear second-order ODE. In fact, we shall prove that such an ODE has at most two linearly independent solutions. Indeed, the technique gave only one solution for Bessel's equation (n an integer). In this section we also develop two methods of obtaining a second independent solution: an integral method and a power series containing a logarithmic term. First, however, we consider the question of independence of a set of functions.

[5]This is why the classification of singularities in Section 7.4 is of vital importance.