which is a separable ODE for $C(x)$ that integrates to yield

$$C(x) = \int^{x} \exp\left(\int^{X} p(Y)dY\right) q(X)dX \quad \text{and} \quad y = C(x)\exp\left(-\int^{x} p(X)dX\right).$$

This particular solution of the inhomogeneous ODE is in agreement with that called y_2 in Eq. (7.17).

Exercises

7.2.1 From Kirchhoff's law the current I in an RC (resistance-capacitance) circuit (Fig. 7.1) obeys the equation

$$R\frac{dI}{dt} + \frac{1}{C}I = 0.$$

(a) Find $I(t)$.

(b) For a capacitance of 10,000 μF charged to 100 V and discharging through a resistance of 1 MΩ, find the current I for $t = 0$ and for $t = 100$ seconds.

Note. The initial voltage is $I_0 R$ or Q/C, where $Q = \int_0^\infty I(t)dt$.

7.2.2 The Laplace transform of Bessel's equation ($n = 0$) leads to

$$(s^2 + 1)f'(s) + sf(s) = 0.$$

Solve for $f(s)$.

7.2.3 The decay of a population by catastrophic two-body collisions is described by

$$\frac{dN}{dt} = -kN^2.$$

This is a first-order, **nonlinear** differential equation. Derive the solution

$$N(t) = N_0\left(1 + \frac{t}{\tau_0}\right)^{-1},$$

where $\tau_0 = (kN_0)^{-1}$. This implies an infinite population at $t = -\tau_0$.

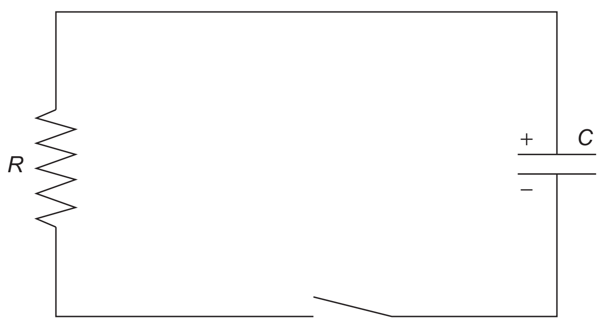

FIGURE 7.1 RC circuit.

7.2.4 The rate of a particular chemical reaction $A + B \to C$ is proportional to the concentrations of the reactants A and B:

$$\frac{dC(t)}{dt} = \alpha[A(0) - C(t)][B(0) - C(t)].$$

(a) Find $C(t)$ for $A(0) \neq B(0)$.

(b) Find $C(t)$ for $A(0) = B(0)$.

The initial condition is that $C(0) = 0$.

7.2.5 A boat, coasting through the water, experiences a resisting force proportional to v^n, v being the boat's instantaneous velocity. Newton's second law leads to

$$m\frac{dv}{dt} = -kv^n.$$

With $v(t = 0) = v_0$, $x(t = 0) = 0$, integrate to find v as a function of time and v as a function of distance.

7.2.6 In the first-order differential equation $dy/dx = f(x, y)$, the function $f(x, y)$ is a function of the ratio y/x:

$$\frac{dy}{dx} = g(y/x).$$

Show that the substitution of $u = y/x$ leads to a separable equation in u and x.

7.2.7 The differential equation

$$P(x, y)dx + Q(x, y)dy = 0$$

is **exact**. Show that its solution is of the form

$$\varphi(x, y) = \int_{x_0}^{x} P(x, y)dx + \int_{y_0}^{y} Q(x_0, y)dy = \text{constant}.$$

7.2.8 The differential equation

$$P(x, y)dx + Q(x, y)dy = 0$$

is **exact**. If

$$\varphi(x, y) = \int_{x_0}^{x} P(x, y)dx + \int_{y_0}^{y} Q(x_0, y)dy,$$

show that

$$\frac{\partial \varphi}{\partial x} = P(x, y), \quad \frac{\partial \varphi}{\partial y} = Q(x, y).$$

Hence, $\varphi(x, y) = \text{constant}$ is a solution of the original differential equation.

7.2.9 Prove that Eq. (7.12) is exact in the sense of Eq. (7.9), provided that $\alpha(x)$ satisfies Eq. (7.14).

7.2.10 A certain differential equation has the form

$$f(x)dx + g(x)h(y)dy = 0,$$

with none of the functions $f(x), g(x), h(y)$ identically zero. Show that a necessary and sufficient condition for this equation to be exact is that $g(x) = \text{constant}$.

7.2.11 Show that

$$y(x) = \exp\left[-\int^{x} p(t)dt\right]\left\{\int^{x} \exp\left[\int^{s} p(t)dt\right] q(s)ds + C\right\}$$

is a solution of

$$\frac{dy}{dx} + p(x)y(x) = q(x)$$

by differentiating the expression for $y(x)$ and substituting into the differential equation.

7.2.12 The motion of a body falling in a resisting medium may be described by

$$m\frac{dv}{dt} = mg - bv$$

when the retarding force is proportional to the velocity, v. Find the velocity. Evaluate the constant of integration by demanding that $v(0) = 0$.

7.2.13 Radioactive nuclei decay according to the law

$$\frac{dN}{dt} = -\lambda N,$$

N being the concentration of a given nuclide and λ, the particular decay constant. In a radioactive series of two different nuclides, with concentrations $N_1(t)$ and $N_2(t)$, we have

$$\frac{dN_1}{dt} = -\lambda_1 N_1,$$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2.$$

Find $N_2(t)$ for the conditions $N_1(0) = N_0$ and $N_2(0) = 0$.

7.2.14 The rate of evaporation from a particular spherical drop of liquid (constant density) is proportional to its surface area. Assuming this to be the sole mechanism of mass loss, find the radius of the drop as a function of time.

7.2.15 In the linear homogeneous differential equation

$$\frac{dv}{dt} = -av$$

the variables are separable. When the variables are separated, the equation is exact. Solve this differential equation subject to $v(0) = v_0$ by the following three methods:

(a) Separating variables and integrating.

(b) Treating the separated variable equation as exact.

(c) Using the result for a linear homogeneous differential equation.

ANS. $v(t) = v_0 e^{-at}$.

7.2.16 (a) Solve Example 7.2.1, assuming that the parachute opens when the parachutist's velocity has reached $v_i = 60$ mi/h (regard this time as $t = 0$). Find $v(t)$.

(b) For a skydiver in free fall use the friction coefficient $b = 0.25$ kg/m and mass $m = 70$ kg. What is the limiting velocity in this case?

7.2.17 Solve the ODE

$$(xy^2 - y)dx + x\,dy = 0.$$

7.2.18 Solve the ODE

$$(x^2 - y^2 e^{y/x})dx + (x^2 + xy)e^{y/x}dy = 0.$$

Hint. Note that the quantity y/x in the exponents is of combined degree zero and does not affect the determination of homogeneity.

7.3 ODEs with Constant Coefficients

Before addressing second-order ODEs, the main topic of this chapter, we discuss a specialized, but frequently occurring class of ODEs that are not constrained to be of specific order, namely those that are linear and whose homogeneous terms have constant coefficients. The generic equation of this type is

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = F(x). \tag{7.18}$$

The homogeneous equation corresponding to Eq. (7.18) has solutions of the form $y = e^{mx}$, where m is a solution of the algebraic equation

$$m^n + a_{n-1}m^{n-1} + \cdots + a_1 m + a_0 = 0,$$

as may be verified by substitution of the assumed form of the solution.

In the case that the m equation has a multiple root, the above prescription will not yield the full set of n linearly independent solutions for the original n th order ODE. If one then considers the limiting process in which two roots approach each other, it is possible to conclude that if e^{mx} is a solution, then so is $d\,e^{mx}/dm = xe^{mx}$. A triple root would have solutions e^{mx}, xe^{mx}, x^2e^{mx}, etc.

Example 7.3.1 Hooke's Law Spring

A mass M attached to a Hooke's Law spring (of spring constant k) is in oscillatory motion. Letting y be the displacement of the mass from its equilibrium position, Newton's law of motion takes the form

$$M\frac{d^2y}{dt^2} = -ky,$$